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Hyperbolic Damped p-System and Diffusion Phenomena

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Dedicated to Professor Kenji Nishihara on his 60th birthday

Abstract

In this survey paper, we review the development and progress of the study on the 2×2 hyperbolic p-system with damping. The damping effort makes such a system to behave as a diffusion equation. The focus in this paper is to show how to find the best asymptotic profile for the damped p-system, and what are the optimal convergent rates. The most new results are reported in this paper.

1 Introduction

For the model of the compressible flow through porous media with dissipative external force field, it can be described in Lagrangian coordinates as the p-system of hyperbolic conservation laws with damping

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u - \beta |u|^{q-1} u, \end{cases} \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+. \tag{1.1}$$

Here, $v = v(x,t) > 0$ is the specific volume, $u = u(x,t)$ is the velocity, the pressure $p(v)$ is a smooth function of v such that $p(v) > 0$, $p'(v) < 0$. As well-known in hyperbolic system, the typical example in the case of a polytropic gas is $p(v) = v^{-\nu}$ with $\nu \geq 1$. The external term $-\alpha u - \beta |u|^{q-1} u$ appears in the momentum equation, where $\alpha > 0$ and β are constants. The term $-\alpha u$ is called the linear damping, and $-\beta |u|^{q-1} u$ with $q \geq 2$ is regarded as a nonlinear source to the linear damping $-\alpha u$. When $\beta > 0$, the term $-\beta |u|^{q-1} u$ is nonlinear damping, while when $\beta < 0$, the term $-\beta |u|^{q-1} u$ is regarded as nonlinear accumulating.

Considered in this paper is the equation (1.1) with the initial value problem (IVP)

$$(v,u)(x,t)|_{t=0} = (v_0, u_0)(x) \to (v_\pm, u_\pm) \quad \text{as } x \to \pm\infty, \tag{1.2}$$

and the initial-boundary value problems (IBVP), respectively,

$$\begin{cases} (v,u)(x,t)|_{t=0} = (v_0, u_0)(x) \to (v_+, u_+) \quad \text{as } x \to +\infty, \ x \in R_+, \\ u|_{x=0} = 0, \end{cases} \tag{1.3}$$

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or

$$\begin{cases} (v,u)(x,t)|_{t=0} = (v_0,u_0)(x) \to (v_+,u_+) \quad \text{as } x \to +\infty, \ x \in R_+, \\ v|_{x=0} = v_-. \end{cases} \tag{1.4}$$

Here $v_\pm > 0$ and $u_\pm$ are the state constants.

When $\beta = 0$, the system (1.1) is linear damping. The asymptotic behavior of the solution for the Cauchy problem or the IVBP for the linear damped 2×2 p-system has been extensively studied. In 1992, Hsiao and Liu [3, 4] first studied the Cauchy problem for the linear damped p-system, and showed that the solution $(v,u)(x,t)$ converges to its diffusion wave $(\bar{v},\bar{u})(x/\sqrt{1+t})$, a self-similar solution to the following porous media equations

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\alpha\bar{u}, \end{cases} \quad \text{or} \quad \begin{cases} \bar{v}_t = -\frac{1}{\alpha}p(\bar{v})_{xx}, \\ p(\bar{v})_x = -\alpha\bar{u}, \end{cases} \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{1.5}$$

in the form of $\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-1/2},t^{-1/2})$. Since then, the convergence have been improved by Nishihara [27, 28] as $\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/4},t^{-5/4})$ for the initial perturbation in H^3, and then by Nishihara, Wang and Yang [32, 36] as $\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-1},t^{-3/2})$ for the initial perturbation in $L^1 \cap H^3$. These convergence results need the initial perturbation around the specified diffusion wave and the wave strength both to be sufficiently small. Such restrictions were then partially released by Zhao [37], where the initial perturbation in L^∞-sense can be arbitrarily large but its first derivative still needs to be small, which implies that the wave must also be weak. For the 2×2 quasi-linear p-system but still with linear damping, the convergence with some decay rates was obtained by Li and Saxton [15]. Furthermore, when $v_+ = v_-$, Nishihara [29] improved the rates as $\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/2}\log t,\ t^{-2}\log t)$. Very recently, when $v_+ \neq v_-$, by a heuristic analysis, Mei [25] pointed out that the best asymptotic profile to the damped p-system is the particular parabolic solution to the corresponding porous media equation with a specific initial data, rather than the self-similar solutions (the so-called nonlinear diffusion waves), and further proved the convergence as $\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/2}\log t,\ t^{-2}\log t)$.

For the initial boundary problem on the quadrant, the convergence to the diffusion waves with different boundary conditions has been studied respectively by Marcati and Mei [20] and by Nishihara and Yang [31] with $\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/4},t^{-5/4})$ for the initial perturbation in H^3, respectively, and then improved to $\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-1},t^{-3/2})$ by Marcati, Mei and Rubino [21] for the initial perturbation in $L^1 \cap H^3$. Inspired by [37], the convergence result has been further improved for the strong diffusion wave by Jiang and Zhu [14]. Very recently, motivated by [25], after looking for the best asymptotic profile to the original IBVP, Ma and Mei [18] obtained a much better convergence rate $\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-\frac{3}{2}-\frac{a}{4}},t^{-2})$ for $0 \le a \le \frac{1}{4}$, in the case of $v_+ = v_-$ and the initial data in $v_0(x) - v_+ \in L^{1,1}$, where $L^{1,1}$ is the weighted L^1 space, for detail we refer to the notations below.

When $\beta \neq 0$, the system (1.1) becomes either nonlinear damping for $\beta > 0$ or nonlinear accumulating for $\beta < 0$. The research related to this topic, so far, is very limited. For the Cauchy problem case, under the stiff condition $u_+ = u_- = 0$, Jiang and Zhu [39, 40] proved the solution to converge the diffusion wave in the form of $\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/4},t^{-5/4})$. Recently, by technically constructing a pair of correction functions, Mei [24] released the condition $u_+ = u_- = 0$ to the general case $u_+ \neq u_-$, and proved the convergence to the diffusion wave with the optimal rates $\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-1},t^{-3/2})$ when the initial perturbation is in $L^1 \cap H^3$.

For the IBVP case, the convergence of the solution has been investigated by Jiang and Zhu in [13] under the condition $u_+ = 0$, and then improved by C.K.Lin, C.T.Lin and Mei [17] for the general case $u_+ \neq 0$ with a much better decay rate $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-1-\frac{\gamma}{2}}, t^{-3/2})$, when the initial perturbation is in $L^{1,\gamma} \cap H^3$ with the best selected number $\gamma = \frac{1}{4}$.

Regarding the multi-dimensional Euler equations with damping, the convergence to the planar waves has been showed by Liao, W.Wang and Yang [16] for $\beta = 0$ and by Huang, Mei and Y.Wang [10] for $\beta \neq 0$, respectively.

For the other interesting studies for the convergence to diffusion waves in many different cases, we refer to [5, 6, 7, 8, 9, 11, 14, 15, 28, 29, 33, 35, 37, 38] and the references therein.

Notations. Throughout the paper, $C > 0$ denotes a generic constant which may change its value from line to line or even in the same line, while $C_i > 0$ $(i = 0, 1, 2, \cdots)$ represents a specific constant. The partial derivatives of f are denoted by f_x, f_{xx}, and so on, or sometimes by $\partial_x^k f$, $k = 0, 1, 2, \cdots$. $L^p(\mathbb{R}_+)$ $(1 \leq p \leq \infty)$ is the usual Lebesque space with the norm $\|f\|_{L^p} = [\int_{\mathbb{R}_+} |f(x)|^p dx]^{1/p}$ for $1 \leq p < \infty$, and $\|f\|_{L^\infty} = \sup_{x \in \mathbb{R}_+} |f(x)|$, where the integral region $\mathbb{R}_+$ will be omitted without any confusion. $L^{p,\gamma}(\mathbb{R}_+)$ with $\gamma > 0$ and $1 \leq p \leq \infty$ is the weighted $L^p(\mathbb{R}_+)$ space with a weight $(1+x)^\gamma$. Its norm is denoted as $\|f\|_{L^{p,\gamma}(\mathbb{R}_+)} = [\int_{\mathbb{R}_+} (1+x)^\gamma |f(x)|^p dx]^{1/p}$, $1 \leq p \leq \infty$. $H^k(\mathbb{R}_+)$ $(k \geq 0)$ is the usual Sobolev space with the norm $\|f\|_{H^k} = [\sum_{i=0}^k \int_{\mathbb{R}_+} |\partial_x^i f|^2 dx]^{1/2}$. For the sake of simplicity, we also denote $\|(f,g,h)\|_{L^2}^2 = \|f\|_{L^2}^2 + \|g\|_{L^2}^2 + \|h\|_{L^2}^2$ and $\|(f,g,h)\|_{H^k}^2 = \|f\|_{H^k}^2 + \|g\|_{H^k}^2 + \|h\|_{H^k}^2$. Let $T > 0$ and let $\mathcal{B}$ be a Banach space. We denote by $C^0([0,T]; \mathcal{B})$ the space of $\mathcal{B}$-valued continuous functions on $[0,T]$, and $L^2([0,T]; \mathcal{B})$ as the space of $\mathcal{B}$-valued L^2-functions on $[0,T]$. The corresponding spaces of $\mathcal{B}$-valued functions on $[0,\infty)$ are defined similarly.

2 $\beta = 0$: Damped p-System without Nonlinear Source

2.1 Best Asymptotical Profile for IVP

In this subsection, we investigate the best asymptotical profile for (1.1) with $\beta = 0$, namely,

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u, (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ (v,u)|_{t=0} = (v_0, u_0)(x) \to (v_\pm, u_\pm) \quad \text{as} \quad x \to \pm. \end{cases} \tag{2.1}$$

In what follows, we are going to make a heuristic analysis, then we will show how to find the best asymptotic profile and what will be the best asymptotic profile. Finally, we will establish the working equations and state our main convergence results with the improved decay rates.

We first investigate the asymptotic behavior of $(v,u)(x,t)$ at $x = \pm\infty$. Let us take the limits as $x \to \pm\infty$ to the damped p-system (1.1), and note that u_x and $p(v)_x$ will vanish at $x = \pm\infty$ due to the boundedness of $(v,u)(x,t)$, then we have $\frac{d}{dt} v(\pm\infty, t) = 0$, $\frac{d}{dt} u(\pm\infty, t) = -\alpha u(\pm\infty, t)$, $(v,u)(\pm\infty, 0) = (v_0, u_0)(\pm\infty) = (v_\pm, u_\pm)$, which can be exactly solved as

$$v(\pm\infty, t) = v_\pm, \quad u(\pm\infty, t) = u_\pm e^{-\alpha t}, \quad t \geq 0. \tag{2.2}$$

By the Darcy's law, the expected asymptotic profile of (1.1) is the (parabolic) porous media equation (1.5). It can be easily verified that the solution $(\bar{v}, \bar{u})$ of (1.5) satisfies $(\bar{v}, \bar{u})(\pm\infty, t) = (v_\pm, 0)$.

Notice that, the solutions $(\bar{v},\bar{u})$ to (1.5) with $(\bar{v},\bar{u})|x=\pm\infty=(v_\pm,0)$ are not unique. These solutions include the so-called diffusion waves (self-similar solutions) $(\bar{v},\bar{u})(x/\sqrt{1+t})$ and the parabolic solutions with given initial data $\bar{v}|_{t=0}=\bar{v}_0(x)$. The natural questions are, which solution is the best asymptotic profile of (1.1) and (1.2), and what is the optimal decay rate. In order to answer these questions, we need to investigate the gap between $(v,u)(x,t)$ and $(\bar{v},\bar{u})(x,t)$.

From $(1.1)_1$ and $(1.5)_1$, we have $(v-\bar{v})_t=(u-\bar{u})_x$. Integrating it with respect to x over $(-\infty,\infty)$, we then get

$$\frac{d}{dt}\int_{-\infty}^{\infty}(v-\bar{v})(x,t)dx=u(+\infty,0)-u(-\infty,t)=(u_+-u_-)e^{-\alpha t}\neq 0.$$

In order to eliminate the gap $u(+\infty,0)-u(-\infty,t)=(u_+-u_-)e^{-\alpha t}$, we need to construct a pair of correction functions $(\hat{v},\hat{u})(x,t)$, which was first introduced by Hsiao and Liu in [3]. Namely, let $\hat{u}(x,t)$ be the solution to the following equation

$$\frac{d}{dt}\hat{u}(x,t)=-\alpha\hat{u}(x,t)\quad\text{with}\quad\hat{u}(\pm\infty,t)=u_\pm e^{-\alpha t},$$

then it can be easily solved as

$$\hat{u}(x,t)=m(x)e^{-\alpha t},\tag{2.3}$$

where $m(x)$ needs to be $m(\pm\infty)=u_\pm$. For this, we construct it as $m(x)=u_-+(u_+-u_-)\int_{-\infty}^{x}m_0(y)dy$, and $m_0(x)\in C_0^\infty(\mathbb{R})$ with $\int_{-\infty}^{\infty}m_0(x)dx=1$. Now setting $\hat{v}(x,t)$ such that $\hat{v}_t=\hat{u}_x$, one then immediately obtains

$$\hat{v}(x,t)=-\frac{u_+-u_-}{\alpha}m_0(x)e^{-\alpha t}.\tag{2.4}$$

Thus, the correction functions $(\hat{v},\hat{u})(x,t)$ satisfy

$$\begin{cases}\hat{v}_t-\hat{u}_x=0,\\ \hat{u}_t=-\alpha\hat{u},\\ (\hat{v},\hat{u})|_{x=\pm\infty}=(0,\ u_\pm e^{-\alpha t}).\end{cases}\tag{2.5}$$

Now we are going to look for the best asymptotic profile $(\bar{v},\bar{u})(x,t)$. Traditionally, we take the self-similar solution $(\bar{v},\bar{u})=(\phi,\psi)((x+\bar{x})/\sqrt{1+t})$ as the asymptotic profile for the solution $(v,u)(x,t)$ for some shift $\bar{x}$. Here, in order to avoid the singularity, we use $(\phi,\psi)(x/\sqrt{1+t})$ to replace $(\phi,\psi)(x/\sqrt{t})$. However, this is not the best asymptotic profile. In fact, as showed in [3, 27, 32], one can expect only

$$\int_{-\infty}^{\infty}(v-\bar{v}-\hat{v})(x,t)dx=0,\ \text{but}\ \int_{-\infty}^{\infty}(u-\bar{u}-\hat{u})(x,t)dx\neq 0.$$

This implies that the selected asymptotic profile $(\bar{v},\bar{u})=(\phi,\psi)((x+\bar{x})/\sqrt{1+t})$ is not optimal. In order to get the best the asymptotic profile $(\bar{v},\bar{u})$, we need technically to construct a particular solution $(\bar{v},\bar{u})(x,t)$ such that, for all $t\geq 0$,

$$\int_{-\infty}^{\infty}(v-\bar{v}-\hat{v})(x,t)dx=0,\ \int_{-\infty}^{\infty}(u-\bar{u}-\hat{u})(x,t)dx=0,\ \int_{-\infty}^{\infty}\int_{-\infty}^{x}(v-\bar{v}-\hat{v})(y,t)dy=0.$$

Let $(\bar{v},\bar{u})(x,t)$ be the expected particular solution of the Cauchy problem

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\alpha\bar{u}, \\ \bar{v}|_{t=0} = \bar{v}_0(x), \end{cases} \quad \text{or equivalently,} \quad \begin{cases} \bar{v}_t = -\frac{1}{\alpha}p(\bar{v})_{xx}, \\ p(\bar{v})_x = -\alpha\bar{u}, \\ \bar{v}|_{t=0} = \bar{v}_0(x), \end{cases} \tag{2.6}$$

where the initial data $\bar{v}_0(x)$ satisfies $\bar{v}_0(x) \to v_\pm$, as $x \to \pm\infty$, and will be specified later. Furthermore, let us take the correction function

$$(\hat{v},\hat{u})(x+x_0,t) = \Big(-\frac{u_+ - u_-}{\alpha} m_0(x+x_0)e^{-\alpha t},\ m(x+x_0)e^{-\alpha t} \Big), \tag{2.7}$$

with a shift x_0 determined by

$$x_0 := \frac{1}{u_+ - u_-}\Big\{ \int_{-\infty}^{\infty} [u_0(x) - m(x)]dx + \frac{1}{\alpha}[p(v_+) - p(v_-)] \Big\}. \tag{2.8}$$

It is verified that

$$\begin{cases} (v - \bar{v} - \hat{v})_t - (u - \bar{u} - \hat{u})_x = 0, \\ (u - \bar{u} - \hat{u})_t + (p(v) - p(\bar{v}))_x = -\alpha(u - \bar{u} - \hat{u}) + \frac{1}{\alpha}p(\bar{v})_{xt}. \end{cases} \tag{2.9}$$

Integrating $(2.9)_2$ with respect to (x,t) over $\mathbb{R} \times [0,t]$, and noting that $p(v) \to p(v_\pm)$, $p(\bar{v}) \to p(v_\pm)$ as $x \to \pm\infty$, and the selection of x_0 mentioned above, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} (u - \bar{u} - \hat{u})(x,t)dx \\ &= e^{-\alpha t}\int_{-\infty}^{\infty} [u_0(x) - \bar{u}(x,0) - \hat{u}(x+x_0,0)]dx \\ &= e^{-\alpha t}\int_{-\infty}^{\infty} \Big[u_0(x) + \frac{1}{\alpha}p(\bar{v}(x,0))_x - m(x+x_0)\Big]dx \\ &= e^{-\alpha t}\Big\{ \int_{-\infty}^{\infty} [u_0(x) - m(x+x_0)]dx + \frac{1}{\alpha}\int_{-\infty}^{\infty} p(\bar{v}(x,0))_x dx \Big\} \\ &= e^{-\alpha t}\Big\{ \Big[\int_{-\infty}^{\infty} [u_0(x) - m(x+x_0)]dx\Big] + \frac{1}{\alpha}[p(v_+) - p(v_-)] \Big\} \\ &= 0. \end{aligned} \tag{2.10}$$

Now we return back to $(2.9)_1$. Integrating it over $(-\infty, x]$ yields

$$\frac{d}{dt}\int_{-\infty}^{x} (v - \bar{v} - \hat{v})(y,t)dy = (u - \bar{u} - \hat{u})(x,t).$$

Again, integrating the above equation over $(-\infty,\infty)$ with respect to x, we have

$$\frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{x} (v - \bar{v} - \hat{v})(y,t)dydx = \int_{-\infty}^{\infty} (u - \bar{u} - \hat{u})(x,t)dx = 0.$$

Then, integrating the above equation with respect to t, we further obtain

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{x} (v - \bar{v} - \hat{v})(y,t)dydx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{x} [v_0(y) - \bar{v}_0(y) - \hat{v}(y,0)]dydx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{x} \Big[v_0(y) - \bar{v}_0(y) + \frac{u_+ - u_-}{\alpha}m_0(y+x_0)\Big]dydx. \end{aligned} \tag{2.11}$$

Now we select the particular initial data $\bar{v}_0(x)$ such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{x}\Big[v_0(y)-\bar{v}_0(y)+\frac{u_+-u_-}{\alpha}m_0(y+x_0)\Big]dydx=0, \tag{2.12}$$

as a particular example, we may take $\bar{v}_0(x):=v_0(x)+\frac{u_+-u_-}{\alpha}m_0(x+x_0)$, then we can expect, from (2.11), that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{x}(v-\bar{v}-\hat{v})(y,t)dydx=0,\quad t\geq 0. \tag{2.13}$$

Defining

$$(V,U)(x,t)=\Big(\int_{-\infty}^{x}\int_{-\infty}^{y}(v-\bar{v}-\hat{v})(z,t)dzdy,\ \int_{-\infty}^{x}(u-\bar{u}-\hat{u})(y,t)dy\Big), \tag{2.14}$$

$$(V_0,U_0)(x)=\Big(\int_{-\infty}^{x}\int_{-\infty}^{y}[v_0(z)-\bar{v}_0(z)-\hat{v}(z,0)]dzdy,\int_{-\infty}^{x}[u_0(y)-\bar{u}(y,0)-\hat{u}(y,0)]dy\Big), \tag{2.15}$$

namely, $V_{xx}=v-\bar{v}-\hat{v}$, $U_x=u-\bar{u}-\hat{u}$, and applying them to (2.9), we finally establish a new working system of equations

$$\begin{cases} V_t-U=0,\\ U_t+p(\bar{v}+\hat{v}+V_{xx})-p(\bar{v})=-\alpha U+\frac{1}{\alpha}p(\bar{v})_t,\\ (V,U)|_{t=0}=(V_0,U_0)(x), \end{cases} \tag{2.16}$$

namely,

$$\begin{cases} V_t-U=0,\\ U_t+(p'(\bar{v})V_x)_x=-\alpha U-F_1-F_2,\\ (V,U)|_{t=0}=(V_0,U_0(x)), \end{cases} \tag{2.17}$$

where

$$F_1:\ =\ -\frac{1}{\alpha}p(\bar{v})_t, \tag{2.18}$$

$$F_2:\ =\ [p(\bar{v}+\hat{v}+V_{xx})-p(\bar{v})-p'(\bar{v})V_{xx}]-p'(\bar{v})_xV_x. \tag{2.19}$$

Our convergence results are as follows.

Theorem 2.1 (Mei [25]) *Let $\bar{v}_0(x)$ be chosen such that (2.12) holds, and let $(V_0,U_0)\in H^3(\mathbb{R})\times H^2(\mathbb{R})$. There exists a number $\varepsilon_1>0$, when the initial perturbation $(V_0,U_0)(x)$ and the wave strength $\delta:=|v_+-v_-|+|u_+-u_-|$ are suitably small such that $\delta+\|V_0\|_{H^3(\mathbb{R})}+\|U_0\|_{H^2(\mathbb{R})}\leq\varepsilon_1$, then the global solution $(V,U)(x,t)$ of (2.16) (or (2.17)) uniquely exists and satisfies*

$$V(x,t)\in C^k(0,\infty;H^{3-k}(\mathbb{R})),\quad k=0,1,2,3,\qquad U(x,t)\in C^k(0,\infty;H^{2-k}(\mathbb{R})),\quad k=0,1,2,$$

and

$$\begin{aligned}&\sum_{k=0}^{3}(1+t)^k\|\partial_x^kV(t)\|^2_{L^2(\mathbb{R})}+\sum_{k=0}^{2}(1+t)^{k+2}\|\partial_x^kU(t)\|^2_{L^2(\mathbb{R})}\\&+\int_0^t\Big[\sum_{k=0}^{3}(1+s)^{k-1}\|\partial_x^kV(s)\|^2_{L^2(\mathbb{R})}+\sum_{k=0}^{2}(1+s)^{k+1}\|\partial_x^kU(s)\|^2_{L^2(\mathbb{R})}\Big]ds\\&\leq C(\|V_0\|^2_{H^3(\mathbb{R})}+\|U_0\|^2_{H^2(\mathbb{R})}+\delta).\end{aligned} \tag{2.20}$$

Moreover, if $(V_0, U_0) \in (H^3(\mathbb{R}) \cap L^1(\mathbb{R})) \times (H^2(\mathbb{R}) \cap (L^1(\mathbb{R}))$, *then the rates can be further improved as follows*

$$\|\partial_x^k V(t)\|_{L^2(\mathbb{R})} \le C(\|V_0\|^2_{H^3(\mathbb{R})} + \|U_0\|_2^2 + \delta)(1+t)^{-\frac{1}{4}-\frac{k}{2}} \log(2+t), \quad k = 0,1,2,3, \tag{2.21}$$

$$\|\partial_x^k U(t)\|_{L^2(\mathbb{R})} \le C(\|V_0\|^2_{H^3(\mathbb{R})} + \|U_0\|_2^2 + \delta)(1+t)^{-\frac{5}{4}-\frac{k}{2}} \log(2+t), \quad k = 0,1,2. \tag{2.22}$$

Corollary 2.2 (Mei [25]) *Under the conditions in Theorem 2.1, the system* (1.1) *and* (1.2) *possesses a uniquely global solution* $(v,u)(x,t)$, *which converges to its best asymptotic profile* $(\bar{v}, \bar{u})(x,t)$ *defined in* (2.6) *with the specified initial data given in* (2.12) *in the form of*

$$\begin{aligned} \|(v-\bar{v})(t)\|_{L^\infty(\mathbb{R})} &= O(1)(1+t)^{-3/2}\log(2+t), &(2.23)\\ \|(u-\bar{u})(t)\|_{L^\infty(\mathbb{R})} &= O(1)(1+t)^{-2}\log(2+t). &(2.24) \end{aligned}$$

2.2 Best Asymptotic Profile with Improved Convergence Rates for IBVP

In this subsection, we consider the following IBVP

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ (v,u)(x,0) = (v_0, u_0)(x) \to (v_+, u_+) \quad \text{as} \quad x \to \infty, \\ v(0,t) = v_-. \end{cases} \tag{2.25}$$

As showed before, the best asymptotic profile for (2.25) is its corresponding IBVP of the porous media equation

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\alpha \bar{u}, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ (\bar{v}, \bar{u})(x,0) = (\bar{v}_0, \bar{u}_0)(x) \to (v_+, 0) \quad \text{as} \quad x \to +\infty, \\ \bar{v}(0,t) = v_-, \end{cases} \tag{2.26}$$

where $\bar{u}_0 = -\frac{1}{\alpha} p(\bar{v}_0)_x$, and $\bar{v}_0(x)$ needs to be specified later.

Let us technically construct the correction function $(\hat{v}, \hat{u})(x,t)$ as follows

$$\begin{cases} \hat{v}(x,t) = -\frac{1}{\alpha}[u_+ m_0(x) + \delta_0 m_0'(x)]e^{-\alpha t}, \\ \hat{u}(x,t) = [u_+ m(x) + \delta_0 m_0(x)]e^{-\alpha t}, \end{cases}$$

which is different from what selected in the previous works for the IBVPs [20, 31, 21]. Here $m_0(x)$ is a smooth and compact supported function $m_0(x) \in C_0^\infty(\mathbb{R}^+)$ satisfying

$$m_0(0) = m_0(\infty) = 0, \quad m_0'(0) = 0, \quad \int_0^\infty m_0(y)dy = 1,$$

and $m(x)$ is defined as

$$m(x) = \int_0^x m_0(y)dy, \quad m(\infty) = 1.$$

δ_0 is a constant given by

$$\delta_0 := \frac{1}{\alpha}[p(v_+) - p(v_-)] + \int_0^\infty [u_0(x) - u_+ m(x)]dx. \tag{2.27}$$

Thus, $(\hat{v},\hat{u})(x,t)$ satisfies

$$\begin{cases} \hat{v}_t - \hat{u}_x = 0, \\ \hat{u}_t = -\alpha\hat{u}, \qquad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ (\hat{v},\hat{u})(x,t) \to (0, u_+ e^{-\alpha t}) \quad \text{as} \quad x \to +\infty. \end{cases} \tag{2.28}$$

Now we are going to determine $\bar{v}_0(x)$ such that the corresponding solution $(\bar{v},\bar{u})(x,t)$ to the system (2.26) is the best asymptotic profile for the original solution $(v,u)(x,t)$, and then we derive the perturbation equations. From (2.25), (2.26) and (2.28), we have

$$\begin{cases} (v-\bar{v}-\hat{v})_t - (u-\bar{u}-\hat{u})_x = 0 \\ (u-\bar{u}-\hat{u})_t + [p(v)-p(\bar{v})]_x = -\alpha(u-\bar{u}-\hat{u}) + \frac{1}{\alpha}p(\bar{v})_{xt}. \end{cases} \tag{2.29}$$

Integrating the second equation of (2.29) with respect to x over $\mathbb{R}^+$ and noting the boundary condition $\bar{v}(0,t) = v(0,t) = v_-$ and $\bar{v}(+\infty,t) = v(+\infty,t) = v_+$ yield

$$\frac{d}{dt}\int_0^\infty [u(x,t)-\bar{u}(x,t)-\hat{u}(x,t)]dx = -\alpha\int_0^\infty [u(x,t)-\bar{u}(x,t)-\hat{u}(x,t)]dx$$

which can be solved as

$$\begin{aligned} &\int_0^\infty [u(x,t)-\bar{u}(x,t)-\hat{u}(x,t)]dx \\ &= e^{-\alpha t}\int_0^\infty [u_0(x)-\bar{u}(x,0)-\hat{u}(x,0)]dx \\ &= e^{-\alpha t}\int_0^\infty [u_0(x)+\frac{1}{\alpha}p(\bar{v}_0(x))_x-\hat{u}(x,0)]dx \\ &= e^{-\alpha t}\Big\{\int_0^\infty [u_0(x)-u_+m(x)]dx + \frac{1}{\alpha}[p(v_+)-p(v_-)] - \delta_0\Big\} \\ &= 0, \end{aligned} \tag{2.30}$$

where (2.27) is used in the last step. Now we turn to the first equation of (2.29) to determine $\bar{v}_0(x)$. Integrating$(2.29)_1$ with respect to x over $[x,\infty)$, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_x^\infty [v(z,t)-\bar{v}(z,t)-\hat{v}(z,t)]dz \\ &= (u-\bar{u}-\hat{u})(z,t)|_{z=\infty} - (u-\bar{u}-\hat{u})(z,t)|_{z=x} \\ &= -[u(x,t)-\bar{u}(x,t)-\hat{u}(x,t)], \end{aligned} \tag{2.31}$$

then integrate the above equation with respect to x over $\mathbb{R}^+$ and use (2.30) to have

$$\frac{d}{dt}\int_0^\infty\int_x^\infty [v(z,t)-\bar{v}(z,t)-\hat{v}(z,t)]dzdx = -\int_0^\infty [u(x,t)-\bar{u}(x,t)-\hat{u}(x,t)]dx = 0,$$

which gives

$$\int_0^\infty\int_x^\infty [v(z,t)-\bar{v}(z,t)-\hat{v}(z,t)]dzdx = \int_0^\infty\int_x^\infty [v_0(z)-\bar{v}_0(z)-\hat{v}(z,0)]dzdx. \tag{2.32}$$

By selecting $\bar{v}_0(x)$ as

$$\bar{v}_0(x) = v_0(x) - \hat{v}(x,0), \tag{2.33}$$

then from (2.32) and (2.33), we obtain

$$\int_0^\infty \int_x^\infty [v(z,t) - \bar{v}(z,t) - \hat{v}(z,t)]dzdx = 0. \tag{2.34}$$

Thus, as explained in Subsection 2.1, the solution $(\bar{v}, \bar{u})(x,t)$ for the system (2.26) with the specified initial data $\bar{v}_0$ in (2.33) is the best asymptotic profile for the original system (2.25). Therefore, let

$$\begin{aligned} V(x,t) :&= \int_x^\infty \int_y^\infty (v - \bar{v} - \hat{v})(z,t)dzdy, \\ U(x,t) :&= \int_0^x (u - \bar{u} - \hat{u})(y,t)dy, \\ V_0(x) :&= \int_x^\infty \int_y^\infty (v_0(z) - \bar{v}_0(z) - \hat{v}(z,0))dzdy = 0, \\ U_0(x) :&= \int_0^x (u_0(y) - \bar{u}(y,0) - \hat{u}(y,0))dy, \end{aligned} \tag{2.35}$$

namely

$$V_{xx} = v - \bar{v} - \hat{v}, \quad U_x = u - \bar{u} - \hat{u}.$$

Then $U(\infty, t) = 0$, the original system can be reformulated as

$$\begin{cases} V_t - U = 0, \\ U_t + p(\bar{v} + \hat{v} + V_{xx}) - p(\bar{v}) = -\alpha U + p(\bar{v})_t, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ (V,U)|_{t=0} = (0, U_0(x)), \\ V(0,t) = 0, \end{cases} \tag{2.36}$$

which can be rewritten as

$$\begin{cases} V_t - U = 0, \\ U_t + (p'(\bar{v})V_x)_x = -\alpha U - F_1 - F_2, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ (V,U)|_{t=0} = (0, U_0(x)), \\ V(0,t) = 0, \end{cases} \tag{2.37}$$

or

$$\begin{cases} V_t - U = 0, \\ U_t + p'(v_+)V_{xx} = -\alpha U - F_1 - F_3, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ (V,U)|_{t=0} = (0, U_0(x)), \\ V(0,t) = 0, \end{cases} \tag{2.38}$$

where

$$\begin{aligned} F_1 :&= -p(\bar{v})_t, \\ F_2 :&= p(\bar{v} + \hat{v} + V_{xx}) - p(\bar{v}) - p'(\bar{v})V_{xx} - p'(\bar{v})_x V_x, \\ F_3 :&= p(\bar{v} + \hat{v} + V_{xx}) - p(\bar{v}) - p'(v_+)V_{xx}. \end{aligned}$$

Here, we mainly consider the case $v_- = v_+$, and for the case $v_- \neq v_+$ we will give a remark at the end of this section. Now we are going to state our convergence results. First of all, we have the following existence and stability of the solution $(\bar{v}, \bar{u})(x,t)$ (the best asymptotic profile) for the system (2.26) with a particular initial data satisfying (2.33).

Theorem 2.3 (Ma-Mei [18]) *Let $v_+ = v_-$ and $l \geq 3$. Suppose $\bar{v}_0 - v_+ \in L^1(\mathbb{R}^+) \cap H^l(\mathbb{R}^+) \cap W^{l-1,1}(\mathbb{R}^+)$, and $\delta_{0\bar{v}} = |\int_0^\infty (\bar{v}_0(x) - v_+)dx|$ is suitably small. Then there exists a unique solution $(\bar{v}, \bar{u})(x,t)$ to (2.26) and (2.33) satisfying the decay properties*

$$\begin{aligned} \|\partial_t^j \partial_x^k (\bar{v} - v_+)(t)\|_{L^p} &\leq C\delta_{0\bar{v}}(1+t)^{-(1-1/p)/2-(k+2j)/2}, \\ & t \geq 0, \quad 1 \leq p \leq \infty, \quad j,k \geq 0, \quad 2j+k \leq l-1, \\ \|\partial_x^k \bar{u}(t)\|_{L^p} &\leq C\delta_{0\bar{v}}(1+t)^{-(1-1/p)/2-(1+k)/2}, \\ & t \geq 0, \quad 1 \leq p \leq \infty, \quad 0 \leq k \leq l-2. \end{aligned} \tag{2.39}$$

Moreover, if $\bar{v}_0 - v_+ \in L^{1,\gamma}(\mathbb{R}^+)$, $0 < \gamma \leq 1$, then $\forall a \in (0, \gamma)$. the solution $\bar{v}(x,t)$ to the system (2.26) and (2.33) satisfies

$$\begin{aligned} \|\partial_t^j \partial_x^k (\bar{v} - v_+)(t)\|_{L^{1,a}} &\leq C(1+t)^{-(2j+k+\gamma-a)/2}, \\ & t \geq 0, \quad j,k \geq 0, \quad 2j+k \leq l-1, \\ \|\partial_t^j \partial_x^k (\bar{v} - v_+)(t)\|_{L^p} &\leq C(1+t)^{-(1-1/p)/2-(k+2j+\gamma)/2}, \\ & t \geq 0, \quad 1 \leq p \leq \infty, \quad j,k \geq 0, \quad 2j+k \leq l-1, \end{aligned} \tag{2.40}$$

Our convergence results are as follows.

Theorem 2.4 (Ma-Mei [18]) *Let $v_+ = v_-$ and $l \geq 3$, $\delta_{0v} := |\int_0^\infty (v_0(x) - v_+)dx|$ and δ_0 be defined as before. Suppose that $v_0 - v_+ \in H^l(\mathbb{R}^+) \cap W^{l-1,1}(\mathbb{R}^+)$ and $U_0(x) \in H^{l-1}(\mathbb{R}^+)$. If $\lambda_l = \|U_0\|_{l-1}^2 + \delta_0 + \delta_{0v}$ is suitably small, then there exists a unique time-global solution $(V,U)(x,t)$ of (2.36)*

$$\begin{aligned} V(x,t) &\in C^k([0,\infty); H^{l-k}), \quad k = 0,1,\cdots,l, \\ U(x,t) &\in C^k([0,\infty); H^{l-1-k}), \quad k = 0,1,\cdots,l-1, \end{aligned}$$

satisfying

$$(1+t)^{k/2+j}\|\partial_x^k \partial_t^j V(t)\| \leq C\lambda_l, \tag{2.41}$$

for $j = 0,1,\cdots,l-2$ and $k = 0,1,\cdots,l-j$,

$$(1+t)^{(k+1)/2+l-2}\|\partial_x^k \partial_t^{l-1} V(t)\| \leq C\lambda_l, \tag{2.42}$$

for $k = 0,1$, and

$$(1+t)^{l-1}\|\partial_t^l V(t)\| \leq C\lambda_l. \tag{2.43}$$

Furthermore, let $l = 4$, if $U_0(x) \in L^1(\mathbb{R}^+)$ and $v_0 - v_+ \in L^{1,\gamma}(\mathbb{R}^+)(0 < \gamma \leq 1)$, then the convergence rates can be further improved as

$$\begin{aligned} \|\partial_x^k V(t)\|_{L^p} &\leq C(\lambda_4 + \|U_0\|_{L^1})(1+t)^{-(\frac{1}{2}-\frac{1}{2p})-\frac{k}{2}}, \quad k = 0,1,2, \\ \|\partial_x^k U(t)\|_{L^p} &\leq C(\lambda_4 + \|U_0\|_{L^1})(1+t)^{-(\frac{3}{2}-\frac{1}{2p})-\frac{k}{2}}, \quad k = 0,1. \end{aligned} \tag{2.44}$$

for $t \geq 0, 2 \leq p \leq \infty$.

Based on Theorem 2.4, we have the following decay properties of the solution $(V,U)(x,t)$ to the system (2.36).

Theorem 2.5 (Ma-Mei [18]) *Let $a \in [0, \frac{1}{2})$. Suppose the conditions in Theorem 2.4 hold, and in addition, $\sum_{k=0}^{2} \|\partial_x^k U_0\|_{2,a}^2 \ll 1$. Then the unique time-global solution $(V,U)(x,t)$ of (2.36) satisfies*

$$
\begin{aligned}
&\sum_{k=0}^{2}(1+t)^k\|\partial_x^k V(t)\|_{2,a}^2 + \sum_{k=0}^{1}(1+t)^{k+1}\|\partial_x^k U(t)\|_{2,a}^2 \\
&+ \int_0^t \Big\{ \sum_{k=1}^{2}(1+s)^{k-1}\|\partial_x^k V(s)\|_{2,a}^2 + \sum_{k=0}^{1}(1+s)^{k+1}\|\partial_x^k U(s)\|_{2,a}^2 \Big\} ds \\
&\le C.
\end{aligned}
\tag{2.45}
$$

Finally, we obtain much better decay rates as follows.

Theorem 2.6 (Ma-Mei [18]) *Let $a \in (0, \frac{1}{4}]$. Suppose the conditions in Theorem 2.4 and Theorem 2.5 hold. In addition, we assume that $v_0 - v_+ \in L^{1,1}(\mathbb{R}^+)$ and $U_0 \in L^{1,a}(\mathbb{R}^+)$. then the decay rates of the solution V to (2.36) can be further improved to be optimal as follows*

$$
\|\partial_x^k V(t)\| \le C(1+t)^{-\frac{2k+1}{4}-\frac{a}{2}}, \quad k = 0,1,2. \tag{2.46}
$$

From Theorem 2.4 and Theorem 2.6, noticing that $\|\partial_x^k(\hat{v},\hat{u})\|_{L^\infty} \le Ce^{-\alpha t}$, we can easily obtain the following decay properties for the solution $(v,u)(x,t)$ of (2.25) to the solution $(\bar{v},\bar{u})(x,t)$ of (2.26).

Corollary 2.7 (Ma-Mei [18]) *Under the conditions in Theorem 2.6, the system (2.25) possesses a unique time-global solution $(v,u)(x,t)$, which converges to its best asymptotic profile $(\bar{v},\bar{u})(x,t)$ defined in (2.26) and (2.33) in the form of*

$$
\begin{aligned}
\|(v-\bar{v}-\hat{v})(t)\| &\le C(1+t)^{-\frac{5}{4}-\frac{a}{2}}, \\
\|(v-\bar{v})(t)\|_{L^\infty} &\le C(1+t)^{-\frac{3}{2}-\frac{a}{4}}, \\
\|(u-\bar{u}-\hat{u})(t)\| &\le C(1+t)^{-\frac{7}{4}}, \\
\|(u-\bar{u})(t)\|_{L^\infty} &\le C(1+t)^{-2}.
\end{aligned}
\tag{2.47}
$$

Finally, we give a remark on the case $v_- \neq v_+$.

Remark 2.8 *For the case $v_- \neq v_+$, $\bar{v}(x,t)$ decays as*

$$
\|\partial_t^j \partial_x^k(\bar{v}-v_+)(t)\|_{L^p} = O(1)(1+t)^{-(1-1/p)/2-(2j+k-1)/2},
$$

even if $\bar{v}_0 - v_+ \in L^{1,1}(\mathbb{R}^+)$. As a result, we can only obtain the following decay properties for the solution $(v,u)(x,t)$ of (2.25),

$$
\begin{aligned}
\|(v-\bar{v}-\hat{v})\| &= O(1)(1+t)^{-\frac{3}{4}}, \\
\|(v-\bar{v})\|_{L^\infty} &= O(1)(1+t)^{-1}, \\
\|(u-\bar{u}-\hat{u})\| &= O(1)(1+t)^{-\frac{5}{4}}, \\
\|(u-\bar{u})\|_{L^\infty} &= O(1)(1+t)^{-\frac{3}{2}}.
\end{aligned}
$$

These rates are exactly same to those obtained by Marcati-Mei-Rubino [21] for the IBVPs.

3 $\beta \neq 0$: Damped p-System with Nonlinear Source

3.1 Initial Value Problem

We first look for the asymptotic profile to (1.1) and (1.2) with $\beta \neq 0$. By setting the following scalings to the variables

$$t = \bar{t}/\varepsilon^2, \quad x = \bar{x}/\varepsilon, \quad v = \bar{v}, \quad u = \varepsilon\bar{u}$$

for $0 < \varepsilon \ll 1$, we then scale the damped p-system (1.1) to the new system (still denote $\bar{t}$ and $\bar{x}$ as t and x, respectively)

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ \varepsilon^2\bar{u}_t + p(\bar{v})_x = -\alpha\bar{u} - \beta\varepsilon^{q-1}|\bar{u}|^{q-1}\bar{u}. \end{cases}$$

Neglecting the small terms $\varepsilon^2\bar{u}_t$ and $-\beta\varepsilon^{q-1}|\bar{u}|^{q-1}\bar{u}$, we derive the asymptotic state equations for (1.1) and (1.2) just same to (1.5), i.e.,

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\alpha\bar{u}. \end{cases}$$

Namely, the diffusion wave $(\bar{v}, \bar{u})(x/\sqrt{1+t})$ is our asymptotic profile for the damped p-system (1.1) with nonlinear source.

Now, we investigate $u(\pm\infty, t)$. Let $u^{\pm}(t) := u(\pm\infty, t) = \lim_{x\to\pm\infty} u(x,t)$. Taking the limits to the second equation of (1.1) as $x \to \pm\infty$, and noting that $p(v)_x$ will be vanishing, then we find that $u^{\pm}(t)$ satisfy formally the following modified Bernoulli's ODEs:

$$\begin{cases} \frac{d}{dt}u^{\pm}(t) = -\alpha u^{\pm}(t) - \beta|u^{\pm}(t)|^{q-1}u^{\pm}(t), \quad t > 0, \\ u^{\pm}(0) = u(\pm\infty, 0) = u_0(\pm\infty) = u_{\pm}. \end{cases} \tag{3.1}$$

Using the method of separation of variables, by a straightforward but tedious calculation, we can exactly solve (3.1) as

$$u^{\pm}(t) = \frac{C_{\pm}e^{-\alpha t}}{\left(1 - \frac{\beta}{\alpha}(|C_{\pm}|e^{-\alpha t})^{q-1}\right)^{\frac{1}{q-1}}}, \tag{3.2}$$

with

$$C_{\pm} = \frac{u_{\pm}}{(1 + \frac{\beta}{\alpha}|u_{\pm}|^{q-1})^{\frac{1}{q-1}}}. \tag{3.3}$$

Here, in order to avoid the blowing-up for the solution, we need

$$1 + \frac{\beta}{\alpha}|u_{\pm}|^{q-1} > 0. \tag{3.4}$$

Note that, when $\beta > 0$, the condition (3.4) automatically holds. While, when $\beta < 0$, (3.4) is also true if we ask

$$|u_{\pm}| < \left(\frac{\alpha}{|\beta|}\right)^{1/(q-1)},$$

which implies that $|u_{\pm}|$ needs to be suitably small. Thus, if $|u_{\pm}| \ll 1$, then (3.4) is always true, and there is no blowing-up for $u^{\pm}(t)$. Substituting (3.3) to (3.2), we obtain

$$u^{\pm}(t) = \frac{u_{\pm}e^{-\alpha t}}{\left(1 + \frac{\beta}{\alpha}|u_{\pm}|^{q-1}[1 - e^{-\alpha(q-1)t}]\right)^{\frac{1}{q-1}}}. \tag{3.5}$$

Obviously, it holds $|u(\pm\infty,t)| = |u^{\pm}(t)| \sim O(1)|u_{\pm}|e^{-\alpha t}$, as $t \to \infty$. Next is to construct the correction functions such that we can eliminate the gap of $u(+\infty,t) - u(-\infty,t)$. Let us consider the function $\hat{u}(x,t)$ such that

$$\begin{cases} \dfrac{d\hat{u}}{dt} = -\alpha\hat{u} - \beta|\hat{u}|^{q-1}\hat{u}, & x \in \mathbb{R},\ t \in \mathbb{R}_+, \\ \hat{u}(x,t) \to u^{\pm}(t) & \text{as } x \to \pm\infty. \end{cases} \tag{3.6}$$

As shown in (3.2), we can similarly solve (3.6) as

$$\hat{u}(x,t) = \frac{m(x)e^{-\alpha t}}{\left(1 - \frac{\beta}{\alpha}[|m(x)|e^{-\alpha t}]^{q-1}\right)^{\frac{1}{q-1}}}, \tag{3.7}$$

where $m(x)$ is an integration constant (with respect to t). Note that $\hat{u}(x,t) \to u^{\pm}(t)$ as $x \to \pm\infty$, we further confirm $m(x) \to C_{\pm}$, as $x \to \pm\infty$. Let $m_0(x) > 0$, $m_0(x) \in C_0^{\infty}(\mathbb{R})$ and $\int_{-\infty}^{\infty} m_0(x)dx = 1$, then we construct the desired function $m(x)$ as

$$m(x) := C_- + (C_+ - C_-)\int_{-\infty}^{x} m_0(y)dy. \tag{3.8}$$

It can be verified that $m(x)$ is sufficiently smooth and satisfies

$$|m(x)| \le \min\{|C_+|, |C_-|\} < \left(\frac{\alpha}{|\beta|}\right)^{\frac{1}{q-1}}, \tag{3.9}$$

which ensures no blowing-up for $\hat{u}(x,t)$.

Technically, we construct

$$\hat{v}(x,t) := -\frac{m'(x)e^{-\alpha t}}{\alpha\left(1 - \frac{\beta}{\alpha}[|m(x)|e^{-\alpha t}]^{q-1}\right)^{\frac{1}{q-1}}}, \tag{3.10}$$

we then have $\hat{v}_t = \hat{u}_x$. Thus, the constructed correction functions $(\hat{v},\hat{u})(x,t)$ satisfy

$$\begin{cases} \hat{v}_t - \hat{u}_x = 0, \\ \hat{u}_t = -\alpha\hat{u} - \beta\hat{u}^q. \end{cases} \tag{3.11}$$

Therefore, from (1.1), (1.5) and (3.11), we get

$$\begin{cases} (v - \bar{v} - \hat{v})_t - (u - \bar{u} - \hat{u})_x = 0, \\ (u - \bar{u} - \hat{u})_t + [p(v) - p(\bar{v})]_x = -\alpha(u - \bar{u} - \hat{u}) - \beta(|u|^{q-1}u - |\hat{u}|^{q-1}\hat{u}) - \bar{u}_t, \end{cases} \tag{3.12}$$

where $(\bar{v},\bar{u})$ is the shifted diffusion wave $(\bar{v},\bar{u})(x + x_0, t)$ with the shift x_0 which is specified as

$$x_0 := \frac{1}{v_+ - v_-}\int_{-\infty}^{\infty}[v_0(x) - \bar{v}(x,0) - \hat{v}(x,0)]dx. \tag{3.13}$$

Then, integrating (3.13) with respect to (x,t) over $R \times [0,t]$ yields

$$\int_{-\infty}^{\infty}[v(x,t) - \bar{v}(x + x_0,t) - \hat{v}(x,t)]dx = \int_{-\infty}^{\infty}[v_0(x) - \bar{v}(x + x_0,0) - \hat{v}(x,0)]dx = 0. \tag{3.14}$$

Thus, we can define

$$\begin{cases} V(x,t) := \int_{-\infty}^{x}[v(y,t) - \bar{v}(y+x_0,t) - \hat{v}(y,t)]dy, \\ z(x,t) := u(x,t) - \bar{u}(x+x_0,t) - \hat{u}(x,t), \end{cases} \tag{3.15}$$

and

$$\begin{cases} V_0(x) := \int_{-\infty}^{x}[v_0(y) - \bar{v}(y+x_0,0) - \hat{v}(y,0)]dy, \\ z_0(x) := u_0(x) - \bar{u}(x+x_0,0) - \hat{u}(x,0), \end{cases} \tag{3.16}$$

we deduce (3.12) into

$$\begin{cases} V_t - z = 0, \\ z_t + (p'(\bar{v})V_x)_x = -\alpha z - F_1 - F_2, \\ (V,z)|_{t=0} = (V_0,z_0)(x), \end{cases} \tag{3.17}$$

where

$$F_1 : = -\frac{1}{\alpha}p(\bar{v})_{xt} + \{p(V_x + \bar{v} + \hat{v}) - p(\bar{v}) - p'(\bar{v})V_x\}_x, \tag{3.18}$$

$$F_2 : = g(z + \bar{u} + \hat{u}) - g(\hat{u}) = g(V_t + \bar{u} + \hat{u}) - g(\hat{u}), \tag{3.19}$$

$$g(u) : = \beta|u|^{q-1}u. \tag{3.20}$$

Theorem 3.1 (Mei [24]) *Let $q > \frac{5}{2}$, $(V_0, z_0)(x)$ be in $H^3(\mathbb{R}) \times H^2(\mathbb{R})$, and $u_\pm$ satisfy*

$$|u_\pm| < \left(\frac{\alpha}{|\beta|}\right)^{1/(q-1)}. \tag{3.21}$$

There exists a number $\varepsilon_1 > 0$, when the initial perturbation and $\delta := |v_+ - v_-| + |u_+| + |u_-|$ are suitably small such that $\delta + \|V_0\|_{H^3(\mathbb{R})} + \|z_0\|_{H^2(\mathbb{R})} \le \varepsilon_1$, then the global solution $(V,z)(x,t)$ of (3.17) uniquely exists and satisfies

$$V(x,t) \in C^k(0,\infty; H^{3-k}(\mathbb{R})), \quad k = 0,1,2,3, \quad z(x,t) \in C^k(0,\infty; H^{2-k}(\mathbb{R})), \quad k = 0,1,2,$$

and

$$\begin{aligned} &\sum_{k=0}^{3}(1+t)^k\|\partial_x^k V(t)\|^2_{L^2(\mathbb{R})} + \sum_{k=0}^{2}(1+t)^{k+2}\|\partial_x^k z(t)\|^2_{L^2(\mathbb{R})} \\ &+ \int_0^t \Big[\sum_{k=0}^{3}(1+s)^{k-1}\|\partial_x^k V(s)\|^2_{L^2(\mathbb{R})} + \sum_{k=0}^{2}(1+s)^{k+1}\|\partial_x^k z(s)\|^2_{L^2(\mathbb{R})}\Big] ds \\ &\le C(\|V_0\|^2_{H^3(\mathbb{R})} + \|z_0\|^2_{H^2(\mathbb{R})} + \delta). \end{aligned} \tag{3.22}$$

Furthermore, if $(V_0, z_0) \in L^1$, (3.22) can be improved as the following optimal convergence rates

$$\|\partial_x^k V(t)\|_{L^2(\mathbb{R})} \le C(\|V_0\|_3^2 + \|z_0\|_2^2 + \delta)(1+t)^{-\frac{1}{4}-\frac{k}{2}}, \quad k = 0,1,2,3, \tag{3.23}$$

$$\|\partial_x^k z(t)\|_{L^2(\mathbb{R})} \le C(\|V_0\|_3^2 + \|z_0\|_2^2 + \delta)(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \quad k = 0,1,2. \tag{3.24}$$

Corollary 3.2 (Mei [24]) *Under the conditions in Theorem 3.1, the system* (1.1) *and* (1.2) *possesses a uniquely global solution* $(v, u)(x, t)$, *which converges to its nonlinear diffusion wave* $(\bar{v}, \bar{u})(x + x_0, t)$ *in the form of*

$$\begin{aligned} \|(v - \bar{v})(t)\|_{L^\infty(\mathbb{R})} &= O(1)(1+t)^{-1}, &(3.25)\\ \|(u - \bar{u})(t)\|_{L^\infty(\mathbb{R})} &= O(1)(1+t)^{-3/2}. &(3.26)\end{aligned}$$

The rates showed in (3.25) *and* (3.26) *are optimal.*

Remark 3.3 *When* $\beta < 0$ *and*

$$|u_\pm| > \left(\frac{\alpha}{|\beta|}\right)^{1/(q-1)}, \tag{3.27}$$

then the solution $(v, u)(x, t)$ *of* (1.1) *and* (1.2) *doses not globally exist, and blows up. In fact, let us consider the following Cauchy problem*

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u - \beta|u|^{q-1}u, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+. \\ (v,u)|_{t=0} = (v_+, u_+), \end{cases}$$

Obviously, it possesses the unique solution

$$\begin{cases} v(x,t) = v_+, \\ u(x,t) = u_+ e^{-\alpha t}\Big(1 + \frac{\beta}{\alpha}|u_+|^{q-1}[1 - e^{-\alpha(q-1)t}]\Big)^{-1/(q-1)}, \end{cases}$$

and $v(x,t) = v_+$ *is never blowing-up, but* $u(x,t)$ *will blow up at* $t_* = \frac{1}{\alpha(q-1)} \ln \frac{|\beta||u_+|^{q-1}}{|\beta||u_+|^{q-1} - \alpha}$ *for* $\beta < 0$ *and* $|\beta| > \frac{\alpha}{|u_+|^{q-1}}$.

3.2 Initial-Boundary Value Problem

In this subsection, we consider the following initial-boundary value problem

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u - \beta|u|^{q-1}u, \end{cases} \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{3.28}$$

with the initial-boundary conditions

$$\begin{cases} (v,u)|_{t=0} = (v_0, u_0)(x) \to (v_+, u_+) \ \text{ as } x \to +\infty, \ x \in \mathbb{R}_+, \\ u|_{x=0} = 0. \end{cases} \tag{3.29}$$

Its best asymptotic profile is expected as the following IBVP to the porous media equation

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\alpha\bar{u}, \\ \bar{v}|_{t=0} = \bar{v}_0(x) \to v_+, \ \text{ as } x \to \infty, \\ \bar{v}_x|_{x=0} = 0, \end{cases} \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{3.30}$$

where $\bar{v}_0(x)$ will be specified.

From the second equation of (3.28), the solution $u(+\infty,t)$ (denoted as $u^+(t)$) satisfies the following Bernoulli's equation

$$\begin{cases} \frac{d}{dt}u^+(t) = -\alpha u^+(t) - \beta|u^+(t)|^{q-1}u^+(t), \\ u^+(0) = u(+\infty,0) = u_0(+\infty) = u_+, \end{cases}$$

which can be solved explicitly as

$$u(+\infty,t) = u^+(t) = \frac{u_+e^{-\alpha t}}{\left(1+\frac{\beta}{\alpha}|u_+|^{q-1}[1-e^{-\alpha(q-1)t}]\right)^{1/(q-1)}}. \tag{3.31}$$

Notice that, when

$$\beta < 0 \ \text{ and } \ |\beta| > \frac{\alpha}{|u_+|^{q-1}}, \tag{3.32}$$

the solution $u^+(t)$ will blow up at $t_* = \frac{1}{\alpha(q-1)}\ln\frac{|\beta||u_+|^{q-1}}{|\beta||u_+|^{q-1}-\alpha}$. So, in order to guarantee the global existence of $u^+(t)$, we need

$$\text{either } \beta > 0, \ \text{ or } \ \beta < 0 \ \text{ but } \ |\beta| < \frac{\alpha}{|u_+|^{q-1}}. \tag{3.33}$$

Since there is a gap between $u(\infty,t)$ and $\bar{u}(\infty,t)=0$, namely,

$$u(\infty,t) - \bar{u}(\infty,t) = u^+(t) - 0 = O(1)|u_+|e^{-\alpha t},$$

which causes that $u-\bar{u}$ is not in $L^2(\mathbb{R}_+)$, thus we need to construct the correction function $\hat{u}(x,t)$ to delete it.

Let $\hat{u}(x,t)$ be such that

$$\begin{cases} \frac{d}{dt}\hat{u} = -\alpha\hat{u} - \beta|\hat{u}|^{q-1}\hat{u}, \ (x,t)\in\mathbb{R}_+\times\mathbb{R}_+, \\ \hat{u}|_{x=\infty} = u^+(t), \\ \hat{u}_x|_{x=0} = 0. \end{cases} \tag{3.34}$$

Similarly, $\hat{u}(x,t)$ can be constructed as

$$\hat{u}(x,t) = \frac{m(x)e^{-\alpha t}}{\left(1-\frac{\beta}{\alpha}[|m(x)|e^{-\alpha t}]^{q-1}\right)^{1/(q-1)}}, \tag{3.35}$$

where $m(x)$ is an integration constant (with respect to t) given by $m(x) = C_+\int_0^x m_0(y)dy$, $m(0)=0$, $m(+\infty)=C_+$. Here,

$$C_+ = \frac{u_+}{\left(1+\frac{\beta}{\alpha}|u_+|^{q-1}\right)^{1/(q-1)}}, \tag{3.36}$$

and $m_0(x)$ satisfies $m_0(x)\geq 0$, $m_0(0)=m_0(+\infty)=0$, $m_0(x)\in C_0^\infty(\mathbb{R}_+)$, and $\int_{\mathbb{R}_+} m_0(x)dx = 1$. Furthermore, let $\hat{v}(x,t)$ be

$$\hat{v}(x,t) = \frac{-m'(x)e^{-\alpha t}}{\alpha\left(1-\frac{\beta}{\alpha}[|m(x)|e^{-\alpha t}]^{q-1}\right)^{1/(q-1)}}. \tag{3.37}$$

Thus, the correction functions $(\hat{v},\hat{u})(x,t)$ satisfy

$$\begin{cases} \hat{v}_t - \hat{u}_x = 0, \\ \hat{u}_t = -\alpha\hat{u} - \beta|\hat{u}|^{q-1}\hat{u}, \\ (\hat{v},\hat{u})|_{x=+\infty} = (0, u^+(t)), \\ \hat{v}|_{x=0} = 0, \\ \hat{u}_x|_{x=0} = 0. \end{cases} \tag{3.38}$$

From $(v-\bar{v}-\hat{v})_t - (u-\bar{u}-\hat{u})_x = 0$, it yields

$$\int_0^\infty [v(x,t)-\bar{v}(x,t)-\hat{v}(x,t)]dx = \int_0^\infty [v_0(x)-\bar{v}_0(x)-\hat{v}(x,0)]dx = 0 \tag{3.39}$$

by selecting the initial data $\bar{v}_0(x)$ as

$$\int_0^\infty [v_0(x)-\bar{v}_0(x)-\hat{v}(x,0)]dx = 0. \tag{3.40}$$

Thus, we can define some possible L^2-functions as

$$(V,U)(x,t) : = \Big(-\int_x^\infty [v(y,t)-\bar{v}(y,t)-\hat{v}(y,t)]dy, u(x,t)-\bar{u}(x,t)-\hat{u}(x,t)\Big), \tag{3.41}$$

$$(V_0,U_0)(x) : = \Big(-\int_x^\infty [v_0(y)-\bar{v}(y,0)-\hat{v}(y,0)]dy, u_0(x)-\bar{u}(x,0)-\hat{u}(x,0)\Big) \tag{3.42}$$

then, from (3.28), (1.5) and (3.38), we can reformulate the system as

$$\begin{cases} V_t - U = 0, \\ U_t + (p'(\bar{v})V_x)_x + \alpha U = -F_1 - F_2, \\ (V,U)|_{t=0} = (V_0,U_0)(x), \\ V|_{x=0} = 0, \end{cases} \quad (x,t)\in\mathbb{R}_+\times\mathbb{R}_+, \tag{3.43}$$

where

$$F_1 : = \frac{1}{\alpha}p(\bar{v})_{xt} + (p(V_x+\bar{v}+\hat{v}) - p(\bar{v}) - p'(\bar{v})V_x)_x, \tag{3.44}$$

$$\begin{aligned} F_2 : &= \beta|U+\bar{u}+\hat{u}|^{q-1}(U+\bar{u}+\hat{u}) - \beta|\hat{u}|^{q-1}\hat{u} \\ &= \beta|V_t+\bar{u}+\hat{u}|^{q-1}(V_t+\bar{u}+\hat{u}) - \beta|\hat{u}|^{q-1}\hat{u}. \end{aligned} \tag{3.45}$$

Theorem 3.4 (Lin-Lin-Mei [17]) *Let β and u_+ satisfy* (3.33), $q\ge 2$, *and $\bar{v}_0(x)$ be chosen such that* (3.40) *holds, and* $\int_0^\infty[\bar{v}_0(x)-v_+]dx = 0$, $\bar{v}_0(x)-v_+ \in L^1(\mathbb{R}_+)\cap H^m(\mathbb{R}_+)$ *with* $m\ge 3$.

1. *If* $(V_0,U_0)\in H^3(\mathbb{R}_+)\times H^2(\mathbb{R}_+)$, *when* $\max_{x\in\mathbb{R}_+}|\bar{v}_0-v_+| + \|V_0\|_{H^3} + \|U_0\|_{H^2} + |u_+| \ll 1$, *then the global solution* $(V,U)(x,t)$ *of* (3.43) *uniquely exists and satisfies*

$$V(x,t)\in\bigcap_{k=0}^{2} C^k(0,\infty;H^{3-k}(\mathbb{R})), \quad U(x,t)\in\bigcap_{k=0}^{1} C^k(0,\infty;H^{2-k}(\mathbb{R})),$$

and

$$\|\partial_x^k V(t)\|_{L^2} = O(1)(1+t)^{-k/2}, \quad k=0,1,2,3, \tag{3.46}$$

$$\|\partial_x^k U(t)\|_{L^2} = O(1)(1+t)^{-(k+2)/2}, \quad k=0,1, \tag{3.47}$$

$$\|\partial_x^k V(t)\|_{L^\infty} = O(1)(1+t)^{-(2k+1)/4}, \quad k=0,1,2, \tag{3.48}$$

$$\|U(t)\|_{L^\infty} = O(1)(1+t)^{-5/4}. \tag{3.49}$$

2. If $(V_0,U_0)\in(L^1(\mathbb{R}_+)\cap H^2(\mathbb{R}_+))\times(L^1(\mathbb{R}_+)\cap H^1(\mathbb{R}_+))$, then

$$\|\partial_x^k V(t)\|_{L^2}=O(1)(1+t)^{-(2k+1)/4},\quad k=0,1,2, \tag{3.50}$$
$$\|U(t)\|_{L^2}=O(1)(1+t)^{-5/4}, \tag{3.51}$$
$$\|\partial_x^k V(t)\|_{L^\infty}=O(1)(1+t)^{-(k+1)/2},\quad k=0,1, \tag{3.52}$$
$$\|U(t)\|_{L^\infty}=O(1)(1+t)^{-3/2}. \tag{3.53}$$

3. If $(V_0,U_0)\in(L^{1,\gamma}(\mathbb{R}_+)\cap H^2(\mathbb{R}_+))\times(L^{1,\gamma}(\mathbb{R}_+)\cap H^1(\mathbb{R}_+))$, where $\gamma=\frac14$, then

$$\|\partial_x^k V(t)\|_{L^2}=O(1)(1+t)^{-\frac{2k+1}{4}-\frac{\gamma}{2}},\quad k=0,1,2, \tag{3.54}$$
$$\|U(t)\|_{L^2}=O(1)(1+t)^{-\frac54-\frac{\gamma}{2}}, \tag{3.55}$$
$$\|\partial_x^k V(t)\|_{L^\infty}=O(1)(1+t)^{-\frac{k+1}{2}-\frac{\gamma}{2}},\quad k=0,1, \tag{3.56}$$
$$\|U(t)\|_{L^\infty}=O(1)(1+t)^{-\frac32}. \tag{3.57}$$

Corollary 3.5 (Lin-Lin-Mei [17]) *Under the conditions in Theorem 2.4, and $(V_0,U_0)(x)\in L^1(\mathbb{R}_+)$, it holds*

$$\|(v-\bar v)(t)\|_{L^\infty} = O(1)(1+t)^{-1}, \tag{3.58}$$
$$\|(u-\bar u)(t)\|_{L^\infty} = O(1)(1+t)^{-3/2}. \tag{3.59}$$

Furthermore, $(V_0,U_0)(x)\in L^{1,\gamma}(\mathbb{R}_+)$ with $\gamma=\frac14$, it holds

$$\|(v-\bar v)(t)\|_{L^\infty} = O(1)(1+t)^{-1-\frac{\gamma}{2}}=O(1)(1+t)^{-\frac98}, \tag{3.60}$$
$$\|(u-\bar u)(t)\|_{L^\infty} = O(1)(1+t)^{-\frac32}. \tag{3.61}$$

Remark 3.6 *From Theorem 3.4 and Corollary 3.5, we get the convergence rates as*

$$\|\partial_x^k V(t)\|_{L^2}=O(1)(1+t)^{-\frac{2k+1}{4}-\frac{\gamma}{2}},\quad k=0,1,2,$$

with the best choice of $\gamma=\frac14$ for $q\geq 2$, which are much better than the existing rates. But, unfortunately we cannot improve $\|U(t)\|_{L^\infty}=\|V_t(t)\|_{L^\infty}=O(1)t^{-\frac32}$ to $O(1)t^{-\frac32-\frac{\gamma}{2}}$ due to the slow decay of $\bar v_{xt}$ in the nonlinear term. These results are also true for the case $\beta=0$, namely, the system (1.1) becomes the linear damping. We notice also that, when $\beta=0$ Said-Houari [34] claimed that he got some better decay rates, for $\gamma\in[0,1]$,

$$\|\partial_x^k V(t)\|_{L^2}=O(1)(1+t)^{-\frac{2k+1}{4}-\frac{\gamma}{2}},\quad k=0,1,2,$$
$$\|\partial_x^k U(t)\|_{L^2}=O(1)(1+t)^{-\frac{2k+3}{4}-\frac{\gamma}{2}},\quad k=0,1,$$

especially, the case of $\frac14<\gamma\leq 1$. However, this is not true, and his proof is wrong. He never checked how the nonlinear term decays, in particular, the term involving $\bar v_{xt}$ in the nonlinear term doesn't give any improved rates in $L^{1,r}(R_+)$, because $\bar v(x,t)$ is the corresponding porous media equation with the Nuemann boundary condition, and the improved rate in the weighted $L^{1,r}(R_+)$ obtained by Ikehata [12] for the Cauchy problem case is failed to the Nuemann boundary case. In another word, the decay rates of the nonlinear term doesn't not decay as faster as we always expect.

Remark 3.7 *When the parameters β and u_+ satisfy* (3.32)*, namely, $\beta < 0$ and $|\beta| > \frac{\alpha}{|u_+|^{q-1}}$, from* (3.31)*, $u(+\infty, t)$ will blow up at the finite time t_*. Thus, the solution $u(x,t)$ of* (3.28) *and* (3.29) *doses not globally exist, and*

$$\lim_{t \to T^*-} \|u(t)\|_{L^\infty} = +\infty, \quad \textit{for } 0 < T^* \leq t_*. \tag{3.62}$$

References

[1] C. Dafermos, A system of hyperbolic conservation laws with frictional damping, *Z. Angew. Math. Phys.*, **46** (1995), 294–307.

[2] C. T. Duyn and L. A. Van Peletier, A class of similarity solutions of the nonlinear diffusion equation, *Nonlinear Anal.*, **1** (1977), 223–233.

[3] L. Hsiao and T.-P. Liu, Convergence to diffusion waves for solutions of a system of hyperbolic conservation laws with damping, *Commun. Math. Phys.*, **143** (1992), 599–605.

[4] L. Hsiao and T.-P. Liu, Nonlinear diffusion phenomena of nonlinear hyperbolic system, *Chinese Annals Math. B*, **14** (1993), 465–480.

[5] L. Hsiao and T. Luo, Nonlinear diffusion phenomena of solutions for the system of compressible adiabatic flow through porous media, *J. Differential Equations*, **125** (1996), 329–365.

[6] L. Hsiao and T. Luo, Nonlinear diffusive phenomena of entropy weak solutions for a system of quasilinear hyperbolic conservation laws with damping, *Quat. Appl. Math.* **56** (1998), 173–189.

[7] L. Hsiao and R. Pan, Initial boundary value problem for the system compressible adiabatic flow through porous media, *J. Differential Equations*, **159** (1999), 380–305.

[8] L. Hsiao and S. Q. Tang, Construction and qualitative behavior of the solution of the perturbed Riemann problem for the system of one-dimensional isentropic flow with damping, *J. Differential Equations*, **123** (1995), 480–503.

[9] F.M. Huang, P. Marcati and R.H. Pan, Convergence to the Barenblatt solution for the compressible Euler equations with damping and vacuum, *Arch. Ration. Mech. Anal.* **176** (2005) 1–24.

[10] F.M Huang, M. Mei and Y. Wang, Convergence to planar waves for multi-dimensional damped Euler equations with nonlinear source, to appear.

[11] F.M. Huang and R.H. Pan, Convergence rate for compressible Euler equations with damping and vacuum, *Arch. Ration. Mech. Anal.* **166** (2003) 359–376.

[12] R. Ikehata, New decay estimates for linear damped wave equations and its application ro nonlinear problem, *Math. Methods Appl. Sci.*, **27** (2004), 865–889.

[13] M. N. Jiang and C. J. Zhu, Convergence rates to nonlinear diffusion waves for p-system with nonlinear damping on quadrant, *Discrete and Continuous Dynamical Systems*, Series A, **23** (2009) 887–918.

[14] M. N. Jiang and C. J. Zhu, Convergence to strong nonlinear diffusion waves for solutions to p-system with damping on quadrant, *J. of Differential Equations*, **246** (2009), 50–77.

[15] H.-L. Li and K. Saxton, Asymptotic behavior of solutions to quasilinear hyperbolic equations with nonlinear damping, *Quart. Appl. Math.*, **61** (2003), 295–313.

[16] J. Liao, W. Wang and T. Yang, L^p convergence rates of planar waves for multi-dimensional Euler equations with damping, *Journal of Differential Equations*, **247** (2009), 303–329.

[17] C.-K. Lin, C.-T. Lin and M. Mei, Asymptotic behavior of solution to nonlinear damped p-system with boundary effect, to appear.

[18] H.-F. Ma and M. Mei, Best asymptotic profile for linear damped p-system with boundary effect, to appear.

[19] P. Marcati and A. Milani, The one-dimensional Darcy's law as the limit of a compressible Euler flow, *J. Differential Equations*, **84** (1990) 129–147.

[20] P. Marcati and M. Mei, Convergence to nonlinear diffusion waves for solutions of the initial boundary problem to the hyperbolic conservation laws with damping, *Quart. Appl. Math.*, **56** (2000), 763–784.

[21] P. Marcati, M. Mei and B. Rubino, Optimal convergence rates to diffusion waves for solutions of the hyperbolic conservation laws with damping, *J. Math. Fluid Mech.*, **7** (2005), supplement, S224-S240.

[22] A. Matsumura, On the asymptotic behavior of solutions of semi-linear wave equation, *Publ. RIMS Kyoto Univ.*, **12** (1976), 169–189.

[23] A. Matsumura, Global existence and asymptotics of the solutions of the second-order quasilinear hyperbolic equation with the first-order dissipation, *Publ. RIMS Kyoto Univ.*, **13** (1977), 349–379.

[24] M. Mei, Nonlinear diffusion waves for hyperbolic p-system with nonlinear damping, *J. Differential Equations*, **247** (2009), 1275–1269.

[25] M. Mei, Best asymptotic profile for hyperbolic p-system with damping, *SIAM J. Math. Anal.*, to appear.

[26] T. Nishida, Nonlinear hyperbolic equations and related topics in fluid dynamics, Publications Mathematiques D'Ocsay 78.02, Department de mathematique, Paris-Sud, 1978.

[27] K. Nishihara, Convergence rates to nonlinear diffusion waves for solutions of system of hyperbolic conservation laws with damping, *J. Differential Equations*, **131** (1996), 171–188.

[28] K. Nishihara, Asymptotic behavior of solutions of quasilinear hyperbolic equations with linear damping, *J. Differential Equations*, **137** (1997), 384–396.

[29] K. Nishihara, Asymptotics toward the diffusion wave for a one-dimensional compressible flow through porous media, *Proc. Roy. Soc. Edinburgh*, Sect. A **133** (2003), 177–196.

[30] K. Nishihara and M. Nishikawa, Asymptotic behavior of solutions to the system of compressible adiabatic flow through porous media, *SIAM J. Math. Anal.*, **33** (2001), 216–239.

[31] K. Nishihara and T. Yang, Boundary effect on asymptotic behavior of solutions to the p-system with linear damping, *J. Differential Equations*, **156** (1999), 439–458.

[32] K. Nishihara, W. K. Wang and T. Yang, L_p-convergence rates to nonlinear diffusion waves for p-system with damping, *J. Differential Equations*, **161** (2000), 191–218.

[33] R.H. Pan, Darcy's law as long-time limit of adiabatic porous media flow, *J. Differential Equations*, **220** (2006) 121–146.

[34] B. Said-Houari, Convergence to strong nonlinear diffusion waves for solutions to p-system with damping, *J. Differential Equations*, **247** (2009), 917–930.

[35] D. Serre and L. Xiao (L. Hsiao), Asymptotic behavior of large weak entropy solutions of the damped P-system, *J. Partial Differential Equations*, **10** (1997), 355–368.

[36] W.K. Wang and T. Yang, Pointwise estimates and L_p convergence to diffusion waves for p-system with damping, *J. Differential Equations*, **187** (2003) 310–336.

[37] H. Zhao, Convergence to strong nonlinear diffusion waves for solutions of p-system with damping, *J. Differential Equations*, **174** (2001), 200–236.

[38] H. Zhao, Asymptotic behavior of solutions of quasilinear hyperbolic equations with damping II, *J. Differential Equations*, **167** (2000), 467–494.

[39] C. J. Zhu, Convergence rates to nonlinear diffusion waves for weak entropy solutions to p-system with damping, *Sciences in China*, Series A, **46** (2003), 562–575.

[40] C. J. Zhu and M. N. Jiang, L^p-decay rates to nonlinear diffusion waves for p-system with nonlinear damping, *Sciences in China*, Series A, **49** (2006), 721–739.