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Eigenvalues of Cartan matrices for group algebras of finite groups

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1 Introduction

Let G be a finite group, k an algebraically closed field of characteristic $p>0$, and let B be a block of the group algebra kG with defect group D. Let C_{B} be the Cartan matrix of B, and let $\rho(C_{B})$ be the largest eigenvalue of C_{B} which is called *Frobenius-Perron eigenvalue*. Let R_{B} and E_{B} be the sets of all eigenvalues and all elementary divisors (with multiplicity) respectively. Let $B_{0}(G)$ be the principal block of kG. Let $O_{p'}(G)$ be the largest normal subgroup of G whose order is prime to p, and let $O^{p'}(G)$ be the smallest normal subgroup of G whose index is prime to p.

The eigenvalues of Cartan matrices of finite groups have been studied by some people around T.Wada. The following conjecture is given in [4]:

Conjecture 1.1 [4, Questions 1 and 2] *The following conditions are equivalent:*

(a) $\rho(C_{B})\in \mathbb{Z}$.
(b) $\rho(C_{B})=|D|$.
(c) $R_{B}=E_{B}$.

It is clear that (c) $\Rightarrow$ (b) $\Rightarrow$ (a). This conjecture is known to be true in the following cases:

- D a cyclic group (see [3, Corollary 4.8] and [4, Proposition 3]),
- D isomorphic to a dihedral, a semidihedral, or a generalized quaternion group (see [4, Proposition 4]),
- The principal block with D abelian when $p=2$ (see [7, Theorem]),

- The principal block with D an elementary abelian group of order 9 (see [9, Theorem C]).

Moreover, in these cases, the conditions in Conjecture 1.1 are equivalent to the condition

(d) B is Morita equivalent to the Brauer correspondent b in $N_G(D)$.

It is known that (d) implies (c) in general (see [4, Proposition 2]), but the converse is not always true (see [4, Example]). Nevertheless, if D is abelian, then all of these conditions are expected to be equivalent:

Conjecture 1.2 *If D is abelian, then the above conditions (a)-(d) are equivalent.*

2 Results

We assert that Conjecture 1.2 is true for the principal block of an arbitrary group with an abelian Sylow 3-subgroup, which is a generalization of Wada's result [9, Theorem C]. This is a joint work with Shigeo Koshitani.

Theorem 2.1 (see [6, Theorem 1.4]) *Let G be a finite group with an abelian Sylow 3-subgroup D, and let B be the principal 3-block of kG. Then the following conditions are equivalent.*

(a) $\rho(C_B) \in \mathbb{Z}$.
(b) $\rho(C_B) = |D|$.
(c) $R_B = E_B$.
(d) *B is Morita equivalent (even stronger Puig equivalent) to the Brauer correspondent b in $N_G(D)$.*
(e) *Set $\tilde{G} = O_{3'}(G/O^{3'}(G))$. Then*

$$\tilde{G} \cong S \times G_1 \times \cdots \times G_r$$

for some positive integer r, where S is an abelian 3-group and each G_i is one of the following nonabelian simple groups:

(1) *nonabelian simple groups with cyclic Sylow 3-groups,*
(2) $\mathrm{PSU}_3(q^2)$ *with* $3|q+1$ *and* $3^2 \nmid q+1$,
(3) $\mathrm{PSp}_4(q)$ *with* $3|q-1$,

(4) $\mathrm{PSL}_5(q)$ *with* $3|q+1$ *and* $3^2 \nmid q+1$,
(5) $\mathrm{PSU}_4(q^2)$ *with* $3|q-1$,
(6) $\mathrm{PSU}_5(q^2)$ *with* $3|q-1$.

The proof is essentially similar to that of [9, Theorem C] for $D \cong C_3 \times C_3$ or [7, Theorem] for $p = 2$. But in our case, some additional tools are needed to prove (a) $\Rightarrow$ (e). The following lemma is useful to reduce the condition (a) to $B_0(\tilde{G})$.

Lemma 2.1 (see [8, Theorem 1.1]) *Let G be a finite group and let H be a normal subgroup of G whose index is not divisible by p. Let b be a (p-)block of H and let B be a block of G covering b. Then $\rho(C_B) = \rho(C_b)$.*

Moreover, since $\rho(C_{B_0(\tilde{G})}) = |S| \times \rho(C_{B_0(G_1)}) \times \cdots \times \rho(C_{B_0(G_r)})$, the following lemma enables us to reduce the condition (a) for $B_0(\tilde{G})$ to each $B_0(G_i)$.

Lemma 2.2 (see [6, Lemma 2.3]) *Let*

$$f(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_1x + a_0,$$

$$g(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0$$

be two $\mathbb{Z}$-polynomials with $m, n \geq 1$. Suppose that $\alpha_1, \cdots, \alpha_m$ and $\beta_1, \cdots, \beta_n$ be the roots of $f(x)$ and $g(x)$. Moreover, we assume that all roots of $f(x)$ and $g(x)$ are real and

$$\alpha_1 > \alpha_2 \geq \cdots \geq \alpha_m > 0 \text{ and } \beta_1 > \beta_2 \geq \cdots \geq \beta_n > 0$$

(but if $m = 1$ (resp. $n = 1$), this means $\alpha_1 > 0$ (resp. $\beta_1 > 0$)). Then, if $\alpha_1\beta_1 \in \mathbb{Z}$, then $\alpha_1 \in \mathbb{Z}$ and $\beta_1 \in \mathbb{Z}$.

In general, it is known that for a finite group G with nontrivial abelian Sylow 3-subgroup such that $O_{3'}(G) = 1$ and $O^{3'}(G) = G$, the form of G is

$$G = S \times G_1 \times \cdots \times G_r,$$

where S is an abelian 3-group and each G_i is one of the nonabelian simple groups in a list (see [5, Proposition 1.1]). So it suffices to check the condition (a) for the groups in the list. But for almost all of the groups the condition (a) is checked in [9], so we only have to check for the principal blocks of the following groups:

- the O'Nan simple group $O'N$,
- $\mathrm{PSL}_4(q)$ with $3|q+1$,
- $\mathrm{PSL}_5(q)$ with $3^2|q+1$,
- $\mathrm{PSp}_4(q)$ with $3|q+1$,
- $\mathrm{PSL}_2(q)$ with $q=3^n$, $n=3,4,5,\cdots$.

It is easy to check it for the principal block of $O'N$. Indeed, we can check directly that the condition (a) does not hold for the block. So we only have to check it for the other groups. In fact, by the result of Alperin [1] and Dade [2] we can reduce to $\mathrm{GL}_4(q)$, $\mathrm{GL}_5(q)$, $\mathrm{Sp}_4(q)$, and $\mathrm{SL}_2(q)$ respectively. Now we have the following result:

Theorem 2.2 *Let B_0 be the principal block of kG, and let D be a Sylow p-subgroup of G.*

(1) *If $G=\mathrm{GL}_4(q)$, p is odd and $p|q+1$, then $\rho(C_{B_0})\not\subseteq\mathbb{Z}$.*

(2) *If $G=\mathrm{GL}_5(q)$, p is odd and $p|q+1$, then $\rho(C_{B_0})\not\subseteq\mathbb{Z}$ except when $p=3$ and $3^2\nmid q+1$.*

(3) *If $G=\mathrm{Sp}_4(q)$, p is odd and $p|q+1$, then $\rho(C_{B_0})\not\subseteq\mathbb{Z}$.*

(4) *If $G=\mathrm{SL}_2(q)$, with $q=3^n$ with $n\geq 2$, then $\rho(C_{B_0})\not\subseteq\mathbb{Z}$.*

References

[1] J.L.Alperin, Isomorphic blocks, J. Algebra 43, 694-698 (1976).

[2] E.C.Dade, Remarks on isomorphic blocks, J.Algebra 45, 254-257 (1977).

[3] M.Kiyota and T.Wada, Some remarks on eigenvalues of the Cartan matrix in finite groups, Comm. Algebra 21, 3839-3860 (1993).

[4] M.Kiyota, M.Murai and T.Wada, Rationality of eigenvalues of Cartan matrices in finite groups, J. Algebra 249, 110-119 (2002).

[5] S.Koshitani, Corrigendum, "Conjectures of Donovan and Puig for principal 3-blocks with abelian defect groups" by Shigeo Koshitani, Comm. Algebra 31, 2229-2243 (2003), Comm. Algebra 32, 391-393 (2004).

[6] S.Koshitani and Y.Yoshii, Eigenvalues of Cartan matrices of principal 3-blocks of finite groups with abelian Sylow 3-subgroups, in preparation, 2009.

[7] N.Kunugi and T.Wada, Eigenvalues of Cartan matrices of principal 2-blocks with abelian defect groups, J. Algebra 319, 4404-4411 (2008).

[8] T.Okuyama and T.Wada, Eigenvalues of Cartan matrices of block in finite groups, preprint, 2009.

[9] T.Wada, Eigenvector matrices of Cartan matrices for finite groups, J. Algebra 308, 629-640 (2007).

[10] Y.Yoshii, On the Frobenius-Perron eigenvalues of Cartan matrices for some finite groups, preprint, 2009.