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NON-EXISTENCE THEOREM EXCEPT THE OUT-OF-PHASE AND IN-PHASE SOLUTIONS IN THE COUPLED VAN DER POL EQUATION SYSTEM

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Abstract
In this paper, we study the coupled van der Pol equation system, which consists of two van der Pol equations connected by the linear terms with each other. We consider that two distinctive solutions: the out-of-phase and in-phase solutions exist in the dynamical system of the coupled equations and we prove the non-existence theorem except the out-of-phase and in-phase solutions in the coupled system.

1 Introduction

We treat the following van der Pol equation system with coupling by the positional difference. Let $y = y(t)$ and $z = z(t)$ two real valued functions and we consider the dynamical system

$$\boldsymbol{\Sigma}_{\varepsilon,k}\begin{cases} y'' - \varepsilon(1-y^2)y' + y = k(y-z), \\ z'' - \varepsilon(1-z^2)z' + z = k(z-y), \quad t_0 \leq t. \end{cases}$$

Here, $'$ denotes the derivative with respect to t. $k, \varepsilon(>0)$ are constants and t_0 indicates an initial time. When $k=0$, the dynamical system $\boldsymbol{\Sigma}_{\varepsilon,0}$ becomes two independent, van der Pol oscillators.[van der Pol, 1926] The single van der Pol oscillator is a well-known, classic problem. Many studies on the van der Pol equation have been carried out and the fact that the van der Pol equation has a unique limit cycle is known and proved by Poincaré-Bendixson's theorem.(see, for example, [Guckenheimer and Holmes, 1983]) However, the coupled van der Pol system, that is, the dynamical system $\boldsymbol{\Sigma}_{\varepsilon,k}$ constructs a three-dimensional manifold. Therefore we cannot apply Poincaré-Bendixson's theorem to the dynamical system $\boldsymbol{\Sigma}_{\varepsilon,k}$ to study the analysis of the system. "Does there exist the limit cycle in $\boldsymbol{\Sigma}_{\varepsilon,k}$?[Nohara and Arimoto, 2008A]", "If there exists the limit cycle, how many limit cycles are there?[Nohara and Arimoto, 2008B]" and "Are the limit cycles 'stable' or 'completely unstable' or 'semistable'?" are still *open* problems we have.

In this paper, we first show the generalized van der Pol equation and analyze it. Then the analysis of the coupled van der Pol equation system is carried out based on the formation of our method after defining the out-of-phase and in-phase solutions, which are new concepts arising when the system is coupled. We consider that there exist two distinctive solutions: the out-of-phase and in-phase solutions in the dynamical system $\boldsymbol{\Sigma}_{\varepsilon,k}$. Finally, we give the answers to some of the above *open* problems.

2 Analysis of the generalized van der Pol equation

Let $\xi_\Sigma(t) = \mathrm{col}\big(y(t), y'(t), z(t), z'(t)\big)$ be a solution of $\boldsymbol{\Sigma}_{\varepsilon,k}$.

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Definition 2.1. *In the dynamical system $\Sigma_{\varepsilon,k}$, the following relation holds*

$$y(t) + z(t) = 0,$$

where $\xi_\Sigma(t)$ not equivalent to 0, then the system is out-of-phase and the non-trivial solutions of $y(t)$ and $z(t)$ are called the out-of-phase solutions.

Definition 2.2. *In the dynamical system $\Sigma_{\varepsilon,k}$, the following relation holds*

$$y(t) - z(t) = 0,$$

where $\xi_\Sigma(t)$ not equivalent to 0, then the system is in-phase and the non-trivial solutions of $y(t)$ and $z(t)$ are called the in-phase solutions.

Here, we consider the differential equation $W_{\varepsilon,m,\phi}$

$$W_{\varepsilon,m,\phi}: \quad w'' - \varepsilon(w' - \phi) + mw = 0, \tag{2.1}$$

where $w = w(t), \phi = \phi(w, w'), 0 < \varepsilon < 2\sqrt{m}$. We call this the generalized van der Pol equation since we obtain the ordinary van der Pol equation if we set $W_{\varepsilon,1,w^2w'}$, that is, $m = 1, \phi(t) = w^2(t)w'(t)$. However, we have no restriction regarding $m \in \mathfrak{R}$ and $\phi = \phi(w, w')$ (but we simply write $\phi = \phi(t)$ instead of $\phi(w, w')$) in this section. We can write this in a matrix form as

$$x_w' = A_w x_w - \varepsilon\xi, \tag{2.2}$$

where

$$A_w = \begin{pmatrix} 0 & 1 \\ -m & \varepsilon \end{pmatrix}, \; x_w = \begin{pmatrix} w \\ w' \end{pmatrix}, \; \xi = \begin{pmatrix} 0 \\ \phi \end{pmatrix}.$$

We know that the solution $x_w(t)$ can be written by

$$x_w(t) = e^{A_w(t-t_0)}x_w(t_0) - \varepsilon \int_{t_0}^{t} e^{A_w(t-s)}\xi(s)ds. \tag{2.3}$$

Here A_w has two eigenvalues r and its complex conjugate $\bar{r}$ as the followings:

$$r = \frac{\varepsilon + \sqrt{4m - \varepsilon^2}\, i}{2}, \; \bar{r} = \frac{\varepsilon - \sqrt{4m - \varepsilon^2}\, i}{2}.$$

We now see that A_w has a spectral representation

$$\begin{aligned} A_w &= rP_1 + \bar{r}P_2, \\ E &= P_1 + P_2, \; P_1P_2 = P_2P_1 = 0. \end{aligned}$$

Hence from the relation

$$rP_1 = A_w - \bar{r}P_2 = A_w - \bar{r}(E - P_1),$$

we obtain

$$P_1 = \frac{1}{r - \bar{r}}(A_w - \bar{r}), \; P_2 = \frac{1}{r - \bar{r}}(r - A_w).$$

Using this, we simply write the exponential function of A_w as follows:

$$e^{A_w t} = e^{rt}P_1 + e^{\bar{r}t}P_2 = \frac{1}{r - \bar{r}}\Big(\big(e^{rt} - e^{\bar{r}t}\big)A_w + \big(re^{\bar{r}t} - \bar{r}e^{rt}\big)\Big).$$

We can easily obtain

$$\frac{e^{rt}-e^{\bar{r}t}}{r-\bar{r}} = e^{\frac{1}{2}\varepsilon t}\frac{\sin(\vartheta t)}{\vartheta},$$
$$\frac{re^{\bar{r}t}-\bar{r}e^{rt}}{r-\bar{r}} = e^{\frac{1}{2}\varepsilon t}\Big(\cos(\vartheta t) - \frac{\varepsilon}{2}\frac{\sin(\vartheta t)}{\vartheta}\Big),$$

where

$$\vartheta = \frac{\sqrt{4m-\varepsilon^2}}{2}. \tag{2.4}$$

Hence we have

$$e^{A_w t} = e^{\frac{1}{2}\varepsilon t}\Big(\frac{\sin(\vartheta t)}{\vartheta}A_w + \cos(\vartheta t) - \frac{\varepsilon}{2}\frac{\sin(\vartheta t)}{\vartheta}\Big)$$

and we see that using Equation (2.3) the solution of Equation (2.2) satisfies

$$\begin{aligned} x_w(t) =& e^{\frac{1}{2}\varepsilon(t-t_0)}\left\{\frac{\sin\big(\vartheta(t-t_0)\big)}{\vartheta}\begin{pmatrix} -\dfrac{\varepsilon}{2} & 1 \\ -m & \dfrac{\varepsilon}{2}\end{pmatrix} + \cos\big(\vartheta(t-t_0)\big)\right\}x_w(t_0) \\ & - \varepsilon\int_{t_0}^{t} e^{\frac{1}{2}\varepsilon(t-s)}\left\{\frac{\sin\big(\vartheta(t-s)\big)}{\vartheta}\begin{pmatrix} -\dfrac{\varepsilon}{2} & 1 \\ -m & \dfrac{\varepsilon}{2}\end{pmatrix} + \cos\big(\vartheta(t-s)\big)\right\}\times \\ & \begin{pmatrix} 0 \\ \phi\big(s;t_0,w(t_0),w'(t_0)\big)\end{pmatrix} ds. \end{aligned} \tag{2.5}$$

In Equation (2.5), $\phi\big(s;t_0,w(t_0),w'(t_0)\big)$ means the function ϕ of s defined by the solution with the initial condition of $w(t_0), w'(t_0)$ at time t_0.

Here we define

$$U_t(\vartheta) := \begin{pmatrix} \cos(\vartheta t) & \dfrac{\sin(\vartheta t)}{\vartheta} \\ -\vartheta\sin(\vartheta t) & \cos(\vartheta t)\end{pmatrix},$$

and give the following lemma:

Lemma 2.1.

$$\frac{\sin(\vartheta t)}{\vartheta}\begin{pmatrix} -\dfrac{\varepsilon}{2} & 1 \\ -m & \dfrac{\varepsilon}{2}\end{pmatrix} + \cos(\vartheta t) = \begin{pmatrix} -\dfrac{1}{\varepsilon} & 0 \\ -\dfrac{1}{2} & \dfrac{1}{\varepsilon}\end{pmatrix}^{-1} U_{-t}(\vartheta)\begin{pmatrix} -\dfrac{1}{\varepsilon} & 0 \\ -\dfrac{1}{2} & \dfrac{1}{\varepsilon}\end{pmatrix}.$$

Here we let $\alpha_{w0} = w(t_0)$, $\beta_{w0} = w'(t_0)$ simpliy and define symbols as follows:

$$I_s(t,t_0;\alpha_{w0},\beta_{w0}) := \int_{t_0}^{t} e^{-\frac{1}{2}\varepsilon(s-t_0)}\frac{\sin(\vartheta s)}{\vartheta}\phi\big(s;t_0,\alpha_{w0},\beta_{w0}\big)ds,$$
$$I_c(t,t_0;\alpha_{w0},\beta_{w0}) := \int_{t_0}^{t} e^{-\frac{1}{2}\varepsilon(s-t_0)}\cos(\vartheta s)\phi\big(s;t_0,\alpha_{w0},\beta_{w0}\big)ds,$$

then we obtain the following equation:

$$
\begin{aligned}
e^{-\frac{1}{2}\varepsilon(t-t_0)} U_{t-t_0}(\vartheta) \begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -\frac{1}{2} & \frac{1}{\varepsilon} \end{pmatrix} x_w(t) \\
= \begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -\frac{1}{2} & \frac{1}{\varepsilon} \end{pmatrix} \begin{pmatrix} \alpha_{w0} \\ \beta_{w0} \end{pmatrix} - U_{-t_0}(\vartheta) \begin{pmatrix} I_s(t, t_0; \alpha_{w0}, \beta_{w0}) \\ I_c(t, t_0; \alpha_{w0}, \beta_{w0}) \end{pmatrix}.
\end{aligned} \tag{2.6}
$$

We utilize the following relations in computing Equation (2.6).

$$
\begin{aligned}
U_t(\vartheta)U_s(\vartheta) =& U_{t+s}(\vartheta), \\
U_t^{-1}(\vartheta) =& U_{-t}(\vartheta), \\
U_0(\vartheta) =& E.
\end{aligned}
$$

Theorem 2.1. *Suppose* $\lim_{t\to\infty} e^{-\frac{1}{2}\varepsilon t} x_w(t) = 0$.

$$
\lim_{t\to\infty} \begin{pmatrix} I_s(t, t_0; \alpha_{w0}, \beta_{w0}) \\ I_c(t, t_0; \alpha_{w0}, \beta_{w0}) \end{pmatrix} = U_{t_0}(\vartheta) \begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -\frac{1}{2} & \frac{1}{\varepsilon} \end{pmatrix} \begin{pmatrix} \alpha_{w0} \\ \beta_{w0} \end{pmatrix}.
$$

Before stating the next theorem, we prepare the following proposition.

Proposition 2.1. (*the property of autonoumous systems*)(*for example, see* [*Braun, 1993*]) *The followings are equivalent. There exists a* $\tau > 0$,
(1) $x_w(t_0 + \tau) = x_w(t_0)$, for some t_0,
(2) $x_w(t + \tau) = x_w(t)$, for any t.

Without warning, we often use this nature hereinafter.

Theorem 2.2. *Let* $x_w(t)$ *be a solution of* $W_{\varepsilon,m,\phi}$. *Then the following statements are equivalent. For some* t_0,
(1)

$$
\begin{pmatrix} I_s(t_0 + \tau, t_0; \alpha_{w0}, \beta_{w0}) \\ I_c(t_0 + \tau, t_0; \alpha_{w0}, \beta_{w0}) \end{pmatrix} = U_{t_0}(\vartheta) \left(1 - e^{-\frac{1}{2}\varepsilon\tau} U_\tau(\vartheta) \right) \begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -\frac{1}{2} & \frac{1}{\varepsilon} \end{pmatrix} \begin{pmatrix} \alpha_{w0} \\ \beta_{w0} \end{pmatrix}. \tag{2.7}
$$

(2)

$$
\begin{aligned}
\begin{pmatrix} I_s(t + \tau, t_0; \alpha_{w0}, \beta_{w0}) \\ I_c(t + \tau, t_0; \alpha_{w0}, \beta_{w0}) \end{pmatrix} =& U_{t_0}(\vartheta) \left(1 - e^{-\frac{1}{2}\varepsilon\tau} U_\tau(\vartheta) \right) \begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -\frac{1}{2} & \frac{1}{\varepsilon} \end{pmatrix} \begin{pmatrix} \alpha_{w0} \\ \beta_{w0} \end{pmatrix} \\
&+ e^{-\frac{1}{2}\varepsilon\tau} U_\tau(\vartheta) \begin{pmatrix} I_s(t, t_0; \alpha_{w0}, \beta_{w0}) \\ I_c(t, t_0; \alpha_{w0}, \beta_{w0}) \end{pmatrix}, \text{ for any } t.
\end{aligned} \tag{2.8}
$$

(3) $x_w(t)$ *is periodic with period* τ.

Proof. (2)⇒(1) Let $t = t_0$ in Equation (2.8), we obtain Equation (2.7).
(1)⇒ (3) We assume Equation (2.7). Letting $t = t_0 + \tau$ in Equation (2.6) and using Equation (2.7) yields $x_w(t_0 + \tau) = x_w(t_0)$. Therefore, by Proposition 2.1, we have $x_w(t + \tau) = x_w(t)$.
(3)⇒(2) Substituting $t + \tau$ into t in Equation (2.6) leads to

$$\begin{aligned} &e^{-\frac{1}{2}\varepsilon(t+\tau-t_0)}U_{t+\tau-t_0}(\vartheta)\begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -\frac{1}{2} & \frac{1}{\varepsilon} \end{pmatrix} x_w(t+\tau) \\ &= \begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -\frac{1}{2} & \frac{1}{\varepsilon} \end{pmatrix} \begin{pmatrix} \alpha_{w0} \\ \beta_{w0} \end{pmatrix} - U_{-t_0}(\vartheta) \begin{pmatrix} I_s(t+\tau, t_0; \alpha_{w0}, \beta_{w0}) \\ I_c(t+\tau, t_0; \alpha_{w0}, \beta_{w0}) \end{pmatrix}. \end{aligned} \tag{2.9}$$

Assuming $x_w(t+\tau) = x_w(t)$ and multiplying both sides of Equation (2.9) by $e^{\frac{1}{2}\varepsilon\tau}U_{-\tau}(\vartheta)$, we have

$$\begin{aligned} &e^{-\frac{1}{2}\varepsilon(t-t_0)}U_{t-t_0}(\vartheta)\begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -\frac{1}{2} & \frac{1}{\varepsilon} \end{pmatrix} x_w(t) \\ &= e^{\frac{1}{2}\varepsilon\tau}U_{-\tau}(\vartheta) \begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -\frac{1}{2} & \frac{1}{\varepsilon} \end{pmatrix} \begin{pmatrix} \alpha_{w0} \\ \beta_{w0} \end{pmatrix} - e^{\frac{1}{2}\varepsilon\tau}U_{-\tau-t_0}(\vartheta) \begin{pmatrix} I_s(t+\tau, t_0; \alpha_{w0}, \beta_{w0}) \\ I_c(t+\tau, t_0; \alpha_{w0}, \beta_{w0}) \end{pmatrix}. \end{aligned} \tag{2.10}$$

Equating both right-hand sides of Equations (2.6) and (2.10) yields Equation (2.8). □

3 Analysis of the coupled van der Pol equation system

3.1 Formation of the fundamental equations for the analysis

Now, we let

$$y(t_0) = \alpha_0, \ y'(t_0) = \beta_0, \ z(t_0) = \lambda_0, \ z'(t_0) = \mu_0,$$

and define some new symbols as follows:

$$\begin{aligned} \phi_\pm(t) &:= y^2(t)y'(t) \pm z^2(t)z'(t), \\ \theta_+ &:= \frac{\sqrt{4-\varepsilon^2}}{2}, \\ \theta_- &:= \frac{\sqrt{4-\varepsilon^2-8k}}{2}, \end{aligned}$$

$$I_s^\pm(t, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) := \int_{t_0}^t e^{\frac{1}{2}\varepsilon(t-s)} \frac{\sin(\theta_\pm(t-s))}{\theta_\pm} \phi_\pm(s; t_0, \alpha_0, \beta_0, \lambda_0, \mu_0) ds,$$

$$I_c^\pm(t, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) := \int_{t_0}^t e^{\frac{1}{2}\varepsilon(t-s)} \cos(\theta_\pm(t-s)) \phi_\pm(s; t_0, \alpha_0, \beta_0, \lambda_0, \mu_0) ds,$$

where a double sign $\pm$ in equations corresponds in order.

For $y, z \in \Sigma_{\varepsilon,k}$, we let $x_+(t) = y(t) + z(t)$, then we have the differential equation $W_{\varepsilon,1,\phi_+}$ corresponding to Equation (2.1) of the previous section, that is,

$$W_{\varepsilon,1,\phi_+}: \ x_+'' - \varepsilon(x_+' - \phi_+) + x_+ = 0.$$

Again we define other symbols as follows:

$$I_{s\pm}(t, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) := \int_{t_0}^{t} e^{-\frac{1}{2}\varepsilon(s-t_0)} \frac{\sin(\theta_\pm s)}{\theta_\pm} \phi_\pm(s; t_0, \alpha_0, \beta_0, \lambda_0, \mu_0) ds,$$

$$I_{c\pm}(t, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) := \int_{t_0}^{t} e^{-\frac{1}{2}\varepsilon(s-t_0)} \cos(\theta_\pm s) \phi_\pm(s; t_0, \alpha_0, \beta_0, \lambda_0, \mu_0) ds.$$

Before obtaining the fundamental equations for the analysis, we prepare the next lemma.

Lemma 3.1.

$$\begin{pmatrix} I_{s\pm}(t, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c\pm}(t, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix} = U_{t_0}(\theta_\pm) \begin{pmatrix} I_{s\pm}(t - t_0, 0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c\pm}(t - t_0, 0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix}$$

Proof. See [Nohara and Arimoto, 2009]. $\square$

As the fundamental equation for $x_+(t)$, that is, $y(t) + z(t)$, we have the following linear system of integral equation using integral symbols defined above, which corresponds to Equation (2.6).

$$\begin{aligned} e^{-\frac{\varepsilon}{2}(t-t_0)} U_{t-t_0}(\theta_+) \begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -\frac{1}{2} & \frac{1}{\varepsilon} \end{pmatrix} \begin{pmatrix} x_+(t) \\ x_+'(t) \end{pmatrix} \\ = \begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -\frac{1}{2} & \frac{1}{\varepsilon} \end{pmatrix} \begin{pmatrix} x_+(t_0) \\ x_+'(t_0) \end{pmatrix} - U_{-t_0}(\theta_+) \begin{pmatrix} I_{s+}(t, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c+}(t, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix}. \end{aligned} \quad (3.1)$$

Here, applying Lemma 3.1 to the above equation, we obtain

$$\begin{aligned} e^{-\frac{\varepsilon}{2}(t-t_0)} U_{t-t_0}(\theta_+) \begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -\frac{1}{2} & \frac{1}{\varepsilon} \end{pmatrix} \begin{pmatrix} x_+(t) \\ x_+'(t) \end{pmatrix} \\ = \begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -\frac{1}{2} & \frac{1}{\varepsilon} \end{pmatrix} \begin{pmatrix} x_+(t_0) \\ x_+'(t_0) \end{pmatrix} - \begin{pmatrix} I_{s+}(t - t_0, 0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c+}(t - t_0, 0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix}. \end{aligned} \quad (3.2)$$

Whereas let $x_-(t) = y(t) - z(t)$ for $y, z \in \Sigma_{\varepsilon,k}$, then we obtain $W_{\varepsilon,1-2k,\phi_-}$, that is,

$$W_{\varepsilon,1-2k,\phi_-}: \ x_-'' - \varepsilon(x_-' - \phi_-) + (1 - 2k)x_- = 0.$$

In the same way, we obtain the linear system of integral equation for $x_-(t) = y(t) -$

$z(t)$

$$
\begin{aligned}
&e^{-\frac{\varepsilon}{2}(t-t_0)}U_{t-t_0}(\theta_-)\begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -\frac{1}{2} & \frac{1}{\varepsilon} \end{pmatrix}\begin{pmatrix} x_-(t) \\ x'_-(t) \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -\frac{1}{2} & \frac{1}{\varepsilon} \end{pmatrix}\begin{pmatrix} x_-(t_0) \\ x'_-(t_0) \end{pmatrix} - \begin{pmatrix} I_{s-}(t-t_0, 0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c-}(t-t_0, 0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix}.
\end{aligned}
$$

3.2 Necessary and sufficient condition for the periodicity of the coupled van der Pol equation system

We give the necessary and sufficient condition for the periodicity of the solutions of the coupled van der Pol equation system in this subsection. First, the following theorem holds in the same way as Theorem 2.1.

Theorem 3.1. *Suppose* $\lim_{t\to\infty} e^{-\frac{1}{2}\varepsilon t}\mathrm{col}\big(x_\pm(t), x'_\pm(t)\big) = 0$.

$$
\lim_{t\to\infty}\begin{pmatrix} I_{s\pm}(t, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c\pm}(t, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix} = U_{t_0}(\theta_\pm)\begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -\frac{1}{2} & \frac{1}{\varepsilon} \end{pmatrix}\begin{pmatrix} x_\pm(t_0) \\ x'_\pm(t_0) \end{pmatrix}.
$$

In this theorem, a double sign $\pm$ corresponds in order.

Next we state some properties when the system has the periodicity. Remember that $\xi_\Sigma(t) = \mathrm{col}\big(y(t), y'(t), z(t), z'(t)\big)$.

Theorem 3.2. *Suppose that* $\xi_\Sigma(t+\tau) = \xi_\Sigma(t)$, *then the followings are equivalent for a fixed* t_0.
(1)

$$
\begin{cases} x_+(t_0) = 0 \\ x'_+(t_0) = 0. \end{cases}
$$

(2)

$$
\begin{cases} I_{s+}(t_0 + n\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) = 0, \\ I_{c+}(t_0 + n\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) = 0, \;\; n = 1, 2, \ldots \end{cases}
$$

Proof. See [Nohara and Arimoto, 2009]. □

Theorem 3.3. *Suppose that* $\xi_\Sigma(t+\tau) = \xi_\Sigma(t)$, *then the followings are equivalent for a fixed* t_0.
(1)

$$
\begin{cases} x_-(t_0) = 0, \\ x'_-(t_0) = 0. \end{cases}
$$

(2)

$$
\begin{cases} I_{s-}(t_0 + n\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) = 0, \\ I_{c-}(t_0 + n\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) = 0, \;\; n = 1, 2, \ldots \end{cases}
$$

Lemma 3.2. *The followings are equivalent.*
(1)
$$\xi_\Sigma(t+\tau) = \xi_\Sigma(t).$$
(2)
$$x_\pm(t+\tau) = x_\pm(t).$$

Theorem 3.4. **(Necessary and sufficient condition for the periodicity)** *The solution of the dynamical system* $\mathbf{\Sigma}_{\varepsilon,k}$ *with the initial condition*
$$y(t_0) = \alpha_0,\ y'(t_0) = \beta_0,\ z(t_0) = \lambda_0,\ z'(t_0) = \mu_0$$
has a period τ *if and only if*
$$\mathbf{F}_\pm(\varepsilon) = 0, \tag{3.3}$$
where,
$$\mathbf{F}_\pm(\varepsilon) = \left(1 - e^{-\frac{\varepsilon}{2}\tau}U_\tau(\theta_\pm)\right)\begin{pmatrix}1 & 0\\ \frac{\varepsilon}{2} & -1\end{pmatrix}\begin{pmatrix}x_\pm(t_0)\\ x'_\pm(t_0)\end{pmatrix} + \varepsilon\begin{pmatrix}I_{s\pm}(\tau,0;\alpha_0,\beta_0,\lambda_0,\mu_0)\\ I_{c\pm}(\tau,0;\alpha_0,\beta_0,\lambda_0,\mu_0)\end{pmatrix}. \tag{3.4}$$

Proof. (proof of the necessary part) $\mathbf{\Sigma}_{\epsilon,k}$ has a period, that is, $\xi_\Sigma(t+\tau) = \xi_\Sigma(t)$ for some $\tau > 0$ inasmuch as $x_\pm(t+\tau) = x_\pm(t)$ and $x'_\pm(t+\tau) = x'_\pm(t)$ from Lemma 3.2. Therefore we have the following equation by the same procedure which yields Equation (2.7).
$$\left(1 - e^{-\frac{\varepsilon}{2}\tau}U_\tau(\theta_\pm)\right)\begin{pmatrix}1 & 0\\ \frac{\varepsilon}{2} & -1\end{pmatrix}\begin{pmatrix}x_\pm(t_0)\\ x'_\pm(t_0)\end{pmatrix} + \varepsilon U_{-t_0}(\theta_\pm)\begin{pmatrix}I_{s\pm}(t_0+\tau,t_0;\alpha_0,\beta_0,\lambda_0,\mu_0)\\ I_{c\pm}(t_0+\tau,t_0;\alpha_0,\beta_0,\lambda_0,\mu_0)\end{pmatrix} = 0. \tag{3.5}$$

The second term is computed by Lemma 3.1 as
$$U_{-t_0}(\theta_\pm)\begin{pmatrix}I_{s\pm}(t_0+\tau,t_0;\alpha_0,\beta_0,\lambda_0,\mu_0)\\ I_{c\pm}(t_0+\tau,t_0;\alpha_0,\beta_0,\lambda_0,\mu_0)\end{pmatrix} = \begin{pmatrix}I_{s\pm}(\tau,0;\alpha_0,\beta_0,\lambda_0,\mu_0)\\ I_{c\pm}(\tau,0;\alpha_0,\beta_0,\lambda_0,\mu_0)\end{pmatrix}, \tag{3.6}$$
then substituting this into Eqaution (3.5) leads to Equations (3.3) and (3.4).

(proof of the sufficient part) Here, we prove that $\mathbf{F}_\pm = 0 \Rightarrow x_\pm(t_0+\tau) = x_\pm(t_0)$, which is equivalent to $x_\pm(t+\tau) = x_\pm(t)$. Using Equation (3.6) in Equation (3.4) we have
$$e^{-\frac{\varepsilon}{2}\tau}U_{t_0+\tau}(\theta_\pm)\begin{pmatrix}1 & 0\\ \frac{\varepsilon}{2} & -1\end{pmatrix}\begin{pmatrix}x_\pm(t_0)\\ x'_\pm(t_0)\end{pmatrix} = U_{t_0}(\theta_\pm)\begin{pmatrix}1 & 0\\ \frac{\varepsilon}{2} & -1\end{pmatrix}\begin{pmatrix}x_\pm(t_0)\\ x'_\pm(t_0)\end{pmatrix} + \varepsilon\begin{pmatrix}I_{s\pm}(t_0+\tau,t_0;\alpha_0,\beta_0,\lambda_0,\mu_0)\\ I_{c\pm}(t_0+\tau,t_0;\alpha_0,\beta_0,\lambda_0,\mu_0)\end{pmatrix}. \tag{3.7}$$

On the other hand, substituting $t = t_0 + \tau$ into Equation (3.1) yields
$$e^{-\frac{\varepsilon}{2}\tau}U_\tau(\theta_+)\begin{pmatrix}1 & 0\\ \frac{\varepsilon}{2} & -1\end{pmatrix}\begin{pmatrix}x_+(t_0+\tau)\\ x'_+(t_0+\tau)\end{pmatrix} = \begin{pmatrix}1 & 0\\ \frac{\varepsilon}{2} & -1\end{pmatrix}\begin{pmatrix}x_+(t_0)\\ x'_+(t_0)\end{pmatrix} + \varepsilon U_{-t_0}(\theta_+)\begin{pmatrix}I_{s+}(t_0+\tau,t_0;\alpha_0,\beta_0,\lambda_0,\mu_0)\\ I_{c+}(t_0+\tau,t_0;\alpha_0,\beta_0,\lambda_0,\mu_0)\end{pmatrix},$$

that is,

$$
\begin{aligned}
e^{-\frac{\varepsilon}{2}\tau} U_{t_0+\tau}(\theta_+) \begin{pmatrix} 1 & 0 \\ \frac{\varepsilon}{2} & -1 \end{pmatrix} \begin{pmatrix} x_+(t_0+\tau) \\ x'_+(t_0+\tau) \end{pmatrix} \\
&= U_{t_0}(\theta_+) \begin{pmatrix} 1 & 0 \\ \frac{\varepsilon}{2} & -1 \end{pmatrix} \begin{pmatrix} x_+(t_0) \\ x'_+(t_0) \end{pmatrix} \\
&\quad + \varepsilon \begin{pmatrix} I_{s+}(t_0+\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c+}(t_0+\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix}.
\end{aligned} \tag{3.8}
$$

Similarly, we have

$$
\begin{aligned}
e^{-\frac{\varepsilon}{2}\tau} U_{t_0+\tau}(\theta_-) \begin{pmatrix} 1 & 0 \\ \frac{\varepsilon}{2} & -1 \end{pmatrix} \begin{pmatrix} x_-(t_0+\tau) \\ x'_-(t_0+\tau) \end{pmatrix} \\
&= U_{t_0}(\theta_-) \begin{pmatrix} 1 & 0 \\ \frac{\varepsilon}{2} & -1 \end{pmatrix} \begin{pmatrix} x_-(t_0) \\ x'_-(t_0) \end{pmatrix} \\
&\quad + \varepsilon \begin{pmatrix} I_{s-}(t_0+\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c-}(t_0+\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix}.
\end{aligned} \tag{3.9}
$$

Subtracting Equation (3.7) from Equations (3.8) and (3.9) leads to

$$
e^{-\frac{\varepsilon}{2}\tau} U_{t_0+\tau}(\theta_\pm) \begin{pmatrix} 1 & 0 \\ \frac{\varepsilon}{2} & -1 \end{pmatrix} \left\{ \begin{pmatrix} x_\pm(t_0+\tau) \\ x'_\pm(t_0+\tau) \end{pmatrix} - \begin{pmatrix} x_\pm(t_0) \\ x'_\pm(t_0) \end{pmatrix} \right\} = 0.
$$

Therefore we obtain

$$
\begin{cases} x_\pm(t_0+\tau) = x_\pm(t_0), \\ x'_\pm(t_0+\tau) = x'_\pm(t_0). \end{cases}
$$

Consequently, $\mathbf{F}_\pm(\varepsilon) = 0 \Rightarrow x_\pm(t_0+\tau) = x_\pm(t_0)$ is proved. □

4 Non-existence theorem of periodic solutions except the out-of-phase and in-phase solutions in $\Sigma_{\varepsilon,k}$

Our objective equation system $\Sigma_{\varepsilon,k}$ is as follows: Let $y = y(t,\varepsilon)$ and $z = z(t,\varepsilon)$ two real valued functions depending on the parameter ε and $0 < \varepsilon < 2$, $0 < k < \dfrac{1}{2} - \dfrac{\varepsilon^2}{8}$.

$$
\Sigma_{\varepsilon,k} \begin{cases} y'' - \varepsilon(1-y^2)y' + y = k(y-z), \\ z'' - \varepsilon(1-z^2)z' + z = k(z-y), \ t_0 \le t, \end{cases}
$$

with the initial condition

$$
y(t_0,\varepsilon) = \alpha_0(\varepsilon), \ y'(t_0,\varepsilon) = \beta_0(\varepsilon), \ z(t_0,\varepsilon) = \lambda_0(\varepsilon), \ z'(t_0,\varepsilon) = \mu_0(\varepsilon),
$$

where the initial condition also depends on the parameter ε inasmush as we write $\alpha_0(\varepsilon), \beta_0(\varepsilon), \lambda_0(\varepsilon)$ and $\mu_0(\varepsilon)$ deliberately.

Here, we give the assumption of periodic solutions of the dynamical system $\Sigma_{\varepsilon,k}$.

Assumption 4.1. Periodic solutions of $\Sigma_{\varepsilon,k}$ *Periodic solutions of $\Sigma_{\varepsilon,k}$ satisfy*

$$y\big(t+\tau(\varepsilon),\varepsilon\big) = y(t,\varepsilon),\ z\big(t+\tau(\varepsilon),\varepsilon\big) = z(t,\varepsilon),\ |\tau(\varepsilon)| < T, \tag{4.1}$$

where τ indicates a period of $\Sigma_{\varepsilon,k}$ and T is independent of the parameter ε. Also periodic solutions and their derivatives satisfy

$$|y(t,\varepsilon)| < M,\ |y'(t,\varepsilon)| < M,\ |z(t,\varepsilon)| < M,\ |z'(t,\varepsilon)| < M, \tag{4.2}$$

where M is independent of the parameter ε and t.

Hereinafter, we consider only periodic solutions restricted by Assumption 4.1. Before stating the main theorem, we prepare the following lemma.

Lemma 4.1. *Let $y(t,\varepsilon), z(t,\varepsilon)$ be a periodic solution of $\Sigma_{\varepsilon,k}$ satisfying Assumption 4.1. We assume that there exists $\lim_{\varepsilon\to 0} x_\pm(t_0,\varepsilon)$ and let $\lim_{\varepsilon\to 0} x_\pm(t_0,\varepsilon) = x_\pm(t_0,0)$. Then there exists a solution $y(t)$ and $z(t)$ of the degenerated system $\Sigma_{0,k}$ such that $\lim_{\varepsilon\to 0} x_\pm(t,\varepsilon) = x_\pm(t,0) = y(t) \pm z(t)$ and $\lim_{\varepsilon\to 0} x'_\pm(t,\varepsilon) = x'_\pm(t,0) = y'(t) \pm z'(t)$. Let $\tau_\pm(\varepsilon)$ and $\tau_\pm(0)$ be periods of $x_\pm(t,\varepsilon)$ created by $\Sigma_{\varepsilon,k}$ and $x_\pm(t,0)$ by $\Sigma_{0,k}$, respectively, then $\lim_{\varepsilon\to 0}\tau_\pm(\varepsilon) = \tau_\pm(0)$.*

Proof. See [Nohara and Arimoto, 2009]. □

We give the next main theorem for $\Sigma_{\varepsilon,k}$.

Theorem 4.1. Non-existence of periodic solutions except the out-of-phase and in-phase solutions *Let $y(t,\varepsilon)$ and $z(t,\varepsilon)$ be a periodic solution of $\Sigma_{\varepsilon,k}$, which is analytic with respect to ε on the segment $[0,\varepsilon_0)$, where $0 < \varepsilon_0 < 2, 0 < k < \dfrac{1}{2} - \dfrac{\varepsilon_0^2}{8}$, and k is irrational. Then this solution is either out-of-phase or in-phase.*

Preparations for the proof We assume that the periodicity is built up and let a period (but unknown) be $\tau(\varepsilon)$, which depends on ε, then we have the following relation from Theorem 3.4.

$$\mathbf{F}_\pm(\varepsilon) = 0,$$

where

$$\mathbf{F}_\pm(\varepsilon) = \left(1 - e^{-\frac{\varepsilon}{2}\tau(\varepsilon)} U_{\tau(\varepsilon)}(\theta_\pm)\right)\begin{pmatrix}1 & 0\\ \frac{\varepsilon}{2} & -1\end{pmatrix}\begin{pmatrix}x_\pm(t_0,\varepsilon)\\ x'_\pm(t_0,\varepsilon)\end{pmatrix} + \varepsilon\begin{pmatrix}I_{s\pm}(\tau(\varepsilon),0;\alpha_0(\varepsilon),\beta_0(\varepsilon),\lambda_0(\varepsilon),\mu_0(\varepsilon))\\ I_{c\pm}(\tau(\varepsilon),0;\alpha_0(\varepsilon),\beta_0(\varepsilon),\lambda_0(\varepsilon),\mu_0(\varepsilon))\end{pmatrix}.$$

First, we take $\varepsilon \to 0$ in $\mathbf{F}_+(\varepsilon) = 0$. Then we have

$$\begin{pmatrix}1-\cos(\tau(0)) & \sin(\tau(0))\\ -\sin(\tau(0)) & 1-\cos(\tau(0))\end{pmatrix}\begin{pmatrix}x_+(t_0,0)\\ x'_+(t_0,0)\end{pmatrix} = 0. \tag{4.3}$$

Here, $\tau(0) = \lim_{\varepsilon\to 0}\tau(\varepsilon)$.

On the other hand, taking $\varepsilon \to 0$ in $\mathbf{F}_-(\varepsilon) = 0$ we have

$$\begin{pmatrix}1-\cos(\sqrt{1-2k}\,\tau(0)) & \dfrac{\sin(\sqrt{1-2k}\,\tau(0))}{\sqrt{1-2k}}\\ -\sqrt{1-2k}\sin(\sqrt{1-2k}\,\tau(0)) & 1-\cos(\sqrt{1-2k}\,\tau(0))\end{pmatrix}\begin{pmatrix}x_-(t_0,0)\\ x'_-(t_0,0)\end{pmatrix} = 0. \tag{4.4}$$

Equations (4.3) and (4.4) must hold simultaneously inasmuch as we have the following results: For each t_0,

(1) We have $\begin{pmatrix} x_+(t_0,0) \\ x'_+(t_0,0) \end{pmatrix} = 0$ or $\tau(0) = 2\pi$ from Equation (4.3). In the latter case, we let $\tau_-(0) = 2\pi$ for the sake of convenience.

(2) Similarly, we have $\begin{pmatrix} x_-(t_0,0) \\ x'_-(t_0,0) \end{pmatrix} = 0$ or $\tau(0) = \dfrac{2\pi}{\sqrt{1-2k}}$ from Equation (4.4). In the latter case, we let $\tau_+(0) = \dfrac{2\pi}{\sqrt{1-2k}}$ for the sake of convenience.

(3) If k is irrational which satisfies $0 < k < \dfrac{1}{2} - \dfrac{\varepsilon^2}{8}$, then $j\tau_+(0) \neq l\tau_-(0)$, $(j, l = 1, 2, 3, \ldots,\ j \neq l)$. Therefore, we obtain following two conditions: a condition is $\begin{pmatrix} x_+(t_0,0) \\ x'_+(t_0,0) \end{pmatrix} = 0$ and $\tau_+(0) = \dfrac{2\pi}{\sqrt{1-2k}}$ and another condition is $\begin{pmatrix} x_-(t_0,0) \\ x'_-(t_0,0) \end{pmatrix} = 0$ and $\tau_-(0) = 2\pi$ since (1) and (2) must hold simultaneously. We take some t_0 in the above consideration, but we find that t_0 can be taken arbitrary in this stage. Consequently, the former condition means out-of-phase and the latter in-phase.

Note that the condition of $\begin{pmatrix} x_+(t_0,0) \\ x'_+(t_0,0) \end{pmatrix} = 0$ and $\begin{pmatrix} x_-(t_0,0) \\ x'_-(t_0,0) \end{pmatrix} = 0$ is $\alpha_0(0) = \beta_0(0) = \lambda_0(0) = \mu_0(0)$, that is, the origin.

Summarizing above, when $\varepsilon = 0$, there exists no periodic solutions except the out-of-phase and in-phase solutions, in which periods are $\tau_+(0) = \dfrac{2\pi}{\sqrt{1-2k}}$ and $\tau_-(0) = 2\pi$, respectively. This fact is consistent with the characteristics of $\mathbf{\Sigma}_{0,k}$.

Before proving the main theorem, we prepare two propositions and give the following definitions in order to prove the propositions using the inductive method.

Definition 4.1. $\mathbf{P}_+(\nu), \nu = 1, 2, 3, \ldots,$ *is defined as:*
If $\begin{pmatrix} x_+(t_0,0) \\ x'_+(t_0,0) \end{pmatrix} = 0$, *then there exist derivatives* $\dfrac{\partial^\nu x_+(t,\varepsilon)}{\partial \varepsilon^\nu}$ *and* $\dfrac{\partial^\nu x'_+(t,\varepsilon)}{\partial \varepsilon^\nu}$, *and* $\dfrac{\partial^\nu x_+(t,\varepsilon)}{\partial \varepsilon^\nu} = 0$ *and* $\dfrac{\partial^\nu x'_+(t,\varepsilon)}{\partial \varepsilon^\nu} = 0$ *at* $\varepsilon = 0$.

Definition 4.2. $\mathbf{P}_-(\nu), \nu = 1, 2, 3, \ldots,$ *is defined as:*
If $\begin{pmatrix} x_-(t_0,0) \\ x'_-(t_0,0) \end{pmatrix} = 0$, *then there exist derivatives* $\dfrac{\partial^\nu x_-(t,\varepsilon)}{\partial \varepsilon^\nu}$ *and* $\dfrac{\partial^\nu x'_-(t,\varepsilon)}{\partial \varepsilon^\nu}$, *and* $\dfrac{\partial^\nu x_-(t,\varepsilon)}{\partial \varepsilon^\nu} = 0$ *and* $\dfrac{\partial^\nu x'_-(t,\varepsilon)}{\partial \varepsilon^\nu} = 0$ *at* $\varepsilon = 0$.

Proposition 4.1. $\mathbf{P}_+(\nu)$ *is true for* $\nu = 1, 2, 3, \ldots$.

Proposition 4.2. $\mathbf{P}_-(\nu)$ *is true for* $\nu = 1, 2, 3, \ldots$.

Proof. We prove only Proposition 4.1 using the inductive method because Proposition 4.2 can be done by the same manner.

(1) $x_+(t,0)$ defined by $\lim_{\varepsilon\to 0} x_+(t,\varepsilon)$ satisfies the differential equations $x''_+(t,0) + x_+(t,0) = 0$ with the initial conditions $x_+(t_0,0)$ and $x'_+(t_0,0)$. By uniqueness of the solution, we must have $\begin{pmatrix} x_+(t,0) \\ x'_+(t,0) \end{pmatrix} \equiv 0$ when $\begin{pmatrix} x_+(t_0,0) \\ x'_+(t_0,0) \end{pmatrix} = 0$. Hence $\lim\limits_{\varepsilon\to 0} \begin{pmatrix} x_+(t,\varepsilon) \\ x'_+(t,\varepsilon) \end{pmatrix} =$

0 from Lemma 4.1. Then we have

$$\begin{aligned}
&y^2(s,\varepsilon)y'(s,\varepsilon) + z^2(s,\varepsilon)z'(s,\varepsilon) \\
&\quad = \big(x_+(s,\varepsilon) - z(s,\varepsilon)\big)^2 x'_+(s,\varepsilon) - z'(s,\varepsilon)\big(y(s,\varepsilon) - z(s,\varepsilon)\big)x_+(s,\varepsilon) \\
&\quad \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0
\end{aligned} \tag{4.5}$$

Since we have $\mathbf{F}_+(\varepsilon) = 0$ by the periodicity condition, that is,

$$\begin{aligned}
&\left(1 - e^{-\frac{\varepsilon}{2}\tau(\varepsilon)}U_{\tau(\varepsilon)}(\theta_+)\right)\begin{pmatrix} 1 & 0 \\ \frac{\varepsilon}{2} & -1 \end{pmatrix}\begin{pmatrix} x_+(t_0,\varepsilon) \\ x'_+(t_0,\varepsilon) \end{pmatrix} \\
&+\varepsilon\begin{pmatrix} \displaystyle\int_0^{\tau(\varepsilon)} e^{-\frac{1}{2}\varepsilon s}\frac{\sin(\theta_+ s)}{\theta_+}\big(y^2(t_0+s,\varepsilon)y'(t_0+s,\varepsilon) + z^2(t_0+s,\varepsilon)z'(t_0+s,\varepsilon)\big)ds \\ \displaystyle\int_0^{\tau(\varepsilon)} e^{-\frac{1}{2}\varepsilon s}\cos(\theta_+ s)\big(y^2(t_0+s,\varepsilon)y'(t_0+s,\varepsilon) + z^2(t_0+s,\varepsilon)z'(t_0+s,\varepsilon)\big)ds \end{pmatrix} = 0.
\end{aligned} \tag{4.6}$$

Dividing Equation (4.6) by ε yields

$$\begin{aligned}
&\left(1 - e^{-\frac{\varepsilon}{2}\tau(\varepsilon)}U_{\tau(\varepsilon)}(\theta_+)\right)\begin{pmatrix} 1 & 0 \\ \frac{\varepsilon}{2} & -1 \end{pmatrix}\begin{pmatrix} \dfrac{x_+(t_0,\varepsilon)}{\varepsilon} \\ \dfrac{x'_+(t_0,\varepsilon)}{\varepsilon} \end{pmatrix} \\
&+\begin{pmatrix} \displaystyle\int_0^{\tau(\varepsilon)} e^{-\frac{1}{2}\varepsilon s}\frac{\sin(\theta_+ s)}{\theta_+}\big(y^2(t_0+s,\varepsilon)y'(t_0+s,\varepsilon) + z^2(t_0+s,\varepsilon)z'(t_0+s,\varepsilon)\big)ds \\ \displaystyle\int_0^{\tau(\varepsilon)} e^{-\frac{1}{2}\varepsilon s}\cos(\theta_+ s)\big(y^2(t_0+s,\varepsilon)y'(t_0+s,\varepsilon) + z^2(t_0+s,\varepsilon)z'(t_0+s,\varepsilon)\big)ds \end{pmatrix} = 0.
\end{aligned} \tag{4.7}$$

We take $\varepsilon \longrightarrow 0$ in Equation (4.7). Then the second term vanishes from Equation (4.5) and there exist the derivatives $\dfrac{\partial x_+(t_0,0)}{\partial\varepsilon} = \lim\limits_{\varepsilon\to 0}\dfrac{x_+(t_0,\varepsilon) - x_+(t_0,0)}{\varepsilon}$ and $\dfrac{\partial x'_+(t_0,0)}{\partial\varepsilon} = \lim\limits_{\varepsilon\to 0}\dfrac{x'_+(t_0,\varepsilon) - x'_+(t_0,0)}{\varepsilon}$. Here we can take arbitrary t_0, therefore we have the derivatives $\dfrac{\partial x_+(t,0)}{\partial\varepsilon}$ and $\dfrac{\partial x'_+(t,0)}{\partial\varepsilon}$. Furthermore we obtain $\dfrac{\partial x_+(t,0)}{\partial\varepsilon} = 0$ and $\dfrac{\partial x'_+(t,0)}{\partial\varepsilon} = 0$.

Note that in the computation of the limit we can exchange the limit and the integral. We show below this fact. The integral of Equation (4.7) is written as follows using T defined in Equation (4.1):

$$\begin{aligned}
&\int_0^{\tau(\varepsilon)} e^{-\frac{1}{2}\varepsilon s}\frac{\sin(\theta_+ s)}{\theta_+}\big(y^2(t_0+s,\varepsilon)y'(t_0+s,\varepsilon) + z^2(t_0+s,\varepsilon)z'(t_0+s,\varepsilon)\big)ds \\
&= \int_0^{T} 1_{\tau(\varepsilon)}(s)e^{-\frac{1}{2}\varepsilon s}\frac{\sin(\theta_+ s)}{\theta_+}\big(y^2(t_0+s,\varepsilon)y'(t_0+s,\varepsilon) + z^2(t_0+s,\varepsilon)z'(t_0+s,\varepsilon)\big)ds,
\end{aligned}$$

where

$$1_{\tau(\varepsilon)}(s) = \begin{cases} 1, & \text{for } s \le \tau(\varepsilon) \\ 0, & \text{for } s > \tau(\varepsilon) \end{cases}$$

Now we find

$$\begin{aligned}&\Big|1_{\tau(\varepsilon)}(s)e^{-\frac{1}{2}\varepsilon s}\frac{\sin(\theta_+ s)}{\theta_+}\big(y^2(t_0+s,\varepsilon)y'(t_0+s,\varepsilon)+z^2(t_0+s,\varepsilon)z'(t_0+s,\varepsilon)\big)\Big|\\&\le\frac{1}{\theta_+}\big|y^2(t_0+s,\varepsilon)y'(t_0+s,\varepsilon)+z^2(t_0+s,\varepsilon)z'(t_0+s,\varepsilon)\big|\le M,\ \text{ for } 0<s<T.\end{aligned}$$

Here we use the same symbol M in the above equation and also in Equation (4.2) for the sake of convenience but these are different from each other. Then we can apply the bounded convergence theorem and we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^{\tau(\varepsilon)} e^{-\frac{1}{2}\varepsilon s}\frac{\sin(\theta_+ s)}{\theta_+}\big(y^2(t_0+s,\varepsilon)y'(t_0+s,\varepsilon)+z^2(t_0+s,\varepsilon)z'(t_0+s,\varepsilon)\big)ds\\&=\int_0^T\lim_{\varepsilon\to 0}1_{\tau(\varepsilon)}(s)e^{-\frac{1}{2}\varepsilon s}\frac{\sin(\theta_+ s)}{\theta_+}\big(y^2(t_0+s,\varepsilon)y'(t_0+s,\varepsilon)+z^2(t_0+s,\varepsilon)z'(t_0+s,\varepsilon)\big)ds\\&=\int_0^T 1_{\tau(0)}(s)\sin s\lim_{\varepsilon\to 0}\big(y^2(t_0+s,\varepsilon)y'(t_0+s,\varepsilon)+z^2(t_0+s,\varepsilon)z'(t_0+s,\varepsilon)\big)ds\\&=0.\end{aligned}$$

In the above equation, we use the relation $\tau(\varepsilon)\to\tau(0)$ as $\varepsilon\to 0$. In fact, we have $\lim_{\varepsilon\to 0}x_\pm\big(t+\tau(\varepsilon),\varepsilon\big)=x_\pm\big(t+\tau(0),0\big)$ from the assumption $\lim_{\varepsilon\to 0}x_\pm(t,\varepsilon)=x_\pm(t,0)=y(t)\pm z(t)$ and the periodicity conditions $\lim_{\varepsilon\to 0}x_\pm\big(t+\tau(\varepsilon),\varepsilon\big)=x_\pm(t,0)$ and $x_\pm\big(t+\tau(0),0\big)=x_\pm(t,0)$.

(2) We assume that $\mathbf{P}_+(\nu),\nu\le n$, that is, there exist $\dfrac{\partial^\nu x_+(t_0,0)}{\partial\varepsilon^\nu}$ and $\dfrac{\partial^\nu x'_+(t_0,0)}{\partial\varepsilon^\nu}$ and $\dfrac{\partial^\nu x_+(t_0,0)}{\partial\varepsilon^\nu}=0$, $\dfrac{\partial^\nu x'_+(t_0,0)}{\partial\varepsilon^\nu}=0$, $\nu=0,1,2,\ldots,n$. Then we show $\mathbf{P}_+(n+1)$. Dividing Equation (4.6) by ε^{n+1} yields

$$\begin{aligned}&\left(1-e^{-\frac{\varepsilon}{2}\tau(\varepsilon)}U_{\tau(\varepsilon)}(\theta_+)\right)\begin{pmatrix}1&0\\\frac{\varepsilon}{2}&-1\end{pmatrix}\begin{pmatrix}\dfrac{x_+(t_0,\varepsilon)-\sum_{\nu=1}^{n}\frac{\partial^\nu x_+(t_0,0)}{\partial\varepsilon^\nu}\varepsilon^\nu}{\varepsilon^{n+1}}\\[2ex]\dfrac{x'_+(t_0,\varepsilon)-\sum_{\nu=1}^{n}\frac{\partial^\nu x'_+(t_0,0)}{\partial\varepsilon^\nu}\varepsilon^\nu}{\varepsilon^{n+1}}\end{pmatrix}\\&+\begin{pmatrix}\displaystyle\int_0^{\tau(\varepsilon)}e^{-\frac{1}{2}\varepsilon s}\frac{\sin(\theta_+ s)}{\theta_+}\Big\{\big(x_+(t_0+s,\varepsilon)-z(t_0+s,\varepsilon)\big)^2\frac{x'_+(t_0+s,\varepsilon)}{\varepsilon^n}\\\qquad-z'(t_0+s,\varepsilon)\big(y(t_0+s,\varepsilon)-z(t_0+s,\varepsilon)\big)\dfrac{x_+(t_0+s,\varepsilon)}{\varepsilon^n}\Big\}ds\\\displaystyle\int_0^{\tau(\varepsilon)}e^{-\frac{1}{2}\varepsilon s}\cos(\theta_+ s)\Big\{\big(x_+(t_0+s,\varepsilon)-z(t_0+s,\varepsilon)\big)^2\frac{x'_+(t_0+s,\varepsilon)}{\varepsilon^n}\\\qquad-z'(t_0+s,\varepsilon)\big(y(t_0+s,\varepsilon)-z(t_0+s,\varepsilon)\big)\dfrac{x_+(t_0+s,\varepsilon)}{\varepsilon^n}\Big\}ds\end{pmatrix}=0.\end{aligned}$$

Here, we take $\varepsilon\to 0$, then the second term vanishes since $\lim_{\varepsilon\to 0}\dfrac{x_+(t_0+s,\varepsilon)}{\varepsilon^n}=0$, $\lim_{\varepsilon\to 0}\dfrac{x'_+(t_0+s,\varepsilon)}{\varepsilon^n}=0$. Therefore we find that there exist the derivatives $\dfrac{\partial^{n+1}x_+(t_0,0)}{\partial\varepsilon^{n+1}}$

and $\dfrac{\partial^{n+1}x'_+(t_0,0)}{\partial\varepsilon^{n+1}}$. Since t_0 is arbitrary, we have the existence of $\dfrac{\partial^{n+1}x_+(t,0)}{\partial\varepsilon^{n+1}}$ and $\dfrac{\partial^{n+1}x'_+(t,0)}{\partial\varepsilon^{n+1}}$. Furthermore, we obtain $\dfrac{\partial^{n+1}x_+(t,0)}{\partial\varepsilon^{n+1}} = 0$, $\dfrac{\partial^{n+1}x'_+(t,0)}{\partial\varepsilon^{n+1}} = 0$.
(3) From (1) and (2), we obtain that $\mathbf{P}_+(\nu)$ is true for $\forall\nu \in \mathcal{N}$.

The - part, that is, $\mathbf{P}_-(\nu)$ is true for $\forall\nu \in \mathcal{N}$ can be also proved in the same way using the relation

$$\begin{aligned} &y^2(s,\varepsilon)y'(s,\varepsilon) - z^2(s,\varepsilon)z'(s,\varepsilon) \\ &\quad = \big(x_-(s,\varepsilon) + z(s,\varepsilon)\big)^2 x'_-(s,\varepsilon) + z'(s,\varepsilon)\big(y(s,\varepsilon) + z(s,\varepsilon)\big)x_-(s,\varepsilon). \end{aligned}$$

□

We obtain the following lemma from Propositions 4.1 and 4.2.

Lemma 4.2. *We assume that $y(t,\varepsilon)$ and $z(t,\varepsilon)$ are analytic with respect to the parameter ε. If $\begin{pmatrix} x_+(t_0,0) \\ x'_+(t_0,0) \end{pmatrix} = 0$, then $\begin{pmatrix} x_+(t_0,\varepsilon) \\ x'_+(t_0,\varepsilon) \end{pmatrix} = 0$. Also, If $\begin{pmatrix} x_-(t_0,0) \\ x'_-(t_0,0) \end{pmatrix} = 0$, then $\begin{pmatrix} x_-(t_0,\varepsilon) \\ x'_-(t_0,\varepsilon) \end{pmatrix} = 0$.*

Proof of Theorem From Lemma 4.2, if $\begin{pmatrix} x_+(t_0,\varepsilon) \\ x'_+(t_0,\varepsilon) \end{pmatrix} \neq 0$ and $\begin{pmatrix} x_-(t_0,\varepsilon) \\ x'_-(t_0,\varepsilon) \end{pmatrix} \neq 0$, then $\begin{pmatrix} x_+(t_0,0) \\ x'_+(t_0,0) \end{pmatrix} \neq 0$ and $\begin{pmatrix} x_-(t_0,0) \\ x'_-(t_0,0) \end{pmatrix} \neq 0$. However, this is inconsistent with the fact of $\mathbf{\Sigma}_{0,k}$ under the assumption which k is irrational, that is, the dynamical system $\mathbf{\Sigma}_{0,k}$ does not have except the out-of-phase and in-phase solutions. Therefore, we have $\begin{pmatrix} x_+(t,\varepsilon) \\ x'_+(t,\varepsilon) \end{pmatrix} = 0$ or $\begin{pmatrix} x_-(t,\varepsilon) \\ x'_-(t,\varepsilon) \end{pmatrix} = 0$. Consequently, the dynamical system $\mathbf{\Sigma}_{\varepsilon,k}$ does not have any other periodic solutions except the out-of-phase and in-phase solutions. □

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