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A remark on hyperfocal subalgebras of blocks of finite groups

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1 The hyperfocal subalgebra of a block

Let $G$ be a finite group and $P$ be a Sylow $p$-subgroup of $G$. Moreover set $Q = O^p(G) \cap P$, which is called the hyperfocal subgroup in [12]. We have

\[ Q = \langle [O^p(N_G(U)), U] \mid U \leq P \rangle \]

(see [1], Lemma 2.2 for a proof). I thank Koshitani who informed me of [1]. In particular $Q = 1$ if and only if $G$ is $p$-nilpotent. If $P$ is abelian, then $Q = [N_G(P), P]$.

Let $(K, \mathcal{O}, k)$ be a sufficiently large $p$-modular system such that $k$ is algebraically closed. Let $G$ be a finite group and $b$ be a block of $OG$ and let $P_\gamma$ be a defect pointed group of a pointed group $G(b)$ on $OG$, that is, $P_\gamma$ is a maximal local pointed group contained in $G(b)$. Let

\[ Q = \langle [O^p(N_G(U_\delta)), U] \mid U_\delta \in \mathcal{S}_P(P_\gamma) \rangle, \]

where $\mathcal{S}_P(P_\gamma)$ is the set of local pointed groups on $OG$ contained in $P_\gamma$. Following [12], $Q$ is called the hyperfocal subgroup of $P_\gamma$. Let $j \in \gamma$ and let $B = j\mathcal{O}_Gj$. $B$ is a source algebra of $b$ and $j$ is called a source idempotent of $b$. By [12], Theorem 1.8, [13], \S13 and \S14, there exists a unique $P$-stable unitary subalgebra $D$ of $B$, up to $(B^P)^\times$-conjugation, which satisfies

\[ D \cap Pj = Qj \quad \text{and} \quad B = \bigoplus_{u \in P/Q} Du \cong D \otimes_{\mathcal{O}G} OP, \]

where $(B^P)^\times$ is the group of invertible elements of $B^P$. $D$ is called a hyperfocal subalgebra of $b$. $D$ becomes an interior $Q$-algebra with a group homomorphism $q \in Q \to qj \in D^\times$. By [12] or [13], Corollary 13.13, $Q = 1$ if and only if $b$ is nilpotent, and in that case $D$ is $\mathcal{O}$-simple, that is, $D$ is isomorphic to a full matrix algebra over $\mathcal{O}$.

We set $R = \mathcal{O}$ or $k$. Let $A$ be an $R$-algebra and $B$ be an interior $A$-algebra, that is, $B$ is an $R$-algebra which is an $A$-bimodule satisfying $(xa)y = x(ay)$ for $a \in A$, $x, y \in B$. Let $\mu_B : B \otimes_A B \to B$ denote the map of $B$-bimodules satisfying $\mu(x \otimes y) = xy$ for $x, y \in B$. Following [6], we say $B$ is a separable interior $A$-algebra if $\mu_B$ splits as a map of $B$-bimodules. By [6], Lemma 4, $B$ is a separable interior $O \mathcal{O}G$-algebra.

Theorem 1 ([18], Theorem 1) $D$ is a separable interior $O \mathcal{O}G$-algebra.

Corollary 1 ([18], Corollary 1) Let $N$ be a finitely generated (left) $D$-module. Then $N$ is a direct summand of $D \otimes_{\mathcal{O}G} N$ as a $D$-module. In particular $\bar{D} = D \otimes_{\mathcal{O}G} k$ is of finite representation type if $Q$ is cyclic.

We recall that if $P$ is abelian and $Q$ is cyclic, then the number of isomorphism classes of irreducible $D$-modules is equal to $|N_G(P_\gamma)/C_G(P_\gamma)|$ by Theorem in [17].
2 Fan’s question

Assume that $P$ is abelian. Then we have $Q = [P, N_G(P_\gamma)]$ ([18]). Let $L = C_P(N_G(P_\gamma))$. Then we have

$$P = Q \times L$$

as is well known. For $x \in OG$ and $X \subseteq OG$, we denote by $\bar{x}$ and $\bar{X}$ the images in $kG$ by the canonical homomorphism from $OG$ onto $kG$. Now $G_{(b)}$ is $Q$-locally controlled by $P_\gamma$ in the sense of Fan [2].

**Question 1** (Fan [2], p. 789) As interior $P$-algebras

$$B \cong D' \otimes_O OL$$

for some interior $P$-algebra $D'$.

This question is true if $P$ is normal in $G$, or $G$ is $p$-solvable (see Remark 1 below). Also Okuyama showed that the question is true for $B = B \otimes_O k$.

**Theorem 2** ([18], Theorem 2) With the above notations, there is a group homomorphism $\rho : P \to D^\times$ such that $\rho(q) = q\bar{j}$ for any $q \in Q$ and that $d^u = d^{\rho(u)}$ for any $d \in D$ and $u \in L$. Moreover, then, there is an interior $P$-algebra isomorphism $\tilde{B} \cong \tilde{D} \otimes_k kL$ mapping $du$ on $d\rho(u) \otimes u$ for any $d \in \tilde{D}$ and $u \in L$ where $\tilde{D}$ is regarded as an interior $P$-algebra with $\rho$ as structural map.

(See also [16].) We will show that if $Q$ is normal in $G$, then Fan’s question is true.

3 The case where $Q$ is normal in $G$

Assume that $P_\gamma$ is associated with the maximal $b$-Brauer pair $(P, b_P)$. We have $N_G(P, b_P) = N_G(P_\gamma)$. Set $b_0 = (b_P)^{N_G(P)}$. Then $b_0$ is a Brauer correspondent of $b$. Let $B$ be a source algebra of $b$ defined in the above and let $B_0$ be a source algebra of $b_0$. Let $E = N/C_G(P)$ be a $p$-complement of $N_G(P_\gamma)/C_G(P)$ and we denote by $|E|$ a set of representatives for the $C_G(P)$-cosets in $N$. For $a \in (OG)^P$, we set $a' = Br_P(a)$. Recall that $ga'g^{-1} = (gag^{-1})'$ ($g \in N_G(P)$).

**Proposition 1** With the above notations, assume that there exists a normal $p$-subgroup $Q$ of $G$ such that $Q \subseteq Z(P)$ and $(b_P)^{C_G(Q)}$ is nilpotent.

(i) $B \cong S \otimes_O B_0$ as interior $P$-algebras, where $S$ is a (primitive) (interior) Dade $P$-algebra.

(ii) If $P$ is abelian, then $B \cong D \otimes_O OL$ as interior $P$-algebras, where $L = C_P(N_G(P_\gamma))$.

(iii) $b$ and $b_0$ are basic Morita equivalent (See [11] for the definition of basic Morita equivalence).

**Remark 1** If $G$ is $p$-solvable and $P$ is abelian, then the above theorem holds without the assumption by Remark 3.6 in [3].

**Remark 2** From the proof of the proposition, if $b$ is a principal block of $G$, then $B \cong B_0$.

For a $p$-subgroup $X$ of $G$, we denote by $LP_{RG}(X)$ the set of local point of $X$ on $RG$. 

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Lemma 1 Let $Q$ be a normal $p$-subgroup of $G$ and set $C = C_G(Q)$. Let $X$ be a $p$-subgroup of $G$ containing $Q$. Then any $\epsilon \in \mathcal{LP}_{RG}(X)$ is contained a uniquely determined $\epsilon' \in \mathcal{LP}_{RG}(X)$. Moreover the map $\epsilon \in \mathcal{LP}_{RG}(X) \mapsto \epsilon' \in \mathcal{LP}_{RG}(X)$ is a bijection.

Proof. Since there is a natural bijection between $\mathcal{LP}_{OG}(X)$ and $\mathcal{LP}_{kG}(X)$, we may assume $\mathcal{K} = k$. Let $\epsilon \in \mathcal{LP}_{kG}(X)$ and let $\iota \in \epsilon$. Suppose that

$$i = i_1 + i_2, \quad i_1i_2 = i_2i_1 = 0$$

for some idempotents $i_1, i_2$ in $(kG)^X$. Since $Q \leq X$, we have $i = Br_Q(i_1) + Br_Q(i_2)$. Since $Br_Q(i_1), Br_Q(i_2) \in (kC)^X$ and since $i$ is primitive in $(kC)^X$, we may assume that $i = Br_B(i_1)$ and $Br_B(i_2) = 0$. So $i_2 \in \text{Ker}(Br_Q) = \sum_{x < Q}(kG)_x^Q$. Since $Q$ is a normal $p$-subgroup of $G$, $\text{Ker}(Br_Q)$ is contained in the radical of $kG$. Therefore $i_2 = 0$. This implies $i$ is primitive in $(kG)^X$. Since $C_G(X) = C_G(X)$ and since there is a canonical bijection between $\mathcal{LP}_{kG}(X)$ and the set of points of $kC_G(X)$, the lemma easily follows. So the proof is complete.

Proof of Proposition 1

(i) Set

$$b_Q = (b_P)^{C_G(Q)} \quad \text{and} \quad C = C_G(Q).$$

Then $b$ is a unique block of $G$ which covers $b_Q$ and $(P, b_P)$ is a maximal $b_Q$-Brauer pair. In order to prove (i), we may assume $b_Q$ is $G$-invariant. By the Frattini argument $G = C_{NG}(P, b_P) = CN$. Since $b_Q$ is nilpotent, $C \cap N = C_G(P)$. Let $P_\delta$ be a defect pointed group of $C(b_Q)$ on $OC$. By Lemma 1, we also may assume $\delta \subseteq \gamma$. Let $i \in \delta$ and set $B_Q = iOCi$, a source algebra of $b_Q$. Note that we may assume $B = iOCi$. Let $S$ be a hyperfocal subalgebra of $b_Q$ contained in $B_Q$ and set $C_B(S) = \{x \in B \mid xs =sx \quad (\forall s \in S)\}$. Then $C_B(S)$ is $P$-stable because $S$ is $P$-stable. We will observe that $C_B(S)$ is a crossed product of $C_{B_Q}(S)$ over $E$, then $C_B(S) \cong B_0$ as interior $P$-algebras.

By [10], Theorem 1.6, $S$ is a (primitive) Dade $P$-algebra. Moreover by [10], 1.8, there is a unique group homomorphism $i : P \rightarrow S^\times$ lifting the action of $P$ on $S$ such that det$(i(u)) = 1$ for any $u \in P$. Set $z_u = i(u^{-1})u = u_i(u^{-1})$. We have $z_uz_v = z_{uv}$ and $z_u \in (C_{B_Q}(S))^P \quad (u \in Z(P))$. Hence $C_B(S)$ becomes an interior $P$-algebra. Moreover

$$B_Q = \bigoplus_{u \in P} Su = \bigoplus_{u \in P} S z_u.$$

Since $S$ is $O$-simple,

$$C_{B_Q}(S) = \bigoplus_{u \in P} O z_u \cong O P.$$

Let $g \in N$. Since $P_\delta$ is $N$-invariant, there is $x_g \in ((OC)^P)^X$ such that $gig^{-1} = x_gi x_g^{-1}$. Set $a_g = (x_g^{-1}g)i = i(x_g^{-1}g) \in B \cap OCg$. Then $(g^{-1}x_g)i = i(g^{-1}x_g)$ is the inverse of $a_g$ in $B$ (cf. [15], (44.2)). It is easy to see that

$$a_g u = a_g u(a_g)^{-1} = (gug^{-1})i \quad (\forall u \in P).$$

Here we note we can take $x_g = cx_g$ and hence $a_g = a_g$ for any $c \in C_G(P)$. From (1), $a_g S$ is a hyperfocal subalgebra of $b_Q$. By [12], 13.3, $S$ is unique up to $(B_Q)^X$-conjugation, and hence we may assume that $S = a_g S$ by replacing $x_g$ by $x_g y_g + (1 - i)$.
where \( y_g \in ((B_{Q})^{P})^\times \). On the other hand, since \( S \) is \( \mathcal{O} \)-simple, there exists \( t_g \in S^\times \) such that
\[
 a_s t_g = t_s t_g \quad (\forall s \in S)
\]
by a theorem of Skolem-Noether. We may assume \( t_g = c_g t_g \) for any \( c \in C_G(P) \). Since \( \iota(u^g)s\iota((u^g)^{-1}) = u^g s(u^g)^{-1} \), we can see
\[
 a_s t_g = t_s t_g \quad \forall s \in S
\]
by a theorem of Skolem-Noether.

Now we can see
\[
 B = \bigoplus_{g \in [E]} B_Q a_g = \bigoplus_{g \in [E]} (B \cap \mathcal{O}Cg).
\]

Set \( c_g = t_g^{-1}a_g \in C_B(S) \cap \mathcal{O}Cg \). We may assume \( c_g = c_{c_g} \) for any \( c \in C_G(P) \). Moreover \( (a_g)^{-1}t_g \) is the inverse of \( c_g \) in \( B \). From (1) and (2) we can see
\[
 a_s z_u = z_{s u}, \quad c_s z_u = z_{c_s u} \quad (g \in N, \ u \in P)
\]
Moreover
\[
 c_g c_h (c_{gh})^{-1} \in (C_{B_Q}(S))^\times.
\]
Since we have
\[
 B = \bigoplus_{g \in [E]} \bigoplus_{u \in P} S z_u c_g,
\]
\[
 C_B(S) = \bigoplus_{g \in [E], u \in P} \mathcal{O}z_u c_g.
\]
Thus \( C_B(S) \) is a crossed product of \( E \) over \( C_{B_Q}(S) \). From (4) and [4], Lemma M, \( C_B(S) \) is a twisted group algebra of \( P \times E \) over \( \mathcal{O} \) (see [7] and [5]). In fact, by replacing \( c_g \) by \( c_g \epsilon_g \) for some \( \epsilon_g \in i + J(Z(\mathcal{O}\tilde{P})) \subseteq (\mathcal{O}C)^P \) if necessary, where \( \tilde{P} = \{z_u | u \in P\} \), we have for some 2-cocycle \( \alpha \in Z^2(E, \mathcal{O}^\times) \)
\[
 c_g c_h (c_{gh})^{-1} = \alpha(g, h) c_{gh} \quad (g, h \in N).
\]
Hence by replacing \( x_g \) by \( \tilde{x}_g := x_g (a_g (\epsilon_g^{-1}) + 1 - i) \), we may assume (6) holds. Then note that we have \( S = (\tilde{x}_g^{-1})^i S \).

Since \( S \) is \( \mathcal{O} \)-simple,
\[
 B \cong S \otimes_{\mathcal{O}} C_B(S)
\]
as interior \( P \)-algebras. In order to complete the proof of (i), by [10], Lemma 7.8, it suffices to show \( C_B(S) \cong B_0 \) as interior \( P \)-algebras assuming \( R = k \).

Set \( N_S(P) = \{ t \in S^\times | t.P = t(\iota(P)) = P.t \} \). By [9], (e) and [10], Theorem 1.6, there is a group homomorphism \( f : N_S(P) \rightarrow S(P)^\times = k^\times i' \) which extends \( Br_P|_{(S^P)^\times} \). Since \( t_g \in N_S^\times \) from (2) we set
\[
 f(t_g) = \delta_g i' \quad (g \in N, \ \delta_g \in k^\times).
\]
Now since \( g_ig^{-1} = x_gi x_g^{-1} \) we have
\[
gi'g^{-1} = x_g'\delta_g'\delta_g^{-1}x_g^{-1}.
\]
We set \( a_g = (\delta_g^{-1}x_g^{J-1}g)i' = i' (\delta_g^{-1}x_g^{-1}g) \in (i'kN_G(P_\gamma)i')^\times \).

We may assume \( a_g = a_{cg} \) for any \( c \in C_G(P) \). Moreover we have
\[
(7) \quad a_g(\nu_i') = \nu_i'(g \in N, \nu \in P).
\]
From (6) we have
\[
\alpha(g, h)i' = Br_P(c_{gh}c_gc_h) = (gh)^{-1}Br_P(x_{gh}t_{gh}t_g^{-1}x_g^{-1}(gt_{h^{-1}}x_{h^{-1}}g^{-1}))gh = (gh)^{-1}x_g'\delta_{gh}\delta_g^{-1}x_g^{-1}(g\delta_{h}^{-1}x_{h^{-1}}'g^{-1})gh = a_{gh^{-1}}a_g a_h,
\]
and hence
\[
(8) \quad a_g a_h = \alpha(g, h)a_{gh} (g, h \in N).
\]
Since \( B_0 = i'kN_G(P_\gamma)i' = \bigoplus_{g \in |E|} \bigoplus_{u \in P} k(u)i' a_g \), from (4), (6), (7) and (8), \( B_0 \cong C_B(S) \) as interior \( P \)-algebras. This proves (i).

(ii) Since \( Q \) is \( N_G(P_\gamma) \)-invariant, from (1), \( D = \bigoplus_{g \in |E|} \bigoplus_{u \in Q}Su a_g = \bigoplus_{g \in |E|} \bigoplus_{u \in Q} Szu c_g \) is \( P \)-stable, and we see \( D \) is a hyperfocal subalgebra of \( b \). On the other hand \( \bigoplus_{r \in L} \mathcal{O}z_r \) is contained in the center \( Z(B) \) and \( B = \bigoplus_{r \in L} \mathcal{O}z_r \). This implies (ii).

(iii). Let \( e \) be a primitive idempotent of \( S \) and set \( V = Se \). Then \( V \) becomes an endo-permutation \( OP \)-module with \( P \neq \text{rank}_\mathcal{O} V \) by [10], Theorem 1.6. Now from (i) and [8], Theorem 3.4, the \( (OGb, ON_G(P)b_0) \)-bimodule
\[
\mathcal{M} = OGi \otimes_{B_0 \cong S \otimes \mathcal{O}P}(V \otimes \mathcal{O}B_0) \otimes_{B_0} ON_G(P)
\]
and the \( (ON_G(P)b_0, OGb) \)-bimodule
\[
\mathcal{N} = ON_G(P) \otimes_{B_0}(B_0 \otimes \mathcal{O}V^*) \otimes_{B_0 \otimes \mathcal{O}S} B \mathcal{O}G
\]
induce a Morita equivalence between \( b \) and \( b_0 \). We notice that \( \mathcal{N} \cong \mathcal{M}^* \). In fact \( \mathcal{N} \cong \text{Hom}_A(\mathcal{M}, A) \cong \mathcal{M}^* \) because \( A \) is symmetric, where \( A = OGb \) (Auslander-Fuller, 22.1). We can see
\[
\mathcal{M} | OGi \otimes_{OP}(V \otimes \mathcal{O}B_0) \otimes_{OP} ON_G(P), \quad V \otimes \mathcal{O}B_0 | \mathcal{M} \mathcal{O}P
\]
because \( B \) and \( B_0 \) are, respectively, separable interior \( OP \)-algebras. Since \( B_0 \) is a permutation \( OP \times P \)-module and \( V \) is an endo-permutation \( OP \)-module, \( V \otimes \mathcal{O}B_0 \) is an endo-permutation \( OP \times P \)-module. This implies \( b \) and \( b_0 \) are basic Morita equivalent. Recall that any indecomposable component of \( B_0 \) is isomorphic to \( \text{Ind}_{P_x}^{GP}(\mathcal{O}) \) for some \( x \in G \), where \( P_x \) denotes the subgroup \( \{(u, x^{-1}u) \in P \times P \mid u \in P \cap xP\} \). Since \( |P| \ | \text{rank}_\mathcal{O}(V \otimes \mathcal{O}B_0) \), we can see \( \Delta P = \{(u, u) \mid u \in P \} \) is a vertex of \( \mathcal{M} \). \( \blacksquare \)

In the above proposition assume that \( P \) is abelian and \( C_G(Q) \cap N_G(P_\gamma) = C_G(Q) \cap N_G(P, b_P) = C_G(P) \). Then \( b_Q \) is nilpotent.
Corollary 2 Assume that $P$ is abelian and let $Q$ be a normal $p$-subgroup of $N_G(P_\gamma)$ such that $C_G(Q) \cap N_G(P_\gamma) = C_G(P)$. Then $(b_P)^{N_G(Q)}$ and $b_0$ are basic Morita equivalent.

Proof. Set 
\[ c = (b_P)^{N_G(Q)}, \quad d = (b_P)^{N_G(Q)\cap N_G(P)}. \]

By the above theorem $c$ and $d$ are basic Morita equivalent. On the other hand $d\mathcal{O}N_G(P)$ realizes a (splendid) Morita equivalent between $d$ and $b_0$. This implies that $c$ and $b_0$ are basic Morita equivalent. 

\[ \begin{array}{c}
N_G(Q) \quad c \\
\uparrow \quad \text{basic Morita eq.} \\
N_G(Q) \cap N_G(P) \quad d \\
\uparrow \\
C_G(P) \quad b_P
\end{array} \]

Corollary 3 Assume that $P$ is abelian. Then $\hat{b}_Q = (b_P)^{N_G(Q)}$ and $b_0$ are basic Morita equivalent. In particular, $b$ and $b_0$ are derived equivalent if and only if $b$ and $\hat{b}_Q$ are derived equivalent.

Corollary 4 (see [14]) Assume that $P$ is abelian and suppose that $Q$ is cyclic, and let $Q_1$ be a non-trivial subgroup of $Q$. Then $(b_P)^{N_G(Q_1)}$ and $b_0$ are basic Morita equivalent.

References


