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Kyoto University
Morita equivalences between blocks of finite group algebras

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1. Introduction and notation

In representation theory of finite groups, particularly, in modular representation theory, studying structure of $p$-blocks (block algebras) of finite groups $G$, where $p$ is a prime number, is one of the most important and interesting things.

Notation 1.1. Throughout this note we use the following notation and terminology. We denote by $G$ always a finite group, and let $p$ be a prime. Then, a triple $(\mathcal{K}, \mathcal{O}, k)$ is so-called a $p$-modular system, which is big enough for all finitely many finite groups which we are looking at, including $G$. Namely, $\mathcal{O}$ is a complete descrete valuation ring, $\mathcal{K}$ is the quotient field of $\mathcal{O}$, $\mathcal{K}$ and $\mathcal{O}$ have characteristic zero, and $k$ is the residue field $\mathcal{O}/\text{rad}(\mathcal{O})$ of $\mathcal{O}$ such that $k$ has characteristic $p$. We mean by "big enough" above that $\mathcal{K}$ and $k$ are both splitting fields for the finite groups mentioned above. Let $A$ be a block of $\mathcal{O}G$ (and sometimes of $kG$) with a defect group $P$. We denote by mod-$kG$ and by mod-$A$ the categories of finitely generated right $kG$- and $A$-modules, respectively. We write $B_0(kG)$ for the principal block algebra of $kG$. For the notation and terminology we shall not explain precisely, see the books of [2] and [3].
Setup 1.2. Throughout this note all the time except in Theorem 2.1 our situation is the following: Namely, $G$ and $H$ are finite groups which have the same Sylow $p$-subgroup $P$, and hence $P \subseteq G \cap H$. Assume that $\tilde{G}$ is a normal subgroup of $G$ and $\tilde{H}$ is a normal subgroup of $H$ such that $\tilde{G}$ and $\tilde{H}$ have the same Sylow $p$-subgroup $\tilde{P}$, and hence $\tilde{P} \subseteq \tilde{G} \cap \tilde{H}$, and moreover that $G/\tilde{G} \cong H/\tilde{H}$.

Remark 1.3. If the factor group $G/\tilde{G}$ is $p'$-groups, then we know essentially by the famous result due to H. Maschke (1898) that the ring extension $k\tilde{G} \subseteq kG$ is a so-called separable extension. Then, roughly speaking, mod-$kG$ and mod-$k\tilde{G}$ are in some sense similar (of course, even the numbers of simples in the two module categories are different, though). Therefore, much more interesting situation should be the case where $|G/\tilde{G}|$ is divisible by $p$. Then, here comes our situation.

Our situation 1.4. We still keep the setup 1.2. In addition we assume that the factor groups $G/\tilde{G} \cong H/\tilde{H}$ are $p$-groups. Surely, the factor groups are isomorphic to $P/\tilde{P}$, too. Then, we naturally come to the following questions.

Questions 1.5. Our main concern in this note is the following:

(i) If there is a nice equivalence between mod-$k\tilde{G}$ and mod-$k\tilde{H}$, can we lift it to a nice equivalence between mod-$kG$ and mod-$kH$?

(ii) If there is a nice equivalence between mod-$kG$ and mod-$kH$, can we descend it to a nice equivalence between mod-$k\tilde{G}$ and mod-$k\tilde{H}$?

2. Results

In this short section we shall list two results which come up from Question 1.5.

Theorem 2.1. Assume 1.4, however, note that we do not assume that $P$ and $\tilde{P}$ are Sylow $p$-subgroups of $G$ and $\tilde{G}$, respectively. Namely, $P$ is just a $p$-subgroup of $G$ and also of $H$, and $\tilde{P}$ is just a $p$-subgroup of $\tilde{G}$ and also of $\tilde{H}$ We assume then that $P$ is a defect group of $A$ and $B$, and $\tilde{P}$ is a defect group of $\tilde{A}$ and $\tilde{B}$. Moreover, we suppose
that the factor groups $Q := G/\tilde{G} \cong H/\tilde{H} \cong P/\tilde{P}$ are just cyclic group $C_p$ of order $p$, and that $A$, $\tilde{A}$, $B$, $\tilde{B}$ respectively are block algebras of $kG$, $k\tilde{G}$, $kH$, $k\tilde{H}$, such that $A$ covers $\tilde{A}$ and $B$ covers $\tilde{B}$. Set $\Delta Q := \{(u, u) \in Q \times Q | u \in Q\}$. We assume, in addition, that $\tilde{A}$ and $\tilde{B}$ are both $\Delta Q$-invariant, that is, they are stable under conjugation action by all elements in $Q$. Set furthermore that $\Delta := (\tilde{G} \times \tilde{H})\Delta Q = (\tilde{G} \times \tilde{H})\Delta P = (\tilde{G} \times \tilde{H})\Delta G = (\tilde{G} \times \tilde{H})\Delta H$. Then, we get the following: Suppose that there is a bounded complex $\tilde{M}^\bullet \in C^b(\mathcal{O}\tilde{A} - \mathcal{O}\tilde{B})$ of finitely generated $(\mathcal{O}\tilde{A}, \mathcal{O}\tilde{B})$-bimodules such that

1. $\tilde{M}^\bullet \otimes_{\mathcal{O}} \mathcal{K}$ induces an isometry $\tilde{I}$ from $\mathbb{Z}\text{Irr}(\tilde{A})$ to $\mathbb{Z}\text{Irr}(\tilde{B})$.

2. $\tilde{M}^\bullet$ is perfect (exact), that is, all terms in the complex $\tilde{M}^\bullet$ are projective as left $\mathcal{O}\tilde{G}$-modules and also as right $\mathcal{O}\tilde{H}$-modules (and hence the isometry $\tilde{I}$ above is perfect).

3. the complex $\tilde{M}^\bullet$ extends from $\tilde{G} \times \tilde{H}$ to $\Delta$.

Then, we can define a bounded complex $M^\bullet := \tilde{M}^\bullet_{\tilde{G} \times \tilde{H} \rightarrow \Delta} \uparrow^{G \times H} \in C^b(\mathcal{O}A - \mathcal{O}B)$, and the new complex $M^\bullet$ induces a perfect isometry from $\mathbb{Z}\text{Irr}(A)$ to $\mathbb{Z}\text{Irr}(B)$, where $M^\bullet := \tilde{M}^\bullet_{\tilde{G} \times \tilde{H} \rightarrow \Delta} \uparrow^{G \times H}$ is an induced complex by applying the functor $- \otimes_{\mathcal{O}\Delta} \mathcal{O}[G \times H]$ to the bounded complex $\tilde{M}^\bullet$.

**Corollary 2.2.** We easily get [1, Example 4.3] in our previous paper by making use of Theorem 2.1.

**Theorem 2.2.** Assume 1.4. Here we assume that $P$ is a Sylow $p$-subgroup of $G$ and $H$, and also $\tilde{P}$ is a Sylow $p$-subgroup of $\tilde{G}$ and $\tilde{H}$. Moreover, we suppose that the factor groups $Q := G/\tilde{G} \cong H/\tilde{H} \cong P/\tilde{P}$ are isomorphic finite $p$-groups, and that $A$, $\tilde{A}$, $B$, $\tilde{B}$ respectively are principal block algebras of $kG$, $k\tilde{G}$, $kH$, $k\tilde{H}$ Set $\Delta P := \{(u, u) \in P \times P | u \in P\}$. Moreover, we denote by Scott($G \times H, \Delta P$) the (Alperin-)Scott module in $G \times H$ with respect to a subgroup $\Delta P$ of $G \times H$, see [2, Chap.4 Theorem 8.4, Corollary 8.5]. Then, we get the following: If $\mathcal{A}M_B := \text{Scott}(G \times H, \Delta P)$ induces a Morita equivalence (and hence it is a Puig equivalence) between $A$ and $B$, then $\mathcal{A}\tilde{M}_B := \text{Scott}(\tilde{G} \times \tilde{H}, \tilde{\Delta} \tilde{P})$...
\( \tilde{H}, \Delta \tilde{P} \) induces a Morita equivalence (and hence it is a Puig equivalence) between \( \tilde{A} \) and \( \tilde{B} \). (Recall that \( A := B_0(kG) = \text{Scott}(G \times G, \Delta \tilde{P}) \) and \( B := B_0(kH) = \text{Scott}(H \times H, \Delta \tilde{P}) \).

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