LITTLEWOOD-RICHARDSON COEFFICIENTS AND EXTREMAL WEIGHT CRYSTALS

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ABSTRACT. We describe the tensor product of two extremal weight crystals of type $A_{+\infty}$ by constructing an explicit bijection between the connected components in the tensor product and a set of quadruples of Littlewood-Richardson tableaux.

1. INTRODUCTION

Let $\mathfrak{g}_{>0}$ be the infinite rank affine Lie algebra of type $A_{+\infty}$ and $U_q(\mathfrak{g}_{>0})$ its quantized enveloping algebra. For an integral weight $\Lambda$, there exists an integrable $U_q(\mathfrak{g}_{>0})$-module called the extremal weight module with extremal weight $\Lambda$. The notion of extremal weight modules introduced by Kashiwara [5] is a generalization of integrable highest weight and lowest weight modules. An extremal weight module has a crystal base, which we call an extremal weight crystal for short, and two extremal weight crystals are isomorphic if their extremal weights are in the same Weyl group orbit.

Let $\mathcal{P}$ be the set of partitions. The Weyl group orbit of $\Lambda$ is naturally in one-to-one correspondence with a pair of partitions $(\mu, \nu) \in \mathcal{P}^2$, where $(\mu, \emptyset)$ (resp. $(\emptyset, \nu)$) corresponds to a dominant (resp. anti-dominant) weight. Let us denote by $B_{\mu, \nu}$ the extremal weight crystal with extremal weight corresponding to $(\mu, \nu) \in \mathcal{P}^2$.

In [9], it is shown that the tensor product of two extremal weight crystals is isomorphic to a finite disjoint union of extremal weight crystals and the Grothendieck ring associated with the category of $\mathfrak{g}_{>0}$-crystals whose object is a finite union of extremal weight crystals, is isomorphic to the Weyl algebra of infinite rank. Using this characterization, it is shown that the multiplicity of $B_{\zeta, \eta}$ in $B_{\mu, \nu} \otimes B_{\sigma, \tau}$ for $(\mu, \nu), (\sigma, \tau), (\zeta, \eta) \in \mathcal{P}^2$ is

\[
\sum_{\alpha, \beta, \gamma \in \mathcal{P}} c^\zeta_{\alpha} c^\mu_{\alpha \beta} c^\tau_{\beta \gamma} c^\eta_{\gamma \nu},
\]

which is a sum of products of four Littlewood-Richardson coefficients.

The main purpose of this note is to construct an explicit crystal isomorphism

\[
B_{\mu, \nu} \otimes B_{\sigma, \tau} \xrightarrow{\sim} \bigsqcup_{(\zeta, \eta) \in \mathcal{P}^2} \bigsqcup_{\alpha, \beta, \gamma \in \mathcal{P}} B_{\zeta, \eta} \times \text{LR}_{\alpha}^\zeta \times \text{LR}_{\beta}^\mu \times \text{LR}_{\gamma}^\tau \times \text{LR}_{\gamma}^\eta,
\]

which gives a bijective proof of (1.1). Here $\text{LR}_\lambda^{\mu \nu}$ denotes the set of Littlewood-Richardson tableaux of shape $\lambda/\mu$ with content $\nu$ for $\lambda, \mu, \nu \in \mathcal{P}$. We remark that the decomposition of $B_{\mu, \nu} \otimes B_{\sigma, \tau}$ is given in [9] by generalizing the insertion algorithm of Stembridge's rational

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tableaux [13, 14] for $\mathfrak{g}l_n$, but the associated recording tableaux which parameterize the connected components in $\mathcal{B}_{\mu,\nu} \otimes \mathcal{B}_{\sigma,\tau}$ do not imply (1.1) directly.

The multiplicity (1.1) has another representation theoretical interpretation, that is, it coincides with a generalization of Littlewood-Richardson coefficients introduced in [2], whose positivity is equivalent to the existence of a long exact sequence of 6 finite abelian $p$-groups with types $\sigma, \zeta, \mu, \tau, \eta, \nu$. The author would like to thank Alexander Yong for pointing out this connection.

This note is organized as follows. In Section 2, we recall briefly the notion of crystals and a combinatorial realization of $\mathcal{B}_{\mu,\nu}$. In Section 3, we review some combinatorics of Littlewood-Richardson tableaux and an insertion algorithm for $\mathcal{B}_{\mu,\nu}$. Finally, in Section 4, we construct the isomorphism (1.2).

2. Extremal weight crystals

2.1. Let $\mathfrak{g}l_{>0}$ denote the Lie algebra of complex matrices $(a_{ij})_{i,j \in \mathbb{N}}$ with finitely many non-zero entries. Let $E_{ij}$ be the elementary matrix with 1 at the $i$-th row and the $j$-th column and zero elsewhere. Then $\{E_{ij} \mid i, j \geq 1\}$ is a linear basis of $\mathfrak{g}l_{>0}$.

Let $\mathfrak{h} = \bigoplus_{i \geq 1} \mathbb{C}E_{ii}$ be the Cartan subalgebra of $\mathfrak{g}l_{0}$ and $\langle \cdot, \cdot \rangle$ the natural pairing on $\mathfrak{h}^* \times \mathfrak{h}$. Let $\Pi' = \{h_i = E_{ii} - E_{i+1,i+1} \mid i \geq 1\}$ be the set of simple coroots and $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i \geq 1\}$ the set of simple roots of $\mathfrak{g}l_{>0}$, where $\varepsilon_i \in \mathfrak{h}^*$ is determined by $\langle \varepsilon_i, E_{jj} \rangle = \delta_{ij}$.

Let $P = \bigoplus_{i \geq 1} \mathbb{Z} \varepsilon_i$ be the weight lattice of $\mathfrak{g}l_{0}$ and $P_+ = \{\Lambda \in P \mid \langle \Lambda, h_i \rangle \geq 0 \ (i \geq 1)\}$ the set of dominant integral weights. The map $\lambda = (\lambda_i)_{i \geq 1} \mapsto \omega_{\lambda} = \sum_{i \geq 1} \lambda_i \varepsilon_i$ gives a bijection between $\mathcal{P}$ and $P_+$, where $\mathcal{P}$ denotes the set of partitions.

For $i \geq 1$, let $r_i$ be the simple reflection given by $r_i(\Lambda) = \Lambda - \langle \Lambda, h_i \rangle \alpha_i$ for $\Lambda \in \mathfrak{h}^*$. Let $W$ be the Weyl group of $\mathfrak{g}l_{>0}$, that is, the subgroup of $GL(\mathfrak{h}^*)$ generated by $r_i$ for $i \geq 1$. Let $P/W$ be the set of $W$-orbits in $P$. For $\Lambda = \sum_{i \geq 1} \lambda_i \epsilon_i \in P$, let $\mu$ and $\nu$ be the partitions determined by $\{\Lambda_i \mid \Lambda_i > 0\}$ and $\{-\Lambda_i \mid \Lambda_i < 0\}$, respectively. Then the map $W\Lambda \mapsto (\mu, \nu)$ is a bijection from $P/W$ to $\mathcal{P}^2$.

2.2. Let us recall briefly the notion of crystals based on [6]. A $\mathfrak{g}l_{>0}$-crystal is a set $B$ together with the maps $\operatorname{wt} : B \to P$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$ and $\tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\}$ ($i \in \mathbb{N}$) such that for $b \in B$

1. $\varphi_i(b) = \langle \operatorname{wt}(b), h_i \rangle + \varepsilon_i(b),$

2. $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1, \operatorname{wt}(\tilde{e}_i b) = \operatorname{wt}(b) + \alpha_i$ if $\tilde{e}_i b \neq 0$, 

3. $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1, \operatorname{wt}(\tilde{f}_i b) = \operatorname{wt}(b) - \alpha_i$ if $\tilde{f}_i b \neq 0$, 

4. $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for $b, b' \in B$, 

5. $\tilde{e}_i b = 0$ if $\varphi_i(b) = -\infty$, 

where 0 is a formal symbol and $-\infty$ is the smallest element in $\mathbb{Z} \cup \{-\infty\}$ such that $-\infty + n = -\infty$ for all $n \in \mathbb{Z}$.
A crystal $B$ is an $N$-colored oriented graph where $b \xrightarrow{i} b'$ if and only if $b' = \tilde{f}_i b$ for $b, b' \in B$ and $i \geq 1$. We say that $B$ is connected if it is connected as a graph and regular if $\varepsilon_i(b) = \max\{ k | \varepsilon_i^k b \neq 0 \}$ and $\varphi_i(b) = \max\{ k | \tilde{f}_i^k b \neq 0 \}$ for $b \in B$ and $i \geq 1$.

The dual crystal $B^\vee$ of $B$ is defined to be the set $\{ b^\vee | b \in B \}$ with
\[
\text{wt}(b^\vee) = -\text{wt}(b),
\varepsilon_i(b^\vee) = \varphi_i(b), \quad \varphi_i(b^\vee) = \varepsilon_i(b),
\tilde{e}_i(b^\vee) = (\tilde{f}_i b)^\vee, \quad \tilde{f}_i(b^\vee) = (\tilde{e}_i b)^\vee,
\]
for $b \in B$ and $i \geq 1$. Here we assume that $0^\vee = 0$.

Let $B_1$ and $B_2$ be crystals. The tensor product of $B_1$ and $B_2$ is defined to be the set $B_1 \otimes B_2 = \{ b_1 \otimes b_2 | b_i \in B_i (i = 1, 2) \}$ with
\[
\text{wt}(b_1 \otimes b_2) = \text{wt}(b_1) + \text{wt}(b_2),
\varepsilon_i(b_1 \otimes b_2) = \max\{ \varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \text{wt}(b_2), h_i \rangle \},
\varphi_i(b_1 \otimes b_2) = \max\{ \varphi_i(b_1) + \langle \text{wt}(b_2), h_i \rangle, \varphi_i(b_2) \},
\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases},
\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases},
\]
for $b_1 \otimes b_2 \in B_1 \otimes B_2$ and $i \geq 1$, where we assume that $0 \otimes b_2 = b_1 \otimes 0 = 0$. Then $B_1 \otimes B_2$ is also a crystal.

A map $\psi : B_1 \to B_2$ is called an isomorphism of crystals if it is a bijection, preserves wt, $\varepsilon_i$ and $\varphi_i$ and commutes with $\tilde{e}_i$, $\tilde{f}_i$ ($i \geq 1$), where we assume that $\psi(0) = 0$.

In this case, we say that $B_1$ is isomorphic to $B_2$ and write $B_1 \simeq B_2$. For example, $(B_1 \otimes B_2)\vee \simeq B_2^\vee \otimes B_1^\vee$, where $(b_1 \otimes b_2)^\vee$ is mapped to $b_2^\vee \otimes b_1^\vee$.

For $b_i \in B_i$ ($i = 1, 2$), we say that $b_1$ is equivalent to $b_2$, and write $b_1 \equiv b_2$ if there exists an isomorphism of crystals $C(b_1) \to C(b_2)$ sending $b_1$ to $b_2$, where $C(b_i)$ denotes the connected component of $B_i$ including $b_i$ ($i = 1, 2$).

2.3. We identify a partition with a Young diagram as usual (see [11]), where we enumerate rows and columns from the top and the left, respectively. Let $\mathcal{A}$ be a linearly ordered set. A tableau $T$ obtained by filling a skew Young diagram $\lambda/\mu$ with entries in $\mathcal{A}$ is called a semistandard tableau of shape $\lambda/\mu$ if the entries in each row are weakly increasing from left to right, and the entries in each column are strictly increasing from top to bottom. We denote by $SST_{\mathcal{A}}(\lambda/\mu)$ the set of all semistandard tableaux of shape $\lambda/\mu$ with entries in $\mathcal{A}$ (cf.[3, 11]).
For $T \in SST_{A}(\lambda/\mu)$, let $w(T)_{\text{col}}$ (resp. $w(T)_{\text{row}}$) denote the word obtained by reading the entries of $T$ column by column (resp. row by row) from right to left (resp. top to bottom), and in each column (resp. row) from top to bottom (resp. right to left). For $a \in A$, we denote by $(a \rightarrow T)$ (resp. $(T \leftarrow a)$) the tableau obtained by the Schensted column (resp. row) insertion (see for example [3, Appendix A.2]). For a finite word $w = w_{1}\ldots w_{r}$ with letters in $A$, we let $(w \rightarrow T) = (w_{r} \rightarrow (\cdots (w_{1} \rightarrow T)\cdots))$ and $(T \leftarrow w) = ((\cdots (T \leftarrow w_{1})\cdots) \leftarrow w_{r})$. For semistandard tableaux $S$ and $T$, we define $(T \rightarrow S)$ (resp. $(S \leftarrow T)$) to be $(w(T)_{\text{col}} \rightarrow S)$ (resp. $S \leftarrow (w(T)_{\text{row}})^{\text{rev}}$) where $w^{\text{rev}}$ is the reverse word of $w$.

We denote by $T^{\vee}$ the tableau obtained from $T$ by $180^{\circ}$-rotation and replacing each entry $t$ with $t^{\vee}$. Then $T^{\vee}$ is a semistandard tableau with entries in $A^{\vee}$, where $A^{\vee} = \{ a^{\vee} | a \in A \}$ and $a^{\vee} < b^{\vee}$ if and only if $b < a$ for $a, b \in A$. Here we use the convention $(t^{\vee})^{\vee} = t$ and hence $(T^{\vee})^{\vee} = T$.

Let $A$ be either $\mathbb{N}$ or $\mathbb{N}^{\vee}$ with the following regular crystal structures
\[
\begin{align*}
1 & \rightarrow 1^{\vee}, \\
2 & \rightarrow 2^{\vee}, \\
& \cdots
\end{align*}
\]
where $\text{wt}(k) = \epsilon_{k}$ and $\text{wt}(k^{\vee}) = -\epsilon_{k}$ for $k \geq 1$. Then the set of all finite words with letters in $A$ is a regular crystal, where we identify each word of length $r$ with an element in $A^{\otimes r} = A \otimes \cdots \otimes A$ ($r$ times). Now, the injective image of $SST_{A}(\lambda/\mu)$ in the set of finite words under the map $T \mapsto w(T)_{\text{col}}$ (or $w(T)_{\text{row}}$) together with $\{ 0 \}$ is invariant under $\tilde{e}_{i}, \tilde{f}_{i}$. Hence $SST_{A}(\lambda/\mu)$ is a regular crystal [8]. Also, the row or column insertion is compatible with the crystal structure on tableaux in the following sense [10];
\[(a \rightarrow T) \equiv T \otimes a, \quad (T \leftarrow a) \equiv a \otimes T,
\]
for $a \in A$ and $T \in SST_{A}(\lambda)$, and hence $(T \rightarrow S) \equiv S \otimes T$, $(S \leftarrow T) \equiv T \otimes S$ for $S \in SST_{A}(\mu)$.

2.4. For $\Lambda \in P$, let $B(\Lambda)$ be the crystal base of the extremal weight $U_{q}(\mathfrak{g}l_{>0})$-module with extremal weight $\Lambda$. Then $B(\Lambda)$ is a regular crystal, and $B(\Lambda) \simeq B(\omega_{\Lambda})$ for $\omega \in W$. Moreover, if $\Lambda \in P_{+}$ (resp. $-\Lambda \in P_{+}$), then $B(\Lambda)$ is isomorphic to the crystal base of the irreducible highest (resp. lowest) weight $U_{q}(\mathfrak{g}l_{>0})$-module with highest (resp. lowest) weight $\Lambda$ (see [5, 7] for detailed exposition of extremal weight modules and their crystal bases).

Recall that for $\lambda \in \mathcal{P}$
\[B(\omega_{\lambda}) \simeq SST_{N}(\lambda), \quad B(-\omega_{\lambda}) \simeq B(\omega_{\lambda})^{\vee} \simeq SST_{N^{\vee}}(\lambda^{\vee}),\]
where $\lambda^{\vee}$ is the skew Young diagram obtained from $\lambda \in \mathcal{P}$ by $180^{\circ}$-rotation, and $SST_{N}(\lambda)$ is connected with a unique highest weight element $H_{\lambda}$, where each $i$-th row is filled with $i$ for $i \geq 1$ [8].
Now, for $\mu, \nu \in \mathcal{P}$, we define $\mathcal{B}_{\mu,\nu}$ to be the set of bitableaux $(S, T)$ such that

(E1) $S \in \text{SST}_{N}(\mu)$ and $T \in \text{SST}_{N^\vee}(\nu^\vee)$,
(E2) for each $k \geq 1$,

$$s(k) + t(k) \leq k$$

where $s(k)$ is the number of entries in the left-most column of $S$ no more than $k$, and $t(k)$ is the number of entries in the right-most column of $T$ no less than $k^\vee$.

Since $\mathcal{B}_{\mu,\nu} \subset \text{SST}_{N}(\mu) \otimes \text{SST}_{N^\vee}(\nu^\vee)$, we can apply $\tilde{e}_i, \tilde{f}_i$ ($i \geq 1$) on $\mathcal{B}_{\mu,\nu}$. Then $\mathcal{B}_{\mu,\nu} \cup \{0\}$ is stable under $\tilde{e}_i, \tilde{f}_i$ ($i \geq 1$) and hence a regular crystal. Moreover, we have the following [9, Theorem 3.5].

**Theorem 2.1.** For $\mu, \nu \in \mathcal{P}$,

1. $\mathcal{B}_{\mu,\nu}$ is connected,
2. $\mathcal{B}_{\mu,\nu} \simeq \mathbb{B}(\Lambda)$, where $W\Lambda \in P/W$ corresponds to $(\mu, \nu) \in \mathcal{P}^2$.

### 3. Insertion Algorithm

3.1. For $\lambda, \mu, \nu \in \mathcal{P}$, let $\text{LR}_{\mu,\nu}^\lambda$ be the set of tableaux $U$ in $\text{SST}_{N}(\lambda/\mu)$ such that for $i \geq 1$

    (LR1) the number of $i$'s in $U$ is $\nu_i$,
    (LR2) the number of $i$'s in $w_1 \cdots w_k$ is no less than that of $i+1$'s in $w_1 \cdots w_k$ for $1 \leq k \leq r$,

where $w(U)_{\text{col}} = w_1 \cdots w_r$.

We call $\text{LR}_{\mu,\nu}^\lambda$ the set of **Littlewood-Richardson tableaux** of shape $\lambda/\mu$ with content $\nu$ and put $c_{\mu,\nu}^\lambda = |\text{LR}_{\mu,\nu}^\lambda|$ [11].

Suppose that $\mathcal{A}$ is a linearly ordered set. For $S \in \text{SST}_{A}(\mu)$ and $T \in \text{SST}_{A}(\nu)$, let $\lambda$ be the shape of $(T \rightarrow S)$ and $w(T)_{\text{col}} = w_1 \cdots w_r$. If $w_i$ is in the $k$th row of $T$ and inserted into $(w_{i-1} \rightarrow (\cdots(w_1 \rightarrow T)))$ to create a node in $\lambda/\mu$, then let us fill the node with $k$. We denote the resulting tableau in $\text{SST}_{N}(\lambda/\mu)$ by $(T \rightarrow S)_R$ and call it the **recording tableau** of $(T \rightarrow S)$. Then we have a bijection

$$\text{SST}_{A}(\mu) \times \text{SST}_{A}(\nu) \xrightarrow{\lambda \in \mathcal{P}} \bigcup_{\lambda \in \mathcal{P}} \text{SST}_{A}(\lambda) \times \text{LR}_{\mu,\nu}^\lambda,$$

where $(S, T)$ corresponds to $((T \rightarrow S), (T \rightarrow S)_R)$ [15]. Moreover, if we assume that $\mathcal{A}$ is either $\mathbb{N}$ or $\mathbb{N}^\vee$, then the above bijection commutes with $\tilde{e}_i$ and $\tilde{f}_i$ ($i \geq 1$) (cf.[4, 10]), where $\tilde{e}_i$ and $\tilde{f}_i$ act on the first component of $\text{SST}_{A}(\lambda) \times \text{LR}_{\mu,\nu}^\lambda$. Summarizing, we have

**Proposition 3.1.** Let $\mu, \nu \in \mathcal{P}$ be given.

1. The map sending $S \otimes T$ to $((T \rightarrow S), (T \rightarrow S)_R)$ is an isomorphism of crystals

$$\text{SST}_{N}(\mu) \otimes \text{SST}_{N}(\nu) \xrightarrow{\sim} \bigcup_{\lambda \in \mathcal{P}} \text{SST}_{N}(\lambda) \times \text{LR}_{\mu,\nu}^\lambda.$$
The map sending $S \otimes T$ to $((S^\vee \to T^\vee)^\vee, (S^\vee \to T^\vee)_R)$ is an isomorphism of crystals

$$SST_{N^\vee}(\mu^\vee) \otimes SST_{N^\vee}(\nu^\vee) \xrightarrow{\sim} \bigsqcup_{\lambda \in \mathcal{P}} SST_{N^\vee}(\lambda^\vee) \times LR_{\nu^\mu}^\lambda.$$  

Remark 3.2. (1) Let $U \in SST_N(\lambda/\mu)$ be given. Then as a crystal element, $U \in LR_{\nu^\mu}^\lambda$ if and only if $U \equiv H_{\lambda}$. 

(2) For $U \in LR_{\nu^\mu}^\lambda$, one may identify $U$ with a unique $T \in SST_N(\nu)$, say $\iota(U)$, such that the number of $k$’s in the $i$-th row of $T$ is equal to the number of $i$’s in the $k$-th row of $\lambda/\mu$ for $i, k \geq 1$. Equivalently, $H_{\lambda} \otimes \iota(U) \equiv H_{\lambda}$ [12].

3.2. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two linearly ordered sets. Let $U$ be a tableau of shape $\lambda/\mu$ with entries in $\mathcal{A} \cup \mathcal{B}$, satisfying the following conditions; 

(S1) if $u, u' \in X$ are entries of $U$ and $u$ is northwest of $u'$, then $u \leq u'$, 

(S2) in each column of $U$, entries in $X$ increase strictly from top to bottom, 

where $X = A$ or $B$, and we say that $u$ is northwest of $u'$ provided the row and column indices of $u$ are no more than those of $u'$. Suppose that $a \in \mathcal{A}$ and $b \in \mathcal{B}$ are two adjacent entries in $U$ such that $a$ is placed above or to the left of $b$. Interchanging $a$ and $b$ is called a switching if the resulting tableau still satisfies the conditions (S1) and (S2).

For $S \in SST_{\mathcal{A}}(\mu)$ and $T \in SST_{\mathcal{B}}(\lambda/\mu)$, we denote by $S \ast T$ the tableau in $SST_{\mathcal{A} \cup \mathcal{B}}(\lambda)$ obtained by gluing $S$ and $T$. Let $U$ be a tableau obtained from $S \ast T$ by applying switching procedures as far as possible. Then it is shown in [1, Theorems 2.2 and 3.1] that 

(1) $U = T' \ast S'$, where $T' \in SST_{\mathcal{B}}(\nu)$ and $S' \in SST_{\mathcal{A}}(\lambda/\nu)$ for some $\nu$, 

(2) $U$ is uniquely determined by $S$ and $T$, 

(3) when $\mathcal{A} = N$, $S' \in LR_{\nu^\mu}^\lambda$ if and only if $S = H_{\mu}$.

Suppose that $\mathcal{A} = N$ and $S = H_{\mu}$. Put 

$$j(T) = T', \quad j(T)_R = S'.$$

Then the map $T \mapsto (j(T), j(T)_R)$ gives a bijection [1]

$$SST_{\mathcal{B}}(\lambda/\mu) \xrightarrow{\sim} \bigsqcup_{\nu \in \mathcal{P}} SST_{\mathcal{B}}(\nu) \times LR_{\nu^\mu}^\lambda.$$  

If $\mathcal{B} = N$, then the map $Q \mapsto j(Q)_R$ restricts to a bijection from $LR_{\mu^\nu}^\lambda$ to $LR_{\nu^\mu}^\lambda$. Moreover, if $\mathcal{B}$ is either $N$ or $N^\vee$, then we can check that $T \equiv j(T)$ and $j(T)_R$ is invariant under $\tilde{e}_i$ and $\tilde{f}_i$ ($i \geq 1$). Hence we have the following.

Proposition 3.3. Suppose that $\mathcal{B}$ is either $N$ or $N^\vee$. For a skew Young diagram $\lambda/\mu$, we have an isomorphism of crystals

$$SST_{\mathcal{B}}(\lambda/\mu) \xrightarrow{\sim} \bigsqcup_{\nu \in \mathcal{P}} SST_{\mathcal{B}}(\nu) \times LR_{\nu^\mu}^\lambda,$$

where $T$ is mapped to $(j(T), j(T)_R)$. 
3.3. Let us review an insertion algorithm for extremal weight crystal elements [9].

3.3.1. Let $\mu, \nu \in \mathcal{P}$ be given. For $a \in \mathbb{N}$ and $(S, T) \in \mathcal{B}_{\mu, \nu}$, we define $(a \rightarrow (S, T))$ in the following way;

Suppose that $S$ is empty and $T$ is a single column tableau. Let $(T', a')$ be the pair obtained by the following process;

1. If $T$ contains $a^\vee, (a+1)^\vee, \ldots, (b-1)^\vee$ but not $b^\vee$, then $T'$ is the tableau obtained from $T$ by replacing $a^\vee, (a+1)^\vee, \ldots, (b-1)^\vee$ with $(a+1)^\vee, (a+2)^\vee, \ldots, b^\vee$, and put $a' = b$.

2. If $T$ does not contain $a^\vee$, then leave $T$ unchanged and put $a' = a$.

Now, we suppose that $S$ and $T$ are arbitrary.

1. Apply the above process to the leftmost column of $T$ with $a$.
2. Repeat (1) with $a'$ and the next column to the right.
3. Continue this process to the right-most column of $T$ to get a tableau $T'$ and $a''$.
4. Define

$$(a \rightarrow (S, T)) = ((a'' \rightarrow S), T').$$

Then $(a \rightarrow (S, T)) \in \mathcal{B}_{\sigma, \nu}$ for some $\sigma \in \mathcal{P}$ with $|\sigma / \mu| = 1$. For a finite word $w = w_1 \ldots w_r$ with letters in $\mathbb{N}$, we let $(w \rightarrow (S, T)) = (w_r \rightarrow (\cdots (w_1 \rightarrow (S, T)) \cdots))$.

3.3.2. For $a \in \mathbb{N}$ and $(S, T) \in \mathcal{B}_{\mu, \nu}$, we define $((S, T) \leftarrow a^\vee)$ to be the pair $(S', T')$ obtained in the following way;

1. If the pair $(S, (T^\vee \leftarrow a)^\vee)$ satisfies the condition (E2) in Section 2.4, then put $S' = S$ and $T' = (T^\vee \leftarrow a)^\vee$.
2. Otherwise, choose the smallest $k$ such that $a_k$ is bumped out of the $k$-th row in the row insertion of $a$ into $T^\vee$ and the insertion of $a_k$ into the $(k+1)$-th row violates the condition (E2) in Section 2.4.

2-a) Stop the row insertion of $a$ into $T^\vee$ when $a_k$ is bumped out and let $T'$ be the resulting tableau after taking $\vee$.

2-b) Remove $a_k$ in the left-most column of $S$, which necessarily exists, and then apply the jeu de taquin (see for example [3, Section 1.2]) to obtain a tableau $S'$.

In this case, $((S, T) \leftarrow a^\vee) \in \mathcal{B}_{\sigma, \tau}$, where either (1) $|\mu / \sigma| = 1$ and $\tau = \nu$, or (2) $\sigma = \mu$ and $|\tau / \nu| = 1$. For a finite word $w = w_1 \ldots w_r$ with letters in $\mathbb{N}^\vee$, we let $((S, T) \leftarrow w) = (((S, T) \leftarrow w_1) \cdots) \leftarrow w_r$.

3.3.3. Let $\mu, \nu, \sigma, \tau \in \mathcal{P}$ be given. For $(S, T) \in \mathcal{B}_{\mu, \nu}$ and $(S', T') \in \mathcal{B}_{\sigma, \tau}$, we define

$$((S', T') \rightarrow (S, T)) = ((w(S')_{\text{col}} \rightarrow (S, T)) \leftarrow w(T')_{\text{col}}).$$
Then \(((S', T') \to (S, T)) \in \mathcal{B}_{\zeta, \eta}\) for some \((\zeta, \eta) \in \mathcal{P}^{2}\). Assume that \(w(S')_{\text{col}} = w_{1} \ldots w_{s}\) and \(w(T')_{\text{col}} = w_{s+1} \ldots w_{s+t}\). For \(1 \leq i \leq s + t\), let

\[
(s^{i}, t^{i}) = \begin{cases} w_{i} \to (\cdots (w_{1} \to (S, T))), & \text{if } 1 \leq i \leq s, \\ ((s^{s}, t^{s}) \leftarrow w_{s+1}) \cdots \leftarrow w_{i}, & \text{if } s + 1 \leq i \leq s + t, \end{cases}
\]

and \((S^{0}, T^{0}) = (S, T)\). We define

\[
((S', T') \to (S, T))_{R} = (U, V),
\]

where \((U, V)\) is the pair of tableaux with entries in \(\mathbb{Z}^{\times} = \mathbb{Z} \setminus \{0\}\) determined by the following process;

1. \(U\) is of shape \(\sigma\) and \(V\) is of shape \(\tau\).
2. Let \(1 \leq i \leq s\). If \(w_{i}\) is inserted into \((S^{i-1}, T^{i-1})\) to create a dot (or box) in the \(k\)-th row of the shape of \(S^{i-1}\), then we fill the dot in \(\sigma\) corresponding to \(w_{i}\) with \(k\).
3. Let \(s + 1 \leq i \leq s + t\). If \(w_{i}\) is inserted into \((S^{i-1}, T^{i-1})\) to create a dot in the \(k\)-th row (from the bottom) of the shape of \(T^{i-1}\), then we fill the dot in \(\tau\) corresponding to \(w_{i}\) with \(-k\). If \(w_{i}\) is inserted into \((S^{i-1}, T^{i-1})\) to remove a dot in the \(k\)-th row of the shape of \(S^{i-1}\), then we fill the corresponding dot in \(\tau\) with \(k\).

We call \(((S', T') \to (S, T))_{R}\) the recording tableau of \(((S', T') \to (S, T))\). By [9, Theorem 4.10], we have the following.

**Proposition 3.4.** Under the above hypothesis, we have

1. \(((S', T') \to (S, T)) \equiv (S, T) \otimes (S', T')\),
2. \(((S', T') \to (S, T))_{R} \in \text{SST}_{\mathbb{N}}(\sigma) \times \text{SST}_{\mathbb{Z}^{\times}}(\tau)\),
3. the recording tableaux are constant on the connected component of \(\mathcal{B}_{\mu, \nu} \otimes \mathcal{B}_{\sigma, \tau}\) including \((S, T) \otimes (S', T')\),

where the linear ordering on \(\mathbb{Z}^{\times}\) is given by \(1 \prec 2 \prec 3 \prec \cdots \prec -3 \prec -2 \prec -1\).

**Example 3.5.** Consider

\[
(S, T) = \begin{pmatrix} 2 & 3 & 4 & 5^\vee & 5^\vee \\ 3 & 5 \end{pmatrix}, \quad (S', T') = \begin{pmatrix} 3 & 3 & 4^\vee \\ 5 \end{pmatrix}.
\]

Since \(w(S')_{\text{col}} = 335\) and \(w(T')_{\text{col}} = 4^\vee 1^\vee 3^\vee\), we have

\[
(w(S')_{\text{col}} \to (S, T)) = \begin{pmatrix} 2 & 3 & 3 & 4 \\ 3 & 5 \\ 4 & 6^\vee & 5^\vee \\ 6 & 4^\vee & 2^\vee \end{pmatrix}.
\]
and
$$((w(S')_{\text{col}} \rightarrow (S, T)) \leftarrow w(T')_{\text{col}}) = \begin{pmatrix} 2 & 3 & 3 & 4 & 5^\vee \\ 3 & 5 & & 6^\vee & 4^\vee \\ 4 & & & 4^\vee & 3^\vee & 1^\vee \end{pmatrix}.$$ Hence,
$$((S', T') \rightarrow (S, T)) = \begin{pmatrix} 3 & 3 & 3 & 4 & 5^\vee \\ 4 & 5 & & 6^\vee & 4^\vee \\ 6 & & & 4^\vee & 3^\vee & 1^\vee \end{pmatrix},$$
$$((S', T') \rightarrow (S, T))_{R} = \begin{pmatrix} 1 & 3 & 4 & -3 \\ 4 & & & -1 \end{pmatrix}.$$

**Remark 3.6.** For $$(U, V) \in \text{SST}_N(\sigma) \times \text{SST}_Z(\tau),$$ an equivalent condition for $$(U, V)$$ to be a recording tableau, that is, $$(U, V) = ((S', T') \rightarrow (S, T))_{R}$$ for some $$(S, T) \in \mathcal{B}_{\mu, \nu}$$ and $$(S', T') \in \mathcal{B}_{\sigma, \tau},$$ can be found in [9, Section 4.3].

4. **Main Theorem**

To prove our main theorem, let us first describe the decompositions of $\text{SST}_N(\nu^\vee) \otimes \text{SST}_N(\mu)$ and $\text{SST}_N(\mu) \otimes \text{SST}_N(\nu^\vee)$ for $\mu, \nu \in \mathcal{P}$.

**Proposition 4.1.** For $\mu, \nu \in \mathcal{P}$, we have an isomorphism of crystals
$$\text{SST}_N(\nu^\vee) \otimes \text{SST}_N(\mu) \rightarrow \mathcal{B}_{\mu, \nu},$$
where $T \otimes S$ is mapped to $((S, \emptyset) \rightarrow (\emptyset, T))_{R}$.

**Proof.** For $T \otimes S \in \text{SST}_N(\nu^\vee) \otimes \text{SST}_N(\mu)$, it follows from Proposition 3.4 (2) that
1. $((S, \emptyset) \rightarrow (\emptyset, T)) \in \mathcal{B}_{\mu, \nu},$
2. $((S, \emptyset) \rightarrow (\emptyset, T))_{R} = (H_{\mu}, \emptyset)$.

Therefore, we have a map
$$\text{SST}_N(\nu^\vee) \otimes \text{SST}_N(\mu) \rightarrow \mathcal{B}_{\mu, \nu} \times \{H_{\mu}, \emptyset\}$$
sending $T \otimes S$ to $\left(((S, \emptyset) \rightarrow (\emptyset, T)), ((S, \emptyset) \rightarrow (\emptyset, T))_{R}\right)$. Since the insertion algorithm is reversible [9, Proposition 4.9], the above map is indeed a bijection and hence an isomorphism of crystals by Proposition 3.4 (1).

Next, suppose that $S \otimes T \in \text{SST}_N(\mu) \otimes \text{SST}_N(\nu^\vee)$ is given. Let $U^{>0}$ (resp. $U^{<0}$) be the subtableau in $((\emptyset, T) \rightarrow (S, \emptyset))_{R}$ consisting of positive (resp. negative) entries. We define
$$\theta(S \otimes T) = (\iota^{-1}(U^{>0}), j(U^{<0})_{R})_{R}$$
(see Remark 3.2 (2) and Section 3.2 (3.2)).
Proposition 4.2. For $\mu, \nu \in \mathcal{P}$, we have an isomorphism of crystals
\[
SST_{\mathbb{N}}(\mu) \otimes SST_{\mathbb{N}^{\vee}}(\nu^{\vee}) \sim \bigsqcup_{\lambda, \sigma, \tau \in \mathcal{P}} \mathcal{B}_{\sigma, \tau} \times LR_{\sigma \lambda}^{\mu} \times LR_{\lambda \tau}^{\nu},
\]
where $S \otimes T$ is mapped to $(((\emptyset, T) \to (S, \emptyset)), \theta(S \otimes T))$.

**Proof.** For $S \otimes T \in SST_{\mathbb{N}}(\mu) \otimes SST_{\mathbb{N}^{\vee}}(\nu^{\vee})$, suppose that $(((\emptyset, T) \to (S, \emptyset)) \in \mathcal{B}_{\sigma, \tau}$ for some $\sigma, \tau \in \mathcal{P}$.

First, note that $U^{>0} \in SST_{\mathbb{N}}(\lambda)$ for some $\lambda \subset \nu$. Then it is not difficult to check that $U^{>-\emptyset} \in LR_{\sigma \lambda}^{\mu}$ (see Remark 3.2). Next, consider $U^{<0} \in SST_{\mathbb{N}^{\vee}}(\nu \lambda)$. Then $(w(U^{<0}), \text{col})^{\text{rev}}$ satisfies (LR1) with respect to $\tau$ and (LR2), ignoring $-$ sign in each letter. Let $L_{\tau}$ be the tableau in $SST_{\mathbb{N}^{\vee}}(\tau)$, where the $i$-th entry from the bottom in each column is $-i$. Considering the Knuth equivalence on the set of words with letters in $\mathbb{Z}_{<0}$ (cf.[3]), we have $j(U^{<0}) = L_{\tau}$ and $j(U^{<0})_{R} \in LR_{\tau \lambda}^{\nu}$ by (3.2). So, we get $j(j(U^{<0})_{R})_{R} \in LR_{\lambda \tau}^{\nu}$.

Now, we have a map
\[
SST_{\mathbb{N}}(\mu) \otimes SST_{\mathbb{N}^{\vee}}(\nu^{\vee}) \to \bigsqcup_{\lambda, \sigma, \tau \in \mathcal{P}} \mathcal{B}_{\sigma, \tau} \times LR_{\sigma \lambda}^{\mu} \times LR_{\lambda \tau}^{\nu},
\]
sending $S \otimes T$ to $(((\emptyset, T) \to (S, \emptyset)), \theta(S \otimes T))$. Since the insertion algorithm is reversible [9, Proposition 4.9], the above map is a bijection and therefore an isomorphism of crystals by Proposition 3.4 (1) and (3).

Now, we are in a position to state our main result in this note.

Theorem 4.3. For $(\mu, \nu), (\sigma, \tau) \in \mathcal{P}^{2}$, we have an isomorphism of crystals
\[
\mathcal{B}_{\mu, \nu} \otimes \mathcal{B}_{\sigma, \tau} \simeq \bigsqcup_{(\zeta, \eta) \in \mathcal{P}^{2}} \mathcal{B}_{\zeta, \eta} \times LR_{\sigma \zeta}^{\mu} \times LR_{\alpha \gamma}^{\nu} \times LR_{\beta \gamma}^{\mu} \times LR_{\beta \gamma}^{\nu}.
\]

**Proof.** Note that $\mathcal{B}_{\mu, \emptyset} = SST_{\mathbb{N}}(\mu)$ and $\mathcal{B}_{\emptyset, \nu} = SST_{\mathbb{N}^{\vee}}(\nu^{\vee})$. Then as a crystal, we have
\[
\mathcal{B}_{\mu, \emptyset} \otimes \mathcal{B}_{\sigma, \emptyset} \\
\simeq \mathcal{B}_{\emptyset, \nu} \otimes \mathcal{B}_{\mu, \emptyset} \otimes \mathcal{B}_{\emptyset, \emptyset} \otimes \mathcal{B}_{\sigma, \emptyset} \\
\simeq \bigsqcup_{\alpha, \beta, \gamma \in \mathcal{P}} (\mathcal{B}_{\emptyset, \nu} \otimes \mathcal{B}_{\alpha, \gamma} \otimes \mathcal{B}_{\sigma, \emptyset} \otimes \mathcal{B}_{\emptyset, \emptyset}) \times LR_{\alpha \beta}^{\mu} \times LR_{\beta \gamma}^{\nu} \\
\simeq \bigsqcup_{\alpha, \beta, \gamma \in \mathcal{P}} (\mathcal{B}_{\emptyset, \nu} \otimes \mathcal{B}_{\emptyset, \gamma} \otimes \mathcal{B}_{\alpha, \emptyset} \otimes \mathcal{B}_{\sigma, \emptyset}) \times LR_{\alpha \beta}^{\mu} \times LR_{\beta \gamma}^{\nu} \\
\simeq \bigsqcup_{(\zeta, \eta) \in \mathcal{P}^{2}} \mathcal{B}_{\zeta, \eta} \times LR_{\sigma \alpha}^{\zeta} \times LR_{\alpha \beta}^{\mu} \times LR_{\beta \gamma}^{\nu} \times LR_{\beta \gamma}^{\eta} \\
\simeq \bigsqcup_{(\zeta, \eta) \in \mathcal{P}^{2}} \mathcal{B}_{\zeta, \eta} \times LR_{\sigma \alpha}^{\zeta} \times LR_{\alpha \beta}^{\mu} \times LR_{\beta \gamma}^{\nu} \times LR_{\gamma \nu}^{\eta} \\
\simeq \bigsqcup_{(\zeta, \eta) \in \mathcal{P}^{2}} \mathcal{B}_{\zeta, \eta} \times LR_{\sigma \alpha}^{\zeta} \times LR_{\alpha \beta}^{\mu} \times LR_{\beta \gamma}^{\nu} \times LR_{\gamma \nu}^{\eta} \\
\simeq \mathcal{B}_{\sigma, \tau} \times LR_{\sigma \lambda}^{\mu} \times LR_{\lambda \tau}^{\nu} \\
\simeq \mathcal{B}_{\sigma, \tau} \times LR_{\sigma \lambda}^{\mu} \times LR_{\lambda \tau}^{\nu}.
\]

\[\square\]
Corollary 4.4. The multiplicity of $\mathcal{B}_{\zeta,\eta}$ in $\mathcal{B}_{\mu,\nu} \otimes \mathcal{B}_{\sigma,\tau}$ is given by

$$\sum_{\alpha,\beta,\gamma \in \mathcal{P}} c_{\sigma}^{\alpha} c_{\mu}^{\alpha} c_{\beta}^{\tau} c_{\gamma}^{\nu}.$$ 

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