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On applications of the cellular algebras

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ABSTRACT. In this report we explain briefly the results of parts of papers [SawS] and [Sa].

1. CELLULAR ALGEBRAS

1.1. **Cellular bases.** We begin with the definition of a cellular basis.

Let R be a commutative domain with 1 and A an associative unital R-algebra which is free as an R-module. Suppose that $(\Lambda, \geq)$ is a (finite) poset and that for each $\lambda \in \Lambda$ there is a finite indexing set $\mathcal{T}(\lambda)$ and elements $c_{\mathfrak{st}}^{\lambda} \in A$ for all $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)$ such that

$$\mathscr{C} = \{c_{\mathfrak{st}}^{\lambda} \mid \lambda \in \Lambda \text{ and } \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)\}$$

is a (free) basis of A. For each $\lambda \in \Lambda$ let $\check{A}^{\lambda}$ be the R-submodule of A with basis $\{c_{\mathfrak{uv}}^{\mu} \mid \mu \in \Lambda,\ \mu > \lambda \ \text{ and } \ \mathfrak{u}, \mathfrak{v} \in \mathcal{T}(\mu)\}$.

The pair $(\mathscr{C}, \Lambda)$ is cellular basis of A if

(i) the R-linear map $* : A \longrightarrow A$ determined by $c_{\mathfrak{st}}^{\lambda}{}^{*} = c_{\mathfrak{ts}}^{\lambda}$, for all $\lambda \in \Lambda$ and all $\mathfrak{s}$ and $\mathfrak{t}$ in $\mathcal{T}(\lambda)$, is an algebra anti-isomorphism of A,

(ii) for any $\lambda \in \Lambda$, $\mathfrak{t} \in \mathcal{T}(\lambda)$ and $a \in A$ there exist $r_{\mathfrak{v}} \in R$ such that for all $\mathfrak{s} \in \mathcal{T}(\lambda)$

$$c_{\mathfrak{st}}^{\lambda} a \equiv \sum_{\mathfrak{v} \in \mathcal{T}(\lambda)} r_{\mathfrak{v}} c_{\mathfrak{sv}}^{\lambda} \mod \check{A}^{\lambda}. \tag{1.1}$$

If A has a cellular basis we say that A is a cellular algebra.

Throughout this section we assume that $(\mathscr{C}, \Lambda)$ is a fixed cellular basis of the algebra A.

For $\lambda \in \Lambda$ let A^{λ} be the R-module with basis the set of $c_{\mathfrak{uv}}^{\mu}$ where $\mu \in \Lambda$, $\mu \geq \lambda$ and $\mathfrak{u}, \mathfrak{v} \in \mathcal{T}(\mu)$. Thus, $\check{A}^{\lambda} \subset A^{\lambda}$ and $A^{\lambda}/\check{A}^{\lambda}$ has basis $c_{\mathfrak{st}}^{\lambda} + \check{A}^{\lambda}$ where $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)$.

Lemma 1.2 (cf. [Ma, Lemma 2.3]). *Let λ be an element of Λ.*

(i) *Suppose that $\mathfrak{s} \in \mathcal{T}(\lambda)$ and $a \in A$. Then for all $\mathfrak{t} \in \mathcal{T}(\lambda)$*

$$a^{*} c_{\mathfrak{st}}^{\lambda} \equiv \sum_{\mathfrak{u} \in \mathcal{T}(\lambda)} r_{\mathfrak{u}} c_{\mathfrak{ut}}^{\lambda} \mod \check{A}^{\lambda}$$

where $r_{\mathfrak{u}}$ is the element of R determined by (1.1) *for each* $\mathfrak{u}$.

(ii) *The R-modules A^{λ} and $\check{A}^{\lambda}$ are two-sided ideals of A.*

(iii) *Suppose that $\mathfrak{s}$ and $\mathfrak{t}$ are elements of $\mathcal{T}(\lambda)$. Then there exists an element $r_{\mathfrak{st}}$ of R such that for any $\mathfrak{u}, \mathfrak{v} \in \mathcal{T}(\lambda)$*

$$c_{\mathfrak{us}}^{\lambda} c_{\mathfrak{tv}}^{\lambda} \equiv r_{\mathfrak{st}} c_{\mathfrak{uv}}^{\lambda} \mod \check{A}^{\lambda}.$$

Fix an element λ of Λ. If $\mathfrak{s} \in \mathcal{T}(\lambda)$ define $C_\mathfrak{s}^\lambda$ to be the R-submodule of $A^\lambda/\check{A}^\lambda$ with basis $\{c_{\mathfrak{st}}^\lambda + \check{A}^\lambda \mid \mathfrak{t} \in \mathcal{T}(\lambda)\}$. Then $C_\mathfrak{s}^\lambda$ is a right A-module by (1.1) and, importantly, the action of A on $C_\mathfrak{s}^\lambda$ is completely independent of $\mathfrak{s}$. That is, $C_\mathfrak{s}^\lambda \cong C_\mathfrak{t}^\lambda$ for any $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)$. This motivates us to define the right cell module C^λ to be the right A-module which is free as an R-module with basis $\{c_\mathfrak{t}^\lambda \mid \mathfrak{t} \in \mathcal{T}(\lambda)\}$ and where for each $a \in A$

$$c_\mathfrak{t}^\lambda a = \sum_{\mathfrak{v}\in\mathcal{T}(\lambda)} r_\mathfrak{v} c_\mathfrak{v}^\lambda \tag{1.2}$$

where $r_\mathfrak{v}$ is the element of R determined by (1.1). Then $C^\lambda \cong C_\mathfrak{s}^\lambda$, for any $\mathfrak{s} \in \mathcal{T}(\lambda)$, via the canonical R-linear map which sends $c_\mathfrak{t}^\lambda$ to $c_{\mathfrak{st}}^\lambda + \check{A}^\lambda$ for all $\mathfrak{t} \in \mathcal{T}(\lambda)$. In particular, (1.2) determines a well-defined action of A on C^λ.

Abusing notation, define the left cell module $C^{*\lambda}$ to be the free R-module with basis $\{c_\mathfrak{t}^\lambda \mid \mathfrak{t} \in \mathcal{T}(\lambda)\}$ and A-action given by

$$a^* c_\mathfrak{t}^\lambda = \sum_{\mathfrak{v}\in\mathcal{T}(\lambda)} r_\mathfrak{v} c_\mathfrak{v}^\lambda$$

for all $a \in A$ and where, once again, $r_\mathfrak{v}$ is given by (1.1). Then $C^{*\lambda}$ is a left A-module and $C^{*\lambda} \cong \operatorname{Hom}_R(C^\lambda, R)$.

Moreover, as (A, A)-bimodules, $A^\lambda/\check{A}^\lambda$ and $C^{*\lambda} \otimes_R C^\lambda$ are canonically isomorphic via the R-linear map determined by $c_{\mathfrak{st}}^\lambda + \check{A}^\lambda \longmapsto c_\mathfrak{s}^\lambda \otimes c_\mathfrak{t}^\lambda$ for all $\mathfrak{s}$ and $\mathfrak{t}$ in $\mathcal{T}(\lambda)$.

Furthermore, as a right A-module,

$$A^\lambda/\check{A}^\lambda \cong C^{*\lambda} \otimes_R C^\lambda \cong \bigoplus_{\mathfrak{s}\in\mathcal{T}(\lambda)} C_\mathfrak{s}^\lambda. \tag{1.3}$$

So, as a right A-module, $A^\lambda/\check{A}^\lambda$ is isomorphic to a direct sum of $|\mathcal{T}(\lambda)|$ copies of C^λ.

By Lemma 1.2 (iii) there is a unique bilinear map $\langle\ ,\ \rangle : C^\lambda \times C^\lambda \longrightarrow R$ such that $\langle c_\mathfrak{s}^\lambda, c_\mathfrak{t}^\lambda\rangle$, for $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)$, is given by

$$\langle c_\mathfrak{s}^\lambda, c_\mathfrak{t}^\lambda\rangle c_{\mathfrak{uv}}^\lambda \equiv c_{\mathfrak{us}}^\lambda c_{\mathfrak{tv}}^\lambda \mod \check{A}^\lambda, \tag{1.4}$$

where $\mathfrak{u}$ and $\mathfrak{v}$ are any elements of $\mathcal{T}(\lambda)$. The bilinear form $\langle\ ,\ \rangle$ is both symmetric and associative.

Let $\operatorname{rad} C^\lambda = \{x \in C^\lambda \mid \langle x, y\rangle = 0 \text{ for all } y \in C^\lambda\}$. One can see that $\operatorname{rad} C^\lambda$ is an A-submodule of C^λ. Accordingly, we define $D^\lambda = C^\lambda/\operatorname{rad} C^\lambda$.

1.2. **Simple modules in a cellular algebra.** We are almost ready to show that every irreducible A-module is isomorphic to D^μ, for some $\mu \in \Lambda$. In this section we also define and describe the decomposition matrix of A. Throughout, we assume that the poset Λ is finite. Thus A is a finite dimensional algebra.

One of the main points of the cellular basis is that it gives rise to many filtrations in A. To formalize this, call a subset Γ of Λ a poset ideal if $\lambda \in \Gamma$ whenever $\lambda > \mu$ for some $\mu \in \Gamma$. If Γ is a poset ideal let $A(\Gamma)$ be the R-submodule of A with basis

$\{c^{\mu}_{\mathfrak{u}\mathfrak{v}} \mid \mu \in \Gamma \text{ and } \mathfrak{u}, \mathfrak{v} \in \mathcal{T}(\mu)\}$. Then $A(\Gamma) = \sum_{\mu\in\Gamma} A^{\mu}$. So $A(\Gamma)$ is a two-sided ideal by Lemma 1.2 (ii).

Lemma 1.3 (cf. [Ma, Lemma 2.14]). *Suppose that Λ is finite and let $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k = \Lambda$ be any maximal chain of ideals in Λ. Then there exists a total ordering $\mu_1, \ldots, \mu_k$ of Λ such that $\Gamma_i = \{\mu_1, \ldots, \mu_i\}$, for all i, and*

$$0 = A(\Gamma_0) \hookrightarrow A(\Gamma_1) \hookrightarrow \cdots \hookrightarrow A(\Gamma_k) = A$$

is a filtration of A with composition factors $A(\Gamma_i)/A(\Gamma_{i-1}) \cong C^{*\mu_i} \otimes_R C^{\mu_i}$.

Let $\Lambda_0 = \{\mu \in \Lambda \mid D^{\mu} \neq 0\}$. Then $\mu \in \Lambda_0$ if and only if the bilinear form $\langle\ ,\ \rangle$ on C^{μ} is non-zero. In principle, the next theorem classifies the simple A-modules. However, in practice, it is often difficult to determine the set Λ_0.

Theorem 1.4 (Graham-Lehrer). *Suppose that R is a field and that Λ is finite. Then $\{D^{\mu} \mid \mu \in \Lambda_0\}$ is a complete set of pairwise inequivalent irreducible A-modules.*

Suppose that $\mu \in \Lambda_0$ and $\lambda \in \Lambda$. Define $d_{\lambda\mu} = [C^{\lambda} : D^{\mu}]$ to be the decomposition number (or composition multiplicity) of the irreducible module D^{μ} in C^{λ}. By the Jordan-Hölder Theorem, $d_{\lambda\mu}$ is well-defined. The matrix $\mathbf{D} = (d_{\lambda\mu})$, where $\lambda \in \Lambda$ and $\mu \in \Lambda_0$, is the so-called decomposition matrix of A.

Corollary 1.5 (cf. [Ma, Corollary 2.17]). *Suppose that R is a field. Then the decomposition matrix $\mathbf{D}$ of A is unitriangular. That is, if $\mu \in \Lambda_0$ and $\lambda \in \Lambda$ then $d_{\mu\mu} = 1$ and $d_{\lambda\mu} \neq 0$ only if $\lambda \geq \mu$.*

The last result in this section connects the theory of quasi-hereditary algebras and cellular algebras. Quasi-hereditary algebras are a very important class of algebras which were introduced by Cline, Parshall and Scott [CPS].

Proposition 1.6 (cf. [Ma, Corollary 2.23]). *Suppose that R is a field. Then the following are equivalent.*

(i) $\Lambda = \Lambda_0$.

(ii) *The decomposition matrix $\mathbf{D}$ is a square unitriangular matrix.*

Furthermore, if these conditions are satisfied then A is quasi-hereditary.

As this criterion indicates, being quasi-hereditary is a non-degeneracy property on A.

2. Preliminaries on Ariki-Koike algebras and Cyclotomic q-Schur algebras

2.1. Fix positive integers r and n and let $\mathfrak{S}_n$ be the symmetric group of degree n. Let R be an integral domain with 1 and $q, Q_1, \ldots, Q_r$ be elements in R, with invertible q. The Ariki-Koike algebra associated to the complex reflection group $W_{n,r} = G(r, 1, n)$, is the associative unital algebra $\mathscr{H} = \mathscr{H}_{n,r}$ over R with generators $T_1, \ldots, T_n$ subject to the following conditions,

$$\begin{aligned}
(T_1 - Q_1)\cdots(T_1 - Q_r) &= 0, && \\
(T_i - q)(T_i + q^{-1}) &= 0 && (i \geq 2), \\
T_1T_2T_1T_2 &= T_2T_1T_2T_1, && \\
T_iT_j &= T_jT_i && (|i-j| \geq 2), \\
T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && (2 \leq i \leq n-1).
\end{aligned}$$

It is known that $\mathscr{H}$ is a free R-module of rank $n!r^n$. The subalgebra $\mathscr{H}(\mathfrak{S}_n)$ of $\mathscr{H}$ generated by $T_2, \ldots, T_n$ is isomorphic to the Iwahori-Hecke algebra $\mathscr{H}_n$ of the symmetric group $\mathfrak{S}_n$.

For $i = 2, \ldots, n$ let s_i be the transposition $(i-1, i)$ in $\mathfrak{S}_n$.Then $\{s_2, \ldots, s_n\}$ generate $\mathfrak{S}_n$. For $w \in \mathfrak{S}_n$, we set $T_w = T_{i_1} \cdots T_{i_k}$ where $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression. Then T_w is independent of the choice of a reduced expression. We also put $L_k = T_k \cdots T_2 T_1 T_2 \cdots T_k$ for $k = 1, 2, \ldots, n$. Note that all $L_1, \ldots, L_n$ commutes. Moreover, these elements produce a basis of $\mathscr{H}$.

Theorem 2.2 ([AK, Theorem 3.10]). *The Ariki-Koike algebra $\mathscr{H}$ is free as an R-module with basis $\{L_1^{a_1} \cdots L_n^{a_n} T_w \mid w \in \mathfrak{S}_n,\ 0 \le a_i < r \text{ for } 1 \le i \le n\}$.*

Recall that a composition of n is sequence $\sigma = (\sigma_1, \sigma_2, \ldots)$ of non-negative integers such that $|\sigma| = \sum_i \sigma_i = n$. σ is a partition if in addition $\sigma_1 \ge \sigma_2 \ge \cdots$. If $\sigma_i = 0$ for all $i > k$ then we write $\sigma = (\sigma_1, \ldots, \sigma_k)$.

An r-composition (or multicomposition) of n is an r-tuple $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of compositions with $\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots)$ such that $|\lambda^{(1)}| + \cdots + |\lambda^{(r)}| = n$. An r-composition λ is an r-partition if each $\lambda^{(i)}$ is a partition. If λ is an r-partition of n then we write $\lambda \vdash n$. The diagram $[\lambda]$ of the r-composition λ is the set $[\lambda] = \{(i,j,s) \mid 1 \le i \le \lambda_j^{(s)}, 1 \le s \le r\}$. The elements of $[\lambda]$ are called nodes. The set of r-compositions of n is partially ordered by dominance, i.e, if λ and μ are two r-compositions then λ dominates μ, and we write $\lambda \trianglerighteq \mu$, if

$$\sum_{c=1}^{s-1} |\lambda^{(c)}| + \sum_{j=1}^{i} |\lambda_j^{(s)}| \ge \sum_{c=1}^{s-1} |\mu^{(c)}| + \sum_{j=1}^{i} |\mu_j^{(s)}|$$

for $1 \le s \le r$ and for all $i \ge 1$. If $\lambda \trianglerighteq \mu$ and $\lambda \ne \mu$ then we write $\lambda \triangleright \mu$.

If λ is an r-composition let $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda^{(1)}} \times \cdots \times \mathfrak{S}_{\lambda^{(r)}}$ be the corresponding Young subgroup of $\mathfrak{S}_n$. Set

$$x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} q^{l(w)} T_w, \qquad u_\lambda^+ = \prod_{s=2}^{r} \prod_{k=1}^{a_s} (L_k - Q_s),$$

where $a_s = |\lambda^{(1)}| + \cdots + |\lambda^{(s-1)}|$ for $2 \le s \le r$. If $s = 1$ then we set $a_s = 0$. Set $m_\lambda = x_\lambda u_\lambda^+ = u_\lambda^+ x_\lambda$ and define M^λ to be the right ideal $M^\lambda = m_\lambda \mathscr{H}$ of $\mathscr{H}$.

For any r-composition μ, a μ-tableau $\mathfrak{t} = (\mathfrak{t}^{(1)}, \ldots, \mathfrak{t}^{(r)})$ is a bijection $\mathfrak{t} : [\mu] \to \{1, 2, \ldots, n\}$, where $\mathfrak{t}^{(i)}$ is a tableau of Shape$(\mathfrak{t}^{(i)}) = \mu^{(i)}$. We write Shape$(\mathfrak{t}) = \mu$ if $\mathfrak{t}$ is a μ-tableau. A μ-tableau $\mathfrak{t}$ is called standard (resp. row standard) if all $\mathfrak{t}^{(i)}$ are standard (resp. row standard). Let Std(λ) be the set of standard λ-tableaux.

For each r-composition μ, let $\mathfrak{t}^\mu$ be the μ-tableau with the numbers $1, 2, \ldots, n$ attached in order from left to right along its rows and from top to bottom, and from $\mu^{(1)}$ to $\mu^{(r)}$. If $\mathfrak{t}$ is any row standard μ-tableau let $d(\mathfrak{t}) \in \mathfrak{S}_n$ be the unique permutation such that $\mathfrak{t} = \mathfrak{t}^\mu d(\mathfrak{t})$. Furthermore, let $* : \mathscr{H} \to \mathscr{H}$ be the anti-isomorphism given by $T_i^* = T_i$ for $i = 1, 2, \ldots, n$, and set $m_{\mathfrak{s}\mathfrak{t}} = T_{d(\mathfrak{s})}^* m_\lambda T_{d(\mathfrak{t})}$.

Theorem 2.3 ([DJM, Theorem 3.26]). *The Ariki-Koike algebra $\mathscr{H}$ is free as an R-module with cellular basis $\{m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \text{Std}(\lambda) \text{ for some } \lambda \vdash n\}$.*

2.4. We can now give a definition of the cyclotomic q-Schur algebras. A set Λ of r-compositions of n is saturated if Λ is finite and whenever λ is an r-partition such that $\lambda \trianglerighteq \mu$ for some $\mu \in \Lambda$ then $\lambda \in \Lambda$. If Λ is a saturated set of r-compositions, we denote by Λ^+ be the set of r-partitions in Λ.

Definition 2.5. *Suppose that Λ is a saturated set of multicompositions of n. The cyclotomic q-Schur algebra with weight poset Λ is the endomorphism algebra*

$$\mathcal{S}(\Lambda) = \mathrm{End}_{\mathscr{H}}(M(\Lambda)), \qquad \text{where } M(\Lambda) = \bigoplus_{\lambda\in\Lambda} M^\lambda.$$

Let λ be an r-partition and μ an r-composition. A λ-Tableau of type μ is a map $T : [\lambda] \to \{(i,s) \mid i \geq 1,\ 1 \leq s \leq r\}$ such that $\mu_i^{(s)} = \sharp\{x \in [\lambda] \mid T(x) = (i,s)\}$ for all $i \geq 1$ and $1 \leq s \leq r$. We regard T as an r-tuple $T = (T^{(1)}, \dots, T^{(r)})$, where $T^{(s)}$ is the $\lambda^{(s)}$-tableau with $T^{(s)}(i,j) = T(i,j,s)$ for all $(i,j,s) \in [\lambda]$. In this way we identify the standard tableaux above with the Tableaux of type $w = ((0), \dots, (1^n))$. If T is a Tableau of type μ then we write $\mathrm{Type}(T) = \mu$.

Given two pairs (i,s) and (j,t) write $(i,s) \preceq (j,t)$ if either $s < t$, or $s = t$ and $i \leq j$.

Definition 2.6. *A Tableau T is (row) semistandard if, for $1 \leq t \leq r$, the entries in $T^{(t)}$ are*

(i) *weakly increasing along the rows with respect to $\preceq$,*
(ii) *strictly increasing down columns,*
(iii) *(i,s) appears in $T^{(t)}$ only if $s \geq t$.*

Let $\mathcal{T}_0(\lambda,\mu)$ be the set of semistandard λ-Tableaux of type μ and let $\mathcal{T}_0(\lambda) = \mathcal{T}_0^\Lambda(\lambda) = \bigcup_{\mu\in\Lambda} \mathcal{T}_0(\lambda,\mu)$. Notice that if $\mathcal{T}_0(\lambda,\mu)$ is non-empty, then $\lambda \trianglerighteq \mu$.

Suppose that $\mathfrak{t}$ is a standard λ-tableau and let μ be an r-composition. Let $\mu(\mathfrak{t})$ be the Tableau obtained from $\mathfrak{t}$ by replacing each entry j with (i,k) if j appears in row i of $(\mathfrak{t}^\mu)^{(k)}$. The tableau $\mu(\mathfrak{t})$ is a λ-Tableau of type μ. It is not necessarily semistandard. If S and T are semistandard λ-Tableaux of type μ and ν respectively, let

$$m_{ST} = \sum_{\substack{\mathfrak{s},\mathfrak{t}\in \mathrm{Std}(\lambda)\\ \mu(\mathfrak{s})=S,\ \nu(\mathfrak{t})=T}} q^{l(d(\mathfrak{s}))+l(d(\mathfrak{t}))} m_{\mathfrak{s}\mathfrak{t}}.$$

For S and T as above we define a map φ_{ST} on $M(\Lambda)$ by $\varphi_{ST}(m_\alpha h) = \delta_{\alpha\nu} m_{ST} h$, for all $h \in \mathscr{H}$ and all $\alpha \in \Lambda$. Here $\delta_{\alpha\nu}$ is the Kronecker delta, i.e, $\delta_{\alpha\nu} = 1$ if $\alpha = \nu$ and it is zero otherwise. Then φ_{ST} is well-defined, and it belongs to $\mathcal{S}(\Lambda)$. Moreover,

Theorem 2.7 ([DJM, Theorem 6.6]). *The cyclotomic q-Schur algebra $\mathcal{S}(\Lambda)$ is free as an R-module with cellular basis $\mathcal{C}(\Lambda) = \{\varphi_{ST} \mid S,T \in \mathcal{T}_0^\Lambda(\lambda)$ for some $\lambda \in \Lambda^+\}$.*

The basis $\{\varphi_{ST}\}$ is called a semistandard basis of $\mathcal{S}(\Lambda)$. Since this basis is cellular, the map $* : \mathcal{S}(\Lambda) \to \mathcal{S}(\Lambda)$ which is determined by $\varphi_{ST}^* = \varphi_{TS}$ is an anti-automorphism of $\mathcal{S}(\Lambda)$. This involution is closely related to the $*$-involution on $\mathscr{H}$. Explicitly, if $\varphi : M^\nu \to M^\mu$ is an $\mathscr{H}$-module homomorphism then $\varphi^* : M^\mu \to M^\nu$ is the homomorphism given by $\varphi^*(m_\mu h) = (\varphi(m_\nu))^* h$, for all $h \in \mathscr{H}$.

For each r-partition $\lambda \in \Lambda^+$, we define $\mathcal{S}^{\vee\lambda} = \mathcal{S}^\vee(\Lambda)^\lambda$ as the R-span of φ_{ST} such that $S,T \in \mathcal{T}_0^\Lambda(\alpha)$ with $\alpha \triangleright \lambda$, which is a two-sided ideal of $\mathcal{S}(\Lambda)$. We define the Weyl module W^λ by the right $\mathcal{S}(\Lambda)$-submodule of $\mathcal{S}(\Lambda)/\mathcal{S}^\vee(\Lambda)^\lambda$ generated by the image $\varphi_\lambda = \varphi_{T^\lambda T^\lambda} \in \mathcal{S}(\Lambda)$ where $T^\lambda = \lambda(\mathfrak{t}^\lambda)$. For each $T \in \mathcal{T}_0^\Lambda(\lambda)$, let φ_T be the image of $\varphi_{T^\lambda T}$ in W^λ. Then the Weyl module W^λ is R-free with basis $\{\varphi_T \mid T \in \mathcal{T}_0^\Lambda(\lambda)\}$. As in the case of Specht modules there is an inner product on W^λ which is determined by

$$\varphi_{T^\lambda S}\varphi_{TT^\lambda} \equiv \langle \varphi_S, \varphi_T\rangle \varphi_{T^\lambda T^\lambda} \mod \mathcal{S}^{\vee\lambda}.$$

Let $\mathrm{rad}W^\lambda = \{x \in W^\lambda \mid \langle x, y\rangle = 0 \text{ for all } y \in W^\lambda\}$. The quotient module $L^\lambda = W^\lambda/\mathrm{rad}W^\lambda$ is absolutely irreducible and $\{L^\lambda \mid \lambda \in \Lambda^+\}$ is a complete set of non-isomorphic irreducible $\mathcal{S}(\Lambda)$-modules.

2.8. For an r-composition μ, we define the type $\alpha = \alpha(\mu)$ of μ by $\alpha = (n_1, \dots, n_r)$ with $n_i = |\mu^{(i)}|$, and the size of μ by $n = \sum_{i=1}^r n_i$. We also define a sequence $\mathbf{a} = \mathbf{a}(\mu) = (a_1, \dots, a_r)$. (Recall that $a_i = \sum_{k=1}^{i-1} |\mu^{(k)}| = \sum_{k=1}^{i-1} n_k$.)

We define a partial order $\geq$ on the set $\mathbb{Z}_{\geq 0}^r$ by $\mathbf{a} \geq \mathbf{a}'$ for $\mathbf{a} = (a_1, \dots, a_r)$, $\mathbf{a}' = (a'_1, \dots, a'_r) \in \mathbb{Z}_{\geq 0}^r$ if $a_i \geq a'_i$ for any i. We write $\mathbf{a} > \mathbf{a}'$ if $\mathbf{a} \geq \mathbf{a}'$ and $\mathbf{a} \neq \mathbf{a}'$. It is clear that

(2.1) If $\lambda \trianglerighteq \mu$, then $\mathbf{a}(\lambda) \geq \mathbf{a}(\mu)$ for r-compositions λ, μ.

Hence if $\mathcal{T}_0(\lambda, \mu)$ is non-empty, then $\lambda \trianglerighteq \mu$, and so we have $\mathbf{a}(\lambda) \geq \mathbf{a}(\mu)$.

For any r-partition λ and r-composition μ, we define a subset $\mathcal{T}_0^+(\lambda, \mu)$ of $\mathcal{T}_0(\lambda, \mu)$ by

$$\mathcal{T}_0^+(\lambda, \mu) = \{S \in \mathcal{T}_0(\lambda, \mu) \mid \mathbf{a}(\lambda) = \mathbf{a}(\mu)\}.$$

Note that the condition $\mathbf{a}(\lambda) = \mathbf{a}(\mu)$ is equivalent to $\alpha(\lambda) = \alpha(\mu)$. Take $S \in \mathcal{T}_0^+(\lambda, \mu)$. Then one can check that $S \in \mathcal{T}_0^+(\lambda, \mu)$ if and only if each entry of $S^{(k)}$ is of the form (i, k) for some i. Hence in this case $S^{(k)}$ can be identified with a semistandard $\lambda^{(k)}$-Tableau of type $\mu^{(k)}$ under the usual definition of the semistandard Tableaux for 1-partitions $\lambda^{(k)}$ and 1-compositions $\mu^{(k)}$. It follows that we have a bijection

$$\mathcal{T}_0^+(\lambda, \mu) \simeq \mathcal{T}_0(\lambda^{(1)}, \mu^{(1)}) \times \cdots \times \mathcal{T}_0(\lambda^{(r)}, \mu^{(r)})$$

via $S \leftrightarrow (S^{(1)}, \dots, S^{(r)})$. Moreover, if $\mathfrak{s} \in \mathrm{Std}(\lambda)$ is such that $\mu(\mathfrak{s}) = S$ with $S \in \mathcal{T}_0^+(\lambda, \mu)$, then the entries of i-th component of $\mathfrak{s}$ consist of numbers $a_i + 1, \dots, a_{i+1}$ for $\mathbf{a}(\lambda) = (a_1, \dots, a_r)$. In particular, $d(\mathfrak{s}) \in \mathfrak{S}_\alpha$ for $\alpha = \alpha(\lambda)$.

Fix an r-tuple $\mathbf{m} = (m_1, \dots, m_r)$ of non-negative integers. Then, an r-composition $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$ with $\mu^{(i)} = (\mu_1^{(i)}, \dots, \mu_{m_i}^{(i)}) \in \mathbb{Z}_{\geq 0}^{m_i}$ is called an $(r, \mathbf{m})$-composition, and $(r, \mathbf{m})$-partition is defined similarly. We denote by $\widetilde{\mathcal{P}}_{n,r} = \widetilde{\mathcal{P}}_{n,r}(\mathbf{m})$ (resp. $\mathcal{P}_{n,r} = \mathcal{P}_{n,r}(\mathbf{m})$) the set of $(r, \mathbf{m})$-compositions (resp. $(r, \mathbf{m})$-partitions) of size n. (Note that $\mathcal{P}_{n,r}(\mathbf{m})$ are naturally identified with each other for any $\mathbf{m}$ such that $m_i \geq n$. However, $\widetilde{\mathcal{P}}_{n,r}$ depends on the choice of $\mathbf{m}$.) Finally, let

$$\begin{aligned}\mathcal{C}^0(\Lambda) = \bigcup_{\mu,\nu\in\Lambda,\ \lambda\in\Lambda^+} \{\varphi_{ST} \in \mathcal{C}(\Lambda) \mid\ & S \in \mathcal{T}_0(\lambda, \mu),\ T \in \mathcal{T}_0(\lambda, \nu), \\ & \mathbf{a}(\lambda) > \mathbf{a}(\mu) \text{ if } \alpha(\mu) \neq \alpha(\nu)\}\end{aligned}$$

and we define $\mathcal{S}^0(\Lambda)$ as the R-submodule of $\mathcal{S}(\Lambda)$ with basis $\mathcal{C}^0(\Lambda)$.

3. The standard basis for $\mathcal{S}^0(\Lambda)$

3.1. First, we prepare some notation. Let

$$\Omega = (\Lambda^+ \times \{0,1\}) \setminus \{(\lambda,1) \mid \mathcal{T}_0(\lambda,\mu) = \emptyset \text{ for any } \mu \in \Lambda \text{ such that } \mathbf{a}(\lambda) > \mathbf{a}(\mu)\}$$

and we define a partial order $(\lambda_1,\varepsilon_1) \geq (\lambda_2,\varepsilon_2)$ on Ω by $(\lambda_1,\varepsilon_1) > (\lambda_2,\varepsilon_2)$ if $\lambda_1 \rhd \lambda_2$, or $\lambda_1 = \lambda_2$ and $\varepsilon_1 > \varepsilon_2$. For a $(\lambda,\varepsilon) \in \Omega$, we define index sets $I(\lambda,\varepsilon)$, $J(\lambda,\varepsilon)$ by

$$I(\lambda,\varepsilon) = \begin{cases} \mathcal{T}_0^+(\lambda) & \text{if } \varepsilon = 0, \\ \bigcup\limits_{\mu\in\Lambda,\ \mathbf{a}(\lambda)>\mathbf{a}(\mu)} \mathcal{T}_0(\lambda,\mu) & \text{if } \varepsilon = 1, \end{cases} \qquad J(\lambda,\varepsilon) = \begin{cases} \mathcal{T}_0^+(\lambda) & \text{if } \varepsilon = 0, \\ \mathcal{T}_0(\lambda) & \text{if } \varepsilon = 1, \end{cases}$$

where $\mathcal{T}_0^+(\lambda) = \bigcup_{\mu\in\Lambda} \mathcal{T}_0^+(\lambda,\mu)$. Then $I(\lambda,\varepsilon)$ and $J(\lambda,\varepsilon)$ are not empty for all $(\lambda,\varepsilon) \in \Omega$. Assume that $(\lambda,\varepsilon) \in \Omega$. We define a subset $\mathcal{C}^0(\lambda,\varepsilon)$ of $\mathcal{S}^0(\Lambda)$ by

$$\mathcal{C}^0(\lambda,\varepsilon) = \{\varphi_{ST} \mid (S,T) \in I(\lambda,\varepsilon) \times J(\lambda,\varepsilon)\}.$$

It is easy to see that

(3.1) $\qquad$ the union $\bigcup\limits_{(\lambda,\varepsilon)\in\Omega} \mathcal{C}^0(\lambda,\varepsilon)$ is disjoint and is equal to the set $\mathcal{C}^0(\Lambda)$.

3.2. For any $(\lambda,\varepsilon) \in \Omega$, we define by $\mathcal{S}_0^{\vee(\lambda,\varepsilon)} = \mathcal{S}^0(\Lambda)(> (\lambda,\varepsilon))$ the R-submodule of $\mathcal{S}^0(\Lambda)$ spanned by φ_{UV} where $(U,V) \in I(\lambda',\varepsilon') \times J(\lambda',\varepsilon')$ for some $(\lambda',\varepsilon') \in \Omega$ with $(\lambda',\varepsilon') > (\lambda,\varepsilon)$. Note that $\mathcal{S}^0(\Lambda) \cap \mathcal{S}^{\vee\lambda} = \mathcal{S}_0^{\vee(\lambda,1)}$ for every $\lambda \in \Lambda^+$. Similarly, we define $\mathcal{S}^0(\Lambda)(\geq (\lambda,\varepsilon))$ as the R-submodule spanned by φ_{UV} with $(\lambda',\varepsilon') \geq (\lambda,\varepsilon)$. We can now state.

Theorem 3.1. *The subalgebra $\mathcal{S}^0(\Lambda)$ is standardly based (in the sense of* [DR]*) on $(\Omega,\geq)$ with standard basis $\mathcal{C}^0(\Lambda)$, that is,*

(i) *The union $\bigcup\limits_{(\lambda,\varepsilon)\in\Omega} \mathcal{C}^0(\lambda,\varepsilon) = \mathcal{C}^0(\Lambda)$ is disjoint and forms an R-basis for $\mathcal{S}^0(\Lambda)$.*

(ii) *For any $\varphi \in \mathcal{S}^0(\Lambda)$, $\varphi_{ST} \in \mathcal{C}^0(\lambda,\varepsilon)$, we have*

$$\begin{aligned} \varphi\cdot\varphi_{ST} &\equiv \sum_{S'\in I(\lambda,\varepsilon)} f_{S',(\lambda,\varepsilon)}(\varphi,S)\cdot\varphi_{S'T} \mod \mathcal{S}_0^{\vee(\lambda,\varepsilon)} \\ \varphi_{ST}\cdot\varphi &\equiv \sum_{T'\in J(\lambda,\varepsilon)} f_{(\lambda,\varepsilon),T'}(T,\varphi)\cdot\varphi_{ST'} \mod \mathcal{S}_0^{\vee(\lambda,\varepsilon)}, \end{aligned} \tag{3.2}$$

where $\varphi_{S'T}, \varphi_{ST'} \in \mathcal{C}^0(\Lambda)$ and $f_{S',(\lambda,\varepsilon)}(\varphi,S)$, $f_{(\lambda,\varepsilon),T'}(T,\varphi) \in R$ are independent of T and S, respectively.

Note that the cellular algebra is a special case of the standardly based.

3.3. Next we introduce the Weyl module for $\mathcal{S}^0(\Lambda)$. By (3.2) in Theorem 3.1, it is easy to see that R-modules $\mathcal{S}^0(\Lambda)(\geq (\lambda,\varepsilon))$ and $\mathcal{S}_0^{\vee(\lambda,\varepsilon)} = \mathcal{S}^0(\Lambda)(> (\lambda,\varepsilon))$ are two-sided ideals of $\mathcal{S}^0(\Lambda)$. Fix a $(\lambda,\varepsilon) \in \Omega$. For $S \in I(\lambda,\varepsilon)$, we define the Weyl module $Z_S^{(\lambda,\varepsilon)}$ for $\mathcal{S}^0(\Lambda)$ by the R-submodule of $\{\mathcal{S}^0(\Lambda)(\geq (\lambda,\varepsilon))\}/\{\mathcal{S}^0(\Lambda)(> (\lambda,\varepsilon))\}$ with basis $\{\varphi_{ST}+\mathcal{S}_0^{\vee(\lambda,\varepsilon)} \mid T\in J(\lambda,\varepsilon)\}$. Moreover, by (3.2), we see that $Z_S^{(\lambda,\varepsilon)}$ is the right $\mathcal{S}^0(\Lambda)$-module and the action of $\mathcal{S}^0(\Lambda)$ on $Z_S^{(\lambda,\varepsilon)}$ is independent of the choice of S, i.e, $Z_{S_1}^{(\lambda,\varepsilon)}\simeq Z_{S_2}^{(\lambda,\varepsilon)}$ for all $S_1,S_2\in I(\lambda,\varepsilon)$. However, since T^λ is not an element in $I(\lambda,1)$ for $(\lambda,1)\in\Omega$, one should pay attention that there is no "canonical"-Weyl module for the case $(\lambda,1)$. (That is, we can not define $Z_{T^\lambda}^{(\lambda,1)}$.) For the convenience sake let $Z^{(\lambda,0)}=Z_{T^\lambda}^{(\lambda,0)}$ and put $\varphi_T^0=\varphi_{T^\lambda T}+\mathcal{S}_0^{\vee(\lambda,\varepsilon)}$ for any $T\in J(\lambda,0)=\mathcal{T}_0^+(\lambda)$.

3.4. Suppose that $S,T\in\mathcal{T}_0^+(\lambda)$. Then there exists an element $r_{ST}\in R$ such that for any $U,V\in\mathcal{T}_0^+(\lambda)$

$$\varphi_{US}\cdot\varphi_{TV}\equiv r_{ST}\cdot\varphi_{UV}\mod \mathcal{S}_0^{\vee(\lambda,0)}.$$

We define a bilinear form $\langle\ ,\ \rangle_0 : Z^{(\lambda,0)}\times Z^{(\lambda,0)}\to R$ by $\langle\varphi_S^0,\varphi_T^0\rangle_0=r_{ST}$. Hence we have

$$\langle\varphi_S^0,\varphi_T^0\rangle_0\cdot\varphi_{UV}\equiv\varphi_{US}\cdot\varphi_{TV}\mod \mathcal{S}_0^{\vee(\lambda,0)}, \tag{3.3}$$

where U and V are any elements of $\mathcal{T}_0^+(\lambda)$. It is easy to see that

$$\langle\varphi_S^0,\varphi_T^0\rangle_0=\langle\varphi_S,\varphi_T\rangle\qquad\text{for every } S,T\in\mathcal{T}_0^+(\lambda). \tag{3.4}$$

Let $\mathrm{rad}Z^{(\lambda,0)}=\{x\in Z^{(\lambda,0)}\mid\langle x,y\rangle_0=0\text{ for all } y\in Z^{(\lambda,0)}\}$.

Lemma 3.2. $\mathrm{rad}Z^{(\lambda,0)}$ *is an* $\mathcal{S}^0(\Lambda)$*-submodule of* $Z^{(\lambda,0)}$.

We put $L_0^\lambda=Z^{(\lambda,0)}/\mathrm{rad}Z^{(\lambda,0)}$. Then we have the following.

Proposition 3.3. *Suppose that R is a field, and $\lambda\in\Lambda^+$. Then*
(i) $L_0^\lambda\neq 0$ *and*
(ii) $\mathrm{rad}Z^{(\lambda,0)}$ *is the unique maximal submodule of* $Z^{(\lambda,0)}$ *and* L_0^λ *is absolutely irreducible. Moreover, the Jacobson radical of* $Z^{(\lambda,0)}$ *is equal to* $\mathrm{rad}Z^{(\lambda,0)}$.

4. A relationship between $\mathcal{S}^\flat(\mathbf{m},n)$ and $\mathcal{S}^0(\Lambda)$

First, we recall the definition of modified Ariki-Koike algebras and their cyclotomic q-Schur algebras ([SawS]).

4.1. From now on, throughout this paper, we consider the following condition on parameters $Q_1,\dots,Q_r$ in R whenever we consider the modified Ariki-Koike algebras (and their cyclotomic q-Schur algebras).

(4.1) Q_i-Q_j are invertible in R for any $i\neq j$.

Let A be a square matrix of degree r whose i-j entry is given by Q_j^{i-1} for $1\le i,j\le r$. Thus A is the Vandermonde matrix, and $\Delta=\det A=\prod_{i>j}(Q_i-Q_j)$ is invertible by (4.1). We express the inverse of A as $A^{-1}=\Delta^{-1}B$ with $B=(h_{ij})$, and define a polynomial $F_i(X)\in R[X]$, for $1\le i\le r$, by $F_i(X)=\sum_{1\le j\le r}h_{ij}X^{j-1}$.

The modified Ariki-Koike algebra $\mathscr{H}^\flat = \mathscr{H}^\flat_{n,r}$ is an associative algebra over R with generators $T_2, \cdots, T_n$ and $\xi_1, \ldots, \xi_n$ and relations

(4.2)

$$
\begin{aligned}
&(T_i - q)(T_i + q^{-1}) = 0 && (2 \le i \le n),\\
&(\xi_i - Q_1)\cdots(\xi_i - Q_r) = 0 && (1 \le i \le n),\\
&T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} && (2 \le i \le n),\\
&T_iT_j = T_jT_i && (|i-j| \ge 2),\\
&\xi_i\xi_j = \xi_j\xi_i && (1 \le i, j \le n),\\
&T_j\xi_j = \xi_{j-1}T_j + \Delta^{-2}\sum_{c_1<c_2}(Q_{c_2} - Q_{c_1})(q - q^{-1})F_{c_1}(\xi_{j-1})F_{c_2}(\xi_j), &&\\
&T_j\xi_{j-1} = \xi_jT_j - \Delta^{-2}\sum_{c_1<c_2}(Q_{c_2} - Q_{c_1})(q - q^{-1})F_{c_1}(\xi_{j-1})F_{c_2}(\xi_j), &&\\
&T_j\xi_k = \xi_jT_j && (k \ne j-1, j).
\end{aligned}
$$

It is known that if $R = \mathbb{Q}(\overline{q}, \overline{Q}_1, \ldots, \overline{Q}_r)$, the field of rational functions with variables $\overline{q}, \overline{Q}_1, \ldots, \overline{Q}_r$, $\mathscr{H}^\flat$ is isomorphic to $\mathscr{H}$, and it gives an alternate presentation of $\mathscr{H}$ apart from 2.1.

The subalgebra $\mathscr{H}^\flat(\mathfrak{S}_n)$ of $\mathscr{H}^\flat$ generated by $T_2, \ldots, T_n$ is isomorphic to $\mathscr{H}_n$, hence it can be naturally identified with the corresponding subalgebra $\mathscr{H}(\mathfrak{S}_n)$ of $\mathscr{H}$. Moreover, it is known by [Sh] that the set $\{\xi_1^{c_1}\cdots\xi_n^{c_n}T_w \mid w \in \mathfrak{S}_n,\ 0 \le c_i < r \text{ for } 1 \le i \le n\}$ gives rise to a basis of $\mathscr{H}^\flat$.

Let $V = \bigoplus_{i=1}^r V_i$ be a free R-module, with rank $V_i = m_i$. We put $m = \sum m_i$. It is known by [SakS] that we can define a right $\mathscr{H}$-module structure on $V^{\otimes n}$. We denote this representation by $\rho : \mathscr{H} \to \operatorname{End} V^{\otimes n}$. Note that this construction works without the condition (4.1). Also it is shown in [Sh] that, under the assumption (4.1), a right action of $\mathscr{H}^\flat$ on $V^{\otimes n}$ can be defined. We denote this representation by $\rho^\flat : \mathscr{H}^\flat \to \operatorname{End} V^{\otimes n}$. By [Sh, Lemma 3.5], we know that $\operatorname{Im}\rho \subset \operatorname{Im}\rho^\flat$.

We consider the condition

(4.3) $\quad m_i \ge n$ for $i = 1, \cdots, r$.

Lemma 4.2 ([SawS, Lemma 1.5]). *Under the conditions* (4.1), (4.3), *there exists an R-algebra homomorphism $\rho_0 : \mathscr{H} \to \mathscr{H}^\flat$ such that ρ_0 induces the identity on $\mathscr{H}_n$. (Here we regard $\mathscr{H}_n \subset \mathscr{H}$, $\mathscr{H}_n \subset \mathscr{H}^\flat$ under the previous identifications.) If $\operatorname{Im}\rho^\flat = \operatorname{Im}\rho$ and R is a field, then $\mathscr{H} \simeq \mathscr{H}^\flat$.*

From now on, throughout the paper, we fix an r-tuple $\mathbf{m} = (m_1, \ldots, m_r)$ of non-negative integers and always assume the condition (4.3) whenever we consider $\mathscr{H}^\flat$.

Any $\mu \in \widetilde{\mathcal{P}}_{n,r}(\mathbf{m})$ may be regarded as an element in $\mathcal{P}_{n,1}$ (i.e, 1-composition) of n by arranging the entries of $\mu = (\mu_j^{(i)})$ in order

$$\mu_1^{(1)}, \ldots, \mu_{m_1}^{(1)}, \mu_1^{(2)}, \ldots, \mu_{m_2}^{(2)}, \ldots, \mu_1^{(r)}, \ldots, \mu_{m_r}^{(r)},$$

which we denote by $\{\mu\}$.

For $\alpha = (n_1, \ldots, n_r) \in \mathbb{Z}_{\ge 0}$ such that $\sum n_i = n$, we define $c(\alpha)$ by

$$c(\alpha) = (\underbrace{r, \ldots, r}_{n_1\text{-times}}, \underbrace{r-1, \ldots, r-1}_{n_2\text{-times}}, \ldots, \underbrace{1, \ldots, 1}_{n_r\text{-times}})$$

and let $c(\alpha) = (c_1, \dots, c_n)$. We define $F_\alpha \in \mathscr{H}^\flat$ by $F_\alpha = \Delta^{-n} F_{c_1}(\xi_1) F_{c_2}(\xi_2) \cdots F_{c_n}(\xi_n)$. For any $\mu \in \widetilde{\mathcal{P}}_{n,r}$, put $m^\flat_\mu = F_{\alpha(\mu)} \cdot m_{\{\mu\}}$ where $m_{\{\mu\}} = \sum_{w \in \mathfrak{S}_{\{\mu\}}} q^{l(w)} T_w \ (= x_\mu) \in \mathscr{H}_n$.

We define an R-linear anti-automorphism $h \to h^*$ on $\mathscr{H}^\flat$ by the condition that $*$ fixes the generators T_i $(2 \le i \le n)$ and ξ_j $(1 \le j \le n)$. As discussed in [SawS, 2.7], this condition induces a well-defined anti-automorphism on $\mathscr{H}^\flat$. Moreover, by Lemma 2.9 in [SawS], we know that $(m^\flat_\mu)^* = m^\flat_\mu$. For $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$ with $\lambda \in \mathcal{P}_{n,r}$, we define an element $m^\flat_{\mathfrak{st}} \in \mathscr{H}^\flat$ by $m^\flat_{\mathfrak{st}} = T^*_{d(\mathfrak{s})} m^\flat_\mu T_{d(\mathfrak{t})}$. By the above fact, we have $(m^\flat_{\mathfrak{st}})^* = m^\flat_{\mathfrak{ts}}$.

Theorem 4.3 ([SawS, Theorem 2.18]). *The modified Ariki-Koike algebra $\mathscr{H}^\flat$ is free as an R-module with cellular basis $\{m^\flat_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$ for some $\lambda \in \mathcal{P}_{n,r}\}$.*

Put $M^\mu_\flat = m^\flat_\mu \mathscr{H}^\flat$ for $\mu \in \widetilde{\mathcal{P}}_{n,r}$. We define a cyclotomic q-Schur algebra $\mathcal{S}^\flat(\mathbf{m}, n)$ as follows.

Definition 4.4. *The cyclotomic q-Schur algebra for $\mathscr{H}^\flat$ with weight poset $\widetilde{\mathcal{P}}_{n,r}$ is the endomorphism algebra*

$$\mathcal{S}^\flat(\mathbf{m}, n) = \mathrm{End}_{\mathscr{H}^\flat}(M^\flat(\widetilde{\mathcal{P}}_{n,r})), \qquad \textit{where } M^\flat(\widetilde{\mathcal{P}}_{n,r}) = \bigoplus_{\mu \in \widetilde{\mathcal{P}}_{n,r}} M^\mu_\flat.$$

For an r-tuples $\alpha \in \widetilde{\mathcal{P}}_{n,1}$, let $M^\alpha_\flat = \bigoplus_{\mu; \alpha(\mu) = \alpha} M^\mu_\flat$. Then by Proposition 5.2 (i) in [SawS], we have $\mathcal{S}^\flat(\mathbf{m}, n) \simeq \bigoplus_{\alpha \in \widetilde{\mathcal{P}}_{n,1}} \mathrm{End}_{\mathscr{H}^\flat} M^\alpha_\flat$ as R-algebras.

Theorem 4.5 ([SawS, Theorem 5.5]). *Let $\mathcal{S}^\flat(\mathbf{m}, n)$ be the cyclotomic q-Schur algebra associated to the modified Ariki-Koike algebra $\mathscr{H}^\flat$ and $\mathcal{S}(m_i, n_i)$ be the q-Schur algebra associated to the Iwahori-Hecke algebra $\mathscr{H}_{n_i}$. Then there exists an isomorphism of R-algebras*

$$\mathcal{S}^\flat(\mathbf{m}, n) \simeq \bigoplus_{\substack{(n_1, \dots, n_r) \\ n = n_1 + \cdots + n_r}} \mathcal{S}(m_1, n_1) \otimes \cdots \otimes \mathcal{S}(m_r, n_r).$$

Let $\mu, \nu \in \widetilde{\mathcal{P}}_{n,r}$ and $\lambda \in \mathcal{P}_{n,r}$. We assume that $\alpha(\mu) = \alpha(\nu) = \alpha(\lambda)$. For $S \in \mathcal{T}^+_0(\lambda, \mu)$ and $T \in \mathcal{T}^+_0(\lambda, \nu)$, put

$$m^\flat_{ST} = \sum_{\substack{\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda) \\ \mu(\mathfrak{s}) = S, \ \nu(\mathfrak{t}) = T}} q^{l(d(\mathfrak{s})) + l(d(\mathfrak{t}))} m^\flat_{\mathfrak{st}}.$$

Moreover, for $S \in \mathcal{T}^+_0(\lambda, \mu)$ and $T \in \mathcal{T}^+_0(\lambda, \nu)$, one can define $\varphi^\flat_{ST} \in \mathcal{S}^\flat(\mathbf{m}, n)$ by $\varphi^\flat_{ST}(m^\flat_\alpha h) = \delta_{\alpha\nu} m^\flat_{ST} h$, for all $h \in \mathscr{H}^\flat$ and all $\alpha \in \widetilde{\mathcal{P}}_{n,r}$.

Theorem 4.6 ([SawS, Theorem 5.9]). *The cyclotomic q-Schur algebra $\mathcal{S}^\flat(\mathbf{m}, n)$ is free as an R-module with cellular basis $\mathcal{C}^\flat(\mathbf{m}, n) = \{\varphi^\flat_{ST} \mid S, T \in \mathcal{T}^+_0(\lambda)$, for some $\lambda \in \mathcal{P}_{n,r}\}$.*

4.2. Let $\mathcal{S}^0(\Lambda)$ be as in Section 3. We describe a relationship between the algebra $\mathcal{S}^0(\Lambda)$ and the cyclotomic q-Schur algebra $\mathcal{S}^\flat(\mathbf{m}, n)$ in the case where $\Lambda = \widetilde{\mathcal{P}}_{n,r}$. But in the moment, we shall consider an arbitrary Λ as in Section 3.

First, let $\mathcal{C}^{00}(\Lambda) = \{\varphi_{ST} \mid (S,T) \in I(\lambda,1) \times J(\lambda,1),\ \lambda \in \Lambda^+\} \subset \mathcal{C}^0(\Lambda)$ and $\mathcal{S}^{00}(\Lambda)$ be the R-span of $\varphi_{ST} \in \mathcal{C}^{00}(\Lambda)$, which is an R-submodule of $\mathcal{S}^0(\Lambda)$. We note that, $\mathcal{S}^{00}(\Lambda)$ is a two-sided ideal of $\mathcal{S}^0(\Lambda)$ by the second and fourth formula in [Sa, Lemma 2.4]. Thus one can define the quotient algebra $\overline{\mathcal{S}^0}(\Lambda) = \mathcal{S}^0(\Lambda)/\mathcal{S}^{00}(\Lambda)$. We write $\overline{x} = x + \mathcal{S}^{00}(\Lambda)$ $(x \in \mathcal{S}^0(\Lambda))$. It is easy to see that $\overline{\mathcal{S}^0}(\Lambda)$ has a free R-basis $\{\overline{\varphi}_{ST} \mid S \in I(\lambda,0),\ T \in J(\lambda,0),\ \lambda \in \Lambda^+\}$. Note that the condition $(S,T) \in I(\lambda,0) \times J(\lambda,0)$ is nothing but $S,T \in \mathcal{T}_0^+(\lambda)$. For $\lambda \in \Lambda^+$, let $\overline{\mathcal{S}_0}^{\vee\lambda} = \overline{\mathcal{S}_0}^{\vee}(\Lambda)^{\lambda}$ be the R-submodule of $\overline{\mathcal{S}^0}(\Lambda)$ spanned by $\overline{\varphi}_{ST}$ with $S,T \in \mathcal{T}_0^+(\alpha)$ for various $\alpha \in \Lambda^+$ such that $\alpha \rhd \lambda$. We show the following.

Theorem 4.7. *The algebra $\overline{\mathcal{S}^0}(\Lambda)$ has a free basis*

$$\overline{\mathcal{C}^0}(\Lambda) = \{\overline{\varphi}_{ST} \mid S,T \in \mathcal{T}_0^+(\lambda),\ \lambda \in \Lambda^+\}$$

satisfying the following properties.

(i) *The R-linear map $*$: $\overline{\mathcal{S}^0}(\Lambda) \to \overline{\mathcal{S}^0}(\Lambda)$ determined by $\overline{\varphi}^*_{ST} = \overline{\varphi}_{TS}$, for all $S,T \in \mathcal{T}_0^+(\lambda)$ and all $\lambda \in \Lambda^+$, is an anti-automorphism of $\overline{\mathcal{S}^0}(\Lambda)$.*

(ii) *Let $T \in \mathcal{T}_0^+(\lambda)$. Then for all $\overline{\varphi} \in \overline{\mathcal{S}^0}(\Lambda)$, and any $V \in \mathcal{T}_0^+(\lambda)$, there exists $r_V \in R$ such that*

$$\overline{\varphi}_{ST} \cdot \overline{\varphi} \equiv \sum_{V \in \mathcal{T}_0^+(\lambda)} r_V \overline{\varphi}_{SV} \mod \overline{\mathcal{S}_0}^{\vee\lambda}$$

for any $S \in \mathcal{T}_0^+(\lambda)$, where r_V is independent of the choice of T.

In particular, $\overline{\mathcal{C}^0}(\Lambda)$ is a cellular basis of $\overline{\mathcal{S}^0}(\Lambda)$.

In the case where $\mathcal{S}^\flat(\mathbf{m},n)$ is defined, $\overline{\mathcal{S}^0}(\Lambda)$ can be identified with $\mathcal{S}^\flat(\mathbf{m},n)$, i.e, we have the following proposition.

Proposition 4.8. *Let $\Lambda = \widetilde{\mathcal{P}}_{n,r}$ and assume that* (4.1) *and* (4.3) *holds. Then there exists an algebra isomorphism $\flat$: $\overline{\mathcal{S}^0}(\Lambda) \to \mathcal{S}^\flat(\mathbf{m},n)$ satisfying the following. For $\overline{\varphi}_{ST} \in \overline{\mathcal{C}^0}(\Lambda)$ such that $S,T \in \mathcal{T}_0^+(\lambda)$ and $\lambda \in \Lambda^+$, we have $(\overline{\varphi}_{ST})^\flat = \varphi^\flat_{ST}$.*

We now return to the general setting, and consider $\overline{\mathcal{S}^0}(\Lambda)$ for arbitrary Λ. The above proposition says that the $\overline{\mathcal{S}^0}(\Lambda)$ is a natural "cover" of the $\mathcal{S}^\flat(\mathbf{m},n)$.

For $\lambda \in \Lambda^+$, $\overline{\varphi}_\lambda = \overline{\varphi}_{T^\lambda T^\lambda}$ is an element in $\overline{\mathcal{S}^0}(\Lambda)$. Hence, by the cellular theory [GL], one can define a Weyl module $\overline{Z}^\lambda$ of $\overline{\mathcal{S}^0}(\Lambda)$ as the right $\overline{\mathcal{S}^0}(\Lambda)$-submodule of $\overline{\mathcal{S}^0}(\Lambda)/\overline{\mathcal{S}_0}^{\vee\lambda}$ spanned by the image of $\overline{\varphi}_\lambda$. We denote by $\overline{\varphi}_T$ the image of $\overline{\varphi}_{T^\lambda T}$ in $\overline{\mathcal{S}^0}(\Lambda)/\overline{\mathcal{S}_0}^{\vee\lambda}$. Then the set $\{\overline{\varphi}_T \mid T \in \mathcal{T}_0^+(\lambda)\}$ is a free R-basis of $\overline{Z}^\lambda$. Define a bilinear form $\langle\ ,\ \rangle_{\overline{0}}$ on $\overline{Z}^\lambda$ by requiring that

$$\overline{\varphi}_{T^\lambda S}\overline{\varphi}_{TT^\lambda} \equiv \langle \overline{\varphi}_S, \overline{\varphi}_T\rangle_{\overline{0}} \cdot \overline{\varphi}_\lambda \mod \overline{\mathcal{S}_0}^{\vee\lambda}$$

for all $S,T \in \mathcal{T}_0^+(\lambda)$. Let $\overline{L}^\lambda = \overline{Z}^\lambda/\mathrm{rad}\overline{Z}^\lambda$, where $\mathrm{rad}\overline{Z}^\lambda = \{x \in \overline{Z}^\lambda \mid \langle x,y\rangle_{\overline{0}} = 0 \text{ for all } y \in \overline{Z}^\lambda\}$. In the case where R is a field, by a general theory of cellular algebras, the set $\{\overline{L}^\lambda \mid \lambda \in \Lambda^+,\ \overline{L}^\lambda \neq 0\}$ gives a complete set of non-isomorphic irreducible $\overline{\mathcal{S}^0}(\Lambda)$-modules. Furthermore, we have the following result.

Proposition 4.9. *Suppose that R is a field. Then $\overline{L}^{\lambda} \neq 0$ for any $\lambda \in \Lambda^+$. Hence, $\{\overline{L}^{\lambda} \mid \lambda \in \Lambda^+\}$ is a complete set of non-isomorphic irreducible $\overline{\mathcal{S}^0}(\Lambda)$-modules. Therefore, $\overline{\mathcal{S}^0}(\Lambda)$ is quasi-hereditary.*

The following result connects the decomposition numbers in $\overline{Z}^{\lambda}$ and in $Z^{(\lambda,0)}$.

Theorem 4.10. *Suppose that R is a field. Then*

(i) *$\{L_0^{\alpha} \mid \alpha \in \Lambda^+,\ \lambda \trianglerighteq \alpha\}$ is a complete set of pairwise inequivalent irreducible $\mathcal{S}^0(\Lambda)$-modules occurring in the composition factors of the $\mathcal{S}^0(\Lambda)$-module $Z^{(\lambda,0)}$.*

(ii) *For $\lambda,\ \mu \in \Lambda^+$, we have*

$$[\overline{Z}^{\lambda} : \overline{L}^{\mu}] = [Z^{(\lambda,0)} : L_0^{\mu}].$$

(iii) *For $\lambda,\ \mu \in \Lambda^+$ such that $\alpha(\lambda) \neq \alpha(\mu)$, we have*

$$[\overline{Z}^{\lambda} : \overline{L}^{\mu}] = 0.$$

5. An estimate for decomposition numbers

We are now ready to estimate the decomposition numbers for the cyclotomic q-Schur algebras.

5.1. We keep the notation in Section 4, and consider the general Λ.

Theorem 5.1. *Suppose that R is a field. Then, for all $\lambda, \mu \in \Lambda^+$ with $\alpha(\lambda) = \alpha(\mu)$,*

$$[\overline{Z}^{\lambda} : \overline{L}^{\mu}] = [Z^{(\lambda,0)} : L_0^{\mu}] = [W^{\lambda} : L^{\mu}].$$

5.8. We return to the setting in 4.1. Let $\Lambda = \widetilde{\mathcal{P}}_{n,r}$ under the condition (4.1) and (4.3). For an r-partition $\lambda \in \mathcal{P}_{n,r}$, we denote by $\mathcal{S}_{\flat}^{\vee\lambda}$ the R-submodule of $\mathcal{S}^{\flat}(\mathbf{m}, n)$ spanned by $\varphi_{ST}^{\flat}$ such that $S, T \in \mathcal{T}_0^+(\alpha)$ with $\alpha \rhd \lambda$. Moreover, for an r-partition $\lambda \in \mathcal{P}_{n,r}$, $T^{\lambda} \in \mathcal{T}_0^+(\lambda, \lambda)$, and in fact T^{λ} is the unique semistandard λ-Tableau of type λ. Moreover, $\mathfrak{t} = \mathfrak{t}^{\lambda}$ is the unique element in $\mathrm{Std}(\lambda)$ such that $\lambda(\mathfrak{t}) = T^{\lambda}$. Thus, $m_{T^{\lambda}T^{\lambda}}^{\flat} = m_{\mathfrak{t}^{\lambda}\mathfrak{t}^{\lambda}}^{\flat} = m_{\lambda}^{\flat}$, and $\varphi_{\lambda}^{\flat} = \varphi_{T^{\lambda}T^{\lambda}}^{\flat}$ is the identity map on $M_{\flat}^{\lambda}$. We define the Weyl module $W_{\flat}^{\lambda}$ as the right $\mathcal{S}^{\flat}(\mathbf{m}, n)$-submodule of $\mathcal{S}^{\flat}(\mathbf{m}, n)/\mathcal{S}_{\flat}^{\vee\lambda}$ spanned by the image of $\varphi_{\lambda}^{\flat}$. For each $T \in \mathcal{T}_0^+(\lambda, \mu)$, we denote by $\varphi_T^{\flat}$ the image of $\varphi_{T^{\lambda}T}^{\flat}$ in $\mathcal{S}^{\flat}(\mathbf{m}, n)/\mathcal{S}_{\flat}^{\vee\lambda}$. Then we know that the Weyl module $W_{\flat}^{\lambda}$ is R-free with basis $\{\varphi_T^{\flat} \mid T \in \mathcal{T}_0^+(\lambda)\}$. The Weyl module $W_{\flat}^{\lambda}$ enjoys an associative symmetric bilinear form, defined by the equation

$$\varphi_{T^{\lambda}S}^{\flat}\varphi_{TT^{\lambda}}^{\flat} \equiv \langle \varphi_S^{\flat}, \varphi_T^{\flat} \rangle_{\flat} \cdot \varphi_{\lambda}^{\flat} \mod \mathcal{S}_{\flat}^{\vee\lambda}$$

for all $S, T \in \mathcal{T}_0^+(\lambda)$. Let $L_{\flat}^{\lambda} = W_{\flat}/\mathrm{rad}W_{\flat}^{\lambda}$, where $\mathrm{rad}W_{\flat}^{\lambda} = \{x \in W_{\flat}^{\lambda} \mid \langle x, y \rangle_{\flat} = 0 \text{ for all } y \in W_{\flat}^{\lambda}\}$. By [SawS, Proposition 5.11], we know that, for all r-partition $\lambda \in \mathcal{P}_{n,r}$, $L_{\flat}^{\lambda}$ is an absolutely irreducible and $\{L_{\flat}^{\lambda} \mid \lambda \in \mathcal{P}_{n,r}\}$ is a complete set of non-isomorphic irreducible $\mathcal{S}^{\flat}(\mathbf{m}, n)$-modules. Furthermore, for $\lambda, \mu \in \mathcal{P}_{n,r}$, we denote by $[W_{\flat}^{\lambda} : L_{\flat}^{\mu}]$ the composition multiplicity of $L_{\flat}^{\mu}$ in $W_{\flat}^{\lambda}$. Note that the above definition of the Weyl module $W_{\flat}^{\lambda}$ coincides with the definition of the Weyl module $\overline{Z}^{\lambda}$ when $\mathcal{S}^{\flat}(\mathbf{m}, n)$ is isomorphic to $\overline{\mathcal{S}^0}(\Lambda)$ under the isomorphism $\flat$ in Proposition 4.8.

Consequently, under the isomorphism $\flat$, we have $[W_\flat^\lambda : L_\flat^\mu] = [\overline{Z}^\lambda : \overline{L}^\mu]$ for every $\lambda, \mu \in \mathcal{P}_{n,r}$. On the other hand, note that in the case where $r = 1$, the notation for $\mathcal{S}^\flat(\mathbf{m}, n)$ coincides with the standard notation for q-Schur algebras discussed as in [Ma, Chapter 4]. So, we use freely such a notation. For $\lambda, \mu \in \mathcal{P}_{n,r}$, we denote by $[W^{\lambda^{(i)}} : L^{\mu^{(i)}}]$ $(1 \le i \le r)$ is defined as the composition multiplicity of $L^{\mu^{(i)}}$ in $W^{\lambda^{(i)}}$ for $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ and $\mu = (\mu^{(i)}, \dots, \mu^{(r)})$.

Proposition 5.2 ([SawS, Proposition 5.14]). *Let $\Lambda = \widetilde{\mathcal{P}}_{n,r}$. Suppose that R is a field, and that* (4.1) *and* (4.3) *are satisfied. Let λ, $\mu \in \mathcal{P}_{n,r}$. Then under the isomorphism in Theorem 4.5, we have*

$$[W^\lambda : L^\mu] = \begin{cases} \prod_{i=1}^r [W^{\lambda^{(i)}} : L^{\mu^{(i)}}] & \text{if } \alpha(\lambda) = \alpha(\mu), \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 5.3. *Let $\Lambda = \widetilde{\mathcal{P}}_{n,r}$. Suppose that R is a field, and that* (4.1) *and* (4.3) *are satisfied. Then, for all $\lambda, \mu \in \mathcal{P}_{n,r}$ with $\alpha(\lambda) = \alpha(\mu)$, we have*

$$[W^\lambda : L^\mu] = \prod_{i=1}^r [W^{\lambda^{(i)}} : L^{\mu^{(i)}}].$$

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