Title: Combinatorial representation of irrational rotations and invariant sets around irrationally indifferent fixed points

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Citation: 数理解析研究所講究録 (2010), 1688: 1-6

Issue Date: 2010-05

URL: http://hdl.handle.net/2433/141517

Type: Departmental Bulletin Paper

Textversion: publisher
Combinatorial representation of irrational rotations and invariant sets around irrationally indifferent fixed points

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Let \( f(z) \) be a holomorphic function defined near \( z = 0 \) with expansion

\[
f(z) = e^{2\pi i \alpha} z + O(z^2),
\]

where \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \). The origin is an irrationally indifferent fixed point. Especially, we are interested in the case where \( f \) is a quadratic polynomial \( f(z) = e^{2\pi i \alpha} z + z^2 \). The irrational number \( \alpha \) can expressed in terms of (fast) continued fraction:

\[
\alpha = a_0 + \frac{\varepsilon_0}{a_1 + \frac{\varepsilon_1}{a_2 + \frac{\varepsilon_2}{\ddots}}}, \quad \text{where} \quad a_n \in \mathbb{Z}, \quad \varepsilon_n = \pm 1 \quad (n = 0, 1, 2, \ldots), \quad a_n \geq 2 \quad (n \geq 1).
\]

We proved in [IS] that there exists a class \( \mathcal{F}_1 \) of holomorphic functions around 0 and a large constant \( N \) such that for \( h(z) = z + O(z^2) \in \mathcal{F}_1 \) and \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) with \( a_n \geq N \), the function \( f = e^{2\pi i \alpha} h \) has a sequence of well-defined \textit{return maps} \( \mathcal{R}^n f \) (which are called \textit{near-parabolic renormalizations}), which have the form \( \mathcal{R}^n f = e^{2\pi i \alpha_n} h_n \) with \( h_n \in \mathcal{F}_1 \). In this paper, we discuss how to derive the properties of an invariant set \( \Lambda_f \) around 0 and how to analyze its combinatorial aspect which is strongly associated with the irrational rotation \( R_\alpha(z) = e^{2\pi i \alpha} z \) on \( S^1 \).

We describe the local dynamics via an infinite systems of open covers of punctured neighborhoods, and in the open sets the dynamics can be conjugated to canonical maps (see Figure). More precisely, we have:

**Theorem.** For \( f \) as above, there exists an infinite sequence of systems

\[
\{ A_n, r_{\alpha,n}, \{ \Omega_{k_1,\ldots,k_n} \}_{(k_1,\ldots,k_n) \in A_n}, \{ \varphi_{k_1,\ldots,k_n} \}_{(k_1,\ldots,k_n) \in A_n}, \{ F_{k_1,\ldots,k_n} \}_{(k_1,\ldots,k_n) \in A_n} \}_{n \in \mathbb{N}}
\]

satisfying:

- The index set is a finite set \( A_n \subset \mathbb{Z}^n \), which inherits the lexicographic order;
- The combinatorial dynamics \( r_{\alpha,n} : A_n \to A_n \) is bijective and preserves the cyclic ordering;
- Open sets \( \Omega_{k_1,\ldots,k_n} \), \( (k_1,\ldots,k_n) \in A_n \) cover a punctured neighborhood of 0, and their order around 0 is the same as the order of the indices in \( A_n \);
- Maps \( \varphi_{k_1,\ldots,k_n} : \Omega_{k_1,\ldots,k_n} \to \Omega_{\text{can}[a_n]} \), where \( \Omega_{\text{can}} \) is so-called the truncated checkerboard pattern (see Figure) and \( \Omega_{\text{can}[a_n]} \) is its truncation according to the coefficient of the continued fraction of \( \alpha \); \( \varphi_{k_1,\ldots,k_n} \) are either holomorphic if \( \delta_n \) (defined later) is +1 or anti-holomorphic otherwise;
Combinatorial representation of irrational rotations

- The model dynamics $F_{k_1,...,k_n} = \varphi_{\alpha,n}(k_1,...,k_n) \circ f \circ \varphi_{k_1,...,k_n}^{-1}$ is $F_{\text{can}}$ if $(k_1,\ldots,k_n) = (0,\ldots,0)$ and it is id otherwise, where $F_{\text{can}}$ is the canonical dynamics on the truncated checkerboard pattern;

- Two open sets $\Omega_{k_1,\ldots,k_n}$ and $\Omega_{\ell_1,\ldots,\ell_n}$ overlap if and only if $(k_1,\ldots,k_n)$ and $(\ell_1,\ldots,\ell_n)$ are (cyclically) adjacent in $A_n$, and the gluing is defined via the $n$-th near-parabolic renormalization $R^n f$ of $f$;

- The $(n+1)$-th system is a "refinement" of $n$-th one: If $(k_1,\ldots,k_n,k_{n+1}) \in A_{n+1}$, then $(k_1,\ldots,k_n) \in A_n$ and $\Omega_{k_1,\ldots,k_n,k_{n+1}} \subset \Omega_{k_1,\ldots,k_n}$; The combinatorial dynamics almost commutes with the projection $\text{proj}_{n+1} : A_{n+1} \to A_n$ in the sense that $\text{proj}_{n+1} \circ r_{\alpha,n+1} = r_{\alpha,n} \circ \text{proj}_{n+1}$ except at one element in $A_{n+1}$.

Figure: Truncated Checkerboard Pattern

$F_{\text{can}}(w) = w/(1-1/w) = w+1+O(1/w)$ is conjugate to $z+z^2$ and has a parabolic fixed point at $\infty$. Its attracting Fatou coordinate $\Phi_{\text{attr}}$ (defined in a right half plane) conjugates $F_{\text{can}}$ to $T(w) = w+1$, and is normalized so that $\Phi_{\text{attr}}(\text{crit.pt}) = 0$. It extends to the whole parabolic basin. The Truncated Checkerboard Pattern $\Omega_{\text{can}}$ is defined to be the union of some inverse images of $\{w : n < Re w < n+1, -2 < Im w < 2\}$ and $\{w : n < Re w < n+1, 2 < Im w\}$ $(n \in \mathbb{Z})$ by $\Phi_{\text{attr}}$ together with boundary curves.

Furthermore, the projective limit $A_{\infty} = \lim A_n$ and the combinatorial dynamics $r_{\alpha,\infty} : A_{\infty} \to A_{\infty}$ are well-defined and there is an order-preserving semi-conjugacy $\pi : A_{\infty} \to \mathbb{S}^1$ from $r_{\alpha,\infty}$ to the $\alpha$-rotation $R_\alpha$ such that two distinct indices $(k_1,k_2,\ldots), (\ell_1,\ell_2,\ldots)$ in $A_{\infty}$ are mapped to the same point if and only if $(k_1,\ldots,k_n)$ and $(\ell_1,\ldots,\ell_n)$ are adjacent in $A_n$ for large $n$. The quotient dynamics $\overline{r}_\alpha$ on $A_{\infty} = A_{\infty}/\sim_{\text{adjacent}}$ is conjugate to $R_\alpha$ on $\mathbb{S}^1$.

The maximal invariant set $\Lambda_f$ covered by these open sets is called the maximal hedgehog:

$$\Lambda_f = \{0\} \cup \bigcup_{n=1}^{\infty} \bigcup_{(k_1,k_2,\ldots,k_n) \in A_n} \Omega_{k_1,k_2,\ldots,k_n} = \{0\} \cup \bigcup_{(k_1,k_2,\ldots) \in \overline{A}_\infty} \bigcap_{n=1}^{\infty} \Omega_{k_1,k_2,\ldots,k_n}.$$  

It can be shown that for each $(k_1,k_2,\ldots) \in \overline{A}_\infty$, $\bigcap_{n=1}^{\infty} \Omega_{k_1,k_2,\ldots,k_n}$ is either empty or an arc tending to 0. The sets $\bigcap_{n=1}^{\infty} \Omega_{k_1,k_2,\ldots,k_n}$ are (cyclically) permuted by the dynamics $f$. The map $\bigcap_{n=1}^{\infty} \Omega_{k_1,k_2,\ldots,k_n} \to (k_1,k_2,\ldots) \to \pi(k_1,k_2,\ldots)$ defines a semi-conjugacy from $f$ on $\Lambda_f \sim \{0\}$ to $R_\alpha$ on $\mathbb{S}^1$ (not necessarily onto).

Rotation combinatorics: The key ingredient in the construction (beside the theorem in [IS]) is the analysis of the combinatorics of the irrational rotation $R_\alpha$. This will naturally gives us the index set $A_n$ and the combinatorial dynamics $r_{\alpha,n}$. Let us first review the fast continued fractions.

**Definition.** For $x \in \mathbb{R}$, $\langle x \rangle$ denotes the closest integer to $x$. If $x = n + \frac{1}{2}$ with $n \in \mathbb{Z}$, we set $\langle x \rangle = n$. Define $||x|| = \text{dist}(x,\mathbb{Z}) = \min\{|x-n| : n \in \mathbb{Z}\} = |x - \langle x \rangle|$. Then we have $0 \le ||x|| \le \frac{1}{2}$ and $x = \langle x \rangle \pm ||x||$. 

Combinatorial representation of irrational rotations

For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, let $\alpha_{-1} = \frac{1}{\alpha}$ and define $a_n \in \mathbb{Z}$, $\alpha_n \in (0, \frac{1}{2}) \subset \mathbb{R}$ and $\epsilon_n = \pm 1$ ($n = 0, 1, 2, \ldots$) as follows:

\[
\alpha_n = \left\lfloor \frac{1}{\alpha_{n-1}} \right\rfloor, \quad \alpha_{n} = \left\| \frac{1}{\alpha_{n-1}} \right\| \quad \text{and} \quad \epsilon_n = \begin{cases} +1 & \text{if } a_n \leq \frac{1}{\alpha_{n-1}} - 1 \\ -1 & \text{if } a_n > \frac{1}{\alpha_{n-1}}. \end{cases} \tag{1}
\]

It immediately follows that

\[
\frac{1}{\alpha_{n-1}} = a_n + \epsilon_n \alpha_n \tag{2}
\]

and that $a_n \geq 2$ for $n \geq 1$. Moreover if $a_n = 2$ ($n \geq 1$), then $\epsilon_n = +1$. A repeated use of (2) shows

\[
\alpha = a_0 + \frac{\epsilon_0}{a_1 + \frac{\epsilon_1}{\ldots + \frac{\epsilon_{n-1}}{a_n + \epsilon_n \alpha_n}}.} \tag{3}
\]

Define the sequence of integers $\{q_n\}_{n=-2}^{\infty}$ and $\{p_n\}_{n=-2}^{\infty}$ by the following

\[
\begin{align*}
q_n &= a_n q_{n-1} + \epsilon_{n-1} q_{n-2} \\
p_n &= a_n p_{n-1} + \epsilon_{n-1} p_{n-2}
\end{align*} \tag{4}
\]

where $q_{-2} = 1$, $p_{-2} = 0$, $q_{-1} = 0$ and $p_{-1} = 1$.

Finally define $\beta_n = |q_n \alpha - p_n|$ and $\delta_n = (-1)^{n-1} \epsilon_0 \ldots \epsilon_{n-1}$ ($n = -1, 0, 1, 2, \ldots$).

Lemma. For $n \geq 0$, we have

\[
\begin{align*}
\beta_n &= (-1)^n \epsilon_0 \ldots \epsilon_n (q_n \alpha - p_n) = \frac{1}{q_n + \epsilon_n q_n \alpha_n + \epsilon_n q_{n+1}} \\
&= \prod_{j=0}^{n} \alpha_j. \tag{5}
\end{align*}
\]

Hence $\beta_{-1} = 1 > \beta_0 = \alpha_0 > \beta_2 > \ldots > \beta_n > \ldots \searrow 0$ and $\frac{p_n}{q_n} \to \alpha$ ($n \to \infty$). Furthermore

\[
\begin{align*}
q_n \beta_{n-1} + \epsilon_n q_{n-1} \beta_n &= 1, \\
a_{n+1} \beta_n + \epsilon_n a_{n+1} \beta_{n+1} &= \beta_{n-1}. \tag{6}
\end{align*}
\]

We can now describe the combinatorics of irrational rotation. For simplicity, we assume that $a_n \geq 5$ ($n \geq 1$). We want to define $A_n, \mathcal{I}_n, \mathcal{R}_{\alpha,n}$ ($n = 1, 2, \ldots$) with following properties ((E)-(H) will be stated later):

(A) $A_n$ is a finite subset of $\mathbb{Z}^n$. $A_n = A_n^0 \cup A_n^1$ (disjoint union), where $A_n^0 = \{(k_1, \ldots, k_n) \in A_n \mid k_n = 0\}$ and $A_n^1 = \{(k_1, \ldots, k_n) \in A_n \mid k_n \neq 0\}$. $\sharp A_n = q_n$ and $\sharp A_n^0 = q_n - 1$.

(B) (Partition) $\mathcal{I}_n = \{I_{k_1, \ldots, k_n} \mid (k_1, \ldots, k_n) \in A_n\}$ is a partition of $[0,1]$, i.e. it is a collection of closed subintervals of $[0,1]$ which have disjoint interior, and their union covers $[0,1]$.

(C) (Length) If $(k_1, \ldots, k_n) \in A_n^0$, then $|I_{k_1, \ldots, k_n}| = \beta_{n-1} + \epsilon_n \beta_n$; if $(k_1, \ldots, k_n) \in A_n^1$, then $|I_{k_1, \ldots, k_n}| = \beta_{n-1}$.

(D) (Order) The correspondence $A_n \ni (k_1, \ldots, k_n) \mapsto I_{k_1, \ldots, k_n} \in \mathcal{I}_n$ is order preserving, where $A_n$ inherits the lexicographic order $<$ from $\mathbb{Z}^n$ and the order among $I_{k_1, \ldots, k_n}$'s comes from the order in $[0,1]$. In particular, adjacent indices in $A_n$ corresponds to the intervals which are adjacent.
Remark. Note that (C) is consistent with the formula (6). In fact, we have $q_{n-1}(\beta_{n-1} + \varepsilon_{n}\beta_{n}) + (q_{n} - q_{n-1})\beta_{n-1} = q_{n}\beta_{n-1} + \varepsilon_{n}q_{n-1}\beta_{n} = 1$. Therefore, once the set $A_{n}$ is given with property (A), it automatically determines the intervals by (B), (C) and (D).

Notation. Let $I_{n} = \{I_{k_{1},\ldots,k_{n}}| (k_{1},\ldots,k_{n}) \in A_{n}^{i}\} (i = 0, 1)$. When it is necessary to indicate the index length $n$ explicitly, we write $I_{n}^{(n)}$. For example, $I_{0,...,0}^{(n)} = I_{0,...,0}^{(0)}$.

Now we define the set $A_{n}$.

Notation. Let $\delta_{n} = (-1)^{n-1}\varepsilon_{0}\ldots\varepsilon_{n-1}$. Note that $\delta_{n}$ is equal to the signature of $(q_{n-1} - p_{n-1})$ and $\delta_{n} = -\delta_{n-1}\varepsilon_{n-1}$. Define

$$
\kappa_{n}(+1) = \left[\frac{a_{n}}{2}\right] \quad \text{and} \quad \kappa_{n}(-1) = -\left[\frac{a_{n} - 1}{2}\right],
$$

(8)

where $[k]$ denotes the largest integer which does not exceed $k$.

Denote $[a, b]_{Z} = [a, b] \cap Z$. We will frequently use the set $[\kappa_{n}(-1), \kappa_{n}(+1)]_{Z}$ below. Note that this set consists of $a_{n}$ elements including 0. In fact, if $a_{n} = 2\ell$, then the set is $[-(\ell - 1), \ell]_{Z}$ and if $a_{n} = 2\ell + 1$, then the set is $[-\ell, \ell]_{Z}$.

Definition. An array $(k_{1},\ldots,k_{n}) \in Z^{n}$ is called allowable (or $\alpha$-allowable) if $k_{1} \in [\kappa_{1}(-1), \kappa_{1}(+1)]_{Z}$ and for $j = 2,\ldots,n$,

$$
k_{j} \in \begin{cases} 
[\kappa_{j}(-1), \kappa_{j}(+1)]_{Z} & (k_{j-1} \neq 0) \\
[\kappa_{j}(-1), \kappa_{j}(+1) - \delta_{j}]_{Z} & (k_{j-1} = 0 \text{ and } \delta_{j-1} = +1) \\
[\kappa_{j}(-1) - \delta_{j}, \kappa_{j}(+1)]_{Z} & (k_{j-1} = 0 \text{ and } \delta_{j-1} = -1).
\end{cases}
$$

(9)

(Note here that $-\delta_{j} = +\varepsilon_{j-1}$ if $\delta_{j-1} = +1$, and $-\delta_{j} = -\varepsilon_{j-1}$ if $\delta_{j-1} = -1$.)

Let $A_{n} = \{(k_{1},\ldots,k_{n}) \in Z^{n} \mid \alpha$-allowable $\}$ for $n \geq 1$. For consistency, we define $A_{0}$ to be a singleton, consisting of an array of length 0. When we need to specify the dependence on $\alpha$, we write $A_{\alpha,n}$. The definition of allowability extends to infinite sequences and defines $A_{\infty} \subset Z^{\infty}$.

For $(k_{1},\ldots,k_{j}) \in A_{j}$, $0 \leq j \leq n$, we denote by $A_{n}(k_{1},\ldots,k_{j})$ the set of arrays in $A_{n}$ that start with $(k_{1},\ldots,k_{j})$.

Lemma 1. The set $A_{n}$ satisfies (A). Moreover $\sharp A_{n+1}(k_{1},\ldots,k_{n}) = a_{n+1} + \varepsilon_{n}$ if $(k_{1},\ldots,k_{n}) \in A_{n}^{0}$ (i.e. $k_{n} = 0$), and $\sharp A_{n+1}(k_{1},\ldots,k_{n}) = a_{n+1}$ if $(k_{1},\ldots,k_{n}) \in A_{n}^{1}$ (i.e. $k_{n} \neq 0$).

Proof. The second half of the statement follows immediately from (9). Let us prove (A) by induction. Clearly $\sharp A_{0} = 1 = q_{0}$, $\sharp A_{1} = a_{1} = q_{1}$ and $\sharp A_{1}^{0} = 1 = q_{0}$. Suppose (A) holds up to $n$ ($n \geq 1$). By decomposing $A_{n+1} = \bigcup_{(k_{1},\ldots,k_{n}) \in A_{n}^{0}} A_{n}(k_{1},\ldots,k_{n}) \cup \bigcup_{(k_{1},\ldots,k_{n}) \in A_{n}^{1}} A_{n}(k_{1},\ldots,k_{n})$, we obtain

$$
\sharp A_{n+1} = q_{n-1}(a_{n+1} + \varepsilon_{n}) + (q_{n} - q_{n-1})a_{n+1} = a_{n+1}q_{n} + \varepsilon_{n}q_{n-1} = q_{n+1}.
$$

Obviously $\sharp A_{n+1}^{0} = \sharp A_{n} = q_{n}$. This proves the assertion. \hfill \square

Although $I_{n}$ can be determined by $A_{n}$, it is important to see the recursive construction of $I_{n}$. For $n = 0$, we set $I_{0} = \{0, 1\}$. For $n = 1$, let $A_{1} = [\kappa_{1}(-1), \kappa_{1}(+1)]_{Z}$ and $I_{1}$ the partition of $[0, 1]$ as in (C) and (D), which is easy to determine. We can define recursively $I_{n}$ by the following rule:
Combinatorial representation of irrational rotations

(E) (Subdivision) Each $I_{k_1,\ldots,k_n} \in \mathcal{I}_n$ is subdivided into a collection of subintervals

$$\mathcal{I}_{n+1}(k_1, \ldots, k_n) = \{I_{k_1,\ldots,k_n,k_{n+1}} \mid k_{n+1} \text{ satisfies } (9) \text{ with } j = n+1\},$$

(10)

which consists of $a_{n+1} + \varepsilon_n$ subintervals if $(k_1, \ldots, k_n) \in A_0^n$, and $a_{n+1}$ subintervals if $(k_1, \ldots, k_n) \in A_1^n$. In particular, $I_{k_1,\ldots,k_n,k_{n+1}} \subset I_{k_1,\ldots,k_n}$. They are ordered within $I_{k_1,\ldots,k_n}$ according to $k_{n+1}$.

If $\mathcal{I}_n$ satisfies (B) and (D), then $\mathcal{I}_{n+1}$ clearly satisfies them also. We only need to check the consistency of the length condition (C). If $(k_1, \ldots, k_n) \in A_1^n$, $|I_{k_1,\ldots,k_n}| = \beta_{n-1}$. By the above rule, $\mathcal{I}_{n+1}(k_1, \ldots, k_n)$ consists of 1 subinterval of length $\beta_n + \varepsilon_{n+1}\beta_{n+1}$ and $a_{n+1} - 1$ subintervals of length $\beta_n$. By (7), we have

$$(\beta_n + \varepsilon_{n+1}\beta_{n+1}) + (a_{n+1} - 1)\beta_n = a_{n+1}\beta_n + \varepsilon_{n+1}\beta_{n+1} = \beta_{n-1}.$$ 

Therefore the length is consistent with (C) and the subdivision (E) is possible.

Now we define $r_{\alpha,n} : A_n \rightarrow A_n$ and make a connection to the irrational rotation $R_{\alpha} : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$.

**Definition.** For $(k_1, \ldots, k_n) \in A_n$ ($n \geq 1$), define $r_{\alpha,n}$ by $r_{\alpha,n}(0, \ldots, 0) = (\delta_1, \ldots, \delta_n)$ and if $1 \leq j \leq n$ and $k_j \neq 0$, then

$$r_{\alpha,n}(0, \ldots, 0, k_j, \ldots, k_n) = (\delta_1, \ldots, \delta_{j-1}, (k_j + \delta_j), k_{j+1}, \ldots, k_n),$$

(11)

except the following exceptional cases.

**Special Case 1**: if $2 \leq j \leq n$, $\varepsilon_{j-1} = +1$, then

$$r_{\alpha,n}(0, \ldots, 0, k_j, \ldots, k_n) = (\delta_1, \ldots, \delta_{j-1}, \kappa_j(-\delta_{j-1}), k_{j+1}, \ldots, k_n),$$

(12)

**Special Case 2**: if $2 \leq j \leq n$, $\varepsilon_{j-1} = -1$, then

$$r_{\alpha,n}(0, \ldots, 0, -k_j, \ldots, k_n) = (\delta_1, \ldots, \delta_{j-1}, \kappa_j(-\delta_{j-1}), k_{j+1}, \ldots, k_n).$$

(13)

**Remark.** By definition, $r_{\alpha,n}(0, \ldots, 0, -\delta_n) = (\delta_1, \ldots, \delta_{n-1}, 0)$ is not a special case. Note also that if the above rule in Special Cases were not applied, i.e. (11) were used instead of (12) and (13), then the images would not be in $A_n$. In fact, for Special Case 1, where $\varepsilon_{j-1} = +1$, the image would be $(\ldots, \delta_{j-1}, \kappa_j(-\delta_{j-1}) + \delta_{j-1}, \ldots)$, but $\kappa_j(-\delta_{j-1}) + \delta_{j-1} \notin [\kappa_j(-1), \kappa_j(+1)]\mathbb{Z}$, since $\delta_j = -\delta_{j-1}\varepsilon_{j-1} = -\delta_{j-1}$. For Special Case 2, the image would be $(\ldots, \delta_{j-2}, 0, \kappa_j(-\delta_{j-1}), \ldots)$ which is not allowed by (9).

We claim:

(F) $r_{\alpha,n}$ is well-defined (i.e. the image belongs to $A_n$) and bijective.

Although one could prove this directly, we will prove it via comparison to the action of the rotation $R_{\alpha}$ on intervals in $\mathcal{I}_n$.

(G) (Dynamics) If $(k_1, \ldots, k_n) \in A_n$ and $(k_1, \ldots, k_n) \neq (0, \ldots, 0, (0, \ldots, 0, -\delta_n)$, then $R_{\alpha}$ maps $I_{k_1,\ldots,k_n}$ bijectively onto $I_{r_{\alpha,n}(k_1,\ldots,k_n)}$. On the other hand, $R_{\alpha}$ maps $I_{0,\ldots,0} \cup I_{0,\ldots,0,-\delta_n}$ bijectively onto $I_{r_{\alpha,n}(0,\ldots,0)} \cup I_{r_{\alpha,n}(0,\ldots,0,-\delta_n)}$. Note here that $I_{0,\ldots,0}$ and $I_{0,\ldots,0,-\delta_n}$ are adjacent, and so are $I_{r_{\alpha,n}(0,\ldots,0)}$ and $I_{r_{\alpha,n}(0,\ldots,0,-\delta_n)}$. 


Combinatorial representation of irrational rotations

(H) (Mismatch) If $\epsilon_n = +1$, then $|I_0^{(n)}| > |I_0^{(n)}|_{0,-\delta_n}$ and

$$R_\alpha(I_0^{(n)}_{0,\ldots,0}) \supsetneq I_{\delta_1,\ldots,\delta_n}^{(n)}, \quad R_\alpha(I_0^{(n)}_{0,\ldots,0,-\delta_n}) \subsetneq I_{\delta_1,\ldots,\delta_n-1,0}^{(n)}$$

and the difference comes from $R_\alpha(I_0^{(n+1)}_{0,\ldots,0,0,\kappa_{n+1}(-\delta_n)}) = I_{\delta_1,\ldots,\delta_n-1,0,\kappa_{n+1}(-\delta_n)}^{(n+1)}$. If $\epsilon_n = -1$, then $|I_0^{(n)}_{0,\ldots,0}| < |I_0^{(n)}_{0,\ldots,0,-\delta_n}|$ and

$$R_\alpha(I_0^{(n)}_{0,\ldots,0}) \subsetneq I_{\delta_1,\ldots,\delta_n}^{(n)}, \quad R_\alpha(I_0^{(n)}_{0,\ldots,0,-\delta_n}) \supsetneq I_{\delta_1,\ldots,\delta_n-1,0}^{(n)}$$

and the difference comes from $R_\alpha(I_0^{(n+1)}_{0,\ldots,0,-\delta_n,\kappa_{n+1}(\delta_n)}) = I_{\delta_1,\ldots,\delta_n-1,\delta_n,\kappa_{n+1}(\delta_n)}^{(n+1)}$.

These properties can be proven by induction. In fact, the induction process corresponds to the subdivision of the intervals. This structure will be reflected in the proof of the main theorem. The key step replaces the open sets given by $n$-th system by smaller open sets for $(n + 1)$-st system and the number and arrangements of the sub-open sets are exactly like the subdivision of the corresponding intervals.