On applications of the cellular algebras

Nobuharu Sawada

Department of Mathematics
Tokyo University of Science

ABSTRACT. In this report we explain briefly the results of parts of papers [SawS] and [Sa].

1. CELLULAR ALGEBRAS

1.1. Cellular bases. We begin with the definition of a cellular basis.

Let $R$ be a commutative domain with 1 and $A$ an associative unital $R$-algebra which is free as an $R$-module. Suppose that $(\Lambda, \geq)$ is a (finite) poset and that for each $\lambda \in \Lambda$ there is a finite indexing set $T(\lambda)$ and elements $c_{st}^\lambda \in A$ for all $s, t \in T(\lambda)$ such that

$$\mathcal{C} = \{c_{st}^\lambda \mid \lambda \in \Lambda \text{ and } s, t \in T(\lambda)\}$$

is a (free) basis of $A$. For each $\lambda \in \Lambda$ let $\check{A}^\lambda$ be the $R$-submodule of $A$ with basis

$$\{c_{uv}^\mu \mid \mu \in \Lambda, \mu > \lambda \text{ and } u, v \in T(\mu)\}.$$

The pair $(\mathcal{C}, \Lambda)$ is cellular basis of $A$ if

(i) the $R$-linear map $*: A \to A$ determined by $c_{st}^\lambda = c_{ts}^\lambda$, for all $\lambda \in \Lambda$ and all $s$ and $t$ in $T(\lambda)$, is an algebra anti-isomorphism of $A$,

(ii) for any $\lambda \in \Lambda$, $t \in T(\lambda)$ and $a \in A$ there exist $r_a \in R$ such that for all $s \in T(\lambda)$

$$c_{st}^\lambda a \equiv \sum_{u \in T(\lambda)} r_u c_{s0}^\lambda \text{ mod } \check{A}^\lambda.$$ 

If $A$ has a cellular basis we say that $A$ is a cellular algebra.

Throughout this section we assume that $(\mathcal{C}, \Lambda)$ is a fixed cellular basis of the algebra $A$.

For $\lambda \in \Lambda$ let $A^\lambda$ be the $R$-module with basis the set of $c_{uv}^\mu$ where $\mu \in \Lambda, \mu \geq \lambda$ and $u, v \in T(\mu)$. Thus, $\check{A}^\lambda \subset A^\lambda$ and $A^\lambda/\check{A}^\lambda$ has basis $c_{st}^\lambda + \check{A}^\lambda$ where $s, t \in T(\lambda)$.

Lemma 1.2 (cf. [Ma, Lemma 2.3]). Let $\lambda$ be an element of $\Lambda$.

(i) Suppose that $s \in T(\lambda)$ and $a \in A$. Then for all $t \in T(\lambda)$

$$a^* c_{st}^\lambda \equiv \sum_{u \in T(\lambda)} r_u c_{st}^\lambda \text{ mod } \check{A}^\lambda$$

where $r_u$ is the element of $R$ determined by (1.1) for each $u$.

(ii) The $R$-modules $A^\lambda$ and $\check{A}^\lambda$ are two-sided ideals of $A$.

(iii) Suppose that $s$ and $t$ are elements of $T(\lambda)$. Then there exists an element $r_{st}$ of $R$ such that for any $u, v \in T(\lambda)$

$$c_{us}^\lambda c_{tv}^\lambda \equiv r_{st} c_{uv}^\lambda \text{ mod } \check{A}^\lambda.$$
Fix an element $\lambda$ of $\Lambda$. If $s \in T(\lambda)$ define $C^\lambda_{s}$ to be the $R$-submodule of $A^\lambda/\check{A}^\lambda$ with basis $\{c_{t}^\lambda + \check{A}^\lambda \mid t \in T(\lambda)\}$. Then $C^\lambda_{s}$ is a right $A$-module by (1.1) and, importantly, the action of $A$ on $C^\lambda_{s}$ is completely independent of $s$. That is, $C^\lambda_{s} \cong C^\lambda_{t}$ for any $s, t \in T(\lambda)$. This motivates us to define the right cell module $C^\lambda$ to be the right $A$-module which is free as an $R$-module with basis $\{c_{t}^\lambda \mid t \in T(\lambda)\}$ and where for each $a \in A$

\begin{equation}
(1.2) \quad c_{t}^\lambda a = \sum_{u \in T(\lambda)} r_{u}c_{u}^\lambda
\end{equation}

where $r_{u}$ is the element of $R$ determined by (1.1). Then $C^\lambda \cong C^\lambda_{s}$, for any $s \in T(\lambda)$, via the canonical $R$-linear map which sends $c_{t}^\lambda$ to $c_{st}^\lambda + \check{A}^\lambda$ for all $t \in T(\lambda)$. In particular, (1.2) determines a well-defined action of $A$ on $C^\lambda$.

Abusing notation, define the left cell module $C^\lambda$ to be the free $R$-module with basis $\{c_{t}^\lambda \mid t \in T(\lambda)\}$ and $A$-action given by

\[ a^*c_{t}^\lambda = \sum_{u \in T(\lambda)} r_{u}c_{u}^\lambda \]

for all $a \in A$ and where, once again, $r_{u}$ is given by (1.1). Then $C^\lambda$ is a left $A$-module and $C^\lambda \cong \text{Hom}_R(C^\lambda, R)$.

Moreover, as $(A, A)$-bimodules, $A^\lambda/\check{A}^\lambda$ and $C^\lambda \otimes_{R} C^\lambda$ are canonically isomorphic via the $R$-linear map determined by $c_{st}^\lambda + \check{A}^\lambda \mapsto c_{s}^\lambda \otimes c_{t}^\lambda$ for all $s$ and $t$ in $T(\lambda)$.

Furthermore, as a right $A$-module,

\begin{equation}
(1.3) \quad A^\lambda/\check{A}^\lambda \cong C^\lambda \otimes_{R} C^\lambda \cong \bigoplus_{s \in T(\lambda)} C^\lambda_{s}.
\end{equation}

So, as a right $A$-module, $A^\lambda/\check{A}^\lambda$ is isomorphic to a direct sum of $|T(\lambda)|$ copies of $C^\lambda$.

By Lemma 1.2 (iii) there is a unique bilinear map $\langle \ , \ \rangle : C^\lambda \times C^\lambda \rightarrow R$ such that $\langle c_{s}^\lambda, c_{t}^\lambda \rangle$, for $s, t \in T(\lambda)$, is given by

\begin{equation}
(1.4) \quad \langle c_{s}^\lambda, c_{t}^\lambda \rangle c_{u}^\lambda = c_{us}^\lambda c_{tu}^\lambda \mod \check{A}^\lambda,
\end{equation}

where $u$ and $v$ are any elements of $T(\lambda)$. The bilinear form $\langle \ , \ \rangle$ is both symmetric and associative.

Let $\text{rad} C^\lambda = \{x \in C^\lambda \mid \langle x, y \rangle = 0 \text{ for all } y \in C^\lambda\}$. One can see that $\text{rad} C^\lambda$ is an $A$-submodule of $C^\lambda$. Accordingly, we define $D^\lambda = C^\lambda/\text{rad} C^\lambda$.

1.2. Simple modules in a cellular algebra. We are almost ready to show that every irreducible $A$-module is isomorphic to $D^\mu$, for some $\mu \in \Lambda$. In this section we also define and describe the decomposition matrix of $A$. Throughout, we assume that the poset $\Lambda$ is finite. Thus $A$ is a finite dimensional algebra.

One of the main points of the cellular basis is that it gives rise to many filtrations in $A$. To formalize this, call a subset $\Gamma$ of $\Lambda$ a poset ideal if $\lambda \in \Gamma$ whenever $\lambda > \mu$ for some $\mu \in \Gamma$. If $\Gamma$ is a poset ideal let $A(\Gamma)$ be the $R$-submodule of $A$ with basis
$\{c_{\mu u}^{\nu} \mid \mu \in \Gamma \text{ and } u, v \in T(\mu)\}$. Then $A(\Gamma) = \sum_{\mu \in \Gamma} A^\mu$. So $A(\Gamma)$ is a two-sided ideal by Lemma 1.2 (ii).

**Lemma 1.3** (cf. [Ma, Lemma 2.14]). Suppose that $\Lambda$ is finite and let $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k = \Lambda$ be any maximal chain of ideals in $\Lambda$. Then there exists a total ordering $\mu_1, \ldots, \mu_k$ of $\Lambda$ such that $\Gamma_i = \{\mu_1, \ldots, \mu_i\}$, for all $i$, and

$$0 = A(\Gamma_0) \hookrightarrow A(\Gamma_1) \hookrightarrow \cdots \hookrightarrow A(\Gamma_k) = A$$

is a filtration of $A$ with composition factors $A(\Gamma_i)/A(\Gamma_{i-1}) \cong C^{*\mu_i} \otimes_R C^{\mu_i}$.

Let $\Lambda_0 = \{\mu \in \Lambda \mid D^\mu \neq 0\}$. Then $\mu \in \Lambda_0$ if and only if the bilinear form $\langle \ , \ \rangle$ on $C^\mu$ is non-zero. In principle, the next theorem classifies the simple $A$-modules. However, in practice, it is often difficult to determine the set $\Lambda_0$.

**Theorem 1.4** (Graham-Lehrer). Suppose that $R$ is a field and that $\Lambda$ is finite. Then $\{D^\mu \mid \mu \in \Lambda_0\}$ is a complete set of pairwise inequivalent irreducible $A$-modules.

Suppose that $\mu \in \Lambda_0$ and $\lambda \in \Lambda$. Define $d_{\lambda \mu} = [C^\lambda : D^\mu]$ to be the decomposition number (or composition multiplicity) of the irreducible module $D^\mu$ in $C^\lambda$. By the Jordan-Hölder Theorem, $d_{\lambda \mu}$ is well-defined. The matrix $D = (d_{\lambda \mu})$, where $\lambda \in \Lambda$ and $\mu \in \Lambda_0$, is the so-called decomposition matrix of $A$.

**Corollary 1.5** (cf. [Ma, Corollary 2.17]). Suppose that $R$ is a field. Then the decomposition matrix $D$ of $A$ is unimatrix. That is, if $\mu \in \Lambda_0$ and $\lambda \in \Lambda$ then $d_{\mu \mu} = 1$ and $d_{\lambda \mu} \neq 0$ only if $\lambda \geq \mu$.

The last result in this section connects the theory of quasi-hereditary algebras and cellular algebras. Quasi-hereditary algebras are a very important class of algebras which were introduced by Cline, Parshall and Scott [CPS].

**Proposition 1.6** (cf. [Ma, Corollary 2.23]). Suppose that $R$ is a field. Then the following are equivalent.

(i) $\Lambda = \Lambda_0$.

(ii) The decomposition matrix $D$ is a square unimatrix.

Furthermore, if these conditions are satisfied then $A$ is quasi-hereditary.

As this criterion indicates, being quasi-hereditary is a non-degeneracy property on $A$.

2. Preliminaries on Ariki-Koike algebras and Cyclotomic $q$-Schur algebras

2.1. Fix positive integers $r$ and $n$ and let $S_n$ be the symmetric group of degree $n$. Let $R$ be an integral domain with 1 and $q, Q_1, \ldots, Q_r$ be elements in $R$, with invertible $q$. The Ariki-Koike algebra associated to the complex reflection group $W_{n,r} = G(r, 1, n)$, is the associative unital algebra $\mathcal{H} = \mathcal{H}_{n,r}$ over $R$ with generators $T_1, \ldots, T_n$ subject to the following conditions,

$$(T_1 - Q_1) \cdots (T_i - Q_r) = 0,$$

$$(T_i - q)(T_i + q^{-1}) = 0 \quad (i \geq 2),$$

$$T_i T_j = T_j T_i \quad (|i - j| \geq 2),$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_i \quad (2 \leq i \leq n - 1).$$
It is known that $\mathcal{H}$ is a free $R$-module of rank $n! r^n$. The subalgebra $\mathcal{H}'(\mathfrak{S}_n)$ of $\mathcal{H}$ generated by $T_2, \ldots, T_n$ is isomorphic to the Iwahori-Hecke algebra $\mathcal{H}_n'$ of the symmetric group $\mathfrak{S}_n$.

For $i = 2, \ldots, n$ let $s_i$ be the transposition $(i-1, i)$ in $\mathfrak{S}_n$. Then $\{s_2, \ldots, s_n\}$ generate $\mathfrak{S}_n$. For $w \in \mathfrak{S}_n$, we set $T_w = T_{s_1} \cdots T_{s_k}$ where $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression. Then $T_w$ is independent of the choice of a reduced expression. We also put $L_k = T_k \cdots T_2 T_1 T_2 \cdots T_k$ for $k = 1, 2, \ldots, n$. Note that all $L_1, \ldots, L_n$ commutes. Moreover, these elements produce a basis of $\mathcal{H}$.

**Theorem 2.2** ([AK, Theorem 3.10]). The Ariki-Koike algebra $\mathcal{H}$ is free as an $R$-module with basis $\{L_1^{a_1} \cdots L_n^{a_n} T_w \mid w \in \mathfrak{S}_n, 0 \leq a_i < r \text{ for } 1 \leq i \leq n\}$.

Recall that a composition of $n$ is sequence $\sigma = (\sigma_1, \sigma_2, \ldots)$ of non-negative integers such that $|\sigma| = \sum \sigma_i = n$. $\sigma$ is a partition if in addition $\sigma_1 \geq \sigma_2 \geq \cdots$. If $\sigma_i = 0$ for all $i > k$ then we write $\sigma = (\sigma_1, \ldots, \sigma_k)$.

An $r$-composition (or multicomposition) of $n$ is an $r$-tuple $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of compositions with $\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots)$ such that $|\lambda^{(1)}| + \cdots + |\lambda^{(r)}| = n$. An $r$-composition $\lambda$ is an $r$-partition if each $\lambda^{(i)}$ is a partition. If $\lambda$ is an $r$-partition of $n$ then we write $\lambda \vdash n$. The diagram $[\lambda]$ of the $r$-composition $\lambda$ is the set $[\lambda] = \{(i, j, s) \mid 1 \leq i \leq \lambda_j^{(s)}, 1 \leq s \leq r\}$. The elements of $[\lambda]$ are called nodes. The set of $r$-compositions of $n$ is partially ordered by dominance, i.e., if $\lambda$ and $\mu$ are two $r$-compositions then $\lambda$ dominates $\mu$, and we write $\lambda \triangleright= \mu$, if

$$\sum_{c=1}^{s-1} |\lambda^{(c)}| + \sum_{j=1}^{i} |\lambda_j^{(s)}| \geq \sum_{c=1}^{s-1} |\mu^{(c)}| + \sum_{j=1}^{i} |\mu_j^{(s)}|$$

for $1 \leq s \leq r$ and for all $i \geq 1$. If $\lambda \triangleright= \mu$ and $\lambda \neq \mu$ then we write $\lambda \triangleright \mu$.

If $\lambda$ is an $r$-composition let $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda^{(1)}} \times \cdots \times \mathfrak{S}_{\lambda^{(r)}}$ be the corresponding Young subgroup of $\mathfrak{S}_n$. Set

$$x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} q^{l(w)} T_w, \quad u_\lambda^+ = \prod_{s=2}^{r} \prod_{k=1}^{a_s} (L_k - Q_s),$$

where $a_s = |\lambda^{(1)}| + \cdots + |\lambda^{(s-1)}|$ for $2 \leq s \leq r$. If $s = 1$ then we set $a_s = 0$. Set $m_\lambda = x_\lambda u_\lambda^+$.

For any $r$-composition $\mu$, a $\mu$-tableau $t = (t^{(1)}, \ldots, t^{(r)})$ is a bijection $t : [\mu] \rightarrow \{1, 2, \ldots, n\}$, where $t^{(i)}$ is a tableau of Shape($t^{(i)}$) = $\mu^{(i)}$. We write Shape($t$) = $\mu$ if $t$ is a $\mu$-tableau. A $\mu$-tableau $t$ is called standard (resp. row standard) if all $t^{(i)}$ are standard (resp. row standard). Let Std($\lambda$) be the set of standard $\lambda$-tableaux.

For each $r$-composition $\mu$, let $t^\ast$ be the $\mu$-tableau with the numbers $1, 2, \ldots, n$ attached in order from left to right along its rows and from top to bottom, and from $\mu^{(1)}$ to $\mu^{(r)}$. If $t$ is any row standard $\mu$-tableau let $d(t) \in \mathfrak{S}_n$ be the unique permutation such that $t = t^\ast d(t)$. Furthermore, let $* : \mathcal{H} \rightarrow \mathcal{H}$ be the anti-isomorphism given by $T_i^* = T_i$ for $i = 1, 2, \ldots, n$, and set $m_{\ast t} = T_{d(t)}^* m_\lambda T_{d(t)}$.

**Theorem 2.3** ([DJM, Theorem 3.26]). The Ariki-Koike algebra $\mathcal{H}$ is free as an $R$-module with cellular basis $\{m_{\ast t} \mid s, t \in \text{Std}(\lambda) \text{ for some } \lambda \vdash n\}$. 
2.4. We can now give a definition of the cyclotomic $q$-Schur algebras. A set $\Lambda$ of $r$-compositions of $n$ is saturated if $\Lambda$ is finite and whenever $\lambda$ is an $r$-partition such that $\lambda \geq \mu$ for some $\mu \in \Lambda$ then $\lambda \in \Lambda$. If $\Lambda$ is a saturated set of $r$-compositions, we denote by $\Lambda^+$ be the set of $r$-partitions in $\Lambda$.

**Definition 2.5.** Suppose that $\Lambda$ is a saturated set of multicompositions of $n$. The cyclotomic $q$-Schur algebra with weight poset $\Lambda$ is the endomorphism algebra

$$S(\Lambda) = \text{End}_\mathcal{H}(M(\Lambda)), \quad \text{where } M(\Lambda) = \bigoplus_{\lambda \in \Lambda} M^\lambda.$$

Let $\lambda$ be an $r$-partition and $\mu$ an $r$-composition. A $\lambda$-Tableau of type $\mu$ is a map $T : [\lambda] \to \{(i, s) \mid i \geq 1, 1 \leq s \leq r\}$ such that $\mu_i^{(s)} = \#\{x \in [\lambda] \mid T(x) = (i, s)\}$ for all $i \geq 1$ and $1 \leq s \leq r$. We regard $T$ as an $r$-tuple $T = (T^{(1)}, \ldots, T^{(r)})$, where $T^{(s)}$ is the $\lambda^{(s)}$-tableau with $T^{(s)}(i, j) = T(i, j, s)$ for all $(i, j, s) \in [\lambda]$. In this way we identify the standard tableaux above with the Tableaux of type $w = ((0), \ldots, (1^n))$. If $T$ is a Tableau of type $\mu$ then we write $\text{Type}(T) = \mu$.

Given two pairs $(i, s)$ and $(j, t)$ write $(i, s) \preceq (j, t)$ if either $s < t$, or $s = t$ and $i \leq j$.

**Definition 2.6.** A Tableau $T$ is (row) semistandard if, for $1 \leq t \leq r$, the entries in $T^{(t)}$ are

(i) weakly increasing along the rows with respect to $\preceq$,
(ii) strictly increasing down columns,
(iii) $(i, s)$ appears in $T^{(t)}$ only if $s \geq t$.

Let $T_0(\lambda, \mu)$ be the set of semistandard $\lambda$-Tableaux of type $\mu$ and let $T_0(\Lambda) = T_0^\Lambda(\lambda) = \bigcup_{\mu \in \Lambda} T_0(\lambda, \mu)$. Notice that if $T_0(\lambda, \mu)$ is non-empty, then $\lambda \geq \mu$.

Suppose that $\lambda$ is a standard $\lambda$-tableau and let $\mu$ be an $r$-composition. Let $\mu(t)$ be the Tableau obtained from $\lambda$ by replacing each entry $j$ with $(i, k)$ if $j$ appears in row $i$ of $(\mu)^{(t)}$. The tableau $\mu(t)$ is a $\lambda$-Tableau of type $\mu$. It is not necessarily semistandard. If $S$ and $T$ are semistandard $\lambda$-Tableaux of type $\mu$ and $\nu$ respectively, let

$$m_{ST} = \sum_{s \in \text{Std}(\lambda), \mu(s) = S, \nu(t) = T} q^{(d(s)) + (d(t))} m_{st}.$$

For $S$ and $T$ as above we define a map $\varphi_{ST}$ on $M(\Lambda)$ by $\varphi_{ST}(m_{\alpha}h) = \delta_{\alpha\nu}m_{ST}h$, for all $h \in \mathcal{H}$ and all $\alpha \in \Lambda$. Here $\delta_{\alpha\nu}$ is the Kronecker delta, i.e., $\delta_{\alpha\nu} = 1$ if $\alpha = \nu$ and it is zero otherwise. Then $\varphi_{ST}$ is well-defined, and it belongs to $S(\Lambda)$. Moreover,

**Theorem 2.7** ([DJM, Theorem 6.6]). The cyclotomic $q$-Schur algebra $S(\Lambda)$ is free as an $R$-module with cellular basis $C(\Lambda) = \{\varphi_{ST} \mid S, T \in T_0^\Lambda(\lambda) \text{ for some } \lambda \in \Lambda^+\}$.

The basis $\{\varphi_{ST}\}$ is called a semistandard basis of $S(\Lambda)$. Since this basis is cellular, the map $*: S(\Lambda) \rightarrow S(\Lambda)$ which is determined by $\varphi_{ST}^* = \varphi_{TS}$ is an anti-automorphism of $S(\Lambda)$. This involution is closely related to the $*$-involution on $\mathcal{H}$. Explicitly, if $\varphi : M^\nu \rightarrow M^\mu$ is an $\mathcal{H}$-module homomorphism then $\varphi^* : M^\mu \rightarrow M^\nu$ is the homomorphism given by $\varphi^*(m_{\alpha}h) = (\varphi(m_{\alpha}))^*h$, for all $h \in \mathcal{H}$.

For each $r$-partition $\lambda \in \Lambda^+$, we define $S^{\nu\lambda} = S^{\nu}(\Lambda)^{\lambda}$ as the $R$-span of $\varphi_{ST}$ such that $S, T \in T_0^\Lambda(\alpha)$ with $\alpha \triangleright \lambda$, which is a two-sided ideal of $S(\Lambda)$. We define the Weyl
module $W^\lambda$ by the right $S(\Lambda)$-submodule of $S(\Lambda)/S^\vee(\Lambda)^{\lambda}$ generated by the image $\varphi_\lambda = \varphi_{T^\lambda T^\lambda} \in S(\Lambda)$ where $T^\lambda = \lambda(t^4)$. For each $T \in T_0^\lambda(\lambda)$, let $\varphi_T$ be the image of $\varphi_{T^\lambda T}$ in $W^\lambda$. Then the Weyl module $W^\lambda$ is $R$-free with basis $\{\varphi_T \mid T \in T_0^\lambda(\lambda)\}$. As in the case of Specht modules there is an inner product on $W^\lambda$ which is determined by

$$\langle \varphi_T, \varphi_S \rangle_{T^\lambda T^\lambda} \equiv (\varphi_S, \varphi_T)_{T^\lambda T^\lambda} \mod S^\vee(\Lambda).$$

Let $\text{rad}W^\lambda = \{x \in W^\lambda \mid \langle x, y \rangle = 0$ for all $y \in W^\lambda\}$. The quotient module $L^\lambda = W^\lambda/\text{rad}W^\lambda$ is absolutely irreducible and $\{L^\lambda \mid \lambda \in \Lambda^+\}$ is a complete set of non-isomorphic irreducible $S(\Lambda)$-modules.

2.8. For an $r$-composition $\mu$, we define the type $\alpha = \alpha(\mu)$ of $\mu$ by $\alpha = (n_1, \ldots, n_r)$ with $n_i = |\mu^{(i)}|$, and the size of $\mu$ by $n = \sum_{i=1}^r n_i$. We also define a sequence $a = a(\mu) = (a_1, \ldots, a_r)$. (Recall that $a_i = \sum_{k=1}^{i-1} |\mu^{(k)}| = \sum_{k=1}^{i-1} n_k$.)

We define a partial order $\geq$ on the set $\mathbb{Z}_{\geq 0}^r$ by $a \geq a'$ for $a = (a_1, \ldots, a_r)$, $a' = (a'_1, \ldots, a'_r) \in \mathbb{Z}_{\geq 0}^r$ if $a_i \geq a'_i$ for any $i$. We write $a > a'$ if $a \geq a'$ and $a \neq a'$. It is clear that

(2.1)

If $\lambda \supseteq \mu$, then $a(\lambda) \geq a(\mu)$ for $r$-compositions $\lambda, \mu$.

Hence if $T_0(\lambda, \mu)$ is non-empty, then $\lambda \supseteq \mu$, and so we have $a(\lambda) \geq a(\mu)$.

For any $r$-partition $\lambda$ and $r$-composition $\mu$, we define a subset $T_0^+(\lambda, \mu)$ of $T_0(\lambda, \mu)$ by

$$T_0^+(\lambda, \mu) = \{S \in T_0(\lambda, \mu) \mid a(\lambda) = a(\mu)\}.$$

Note that the condition $a(\lambda) = a(\mu)$ is equivalent to $\alpha(\lambda) = \alpha(\mu)$. Take $S \in T_0^+(\lambda, \mu)$. Then one can check that $S \in T_0^+(\lambda, \mu)$ if and only if each entry of $S^{(k)}$ is of the form $(i, k)$ for some $i$. Hence in this case $S^{(k)}$ can be identified with a semistandard $\lambda^{(k)}$-Tableau of type $\mu^{(k)}$ under the usual definition of the semistandard Tableaux for $1$-partitions $\lambda^{(k)}$ and $1$-compositions $\mu^{(k)}$. It follows that we have a bijection

$$T_0^+(\lambda, \mu) \simeq T_0(\lambda^{(1)}, \mu^{(1)}) \times \cdots \times T_0(\lambda^{(r)}, \mu^{(r)})$$

via $S \mapsto (S^{(1)}, \ldots, S^{(r)})$. Moreover, if $s \in \text{Std}(\lambda)$ is such that $\mu(s) = S$ with $S \in T_0^+(\lambda, \mu)$, then the entries of $i$-th component of $s$ consist of numbers $a_i + 1, \ldots, a_{i+1}$ for $a(\lambda) = (a_1, \ldots, a_r)$. In particular, $d(s) \in \mathcal{S}_\alpha$ for $\alpha = \alpha(\lambda)$.

Fix an $r$-tuple $m = (m_1, \ldots, m_r)$ of non-negative integers. Then, an $r$-composition $\mu = (\mu^{(1)}, \ldots, \mu^{(r)})$ with $\mu^{(i)} = (\mu_1^{(i)}, \ldots, \mu_m^{(i)}) \in \mathbb{Z}_{\geq 0}^m$ is called an $(r, m)$-composition, and $(r, m)$-partition is defined similarly. We denote by $\mathcal{P}_{n, r} = \mathcal{P}_{n, r}(m)$ (resp. $\mathcal{P}_{n, r} = \mathcal{P}_{n, r}(m)$) the set of $(r, m)$-compositions (resp. $(r, m)$-partitions) of size $n$. (Note that $\mathcal{P}_{n, r}(m)$ are naturally identified with each other for any $m$ such that $m_i \geq n$. However, $\mathcal{P}_{n, r}$ depends on the choice of $m$.) Finally, let

$$C^0(\Lambda) = \bigcup_{\mu, \nu \in \Lambda, \lambda \in \Lambda^+} \{\varphi_{ST} \in C(\Lambda) \mid S \in T_0(\lambda, \mu), \nu \in T_0(\lambda, \mu), a(\lambda) > a(\mu) \text{ if } \alpha(\mu) \neq \alpha(\nu)\}$$

and we define $S^0(\Lambda)$ as the $R$-submodule of $S(\Lambda)$ with basis $C^0(\Lambda)$.
3. The Standard Basis for $S^0(\Lambda)$

3.1. First, we prepare some notation. Let

$$\Omega = (\Lambda^+ \times \{0, 1\}) \setminus \{(\lambda, 1) \mid T_0(\lambda, \mu) = \emptyset \text{ for any } \mu \in \Lambda \text{ such that } a(\lambda) > a(\mu)\}$$

and we define a partial order $(\lambda_1, \epsilon_1) \geq (\lambda_2, \epsilon_2)$ on $\Omega$ by $(\lambda_1, \epsilon_1) > (\lambda_2, \epsilon_2)$ if $\lambda_1 \triangleright \lambda_2$, or $\lambda_1 = \lambda_2$ and $\epsilon_1 > \epsilon_2$. For a $(\lambda, \epsilon) \in \Omega$, we define index sets $I(\lambda, \epsilon), J(\lambda, \epsilon)$ by

$$I(\lambda, \epsilon) = \left\{ \begin{array}{ll} T_0^+(\lambda) & \text{if } \epsilon = 0, \\ \bigcup_{\mu \in \Lambda, a(\lambda) > a(\mu)} T_0(\lambda, \mu) & \text{if } \epsilon = 1, \end{array} \right.$$  

$$J(\lambda, \epsilon) = \left\{ \begin{array}{ll} T_0^+(\lambda) & \text{if } \epsilon = 0, \\ T_0(\lambda) & \text{if } \epsilon = 1, \end{array} \right.$$  

where $T_0^+(\lambda) = \bigcup_{\mu \in \Lambda} T_0^+(\lambda, \mu)$. Then $I(\lambda, \epsilon)$ and $J(\lambda, \epsilon)$ are not empty for all $(\lambda, \epsilon) \in \Omega$. Assume that $(\lambda, \epsilon) \in \Omega$. We define a subset $C^0(\lambda, \epsilon)$ of $S^0(\Lambda)$ by

$$C^0(\lambda, \epsilon) = \{ \varphi_{ST} \mid (S, T) \in I(\lambda, \epsilon) \times J(\lambda, \epsilon) \}.$$  

It is easy to see that

(3.1) the union $\bigcup_{(\lambda, \epsilon) \in \Omega} C^0(\lambda, \epsilon)$ is disjoint and is equal to the set $C^0(\Lambda)$.

3.2. For any $(\lambda, \epsilon) \in \Omega$, we define by $S^0(\lambda, \epsilon) = S^0(\Lambda)(> (\lambda, \epsilon))$ the $R$-submodule of $S^0(\Lambda)$ spanned by $\varphi_{UV}$ where $(U, V) \in I(\lambda', \epsilon') \times J(\lambda', \epsilon')$ for some $(\lambda', \epsilon') \in \Omega$ with $(\lambda', \epsilon') > (\lambda, \epsilon)$. Note that $S^0(\Lambda) \cap S_{\lambda}^\vee = S^0(\Lambda)(\geq (\lambda, \epsilon))$. Similarly, we define $S^0(\Lambda)(\geq (\lambda, \epsilon))$ as the $R$-submodule spanned by $\varphi_{UV}$ with $(\lambda', \epsilon') \geq (\lambda, \epsilon)$. We can now state.

Theorem 3.1. The subalgebra $S^0(\Lambda)$ is standardly based (in the sense of [DR]) on $\Omega$ with standard basis $C^0(\Lambda)$, that is,

(i) The union $\bigcup_{(\lambda, \epsilon) \in \Omega} C^0(\lambda, \epsilon) = C^0(\Lambda)$ is disjoint and forms an $R$-basis for $S^0(\Lambda)$.

(ii) For any $\varphi \in S^0(\Lambda)$, $\varphi_{ST} \in C^0(\lambda, \epsilon)$, we have

$$\varphi \cdot \varphi_{ST} \equiv \sum_{S' \in I(\lambda, \epsilon)} f_{S', (\lambda, \epsilon)}(\varphi, S) \cdot \varphi_{ST} \mod S^0(\lambda, \epsilon),$$

$$\varphi_{ST} \cdot \varphi \equiv \sum_{T' \in J(\lambda, \epsilon)} f_{(\lambda, \epsilon), T'}(\varphi) \cdot \varphi_{ST'} \mod S^0(\lambda, \epsilon),$$

where $\varphi_{ST}, \varphi_{ST'} \in C^0(\Lambda)$ and $f_{S', (\lambda, \epsilon)}(\varphi, S), f_{(\lambda, \epsilon), T'}(\varphi) \in R$ are independent of $T$ and $S$, respectively.

Note that the cellular algebra is a special case of the standardly based.

3.3. Next we introduce the Weyl module for $S^0(\Lambda)$. By (3.2) in Theorem 3.1, it is easy to see that $R$-modules $S^0(\Lambda)(\geq (\lambda, \epsilon))$ and $S^0(\lambda, \epsilon) = S^0(\Lambda)(> (\lambda, \epsilon))$ are two-sided ideals of $S^0(\Lambda)$. Fix a $(\lambda, \epsilon) \in \Omega$. For $S \in I(\lambda, \epsilon)$, we define the Weyl module $Z^\lambda_S$ for $S^0(\Lambda)$ by the $R$-submodule of $\{S^0(\Lambda)(\geq (\lambda, \epsilon))\}/\{S^0(\Lambda)(> (\lambda, \epsilon))\}$ with
basis \{ \varphi_{ST} + \delta_{0}^{\nu}(\lambda, \epsilon) \mid T \in J(\lambda, \epsilon) \}. Moreover, by (3.2), we see that \( Z_{S}^{(\lambda, \epsilon)} \) is the right \( S^{0}(\Lambda) \)-module and the action of \( S^{0}(\Lambda) \) on \( Z_{S}^{(\lambda, \epsilon)} \) is independent of the choice of \( S \), i.e., \( Z_{S_{1}}^{(\lambda, \epsilon)} \cong Z_{S_{2}}^{(\lambda, \epsilon)} \) for all \( S_{1}, S_{2} \in I(\lambda, \epsilon) \). However, since \( T^{\lambda} \) is not an element in \( I(\lambda, 1) \) for \((\lambda, 1) \in \Omega \), one should pay attention that there is no "canonical"-Weyl module for the case \((\lambda, 1) \). (That is, we can not define \( Z_{T^{\lambda}}^{(\lambda, 1)} \).) For the convenience sake let \( Z^{(\lambda, 0)} = Z_{T^{\lambda}}^{(\lambda, 0)} \) and put \( \varphi_{T}^{0} = \varphi_{T^{\lambda}T} + \delta_{0}^{\nu}(\lambda, \epsilon) \) for any \( T \in J(\lambda, 0) = T^{0}(\lambda) \).

3.4. Suppose that \( S, T \in T^{0}_{0}(\lambda) \). Then there exists an element \( r_{ST} \in R \) such that for any \( U, V \in T^{0}_{0}(\lambda) \)

\[
\varphi_{US} \cdot \varphi_{TV} \equiv r_{ST} \cdot \varphi_{UV} \mod \delta_{0}^{\nu}(\lambda, 0).
\]

We define a bilinear form \( (\cdot, \cdot)_{0} : Z^{(\lambda, 0)} \times Z^{(\lambda, 0)} \rightarrow R \) by \( (\varphi_{S}^{0}, \varphi_{T}^{0})_{0} = r_{ST} \). Hence we have

\[
(3.3) \quad (\varphi_{S}^{0}, \varphi_{T}^{0})_{0} \cdot \varphi_{UV} \equiv \varphi_{US} \cdot \varphi_{TV} \mod \delta_{0}^{\nu}(\lambda, 0),
\]

where \( U \) and \( V \) are any elements of \( T^{0}_{0}(\lambda) \). It is easy to see that

\[
(3.4) \quad (\varphi_{S}^{0}, \varphi_{T}^{0})_{0} = (\varphi_{S}, \varphi_{T}) \quad \text{for every} \quad S, T \in T^{0}_{0}(\lambda).
\]

Let \( \text{rad} Z^{(\lambda, 0)} = \{ x \in Z^{(\lambda, 0)} \mid (x, y)_{0} = 0 \text{ for all } y \in Z^{(\lambda, 0)} \} \).

**Lemma 3.2.** \( \text{rad} Z^{(\lambda, 0)} \) is an \( S^{0}(\Lambda) \)-submodule of \( Z^{(\lambda, 0)} \).

We put \( L^{\lambda}_{0} = Z^{(\lambda, 0)}/\text{rad} Z^{(\lambda, 0)} \). Then we have the following.

**Proposition 3.3.** Suppose that \( R \) is a field, and \( \lambda \in \Lambda^{+} \). Then

(i) \( L^{\lambda}_{0} \neq 0 \) and

(ii) \( \text{rad} Z^{(\lambda, 0)} \) is the unique maximal submodule of \( Z^{(\lambda, 0)} \) and \( L^{\lambda}_{0} \) is absolutely irreducible. Moreover, the Jacobson radical of \( Z^{(\lambda, 0)} \) is equal to \( \text{rad} Z^{(\lambda, 0)} \).

4. A RELATIONSHIP BETWEEN \( S^{0}(m, n) \) AND \( S^{0}(\Lambda) \)

First, we recall the definition of modified Ariki-Koike algebras and their cyclotomic \( q \)-Schur algebras ([SawS]).

4.1. From now on, throughout this paper, we consider the following condition on parameters \( Q_{1}, \ldots, Q_{r} \) in \( R \) whenever we consider the modified Ariki-Koike algebras (and their cyclotomic \( q \)-Schur algebras).

(4.1) \( Q_{i} - Q_{j} \) are invertible in \( R \) for any \( i \neq j \).

Let \( A \) be a square matrix of degree \( r \) whose \( i-j \) entry is given by \( Q_{ij}^{i-1} \) for \( 1 \leq i, j \leq r \). Thus \( A \) is the Vandermonde matrix, and \( \Delta = \det A = \prod_{i>j} (Q_{i} - Q_{j}) \) is invertible by (4.1). We express the inverse of \( A \) as \( A^{-1} = \Delta^{-1} B \) with \( B = (h_{ij}) \), and define a polynomial \( F_{i}(X) \in R[X] \), for \( 1 \leq i \leq r \), by \( F_{i}(X) = \sum_{1 \leq j \leq r} h_{ij} X^{j-1} \).
The modified Ariki-Koike algebra $\mathcal{H}^b = \mathcal{H}^b_{n,r}$ is an associative algebra over $R$ with generators $T_2, \cdots, T_n$ and $\xi_1, \ldots, \xi_n$ and relations

\begin{align}
(T_i - q)(T_i + q^{-1}) &= 0 \quad (2 \leq i \leq n), \\
(\xi_i - Q_1) \cdots (\xi_i - Q_r) &= 0 \quad (1 \leq i \leq n), \\
T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \quad (2 \leq i \leq n), \\
T_iT_j &= T_jT_i \quad (|i - j| \geq 2), \\
\xi_i\xi_j &= \xi_j\xi_i \quad (1 \leq i, j \leq n), \\
T_j\xi_j &= \xi_jT_j + \Delta^{-2} \sum_{c_1 < c_2} (Q_{c_2} - Q_{c_1})(q - q^{-1})F_{c_1}(\xi_j-1)F_{c_2}(\xi_j), \\
T_j\xi_{j-1} &= \xi_jT_j - \Delta^{-2} \sum_{c_1 < c_2} (Q_{c_2} - Q_{c_1})(q - q^{-1})F_{c_1}(\xi_j-1)F_{c_2}(\xi_j), \\
T_j\xi_k &= \xi_jT_j \quad (k \neq j - 1, j).
\end{align}

It is known that if $R = \mathbb{Q}(\overline{q}, \overline{Q}_1, \ldots, \overline{Q}_r)$, the field of rational functions with variables $\overline{q}, \overline{Q}_1, \ldots, \overline{Q}_r$, $\mathcal{H}^b$ is isomorphic to $\mathcal{H}$, and it gives an alternate presentation of $\mathcal{H}$ apart from 2.1.

The subalgebra $\mathcal{H}^b(\mathfrak{S}_n)$ of $\mathcal{H}^b$ generated by $T_2, \ldots, T_n$ is isomorphic to $\mathcal{H}_n$, hence it can be naturally identified with the corresponding subalgebra $\mathcal{H}(\mathfrak{S}_n)$ of $\mathcal{H}$. Moreover, it is known by [Sh] that the set $\{\xi_1^{c_1} \cdots \xi_n^{c}T_w | w \in \mathfrak{S}_n, 0 \leq c_i < r \text{ for } 1 \leq i \leq n\}$ gives rise to a basis of $\mathcal{H}^b$.

Let $V = \bigoplus_{i=1}^{r} V_i$ be a free $R$-module, with rank $V_i = m_i$. We put $m = \sum m_i$. It is known by [SakS] that we can define a right $\mathcal{H}$-module structure on $V \otimes \mathfrak{S}_n$. We denote this representation by $\rho : \mathcal{H} \rightarrow \operatorname{End} V \otimes \mathfrak{S}_n$. Note that this construction works without the condition (4.1). Also it is shown in [Sh] that, under the assumption (4.1), a right action of $\mathcal{H}^b$ on $V \otimes \mathfrak{S}_n$ can be defined. We denote this representation by $\rho^b : \mathcal{H}^b \rightarrow \operatorname{End} V \otimes \mathfrak{S}_n$. By [Sh, Lemma 3.5], we know that $\operatorname{Im} \rho \subset \operatorname{Im} \rho^b$.

We consider the condition

\begin{equation}
m_i \geq n \text{ for } i = 1, \ldots, r.
\end{equation}

**Lemma 4.2** ([SawS, Lemma 1.5]). Under the conditions (4.1), (4.3), there exists an $R$-algebra homomorphism $\rho_0 : \mathcal{H} \rightarrow \mathcal{H}^b$ such that $\rho_0$ induces the identity on $\mathcal{H}_n$. (Here we regard $\mathcal{H}_n \subset \mathcal{H}$, $\mathcal{H}_n \subset \mathcal{H}^b$ under the previous identifications.) If $\operatorname{Im} \rho^b = \operatorname{Im} \rho$ and $R$ is a field, then $\mathcal{H} \simeq \mathcal{H}^b$.

From now on, throughout the paper, we fix an $r$-tuple $m = (m_1, \ldots, m_r)$ of nonnegative integers and always assume the condition (4.3) whenever we consider $\mathcal{H}^b$.

Any $\mu \in \mathcal{P}_{n,r}(m)$ may be regarded as an element in $\mathcal{P}_{n,1}$ (i.e., 1-composition) of $n$ by arranging the entries of $\mu = (\mu_j^{(i)})$ in order

$$
\mu_1^{(1)}, \ldots, \mu_{m_1}^{(1)}, \mu_1^{(2)}, \ldots, \mu_{m_2}^{(2)}, \ldots, \mu_1^{(r)}, \ldots, \mu_{m_r}^{(r)}
$$

which we denote by $\{\mu\}$.

For $\alpha = (n_1, \ldots, n_r) \in \mathbb{Z}_{\geq 0}$ such that $\sum n_i = n$, we define $c(\alpha)$ by

$$
c(\alpha) = \left(\underbrace{r, \ldots, r}_{n_1 \text{-times}}, \underbrace{r - 1, \ldots, r - 1}_{n_2 \text{-times}}, \ldots, \underbrace{1, \ldots, 1}_{n_r \text{-times}}\right)
$$
and let $c(\alpha) = (c_1, \ldots, c_n)$. We define $F_\alpha \in \mathcal{H}^b$ by $F_\alpha = \Delta^{-1} F_{c_1}(\xi_1) F_{c_2}(\xi_2) \cdots F_{c_n}(\xi_n)$. For any $\mu \in \tilde{P}_{n,r}$, put $m^b_{\mu} = F_{\alpha(\mu)} \cdot m_{\mu}$ where $m_{\mu} = \sum_{w \in S_{\mu}} q^{l(w)} T_w = x_\mu \in \mathcal{H}_n$.

We define an $R$-linear anti-automorphism $h \to h^*$ on $\mathcal{H}^b$ by the condition that $*$ fixes the generators $T_i \ (2 \leq i \leq n)$ and $\xi_j \ (1 \leq j \leq n)$. As discussed in [SawS, 2.7], this condition induces a well-defined anti-automorphism on $\mathcal{H}^b$. Moreover, by Lemma 2.9 in [SawS], we know that $(m^b_{\mu})^* = m^b_{\mu}$. For $s, t \in \text{Std}(\lambda)$ with $\lambda \in P_{n,r}$, we define an element $m^b_{st} \in \mathcal{H}^b$ by $m^b_{st} = T^*_d(s) m^b_t T_d(t)$. By the above fact, we have $(m^b_{st})^* = m^b_{ts}$.

**Theorem 4.3** ([SawS, Theorem 2.18]). The modified Ariki-Koike algebra $\mathcal{H}^b$ is free as an $R$-module with cellular basis \{m^b_s \mid s, t \in \text{Std}(\lambda) \text{ for } s, t \in P_{n,r}\}.

Put $M^b_\mu = m^b_{\mu} \mathcal{H}^b$ for $\mu \in \tilde{P}_{n,r}$. We define a cyclotomic $q$-Schur algebra $S^b(m, n)$ as follows.

**Definition 4.4.** The cyclotomic $q$-Schur algebra for $\mathcal{H}^b$ with weight poset $\tilde{P}_{n,r}$ is the endomorphism algebra

$$S^b(m, n) = \text{End}_{\mathcal{H}^b}(M^b(\tilde{P}_{n,r})), \quad \text{where } M^b(\tilde{P}_{n,r}) = \bigoplus_{\mu \in \tilde{P}_{n,r}} M^b_\mu.$$

For an $r$-tuples $\alpha \in \tilde{P}_{n,1}$, let $M^\alpha = \bigoplus_{\mu, \alpha(\mu) = \alpha} M^b_\mu$. Then by Proposition 5.2 (i) in [SawS], we have $S^b(m, n) \cong \bigoplus_{\alpha \in \tilde{P}_{n,1}} \text{End}_{\mathcal{H}^b} M^\alpha$ as $R$-algebras.

**Theorem 4.5** ([SawS, Theorem 5.5]). Let $S^b(m, n)$ be the cyclotomic $q$-Schur algebra associated to the modified Ariki-Koike algebra $\mathcal{H}^b$ and $S(m_1, n_1)$ be the $q$-Schur algebra associated to the Iwahori-Hecke algebra $\mathcal{H}_{n_1}$. Then there exists an isomorphism of $R$-algebras

$$S^b(m, n) \cong \bigoplus_{(n_1, \ldots, n_r), n = n_1 + \cdots + n_r} S(m_1, n_1) \otimes \cdots \otimes S(m_r, n_r).$$

Let $\mu, \nu \in \tilde{P}_{n,r}$ and $\lambda \in P_{n,r}$. We assume that $\alpha(\mu) = \alpha(\nu) = \alpha(\lambda)$. For $S \in T^+_0(\lambda, \mu)$ and $T \in T^+_0(\lambda, \nu)$, put

$$m^b_{ST} = \sum_{\substack{s,t \in \text{Std}(\lambda) \mu(s) = S, \nu(t) = T}} q^{l(d(s)) + l(d(t))} m^b_{st}.$$

Moreover, for $S \in T^+_0(\lambda, \mu)$ and $T \in T^+_0(\lambda, \nu)$, one can define $\varphi^b_{ST} \in S^b(m, n)$ by $\varphi^b_{ST}(m^b_{a}h) = \delta_{\alpha \nu} m^b_{ST} h$, for all $h \in \mathcal{H}^b$ and all $\alpha \in \tilde{P}_{n,r}$.

**Theorem 4.6** ([SawS, Theorem 5.9]). The cyclotomic $q$-Schur algebra $S^b(m, n)$ is free as an $R$-module with cellular basis $\mathcal{C}^b(m, n) = \{\varphi^b_{ST} \mid S, T \in T^+_0(\lambda), \text{ for some } \lambda \in P_{n,r}\}.$

4.2. Let $S^0(\Lambda)$ be as in Section 3. We describe a relationship between the algebra $S^0(\Lambda)$ and the cyclotomic $q$-Schur algebra $S^b(m, n)$ in the case where $\Lambda = \tilde{P}_{n,r}$. But in the moment, we shall consider an arbitrary $\Lambda$ as in Section 3.
First, let $C^0(\Lambda) = \{ \varphi_{ST} \mid (S, T) \in I(\lambda, 1) \times J(\lambda, 1), \lambda \in \Lambda^+ \} \subset C^0(\Lambda)$ and $S^0(\Lambda)$ be the $R$-span of $\varphi_{ST} \in C^0(\Lambda)$, which is an $R$-submodule of $S^0(\Lambda)$. We note that, $S^0(\Lambda)$ is a two-sided ideal of $S^0(\Lambda)$ by the second and fourth formula in [Sa, Lemma 2.4]. Thus one can define the quotient algebra $\overline{S}^0(\Lambda) = S^0(\Lambda)/S^0(\Lambda)$. We write $\overline{x} = x + S^0(\Lambda)$ ($x \in S^0(\Lambda)$). It is easy to see that $\overline{S}^0(\Lambda)$ has a free $R$-basis $\{ \overline{\varphi}_{ST} \mid S \in I(\lambda, 0), T \in J(\lambda, 0), \lambda \in \Lambda^+ \}$. Note that the condition $(S, T) \in I(\lambda, 0) \times J(\lambda, 0)$ is nothing but $S, T \in T_0^+(\lambda)$. For $\lambda \in \Lambda^+$, let $\overline{S}_{0}^{\vee \lambda} = \overline{S}_{0}^{\vee} (\Lambda)^{\lambda}$ be the $R$-submodule of $\overline{S}^0(\Lambda)$ spanned by $\overline{\varphi}_{ST}$ with $S, T \in T_0^+(\lambda)$ for various $\alpha \in \Lambda^+$ such that $\alpha \succ \lambda$. We show the following.

**Theorem 4.7.** The algebra $\overline{S}^0(\Lambda)$ has a free basis

$$\overline{C}^0(\Lambda) = \{ \overline{\varphi}_{ST} \mid S, T \in T_0^+(\lambda), \lambda \in \Lambda^+ \}$$

satisfying the following properties.

(i) The $R$-linear map $\ast : \overline{S}^0(\Lambda) \rightarrow \overline{S}^0(\Lambda)$ determined by $\overline{\varphi}_{ST} = \overline{\varphi}_{TS}$, for all $S, T \in T_0^+(\lambda)$ and all $\lambda \in \Lambda^+$, is an anti-automorphism of $\overline{S}^0(\Lambda)$.

(ii) Let $T \in T_0^+(\lambda)$. Then for all $\overline{\varphi} \in \overline{S}^0(\Lambda)$, and any $V \in T_0^+(\lambda)$, there exists $\lambda \in \Lambda^+$ such that $\overline{\varphi}_{ST} \cdot \overline{\varphi} \equiv \sum_{V \in T_0^+(\lambda)} r_{\lambda} \overline{\varphi}_{SV} \mod S_{0,\lambda}$

for any $S \in T_0^+(\lambda)$, where $r_{\lambda}$ is independent of the choice of $T$.

In particular, $\overline{C}^0(\Lambda)$ is a cellular basis of $\overline{S}^0(\Lambda)$.

In the case where $S^b(m, n)$ is defined, $\overline{S}^0(\Lambda)$ can be identified with $S^b(m, n)$, i.e, we have the following proposition.

**Proposition 4.8.** Let $\Lambda = \tilde{P}_{n,r}$ and assume that (4.1) and (4.3) holds. Then there exists an algebra isomorphism $b : \overline{S}^0(\Lambda) \rightarrow S^b(m, n)$ satisfying the following. For $\overline{\varphi}_{ST} \in \overline{C}^0(\Lambda)$ such that $S, T \in T_0^+(\lambda)$ and $\lambda \in \Lambda^+$, we have $(\overline{\varphi}_{ST})^b = \varphi_{ST}$.

We now return to the general setting, and consider $\overline{S}^0(\Lambda)$ for arbitrary $\Lambda$. The above proposition says that the $\overline{S}^0(\Lambda)$ is a natural “cover” of the $S^b(m, n)$.

For $\lambda \in \Lambda^+$, $\overline{\varphi}_\lambda = \overline{\varphi}_{T^\lambda T^\lambda}$ is an element in $\overline{S}^0(\Lambda)$. Hence, by the cellular theory [GL], one can define a Weyl module $\overline{Z}^\lambda$ of $\overline{S}^0(\Lambda)$ as the right $\overline{S}^0(\Lambda)$-submodule of $\overline{S}^0(\Lambda)/\overline{S}_{0,\lambda}^{\vee}$ spanned by the image of $\overline{\varphi}_\lambda$. We denote by $\overline{\varphi}_T$ the image of $\overline{\varphi}_{T^\lambda T^\lambda}$ in $\overline{S}^0(\Lambda)/\overline{S}_{0,\lambda}^{\vee}$. Then the set $\{ \overline{\varphi}_T \mid T \in T_0^+(\lambda) \}$ is a free $R$-basis of $\overline{Z}^\lambda$. Define a bilinear form $(\ ,\ )_\overline{0}$ on $\overline{Z}^\lambda$ by requiring that

$$\overline{\varphi}_{T^\lambda S^\lambda} \overline{\varphi}_{T^\lambda T^\lambda} \equiv (\overline{\varphi}_S, \overline{\varphi}_T)_\overline{0} \cdot \overline{\varphi}_\lambda \mod \overline{S}_{0,\lambda}^{\vee}$$

for all $S, T \in T_0^+(\lambda)$. Let $\overline{L}^\lambda = \overline{Z}^\lambda/\mathfrak{rad} \overline{Z}^\lambda$, where $\mathfrak{rad} \overline{Z}^\lambda = \{ x \in \overline{Z}^\lambda \mid (x, y)_\overline{0} = 0 \mbox{ for all } y \in \overline{Z}^\lambda \}$. In the case where $R$ is a field, by a general theory of cellular algebras, the set $\{ \overline{L}^\lambda \mid \lambda \in \Lambda^+, \overline{L}^\lambda \neq 0 \}$ gives a complete set of non-isomorphic irreducible $\overline{S}^0(\Lambda)$-modules. Furthermore, we have the following result.
Proposition 4.9. Suppose that $R$ is a field. Then $\overline{L}^\lambda \neq 0$ for any $\lambda \in \Lambda^+$. Hence, $\{\overline{L}^\lambda \mid \lambda \in \Lambda^+\}$ is a complete set of non-isomorphic irreducible $\overline{S}^0(\Lambda)$-modules. Therefore, $\overline{S}^0(\Lambda)$ is quasi-hereditary.

The following result connects the decomposition numbers in $\overline{Z}^\lambda$ and in $Z^{(\lambda,0)}$.

Theorem 4.10. Suppose that $R$ is a field. Then

(i) $\{L_0^\alpha \mid \alpha \in \Lambda^+, \, \lambda \supseteq \alpha\}$ is a complete set of pairwise inequivalent irreducible $S^0(\Lambda)$-modules occurring in the composition factors of the $S^0(\Lambda)$-module $Z^{(\lambda,0)}$.

(ii) For $\lambda$, $\mu \in \Lambda^+$, we have

$$[\overline{Z}^\lambda : \overline{L}^\mu] = [Z^{(\lambda,0)} : L_0^\mu].$$

(iii) For $\lambda$, $\mu \in \Lambda^+$ such that $\alpha(\lambda) \neq \alpha(\mu)$, we have

$$[\overline{Z}^\lambda : \overline{L}^\mu] = 0.$$

5. AN ESTIMATE FOR DECOMPOSITION NUMBERS

We are now ready to estimate the decomposition numbers for the cyclotomic $q$-Schur algebras.

5.1. We keep the notation in Section 4, and consider the general $\Lambda$.

Theorem 5.1. Suppose that $R$ is a field. Then, for all $\lambda, \mu \in \Lambda^+$ with $\alpha(\lambda) = \alpha(\mu)$,

$$[\overline{Z}^\lambda : \overline{L}^\mu] = [Z^{(\lambda,0)} : L_0^\mu] = [W^\lambda : L^\mu].$$

5.8. We return to the setting in 4.1. Let $\Lambda = \tilde{\mathcal{P}}_{n,r}$ under the condition (4.1) and (4.3). For an $r$-partition $\lambda \in \mathcal{P}_{n,r}$, we denote by $S^0_{\lambda}^{(\lambda)}$ the $R$-submodule of $S^0(m,n)$ spanned by $\varphi_{ST}^b$ such that $S, T \in T_0^+(\alpha)$ with $\alpha \triangleright \lambda$. Moreover, for an $r$-partition $\lambda \in \mathcal{P}_{n,r}$, $T^\lambda \in T_0^+(\lambda, \lambda)$, and in fact $T^\lambda$ is the unique semistandard $\lambda$-Tableau of type $\lambda$. Moreover, $t = t^\lambda$ is the unique element in $\text{Std}(\lambda)$ such that $\lambda(t) = T^\lambda$. Thus, $m_{T^\lambda T^\lambda}^\lambda = m_{T^\lambda T^\lambda}^\lambda$, and $\varphi_{T^\lambda T^\lambda}^\lambda$ is the identity map on $M^\lambda_{\lambda}$. We define the Weyl module $W^\lambda_{\lambda}$ as the right $S^0(m,n)$-submodule of $S^0(m,n)/S^0_{\lambda}^{(\lambda)}$ spanned by the image of $\varphi_{T}^\lambda$. For each $T \in T_0^+(\lambda, \mu)$, we denote by $\varphi_{T}^\lambda$ the image of $\varphi_{T^\lambda T}^b$ in $S^0(m,n)/S^0_{\lambda}^{(\lambda)}$. Then we know that the Weyl module $W^\lambda_{\lambda}$ is $R$-free with basis $\{\varphi_T^\lambda \mid T \in T_0^+(\lambda)\}$. The Weyl module $W^\lambda_{\lambda}$ enjoys an associative symmetric bilinear form, defined by the equation

$$\varphi_{T^\lambda S}^b \varphi_{T^\lambda T}^b \equiv (\varphi_{S}^b, \varphi_{T}^b)_b \cdot \varphi_{T}^\lambda \mod S^0_{\lambda}^{(\lambda)}$$

for all $S, T \in T_0^+(\lambda)$. Let $L^\lambda_{\lambda} = W_{\lambda}/\text{rad}W_{\lambda}$, where $\text{rad}W_{\lambda} = \{x \in W^\lambda_{\lambda} \mid (x, y)_b = 0 \text{ for all } y \in W_{\lambda}^\lambda\}$. By [SawS, Proposition 5.11], we know that, for all $r$-partition $\lambda \in \mathcal{P}_{n,r}$, $L^\lambda_{\lambda}$ is an absolutely irreducible and $\{L^\lambda_{\lambda} \mid \lambda \in \mathcal{P}_{n,r}\}$ is a complete set of non-isomorphic irreducible $S^0(m,n)$-modules. Furthermore, for $\lambda, \mu \in \mathcal{P}_{n,r}$, we denote by $[W^\lambda_{\lambda} : L^\mu_{\lambda}]$ the composition multiplicity of $L^\mu_{\lambda}$ in $W^\lambda_{\lambda}$. Note that the above definition of the Weyl module $W^\lambda_{\lambda}$ coincides with the definition of the Weyl module $\overline{Z}^\lambda$ when $S^0(m,n)$ is isomorphic to $\overline{S}^0(\Lambda)$ under the isomorphism $b$ in Proposition 4.8.
Consequently, under the isomorphism $b$, we have $[W^\lambda_b : L^\mu_b] = [\overline{Z}^\lambda : \overline{L}^\mu]$ for every $\lambda, \mu \in \mathcal{P}_{n,r}$. On the other hand, note that in the case where $r = 1$, the notation for $S^b(m, n)$ coincides with the standard notation for $q$-Schur algebras discussed as in [Ma, Chapter 4]. So, we use freely such a notation. For $\lambda, \mu \in \mathcal{P}_{n,r}$, we denote by $[W^\lambda_{\mu^{(i)}} : L^{\mu^{(i)}}]$ for $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ and $\mu = (\mu^{(1)}, \ldots, \mu^{(r)})$.

**Proposition 5.2** ([SawS, Proposition 5.14]). Let $\Lambda = \tilde{\mathcal{P}}_{n,r}$. Suppose that $R$ is a field, and that (4.1) and (4.3) are satisfied. Let $\lambda, \mu \in \mathcal{P}_{n,r}$. Then under the isomorphism in Theorem 4.5, we have

$$[W^\lambda : L^\mu] = \begin{cases} \prod_{i=1}^{r}[W^\lambda_{\mu^{(i)}} : L^{\mu^{(i)}}] & \text{if } \alpha(\lambda) = \alpha(\mu), \\ 0 & \text{otherwise}. \end{cases}$$

**Corollary 5.3.** Let $\Lambda = \tilde{\mathcal{P}}_{n,r}$. Suppose that $R$ is a field, and that (4.1) and (4.3) are satisfied. Then, for all $\lambda, \mu \in \mathcal{P}_{n,r}$ with $\alpha(\lambda) = \alpha(\mu)$, we have

$$[W^\lambda : L^\mu] = \prod_{i=1}^{r}[W^\lambda_{\mu^{(i)}} : L^{\mu^{(i)}}].$$

**REFERENCES**


