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Primitive derivation, Coxeter multiarrangements and some examples

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Abstract

This is a survey on recent developments about Coxeter multiarrangements, its freeness and the relation with the primitive derivation. Also, we give some examples on free and non-free Coxeter multiarrangements.

0 Introduction

Let $V = V^\ell$ be an ℓ-dimensional vector space over a field $\mathbb{R}$, $\{x_1, \ldots, x_\ell\}$ a basis for the dual vector space V^* and $S := \mathrm{Sym}(V^*) \otimes \mathbb{C} \simeq \mathbb{C}[x_1, \ldots, x_\ell]$. Fix an inner product $I^* : V^* \times V^* \to \mathbb{R}$. Let Der (S) denote the S-module of $\mathbb{C}$-linear derivations of S and Ω_V^1 the S-module of differential 1-forms, i.e., Der $(S) = \oplus_{i=1}^\ell S \cdot \partial_{x_i}$ and $\Omega_S^1 := \oplus_{i=1}^\ell S \cdot dx_i$. Also, for the quotient field F of S, Der F and Ω_F denote the derivation and differential modules with coefficients in F. Note that I^* can be canonically extended to $I^* : \Omega_F^1 \times \Omega_F^1 \to F$. A non-zero element $\theta = \sum_{i=1}^\ell f_i \partial_{x_i} \in$ Der (F) (resp. $\omega = \sum_{i=1}^\ell g_i dx_i \in \Omega_F^1$) is *homogeneous of degree* p if f_i (resp. g_i) is zero or homogeneous of degree p for each i.

A *hyperplane arrangement* $\mathcal{A}$ (or simply an *arrangement*) is a finite collection of affine hyperplanes in V. If each hyperplane in $\mathcal{A}$ contains the origin, we say that $\mathcal{A}$ is *central*. In this article we assume that all arrangements are central unless otherwise specified. A *multiplicity* m on an arrangement $\mathcal{A}$ is a map $m : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ and a pair $(\mathcal{A}, m)$ is called a *multiarrangement*. Let $|m|$ denote the sum of the multiplicities $\sum_{H \in \mathcal{A}} m(H)$. When $m \equiv 1$, $(\mathcal{A}, m)$ is the same as the hyperplane arrangement $\mathcal{A}$ and sometimes called a *simple arrangement*. For each hyperplane $H \in \mathcal{A}$ fix a linear form $\alpha_H \in V^*$ such

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that $\ker(\alpha_H) = H$. Put $Q(\mathcal{A}, m) := \prod_{H\in\mathcal{A}} \alpha_H^{m(H)}$. Also, for $k \in \mathbb{Z}$ let $\mathbf{k}$ denote the constant multiplicity on $\mathcal{A}$:

$$\mathbf{k}(H) = k \ (\forall H \in \mathcal{A}).$$

The main objects in this article are the *logarithmic derivation module* $D(\mathcal{A}, m)$ of $(\mathcal{A}, m)$ defined by

$$D(\mathcal{A}, m) := \{\theta \in \mathrm{Der}\ (S) | \theta(\alpha_H) \in S \cdot \alpha_H^{m(H)} \text{ (for all } H \in \mathcal{A})\},$$

and the *logarithmic differential module* $\Omega^1(\mathcal{A}, m)$ of $(\mathcal{A}, m)$ defined by

$$\Omega^1(\mathcal{A}, m) := \{\omega \in \frac{1}{Q(\mathcal{A}, m)}\Omega^1_V | d\alpha_H \wedge \omega \text{ is regular along } H \text{ for all } H \in \mathcal{A}\}.$$

It is well-known that $D(\mathcal{A}, m)$ and $\Omega^1(\mathcal{A}, m)$ are S-dual modules, and hence reflexive in general. A multiarrangement $(\mathcal{A}, m)$ is *free* if $D(\mathcal{A}, m)$ is a free S-module of rank ℓ. If $(\mathcal{A}, m)$ is free, then there exists a homogeneous free basis $\{\theta_1, \ldots, \theta_\ell\}$ for $D(\mathcal{A}, m)$. Then we define the *exponents* of a free multiarrangement $(\mathcal{A}, m)$ by $\exp(\mathcal{A}, m) := (\deg(\theta_1), \ldots, \deg(\theta_\ell))$. The exponents are independent of a choice of a basis. When $m \equiv 1$, $\exp(\mathcal{A}, \mathbf{1})$ is denoted by $\exp(\mathcal{A})$.

A multiarrangement was introduced by Ziegler in [23], and after that, there have been a lot of studies and results related to it. Especially, Yoshinaga's two works [20] and [21] have made the study of multiarrangements and their freeness more interesting and important. For example, [6] generalized a characteristic polynomial and [7] the addition-deletion theorems for multiarrangements. Also, the totally non-freeness of generic arrangements was proved in [22], the first complete classification of an arrangement which admits both free and non-free multiplicities was done in [1], and the geometric characterization of totally free arrangements was obtained in [8]. More recently, the author generalized the definition of multiarrangements to that with multiplicities consisting of positive and negative integers, and gave a definition of a logarithmic module associated to them in [2]. In other words, a *(generalized) multiplicity* is a function $m : \mathcal{A} \to \mathbb{Z}$ and the corresponding *logarithmic module* $D\Omega(\mathcal{A}, m)$ is defined by

$$D\Omega(\mathcal{A}, m) := \frac{D(\mathcal{A}_+, m_+)}{Q_-} \bigcap I^*(\Omega^1(\mathcal{A}_-, -m_-)).$$

For details, see [2], in which a theory of generalized multiarrangements and their logarithmic modules had been constructed. For example, the freeness, exponents, reflexivity (duality) and Saito's criterion hold true for $D\Omega(\mathcal{A}, m)$ as for usual multiarrangements, and a duality of Coxeter multiarrangements was generalized by using this module and Kyoji Saito's primitive derivation.

As we could see in the above, the theory of free multiarrangements and its investigations are developing very rapidly, and producing several new concepts and ideas which can be used in the study of simple arrangements. Among them, one of the most important roles is played by the research of Coxeter multiarrangements, initiated by Solomon and Terao in [15], developed by Terao in [16], and still being studied intensively by Terao, Wakamiko, Wakefield, Yuzvinsky, Yoshinaga and the author. This note is devoted for the survey of these studies and results with some new examples which are related to free Coxeter arrangements.

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1 Coxeter multiarrangements

In this section we review results on free Coxeter multiarrangements. It has been very difficult to determine whether given multiarrangements are free or not, even for simple arrangements. One of the reasons why this problem is difficult is there are only few tools or methods to study the freeness. Though we have addition-deletion theorems ([7]) and characteristic polynomials for non-freeness criterion ([6]), there are no sufficient freeness criterions so far.

On the other hand, if we go back to the original point of free arrangements, we can obtain very strong theory for the freeness, i.e., the invariant theory of Coxeter groups. To see it, recall the most important theorem by Chevalley:

Theorem 1.1 ([10])
For the invariant ring $R := S^W$, there exist homogeneous basic invariants $P_1, \ldots, P_\ell \in R$ such that

$$R = S^W = \mathbb{R}[P_1, \ldots, P_\ell].$$

Moreover, if we put $\deg P_i = m_i + 1$ with $m_1 \leq m_2 \leq \cdots \leq m_\ell$, then

$$\exp(W) = (m_1, \ldots, m_\ell)$$

and $1 = m_1 < m_2 \leq \cdots \leq m_{\ell-1} < m_\ell = h - 1$ with the Coxeter number h of W.

The following is due to Kyoji Saito, which is the starting point of free arrangement theory.

Theorem 1.2 (K. Saito)
For a Coxeter arrangement $\mathcal{A}$, $\{I^*\partial_{P_1}, \ldots, I^*\partial_{P_\ell}\}$ *forms an* W*-invariant basis for* $\Omega^1(\mathcal{A}, 1)$, *and* $\{IdP_1, \ldots, IdP_\ell\}$ *forms that for* $D(\mathcal{A}, 1)$. *In particular,*

$$\exp(\mathcal{A}) = \exp(W).$$

Also, K. Saito gives us a very interesting derivation for the research of free Coxeter arrangements.

Definition 1.3 (K. Saito)
The lowest degree derivation $D := \partial_{P_\ell} \in \mathrm{Der}(R)$ *(up to scalors) is called a primitive derivation.*

The primitive derivation will play, in the rest of this article, a key role to construct free bases for Coxeter multiarrangements. Also, the primitive derivation is essential to construct the Hodge filtration (see [14] for example), and primitive filtration ([4] and [5]), which we do not see in this article.

Now it is natural to consider the freeness of Coxeter multiarrangements. The first research on this problem is done by Solomon and Terao in [15].

Theorem 1.4 ([15])
Let $\mathcal{A}$ *be a Coxeter arrangement with the Coxeter number* h. *Then the multiarrangement* $(\mathcal{A}, \mathbf{2})$ *is free with* $\exp(\mathcal{A}, \mathbf{2}) = (h, \ldots, h)$.

Theorem 1.4 may seem to be a very special case. However, the following work by Terao shows that Theorems 1.2 and 1.4 are very essential in the theory of free Coxeter multiarrangements.

Theorem 1.5 ([16])
Let $\mathcal{A}$ *be a Coxeter arrangement with the Coxeter number* h.

(1) *The multiarrangement* $(\mathcal{A}, \mathbf{2k})$ *is free with* $\exp(\mathcal{A}, \mathbf{2k}) = (kh, \ldots, kh)$.

(2) *The multiarrangement* $(\mathcal{A}, \mathbf{2k+1})$ *is free with* $\exp(\mathcal{A}, \mathbf{2k+1}) = (kh + m_1, \ldots, kh + m_\ell)$.

Combining these theorems, we can see that there seems to be a shifting of exponents and freeness. The shifting phenomenon will be studied soon with more strong statement. We note that the results above are proved by constructing explicit bases by using the invariant theory and primitive derivations.

2 Shifting isomorphism

In this section we introduce a different method from that in the last section for constructing bases for Coxeter multiarrangements.

This section aims at constructing an isomorphism

$$\Phi_k : D(\mathcal{A}, m) \to D(\mathcal{A}, \mathbf{2k} + m),$$

which gives the shifting of multiplicity $2k$. Before explaining how to construct, we concentrate our interest on the starting module $D(\mathcal{A}, m)$. We use the multiplicity $m : \mathcal{A} \to \{+1, 0, -1\}$ with the logarithmic module $D\Omega(\mathcal{A}, m)$ defined in the introduction.

Remark 2.1
However, even for a generalized multiplicity $m : \mathcal{A} \to \mathbb{Z}$, we use the notation $D(\mathcal{A}, m)$ not $D\Omega(\mathcal{A}, m)$ in this article, different from the original paper [2].

First for the construction, we have to find a "multi-Euler" derivation. To construct a basis for the logarithmic derivation module, the Euler derivation $\theta_E := \sum_{i=1}^{\ell} x_i \partial_{x_i}$ plays the key role in the sense that θ_E is tangent to any hyperplanes with the multiplicity one:

$$\theta_E(\alpha_H) = \alpha_H \in S \cdot \alpha_H \setminus S \cdot \alpha_H^2 \ (\forall \alpha_H \in V^*).$$

This is the special derivation, and we cannot expect the existence of such a kind of derivations for multiarrangements. However, Yoshinaga noticed that we can find a derivation similar to the Euler derivation associated to the fixed Coxeter multiarrangements. Note that the primitive derivation D induces the $T := \ker(D : R \to R)$- isomorphism $\nabla_D : \mathrm{Der}(S^W) \to \mathrm{Der}\, R$ (e.g., see [14]) and

$$\cdots \subset D(\mathcal{A}, \mathbf{2k+1})^W \subset D(\mathcal{A}, \mathbf{2k-1})^W \subset \cdots \subset D(\mathcal{A}, \mathbf{1})^W \subset \mathrm{Der}\, R.$$

Thus we can always define the derivation $E_k := \nabla_D^{-k} \theta_E \in D(\mathcal{A})^W$. Then the following fact, proved by Yoshinaga in [19] and generalized in [9], is very important.

Proposition 2.2
The derivation $E_k \in D(\mathcal{A}, \mathbf{2k+1})^W$.

Hence we can use E_k as a kind of the "multi-Euler" derivation. Now let us construct a map $\Phi = \Phi_k$ by

$$D(\mathcal{A}, m) \ni \theta \mapsto \Phi_k(\theta) := \nabla_\theta E_k.$$

To prove that Φ_k gives rise to an S-module isomorphism

$$D(\mathcal{A}, m) \simeq D(\mathcal{A}, \mathbf{2k} + m),$$

we have to check the following three points:

Point 1. Φ_k is well-defined.
Point 2. Φ_k is injective.
Point 3. Φ_k is surjective.

We do not explain details on the above three. For details see [2]. We just point out what plays the key role to check above three points.

Point 1. To prove this point, we use the W-invariance of E_k and count orders of poles along $H \in \mathcal{A}$ carefully.

Point 2. This part can be proved by using the totally same argument as in [9], which is essentially proved in [16].

Point 3. We prove in two steps. First we prove when $(\mathcal{A}, m)$ is free. Then, since $D(\mathcal{A}, -1)$ is free, we can use the structure of this free module to prove a general $(\mathcal{A}, m)$, for $D(\mathcal{A}, m) \subset D(\mathcal{A}, -1)$.

Summarizing these, we obtain the following:

Theorem 2.3 ([2])
The map Φ_k gives rise to an S-module isomorphism

$$\Phi_k : D(\mathcal{A}, m) \to D(\mathcal{A}, \mathbf{2k} + m).$$

In particular, for a free $D(\mathcal{A}, m)$ with basis $\theta_1, \dots, \theta_\ell$, $D(\mathcal{A}, \mathbf{2k} + m)$ is free with basis

$$\nabla_{\theta_1} E_k, \dots, \nabla_{\theta_\ell} E_k.$$

Remark 2.4
Though Theorem 2.3 gives an explicit form, to compute $D(\mathcal{A}, m)$ itself for $m : \mathcal{A} \to \{+1, 0, -1\}$ is not easy, even for a Coxeter arrangements of type A_3. We will see it in the next section.

3 Remarks and Examples

In the previous section we restricted our interest on multiplicities $m : \mathcal{A} \to \{+1, 0, -1\}$. Then it is natural to ask whether these multiplicities can be extended or not. For example, do the results in the previous section hold true for multiplicities like $m : \mathcal{A} \to \{+2, +1, 0, -1, -2\}$? The answer is NO, see the following example:

Example 3.1
Consider a Coxeter arrangement $\mathcal{A}$ of type A_3 defined by $xyz(x-y)(x-z)(y-z)=0$. Note that we use a notation $x_1=x, x_2=y, x_3=z, x_4=w$ in this section. Consider the following two multiarrangements:

$$(\mathcal{A},m_1) : x^2y^2z^2(x-y)^2(x-z)^0(y-z)^0=0,$$
$$(\mathcal{A},m_2) : x^4y^4z^4(x-y)^4(x-z)^2(y-z)^2=0.$$

These multiplicities on $\mathcal{A}$ can be expressed in the same way as in the previous section with $m : \mathcal{A} \to \{+2,+1,0,-1,-2\}$. If the same result holds true, then $D(\mathcal{A},m_1)$ and $D(\mathcal{A},m_2)$ have to be isomorphic, but they do not. In fact, we can prove the following:

Lemma 3.2
A multiarrangement $(\mathcal{A},m)$ defined by

$$x^{2k}y^{2k}z^{2k}(x-y)^{2k}(x-z)^{2k-2}(y-z)^{2k-2}=0 \ (k \in \mathbb{Z}_{\geq 1})$$

is free if and only if $k=1$.

Hence the condition on multiplicities is essential.

Proof. When $k=1$ then the multiarrangement is supersolvable in the sense of [7], hence free. Assume that $k>1$ and $(\mathcal{A},m)$ is free. Compute the upper bound of GMP, global mixed product. Note that $|m| = 12k-4$. Hence the upper bound of $GMP(\mathcal{A},m)$ is attained when the exponents are $(4k-2, 4k-1, 4k-1)$. Hence

$$GMP(\mathcal{A},m) \geq 2(4k-2)(4k-1)+(4k-1)^2 = 48k^2-32k+5.$$

Next let us compute the LMP, local mixed product. It is easy to get, when $k>1$,

$$LMP(\mathcal{A},m) = 48k^2-32k+6 > GML(\mathcal{A},m),$$

which is a contradiction. □

It seems to the author that the condition on multiplicities occurs from the invariant theory. More explicitly, for a Coxeter arrangement $\mathcal{A}$ with the Coxeter group W, it holds that

$$D(\mathcal{A},\mathbf{2k}) \subsetneq D(\mathcal{A},\mathbf{2k+1}),$$

but

$$D(\mathcal{A},\mathbf{2k})^W = D(\mathcal{A},\mathbf{2k+1})^W.$$

Hence the result in the previous section is to investigate the derivation modules which vanish in the invariant part. Also,

$$D(\mathcal{A}, -\mathbf{2k}) \supsetneq D(\mathcal{A}, -\mathbf{2k}+\mathbf{1}),$$

but

$$D(\mathcal{A}, -\mathbf{2k})^W = D(\mathcal{A}, -\mathbf{2k}+\mathbf{1})^W.$$

Since $D(\mathcal{A}, m)$ with $m : \mathcal{A} \to \{+1, 0, -1\}$ is a mixed module of these two modules, such a multiplicity seems not to be extended.

Let us give more examples of free bases for the A_3-type arrangement with a non-constant $m : \mathcal{A} \to \{+1, -1\}$. Bases for constant multiplicities can be obtained by the invariant theory.

First consider the multiarrangement defined by

$$\frac{(y-z)(z-x)(x-w)(y-w)(z-w)}{(x-y)} = 0.$$

By [3] and [2] this multiarrangement is free with exponents $(0, 1, 2, 1)$. We can obtain the explicit basis as follows:

$$\begin{aligned}
\theta_0 &= \partial_x + \partial_y + \partial_z + \partial_w, \\
\theta_1 &= x\partial_x + y\partial_y + z\partial_z + w\partial_w, \\
\theta_2 &= x^2\partial_x + y^2\partial_y + z^2\partial_z + w^2\partial_w, \\
\theta_3 &= \frac{(x-z)(x-w)\partial_x - (y-z)(y-w)\partial_y}{x-y}
\end{aligned}$$

Second consider the arrangement defined by

$$\frac{(x-y)(y-z)(z-x)(x-w)}{(y-w)(z-w)} = 0.$$

By [3] and [2] this multiarrangement is free with exponents $(0, 1, 0, 1)$. We can obtain the explicit basis as follows:

$$\begin{aligned}
\theta_0 &= \partial_x + \partial_y + \partial_z + \partial_w, \\
\theta_1 &= x\partial_x + y\partial_y + z\partial_z + w\partial_w, \\
\theta_2 &= (x-w)\partial_w, \\
\theta_3 &= \partial_x + \frac{x-w}{y-w}\partial_y + \frac{x-w}{z-w}\partial_z + \frac{-w^2 - xy - xz + 2xw + yz}{(y-w)(z-w)}\partial_w.
\end{aligned}$$

The basis of a multiarrangement defined by

$$\frac{(y-z)(y-w)(z-w)}{(x-y)(x-z)(x-w)} = 0$$

can be found in [2].

Next consider the arrangement defined by

$$\frac{(x-y)(x-z)(x-w)}{(y-z)(y-w)(z-w)}=0.$$

By [3] and [2] this multiarrangement is free with exponents $(0,1,0,-1)$. We can obtain the explicit basis as follows:

$$\begin{aligned}
\theta_0 &= \partial_x+\partial_y+\partial_z+\partial_w,\\
\theta_1 &= x\partial_x+y\partial_y+z\partial_z+w\partial_w,\\
\theta_2 &= \frac{x-y}{(y-z)(y-w)}\partial_y-\frac{x-z}{(y-z)(z-w)}\partial_z+\frac{x-w}{(y-w)(z-w)}\partial_w,\\
\theta_3 &= \frac{(x-y)(z-w)y\partial_y-(x-z)(y-w)z\partial_z+(x-w)(y-z)w\partial_w}{(y-z)(y-w)(z-w)}.
\end{aligned}$$

Next consider the arrangement defined by

$$\frac{(x-z)(x-w)}{(x-y)(y-z)(y-w)(z-w)}=0.$$

By [3] and [2] this multiarrangement is free with exponents $(0,-1,0,-1)$. We can obtain the explicit basis as follows:

$$\begin{aligned}
\theta_0 &= \partial_x+\partial_y+\partial_z+\partial_w,\\
\theta_1 &= \frac{1}{x-y}\partial_x-\left(\frac{1}{x-y}-\frac{1}{y-z}-\frac{1}{y-w}\right)\partial_y-\frac{1}{y-z}\partial_z-\frac{1}{y-w}\partial_w,\\
\theta_2 &= \frac{x-y}{(y-z)(y-w)}\partial_y-\frac{x-z}{(y-z)(z-w)}\partial_z+\frac{x-w}{(y-w)(z-w)}\partial_w,\\
\theta_3 &= \frac{(x-y)(z-w)y\partial_y-(x-z)(y-w)z\partial_z+(x-w)(y-z)w\partial_w}{(y-z)(y-w)(z-w)}.
\end{aligned}$$

Finally consider the arrangement defined by

$$\frac{x-w}{(x-y)(x-z)(y-z)(y-w)(z-w)}=0.$$

By [3] and [2] this multiarrangement is free with exponents $(0,-1,-1,-2)$. We can obtain the explicit basis as follows:

$$
\begin{aligned}
\theta_0 &= \partial_x+\partial_y+\partial_z+\partial_w,\\
\theta_1 &= \frac{1}{x-y}\partial_x-(\frac{1}{x-y}-\frac{1}{y-z}-\frac{1}{y-w})\partial_y-\frac{1}{y-z}\partial_z-\frac{1}{y-w}\partial_w,\\
\theta_2 &= \frac{x-y}{(y-z)(y-w)}\partial_y-\frac{x-z}{(y-z)(z-w)}\partial_z+\frac{x-w}{(y-w)(z-w)}\partial_w,\\
\theta_3 &= \frac{1}{Q_-}\{(y-z)(y-w)(z-w)\partial_x+(x-z)(z-w)(x-2y+w)\partial_y\\
&\quad +(x-y)(y-w)(-x+2z-w)\partial_z+(x-y)(x-z)(y-z)\partial_w\},
\end{aligned}
$$

where

$$Q_- := (x-y)(x-z)(y-z)(y-w)(z-w).$$

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