数理解析研究所講究録
第 1687 巻 2010 年 99-102

ON EXISTENCE OF A CLASS OF NON-COMMUTATIVE ASSOCIATION SCHEMES

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ABSTRACT. We investigate a relationship between symmetric generalized conference matrices and association schemes with some conditions.

1. INTRODUCTION

Let X be a finite set and G a set of binary relations on X which partitions $X \times X$. For each $g \in G$ we set

$$g^* := \{(x, y) \mid (y, x) \in g\}.$$

For each $x \in X$ and $g \in G$ we set

$$xg := \{y \in X \mid (x, y) \in g\}.$$

We say that (X, G) is an *association scheme* (or shortly, *scheme*) if it satisfies the following conditions:

(i) $1_X := \{(x, x) \mid x \in X\}$ is a member of G;
(ii) For each $g \in G$ g^* is a member of G;
(iii) For all d, e, $f \in G$ $|xd \cap ye^*|$ is constant whenever $(x, y) \in f$.

The constant is denoted by a_{def}, and $\{a_{def}\}_{d,e,f \in G}$ are called the *intersection numbers* of G. For each $g \in G$ we abbreviate $a_{gg^*1_X}$ as n_g, which is called the *valency* of g. In particular, G is called *thin* if every valency of G is one.

We define TS to be the set of all elements g in G such that there exist elements r in T and s in S with $a_{rsg} \neq 0$. The set TS is called the *complex product* of T and S.

A subset H of G is called *closed* if $HH \subseteq H$, *normal* if $gH = Hg$ for each $g \in G$.

Let H be a closed subset of S. According to [3] we say that $Y \subseteq X$ is a *transversal* of H in X if $|xH \cap Y| = 1$ for each $x \in X$.

For each $g \in G$, we define the *adjacency matrix* of g as follows:

$$(\sigma_g)_{x,y} := \begin{cases} 1 & \text{if } (x, y) \in g; \\ 0 & \text{otherwise} \end{cases}$$

where the rows and columns of σ_g are indexed by the elements of X.

A generalized conference matrix over a finite group F of order f is a $(nf+2)\times(nf+2)$ matrix $C=[c_{ij}]$ with $c_{ii}=0$ and $c_{ij}\in F$ such that for distinct i and h, the multiset $\{c_{ij}c_{hj}^{-1}\mid j\neq i, j\neq h\}$ contains n-copies of every element of F.

2. Construction of symmetric conference matrices from association schemes

Let (X,G) be an association scheme of order $p(np+2)$, where p is an odd prime and n is a positive integer.

Suppose that there exists a normal thin-closed H of G such that

$\sigma_{g_i}\sigma_{h_1}=\sigma_{g_{i+1}}, \quad \sigma_{h_1}\sigma_{g_{i+1}}=\sigma_{g_i}, \quad \sigma_{g_i}=\sigma_{g_i^*}$ and
$\sigma_{g_i}\sigma_{g_j}=(np+1)\sigma_{h_1}^{j-i}+n(\sigma_{g_0}+\cdots+\sigma_{g_{p-1}})$
, where $H=\{h_0,h_1,\ldots,h_{p-1}\}$, $G-H=\{g_0,g_1,\ldots,g_{p-1}\}$.

Then we can define a symmetric generalized conference matrix as follows:

Let Y be a transversal of H in X. For distinct x, y in Y, there exists a element i_{xy} of Z_p such that $(xH\times yH)\cap g_0=\{(xh_1^a, yh_1^{i_{xy}-a})\mid a\in Z\}$.

Define a $\mid Y\mid\times\mid Y\mid$ matrix M such that $M_{xx}=0$ and $M_{xy}=\epsilon^{i_{xy}}$, where ϵ is a primitive p-th root of unity.

Then M is a symmetric generalized conference matrix.

3. Construction of association schemes from symmetric conference matrices

Suppose that M is a $(np+2)\times(np+2)$ symmetric generalized conference matrix such that $M_{xx}=0$ and $M_{xy}=\epsilon^{i_{xy}}$, where ϵ is a primitive p-th root of unity and i_{xy} is a element of Z_p.

Define $\sigma_{h_1}:=I_{np+2}\otimes P$ and $\sigma_{g_0}:=[B_{xy}]$, where P and B_{xy} are permutation matrices of Z_p defined by $a\mapsto a+1$ and $a\mapsto i_{xy}-a$, respectively.

Then there exist an association scheme of order $p(np+2)$.

Remark 3.1. In [2], it is known that there exist some symmetric conference matrices.

4. Observation of character table

In this section, we investigate algebraic aspect of an association scheme defined in section 2.

In [4], it is well-known that the matrix $(\sum_{g\in G} \frac{a_{g^* e g f}}{n_g})_{ef}$ has rank $\mid Irr(RG) \mid$, where R is algebraically closed.

This fact implies that the number of $Irr(RG)$ is $2 + \frac{p-1}{2}$.

Central primitive idempotents of RG are as follows.

$$b_1 = \frac{1}{p(np+2)}\sigma_G,\ b_2 = \frac{1}{p}\sigma_H - \frac{1}{p(np+2)}\sigma_G$$
$$c_1 = e_1 + e_{p-1},\quad c_2 = e_2 + e_{p-2},\quad \cdots,\quad c_{\frac{p-1}{2}} = e_{\frac{p-1}{2}} + e_{\frac{p+1}{2}}$$
,where $e_1 = \frac{1}{p}(\sigma_{h_0} + \varepsilon\sigma_{h_1} + \ldots + \varepsilon^{p-1}\sigma_{h_{p-1}})$
$$e_2 = \frac{1}{p}(\sigma_{h_0} + \varepsilon^2\sigma_{h_1} + \ldots + \varepsilon^{p-2}\sigma_{h_{p-1}})$$
....
$$e_{p-2} = \frac{1}{p}(\sigma_{h_0} + \varepsilon^{p-2}\sigma_{h_1} + \ldots + \varepsilon^2\sigma_{h_{p-1}})$$
$$e_{p-1} = \frac{1}{p}(\sigma_{h_0} + \varepsilon^{p-1}\sigma_{h_1} + \ldots + \varepsilon\sigma_{h_{p-1}})$$

Character table of (X, G)

	h_0	h_1	$\ldots$	h_{p-1}	g_0	$\ldots$	g_{p-1}	m
χ_1	1	1	$\ldots$	1	$np+1$	$\ldots$	$np+1$	1
χ_2	1	1	$\ldots$	1	-1	$\ldots$	-1	$np+1$
ψ_1	2	$\varepsilon + \varepsilon^{p-1}$	$\ldots$	$\varepsilon^{p-1} + \varepsilon$	0	$\ldots$	0	$np+2$
ψ_2	2	$\varepsilon^2 + \varepsilon^{p-2}$	$\ldots$	$\varepsilon^{p-2} + \varepsilon^2$	0	$\ldots$	0	$np+2$
ψ_3	2	$\varepsilon^3 + \varepsilon^{p-3}$	$\ldots$	$\varepsilon^{p-3} + \varepsilon^3$	0	$\ldots$	0	$np+2$
.	.	.	$\ldots$	.	.	.	$\ldots$	.
$\psi_{\frac{p-3}{2}}$	2	$\varepsilon^{\frac{p-3}{2}} + \varepsilon^{\frac{p+3}{2}}$	$\ldots$	$\varepsilon^{\frac{p+3}{2}} + \varepsilon^{\frac{p-3}{2}}$	0	$\ldots$	0	$np+2$
$\psi_{\frac{p-1}{2}}$	2	$\varepsilon^{\frac{p-1}{2}} + \varepsilon^{\frac{p+1}{2}}$	$\ldots$	$\varepsilon^{\frac{p+1}{2}} + \varepsilon^{\frac{p-1}{2}}$	0	$\ldots$	0	$np+2$

Acknowledgments

This work was supported by the National Research Foundation of Korea Grant funded by the Korean Government[NRF-2009-352-C00003].

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