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Combinatorial representation of irrational rotations and invariant sets around irrationally indifferent fixed points

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Let $f(z)$ be a holomorphic function defined near $z=0$ with expansion

$$f(z)=e^{2\pi i\alpha}z+O(z^{2}),$$

where $\alpha\in\mathbb{R}\backslash\mathbb{Q}$. The origin is an *irrationally indifferent fixed point.* Especially, we are interested in the case where f is a quadratic polynomial $f(z)=e^{2\pi i\alpha}z+z^{2}$. The irrational number α can expressed in terms of (fast) continued fraction:

$$\alpha=a_{0}+\cfrac{\varepsilon_{0}}{a_{1}+\cfrac{\varepsilon_{1}}{a_{2}+\cfrac{\varepsilon_{2}}{\ddots}}}, \quad \text{where} \quad \begin{array}{l} a_{n}\in\mathbb{Z},\ \ \varepsilon_{n}=\pm 1\ (n=0,1,2,\ldots), \\ a_{n}\geq 2\ (n\geq 1). \end{array}$$

We proved in [IS] that there exists a class $\mathcal{F}_{1}$ of holomorphic functions around 0 and a large constant N such that for $h(z)=z+O(z^{2})\in \mathcal{F}_{1}$ and $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ with $a_{n}\geq N$, the function $f=e^{2\pi i\alpha}h$ has a sequence of well-defined "return maps" $\mathcal{R}^{n}f$ (which are called *near-parabolic renormalizations*), which have the form $\mathcal{R}^{n}f=e^{2\pi i\alpha_{n}}h_{n}$ with $h_{n}\in \mathcal{F}_{1}$. In this paper, we discuss how to derive the properties of an invariant set Λ_{f} around 0 and how to analyze its combinatorial aspect which is strongly associated with the irrational rotation $R_{\alpha}(z)=e^{2\pi i\alpha}z$ on $\mathbb{S}^{1}$.

We describe the local dynamics via an infinite systems of open covers of punctured neighborhoods, and in the open sets the dynamics can be conjugated to canonical maps (see Figure). More precisely, we have:

Theorem. *For f as above, there exists an infinite sequence of systems*

$$\{A_{n}, r_{\alpha,n}, \{\Omega_{k_{1},\ldots,k_{n}}\}_{(k_{1},\ldots,k_{n})\in A_{n}}, \{\varphi_{k_{1},\ldots,k_{n}}\}_{(k_{1},\ldots,k_{n})\in A_{n}}, \{F_{k_{1},\ldots,k_{n}}\}_{(k_{1},\ldots,k_{n})\in A_{n}}\}_{n\in\mathbb{N}}$$

satisfying:

- *The index set is a finite set $A_{n}\subset\mathbb{Z}^{n}$, which inherits the lexicographic order;*
- *The combinatorial dynamics $r_{\alpha,n}:A_{n}\rightarrow A_{n}$ is bijective and preserves the cyclic ordering;*
- *Open sets $\Omega_{k_{1},\ldots,k_{n}}$ $((k_{1},\ldots,k_{n})\in A_{n})$ cover a punctured neighborhood of 0, and their order around 0 is the same as the order of the indices in A_{n};*
- *Maps $\varphi_{k_{1},\ldots,k_{n}}:\Omega_{k_{1},\ldots,k_{n}}\rightarrow\Omega_{can}[a_{n}]$, where Ω_{can} is so-called the truncated checkerboard pattern (see Figure) and $\Omega_{can}[a_{n}]$ is its truncation according to the coefficient of the continued fraction of α; $\varphi_{k_{1},\ldots,k_{n}}$ are either holomorphic if δ_{n} (defined later) is $+1$ or anti-holomorphic otherwise;*

- *The model dynamics* $F_{k_1,\ldots,k_n} = \varphi_{r_{\alpha,n}(k_1,\ldots,k_n)} \circ f \circ \varphi^{-1}_{k_1,\ldots,k_n}$ *is* F_{can} *if* $(k_1,\ldots,k_n) = (0,\ldots,0)$ *and it is id otherwise, where* F_{can} *is the canonical dynamics on the truncated checkerboard pattern;*

- *Two open sets* $\Omega_{k_1,\ldots,k_n}$ *and* $\Omega_{\ell_1,\ldots,\ell_n}$ *overlap if and only if* $(k_1,\ldots,k_n)$ *and* $(\ell_1,\ldots,\ell_n)$ *are (cyclically) adjacent in* A_n*, and the gluing is defined via the n-th near-parabolic renormalization* $\mathcal{R}^n f$ *of* f*;*

- *The* $(n+1)$*-th system is a "refinement" of n-th one; If* $(k_1,\ldots,k_n,k_{n+1}) \in A_{n+1}$*, then* $(k_1,\ldots,k_n) \in A_n$ *and* $\Omega_{k_1,\ldots,k_n,k_{n+1}} \subset \Omega_{k_1,\ldots,k_n}$*; The combinatorial dynamics almost commutes with the projection* $proj_{n+1} : A_{n+1} \to A_n$ *in the sense that* $proj_{n+1} \circ r_{\alpha,n+1} = r_{\alpha,n} \circ proj_{n+1}$ *except at one element in* A_{n+1}*.*

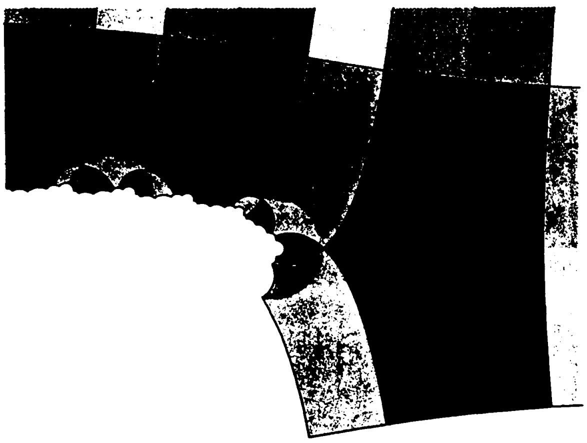

Figure: Truncated Checkerboard Pattern

$F_{can}(w) = w/(1-\frac{1}{w}) = w+1+O(\frac{1}{w})$ is conjugate to $z+z^2$ and has a parabolic fixed point at ∞. Its attracting Fatou coordinate Φ_{attr} (defined in a right half plane) conjugates F_{can} to $T(w) = w+1$, and is normalized so that $\Phi_{attr}(crit.pt) = 0$. It extends to the whole parabolic basin. The Truncated Checkerboard Pattern Ω_{can} is defined to be the union of some inverse images of $\{w : n < Re\ w < n+1,\ -2 < Im\ w < 2\}$ and $\{w : n < Re\ w < n+1,\ 2 < Im\ w\}$ $(n \in \mathbb{Z})$ by Φ_{attr} together with boundary curves.

Furthermore, the projective limit $A_\infty = \varprojlim A_n$ and the combinatorial dynamics $r_{\alpha,\infty} : A_\infty \to A_\infty$ are well-defined and there is an order-preserving semi-conjugacy $\pi : A_\infty \to \mathbb{S}^1$ from $r_{\alpha,\infty}$ to the α-rotation R_α such that two distinct indices $(k_1,k_2,\ldots),(\ell_1,\ell_2,\ldots)$ in A_∞ are mapped to the same point if and only if $(k_1,\ldots,k_n)$ and $(\ell_1,\ldots,\ell_n)$ are adjacent in A_n for large n. The quotient dynamics $\overline{r}_\alpha$ on $\overline{A}_\infty = A_\infty / \sim_{adjacent}$ is conjugate to R_α on $\mathbb{S}^1$.

The maximal invariant set Λ_f covered by these open sets is called the *maximal hedgehog*:

$$\Lambda_f = \{0\} \cup \bigcap_{n=1}^{\infty} \bigcup_{(k_1,k_2,\ldots,k_n)\in A_n} \Omega_{k_1,k_2,\ldots,k_n} = \{0\} \cup \bigcup_{(k_1,k_2,\ldots)\in \overline{A}_\infty} \bigcap_{n=1}^{\infty} \Omega_{k_1,k_2,\ldots,k_n}.$$

It can be shown that for each $(k_1,k_2,\ldots) \in \overline{A}_\infty$, $\bigcap_{n=1}^\infty \Omega_{k_1,k_2,\ldots,k_n}$ is either empty or an arc tending to 0. The sets $\bigcap_{n=1}^\infty \Omega_{k_1,k_2,\ldots,k_n}$ $((k_1,k_2,\ldots) \in \overline{A}_\infty)$ are (cyclically) permuted by the dynamics f. The map $\bigcap_{n=1}^\infty \Omega_{k_1,k_2,\ldots,k_n} \longrightarrow (k_1,k_2,\ldots) \longrightarrow \pi(k_1,k_2,\ldots)$ defines a semi-conjugacy from f on $\Lambda_f \setminus \{0\}$ to R_α on $\mathbb{S}^1$ (not necessarily onto).

Rotation combinatorics: The key ingredient in the construction (beside the theorem in [IS]) is the analysis of the combinatorics of the irrational rotation R_α. This will naturally gives us the index set A_n and the combinatorial dynamics $r_{\alpha,n}$. Let us first review the fast continued fractions.

Definition. For $x \in \mathbb{R}$, $\langle x \rangle$ denotes the closest integer to x. If $x = n + \frac{1}{2}$ with $n \in \mathbb{Z}$, we set $\langle x \rangle = n$. Define $||x|| = dist(x,\mathbb{Z}) = \min\{|x-n| : n \in \mathbb{Z}\} = |x - \langle x \rangle|$. Then we have $0 \le ||x|| \le \frac{1}{2}$ and $x = \langle x \rangle \pm ||x||$.

For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, let $\alpha_{-1} = \frac{1}{\alpha}$ and define $a_n \in \mathbb{Z}$, $\alpha_n \in (0, \frac{1}{2}] \subset \mathbb{R}$ and $\varepsilon_n = \pm 1$ $(n = 0, 1, 2, \dots)$ as follows:

$$a_n = \left\langle \frac{1}{\alpha_{n-1}} \right\rangle, \ \alpha_n = \left\| \frac{1}{\alpha_{n-1}} \right\| \ \text{ and } \ \varepsilon_n = \begin{cases} +1 & \text{if } a_n \le \frac{1}{\alpha_{n-1}} \\ -1 & \text{if } a_n > \frac{1}{\alpha_{n-1}}. \end{cases} \tag{1}$$

It immediately follows that

$$\frac{1}{\alpha_{n-1}} = a_n + \varepsilon_n \alpha_n \tag{2}$$

and that $a_n \ge 2$ for $n \ge 1$. Moreover if $a_n = 2$ $(n \ge 1)$, then $\varepsilon_n = +1$. A repeated use of (2) shows

$$\alpha = a_0 + \cfrac{\varepsilon_0}{a_1 + \cfrac{\varepsilon_1}{\ddots + \cfrac{\varepsilon_{n-1}}{a_n + \varepsilon_n \alpha_n}}}. \tag{3}$$

Define the sequence of integers $\{q_n\}_{n=-2}^{\infty}$ and $\{p_n\}_{n=-2}^{\infty}$ by the following

$$\begin{cases} q_n = a_n q_{n-1} + \varepsilon_{n-1} q_{n-2} \\ p_n = a_n p_{n-1} + \varepsilon_{n-1} p_{n-2} \end{cases}, \tag{4}$$

where $q_{-2} = 1$, $p_{-2} = 0$, $q_{-1} = 0$ and $p_{-1} = 1$.

Finally define $\beta_n = |q_n \alpha - p_n|$ and $\delta_n = (-1)^{n-1} \varepsilon_0 \dots \varepsilon_{n-1}$ $(n = -1, 0, 1, 2, \dots)$.

Lemma. *For $n \ge 0$, we have*

$$\beta_n = (-1)^n \varepsilon_0 \dots \varepsilon_n (q_n \alpha - p_n) = \frac{1}{q_{n+1} + \varepsilon_{n+1} q_n \alpha_{n+1}} = \prod_{j=0}^{n} \alpha_j. \tag{5}$$

Hence $\beta_{-1} = 1 > \beta_0 = \alpha_0 > \beta_2 > \cdots > \beta_n > \cdots \searrow 0$ *and* $\frac{p_n}{q_n} \to \alpha$ $(n \to \infty)$. *Furthermore*

$$q_n \beta_{n-1} + \varepsilon_n q_{n-1} \beta_n = 1, \tag{6}$$

$$a_{n+1} \beta_n + \varepsilon_{n+1} \beta_{n+1} = \beta_{n-1}. \tag{7}$$

We can now describe the combinatorics of irrational rotation. For simplicity, we assume that $a_n \ge 5$ $(n \ge 1)$. We want to define A_n, $\mathcal{I}_n$, $r_{\alpha,n}$ $(n = 1, 2, \dots)$ with following properties ((E)-(H) will be stated later):

(A) A_n is a finite subset of $\mathbb{Z}^n$. $A_n = A_n^0 \cup A_n^1$ (disjoint union), where $A_n^0 = \{(k_1, \dots, k_n) \in A_n \mid k_n = 0\}$ and $A_n^1 = \{(k_1, \dots, k_n) \in A_n \mid k_n \ne 0\}$. $\sharp A_n = q_n$ and $\sharp A_n^0 = q_{n-1}$.

(B) (Partition) $\mathcal{I}_n = \{I_{k_1,\dots,k_n} \mid (k_1, \dots, k_n) \in A_n\}$ is a partition of $[0,1]$, i.e. it is a collection of closed subintervals of $[0,1]$ which have disjoint interior, and their union covers $[0,1]$.

(C) (Length) If $(k_1, \dots, k_n) \in A_n^0$, then $|I_{k_1,\dots,k_n}| = \beta_{n-1} + \varepsilon_n \beta_n$; if $(k_1, \dots, k_n) \in A_n^1$, then $|I_{k_1,\dots,k_n}| = \beta_{n-1}$.

(D) (Order) The correspondence $A_n \ni (k_1, \dots, k_n) \longmapsto I_{k_1,\dots,k_n} \in \mathcal{I}_n$ is order preserving, where A_n inherits the lexicographic order $<$ from $\mathbb{Z}^n$ and the order among $I_{k_1,\dots,k_n}$'s comes from the order in $[0,1]$. In particular, adjacent indices in A_n corresponds to the intervals which are adjacent.

Remark. Note that (C) is consistent with the formula (6). In fact, we have $q_{n-1}(\beta_{n-1}+\varepsilon_n\beta_n)+(q_n-q_{n-1})\beta_{n-1} = q_n\beta_{n-1}+\varepsilon_n q_{n-1}\beta_n = 1$. Therefore, once the set A_n is given with property (A), it automatically determines the intervals by (B), (C) and (D).

Notation. Let $\mathcal{I}_n^i = \{I_{k_1,\dots,k_n} \mid (k_1,\dots,k_n) \in A_n^i\}$ $(i=0,1)$. When it is necessary to indicate the index length n explicitly, we write $I^{(n)}_{k_1,\dots,k_n}$ for $\mathcal{I}_n$. For example, $I^{(n)}_{0,\dots,0} = I_{\underbrace{0,\dots,0}_{n}}$.

Now we define the set A_n.

Notation. Let $\delta_n = (-1)^{n-1}\varepsilon_0\dots\varepsilon_{n-1}$. Note that δ_n is equal to the signature of $(q_{n-1}\alpha - p_{n-1})$ and $\delta_n = -\delta_{n-1}\varepsilon_{n-1}$. Define

$$\kappa_n(+1) = \left[\frac{a_n}{2}\right] \quad \text{and} \quad \kappa_n(-1) = -\left[\frac{a_n-1}{2}\right], \tag{8}$$

where $[k]$ denotes the largest integer which does not exceed k.

Denote $[a,b]_{\mathbb{Z}} = [a,b]\cap\mathbb{Z}$. We will frequently use the set $[\kappa_n(-1),\kappa_n(+1)]_{\mathbb{Z}}$ below. Note that this set consists of a_n elements including 0. In fact, if $a_n = 2\ell$, then the set is $[-(\ell-1),\ell]_{\mathbb{Z}}$ and if $a_n = 2\ell+1$, then the set is $[-\ell,\ell]_{\mathbb{Z}}$.

Definition. An array $(k_1,\dots,k_n) \in \mathbb{Z}^n$ is called *allowable* (or *α-allowable*) if $k_1 \in [\kappa_1(-1),\kappa_1(+1)]_{\mathbb{Z}}$ and for $j = 2,\dots,n$,

$$k_j \in \begin{cases} [\kappa_j(-1),\kappa_j(+1)]_{\mathbb{Z}} & (k_{j-1}\neq 0) \\ [\kappa_j(-1),\kappa_j(+1)-\delta_j]_{\mathbb{Z}} & (k_{j-1}=0 \text{ and } \delta_{j-1}=+1) \\ [\kappa_j(-1)-\delta_j,\kappa_j(+1)]_{\mathbb{Z}} & (k_{j-1}=0 \text{ and } \delta_{j-1}=-1). \end{cases} \tag{9}$$

(Note here that $-\delta_j = +\varepsilon_{j-1}$ if $\delta_{j-1}=+1$, and $-\delta_j = -\varepsilon_{j-1}$ if $\delta_{j-1}=-1$.)

Let $A_n = \{(k_1,\dots,k_n)\in\mathbb{Z}^n,\ \alpha\text{-allowable}\}$ for $n\geq 1$. For consistency, we define A_0 to be a singleton, consisting of an array of length 0. When we need to specify the dependence on α, we write $A_{\alpha,n}$. The definition of allowability extends to infinite sequences and defines $A_\infty \subset \mathbb{Z}^{\mathbf{N}}$.

For $(k_1,\dots,k_j)\in A_j$, $0\leq j\leq n$, we denote by $A_n(k_1,\dots,k_j)$ the set of arrays in A_n that start with $(k_1,\dots,k_j)$.

Lemma 1. *The set A_n satisfies (A). Moreover $\sharp A_{n+1}(k_1,\dots,k_n) = a_{n+1}+\varepsilon_n$ if $(k_1,\dots,k_n)\in A_n^0$ (i.e. $k_n=0$), and $\sharp A_{n+1}(k_1,\dots,k_n) = a_{n+1}$ if $(k_1,\dots,k_n)\in A_n^1$ (i.e. $k_n\neq 0$).*

Proof. The second half of the statement follows immediately from (9). Let us prove (A) by induction. Clearly $\sharp A_0 = 1 = q_0$, $\sharp A_1 = a_1 = q_1$ and $\sharp A_1^0 = 1 = q_0$. Suppose (A) holds up to n $(n\geq 1)$. By decomposing $A_{n+1} = \cup_{(k_1,\dots,k_n)\in A_n^0} A_n(k_1,\dots,k_n) \cup \cup_{(k_1,\dots,k_n)\in A_n^1} A_n(k_1,\dots,k_n)$, we obtain

$$\sharp A_{n+1} = q_{n-1}(a_{n+1}+\varepsilon_n) + (q_n-q_{n-1})a_{n+1} = a_{n+1}q_n + \varepsilon_n q_{n-1} = q_{n+1}.$$

Obviously $\sharp A_{n+1}^0 = \sharp A_n = q_n$. This proves the assertion. □

Although $\mathcal{I}_n$ can be determined by A_n, it is important to see the recursive construction of $\mathcal{I}_n$. For $n=0$, we set $\mathcal{I}_0 = \{[0,1]\}$. For $n=1$, let $A_1 = [\kappa_1(-1),\kappa_1(+1)]_{\mathbb{Z}}$ and $\mathcal{I}_1$ the partition of $[0,1]$ as in (C) and (D), which is easy to determine. We can define recursively $\mathcal{I}_n$ by the following rule:

(E) (Subdivision) Each $I_{k_1,\ldots,k_n} \in \mathcal{I}_n$ is subdivided into a collection of subintervals

$$\mathcal{I}_{n+1}(k_1,\ldots,k_n) = \{I_{k_1,\ldots,k_n,k_{n+1}} \mid k_{n+1} \text{ satisfies (9) with } j = n+1\}, \tag{10}$$

which consists of $a_{n+1} + \varepsilon_n$ subintervals if $(k_1,\ldots,k_n) \in A_n^0$, and a_{n+1} subintervals if $(k_1,\ldots,k_n) \in A_n^1$. In particular, $I_{k_1,\ldots,k_n,k_{n+1}} \subset I_{k_1,\ldots,k_n}$. They are ordered within $I_{k_1,\ldots,k_n}$ according to k_{n+1}.

If $\mathcal{I}_n$ satisfies (B) and (D), then $\mathcal{I}_{n+1}$ clearly satisfies them also. We only need to check the consistency of the length condition (C). If $(k_1,\ldots,k_n) \in A_n^1$, $|I_{k_1,\ldots,k_n}| = \beta_{n-1}$. By the above rule, $\mathcal{I}_{n+1}(k_1,\ldots,k_n)$ consists of 1 subinterval of length $\beta_n + \varepsilon_{n+1}\beta_{n+1}$ and $a_{n+1} - 1$ subintervals of length β_n. By (7), we have

$$(\beta_n + \varepsilon_{n+1}\beta_{n+1}) + (a_{n+1} - 1)\beta_n = a_{n+1}\beta_n + \varepsilon_{n+1}\beta_{n+1} = \beta_{n-1}.$$

Therefore the length is compatible with (C) and the subdivision (E) is possible.

Now we define $r_{\alpha,n} : A_n \to A_n$ and make a connection to the irrational rotation $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$.

Definition. For $(k_1,\ldots,k_n) \in A_n$ $(n \geq 1)$, define $r_{\alpha,n}$ by $r_{\alpha,n}(0,\ldots,0) = (\delta_1,\ldots,\delta_n)$ and if $1 \leq j \leq n$ and $k_j \neq 0$, then

$$r_{\alpha,n}(\underbrace{0,\ldots,0}_{j-1}, k_j,\ldots,k_n) = (\delta_1,\ldots,\delta_{j-1},(k_j+\delta_j),k_{j+1},\ldots,k_n), \tag{11}$$

except the following exceptional cases.

Special Case 1: if $2 \leq j \leq n$, $\varepsilon_{j-1} = +1$, then

$$r_{\alpha,n}(\underbrace{0,\ldots,0}_{j-1}, \kappa_j(-\delta_{j-1}),k_{j+1},\ldots,k_n) = (\delta_1,\ldots,\delta_{j-2},0,\kappa_j(\delta_{j-1})-\delta_j,k_{j+1},\ldots,k_n). \tag{12}$$

Special Case 2: if $2 \leq j \leq n$, $\varepsilon_{j-1} = -1$, then

$$r_{\alpha,n}(\underbrace{0,\ldots,0}_{j-2}, -\delta_{j-1},\kappa_j(\delta_{j-1}),k_{j+1},\ldots,k_n) = (\delta_1,\ldots,\delta_{j-2},\delta_{j-1},\kappa_j(-\delta_{j-1}),k_{j+1},\ldots,k_n). \tag{13}$$

Remark. By definition, $r_{\alpha,n}(0,\ldots,0,-\delta_n) = (\delta_1,\ldots,\delta_{n-1},0)$ is not a special case. Note also that if the above rule in Special Cases were not applied, i.e. (11) were used instead of (12) and (13), then the images would not be in A_n. In fact, for Special Case 1, where $\varepsilon_{j-1} = +1$, the image would be $(\ldots,\delta_{j-1},\kappa_j(-\delta_{j-1})+\delta_j,\ldots)$, but $\kappa_j(-\delta_{j-1})+\delta_j \notin [\kappa_j(-1),\kappa_j(+1)]_{\mathbb{Z}}$, since $\delta_j = -\delta_{j-1}\varepsilon_{j-1} = -\delta_{j-1}$. For Special Case 2, the image would be $(\ldots,\delta_{j-2},0,\kappa_j(\delta_{j-1}),\ldots)$ which is not allowed by (9).

We claim:

(F) $r_{\alpha,n}$ is well-defined (i.e. the image belongs to A_n) and bijective.

Although one could prove this directly, we will prove it via comparison to the action of the rotation R_α on intervals in $\mathcal{I}_n$.

(G) (Dynamics) If $(k_1,\ldots,k_n) \in A_n$ and $(k_1,\ldots,k_n) \neq (0,\ldots,0),(0,\ldots,0,-\delta_n)$, then R_α maps $I^{(n)}_{k_1,\ldots,k_n}$ bijectively onto $I^{(n)}_{r_{\alpha,n}(k_1,\ldots,k_n)}$. On the other hand, R_α maps $I^{(n)}_{0,\ldots,0} \cup I^{(n)}_{0,\ldots,0,-\delta_n}$ bijectively onto $I^{(n)}_{\delta_1,\ldots,\delta_n} \cup I^{(n)}_{\delta_1,\ldots,\delta_{n-1},0}$. Note here that $I^{(n)}_{0,\ldots,0}$ and $I^{(n)}_{0,\ldots,0,-\delta_n}$ are adjacent, and so are $I^{(n)}_{\delta_1,\ldots,\delta_n} = I^{(n)}_{r_{\alpha,n}(0,\ldots,0)}$ and $I^{(n)}_{\delta_1,\ldots,\delta_{n-1},0} = I^{(n)}_{r_{\alpha,n}(0,\ldots,0,-\delta_n)}$.

(H) (Mismatch) If $\varepsilon_n = +1$, then $|I^{(n)}_{0,\dots,0}| > |I^{(n)}_{0,\dots,0,-\delta_n}|$ and

$$R_\alpha(I^{(n)}_{0,\dots,0}) \supsetneqq I^{(n)}_{\delta_1,\dots,\delta_n}, \quad R_\alpha(I^{(n)}_{0,\dots,0,-\delta_n}) \subsetneqq I^{(n)}_{\delta_1,\dots,\delta_{n-1},0}$$

and the difference comes from $R_\alpha(I^{(n+1)}_{0,\dots,0,\kappa_{n+1}(-\delta_n)}) = I^{(n+1)}_{\delta_1,\dots,\delta_{n-1},0,\kappa_{n+1}(\delta_n)-\delta_{n+1}}$.
If $\varepsilon_n = -1$, then $|I_{0,\dots,0}| < |I_{0,\dots,0,-\delta_n}|$ and

$$R_\alpha(I^{(n)}_{0,\dots,0}) \subsetneqq I^{(n)}_{\delta_1,\dots,\delta_n}, \quad R_\alpha(I^{(n)}_{0,\dots,0,-\delta_n}) \supsetneqq I^{(n)}_{\delta_1,\dots,\delta_{n-1},0}$$

and the difference comes from $R_\alpha(I^{(n+1)}_{0,\dots,0,-\delta_n,\kappa_{n+1}(\delta_n)}) = I^{(n+1)}_{\delta_1,\dots,\delta_{n-1},\delta_n,\kappa_{n+1}(-\delta_n)}$.

These properties can be proven by induction. In fact, the induction process corresponds to the subdivision of the intervals. This structure will be reflected in the proof of the main theorem. The key step replaces the open sets given by n-th system by smaller open sets for $(n+1)$-st system and the number and arrangements of the sub-open sets are exactly like the subdivision of the corresponding intervals.

Reference: [IS] H. Inou an d M. Shishikura, The renormalization for parabolic fixed points and their perturbation, preprint.