ON EXISTENCE OF A CLASS OF NON-COMMUTATIVE ASSOCIATION SCHEMES

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ABSTRACT. We investigate a relationship between symmetric generalized conference matrices and association schemes with some conditions.

1. INTRODUCTION

Let $X$ be a finite set and $G$ a set of binary relations on $X$ which partitions $X \times X$. For each $g \in G$ we set

$$g^* := \{(x, y) \mid (y, x) \in g\}.$$ 

For each $x \in X$ and $g \in G$ we set

$$xg := \{y \in X \mid (x, y) \in g\}.$$ 

We say that $(X, G)$ is an association scheme (or shortly, scheme) if it satisfies the following conditions:

(i) $1_X := \{(x, x) \mid x \in X\}$ is a member of $G$;
(ii) For each $g \in G$ $g^*$ is a member of $G$;
(iii) For all $d, e, f \in G$ $|xd \cap ye^*|$ is constant whenever $(x, y) \in f$.

The constant is denoted by $a_{def}$, and $\{a_{def}\}_{d, e, f \in G}$ are called the intersection numbers of $G$. For each $g \in G$ we abbreviate $a_{gg^*1_X}$ as $n_g$, which is called the valency of $g$. In particular, $G$ is called thin if every valency of $G$ is one.

We define $TS$ to be the set of all elements $g$ in $G$ such that there exist elements $r$ in $T$ and $s$ in $S$ with $a_{rs} \neq 0$. The set $TS$ is called the complex product of $T$ and $S$.

A subset $H$ of $G$ is called closed if $HH \subseteq H$, normal if $gH = Hg$ for each $g \in G$.

Let $H$ be a closed subset of $S$. According to [3] we say that $Y \subseteq X$ is a transversal of $H$ in $X$ if $xH \cap Y = 1$ for each $x \in X$.

For each $g \in G$, we define the adjacency matrix of $g$ as follows:

$$(\sigma_g)_{x,y} := \begin{cases} 
1 & \text{if } (x, y) \in g; \\
0 & \text{otherwise}
\end{cases}$$

where the rows and columns of $\sigma_g$ are indexed by the elements of $X$. 
A generalized conference matrix over a finite group $F$ of order $f$ is a $(nf+2) \times (nf+2)$ matrix $C = [c_{ij}]$ with $c_{ii} = 0$ and $c_{ij} \in F$ such that for distinct $i$ and $h$, the multiset $\{c_{ij}c_{hj}^{-1} \mid j \neq i, j \neq h\}$ contains $n$-copies of every element of $F$.

2. CONSTRUCTION OF SYMMETRIC CONFERENCE MATRICES FROM ASSOCIATION SCHEMES

Let $(X, G)$ be an association scheme of order $p(np + 2)$, where $p$ is an odd prime and $n$ is a positive integer.

Suppose that there exists a normal thin-closed $H$ of $G$ such that

$$
\sigma_{g_i} \sigma_{h_1} = \sigma_{g_{i+1}}, \quad \sigma_{h_1} \sigma_{g_{i+1}} = \sigma_{g_i}, \quad \sigma_{g_i} = \sigma_{g_i}^{*} \quad \text{and} \\
\sigma_{g_i} \sigma_{g_j} = (np + 1)\sigma_{h_1}^{j-i} + n(\sigma_{g_0} + \cdots + \sigma_{g_{p-1}}),
$$

where $H = \{h_0, h_1, \ldots, h_{p-1}\}$, $G - H = \{g_0, g_1, \ldots, g_{p-1}\}$.

Then we can define a symmetric generalized conference matrix as follows:

Let $Y$ be a transversal of $H$ in $X$. For distinct $x$, $y$ in $Y$, there exists an element $i_{xy}$ of $Z_p$ such that $(xH \times yH) \cap g_0 = \{(xh_1^a, yh_1^{i_{xy}-a}) \mid a \in Z\}$.

Define a $|Y| \times |Y|$ matrix $M$ such that $M_{xx} = 0$ and $M_{xy} = \epsilon^{i_{xy}}$, where $\epsilon$ is a primitive $p$-th root of unity.

Then $M$ is a symmetric generalized conference matrix.

3. CONSTRUCTION OF ASSOCIATION SCHEMES FROM SYMMETRIC CONFERENCE MATRICES

Suppose that $M$ is a $(np + 2) \times (np + 2)$ symmetric generalized conference matrix such that $M_{xx} = 0$ and $M_{xy} = \epsilon^{i_{xy}}$, where $\epsilon$ is a primitive $p$-th root of unity and $i_{xy}$ is an element of $Z_p$.

Define $\sigma_{h_1} := I_{np+2} \otimes P$ and $\sigma_{g_0} := [B_{xy}]$, where $P$ and $B_{xy}$ are permutation matrices of $Z_p$ defined by $a \mapsto a + 1$ and $a \mapsto i_{xy} - a$, respectively.

Then there exist an association scheme of order $p(np + 2)$.

Remark 3.1. In [2], it is known that there exist some symmetric conference matrices.

4. OBSERVATION OF CHARACTER TABLE

In this section, we investigate algebraic aspect of an association scheme defined in section 2.
In [4], it is well-known that the matrix \( \left( \sum_{g \in G} \frac{a_{g^{*}egf}}{n_g} \right)_{ef} \) has rank \( |Irr(RG)| \), where \( R \) is algebraically closed.

This fact implies that the number of \( Irr(RG) \) is \( 2 + \frac{p-1}{2} \).

Central primitive idempotents of \( RG \) are as follows.

\[
\begin{align*}
b_1 &= \frac{1}{p(np+2)} \sigma_G, \quad b_2 = \frac{1}{p} \sigma_H - \frac{1}{p(np+2)} \sigma_G \\
c_1 &= e_1 + e_{p-1}, \quad c_2 = e_2 + e_{p-2}, \quad \cdots, \quad c_{\frac{p-1}{2}} = e_{\frac{p-1}{2}} + e_{\frac{p+1}{2}} \\
e_1 &= \frac{1}{p} (\sigma_{h_0} + \epsilon \sigma_{h_1} + \ldots + \epsilon^{p-1} \sigma_{h_{p-1}}) \\
e_2 &= \frac{1}{p} (\sigma_{h_0} + \epsilon^2 \sigma_{h_1} + \ldots + \epsilon^{p-2} \sigma_{h_{p-1}}) \\
\cdots \\
e_{p-2} &= \frac{1}{p} (\sigma_{h_0} + \epsilon^{p-2} \sigma_{h_1} + \ldots + \epsilon^{2} \sigma_{h_{p-1}}) \\
e_{p-1} &= \frac{1}{p} (\sigma_{h_0} + \epsilon^{p-1} \sigma_{h_1} + \ldots + \epsilon \sigma_{h_{p-1}})
\end{align*}
\]

Character table of \((X, G)\)

\[
\begin{array}{cccccccc}
& h_0 & h_1 & \ldots & h_{p-1} & g_0 & \ldots & g_{p-1} & m \\
\chi_1 & 1 & 1 & \ldots & 1 & np + 1 & \ldots & np + 1 & 1 \\
\chi_2 & 1 & 1 & \ldots & 1 & -1 & \ldots & -1 & np + 1 \\
\psi_1 & 2 & \epsilon + \epsilon^{p-1} & \ldots & \epsilon^{p-1} + \epsilon & 0 & \ldots & 0 & np + 2 \\
\psi_2 & 2 & \epsilon^2 + \epsilon^{p-2} & \ldots & \epsilon^{p-2} + \epsilon^2 & 0 & \ldots & 0 & np + 2 \\
\psi_3 & 2 & \epsilon^3 + \epsilon^{p-3} & \ldots & \epsilon^{p-3} + \epsilon^3 & 0 & \ldots & 0 & np + 2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\psi_{\frac{p-3}{2}} & 2 & \epsilon^{\frac{p-3}{2}} + \epsilon^{\frac{p+3}{2}} & \ldots & \epsilon^{\frac{p+3}{2}} + \epsilon^{\frac{p-3}{2}} & 0 & \ldots & 0 & np + 2 \\
\psi_{\frac{p-1}{2}} & 2 & \epsilon^{\frac{p-1}{2}} + \epsilon^{\frac{p+1}{2}} & \ldots & \epsilon^{\frac{p+1}{2}} + \epsilon^{\frac{p-1}{2}} & 0 & \ldots & 0 & np + 2 \\
\end{array}
\]

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