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Asymptotic behavior of solutions for the damped wave equation with absorbing semilinear term

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1 Introduction

In this note we consider the Cauchy problems for the wave equations with space- or time-dependent damping and asborbing semilinear term

$$u_{tt}-\Delta u+(|x|^{2}+1)^{-\frac{\alpha}{2}}u_{t}+|u|^{\rho-1}u=0, \quad (t,x)\in\mathbf{R}_{+}\times\mathbf{R}^{N}, \tag{1.1}$$

and

$$u_{tt}-\Delta u+(t+1)^{-\beta}u_{t}+|u|^{\rho-1}u=0, \quad (t,x)\in\mathbf{R}_{+}\times\mathbf{R}^{N}, \tag{1.2}$$

with data

$$u(0,x)=u_{0}(x), \quad u_{t}(0,x)=u_{1}(x), \quad x\in\mathbf{R}^{N}. \tag{1.3}$$

Here, $\rho>1$, and $0\leq\alpha,\beta<1$ are constants, and the initial data in (1.3) are assumed to be in $H^{1}\times L^{2}$ with compact support. Note that the semilinear term works as absorbing, and the smallness of the data is not assumed.

When $\alpha=\beta=0$, the coefficient of damping is constant, and the solution of the Cauchy problem of (1.1) or (1.2) is expected to behave as that of the corresponding diffusion equations:

(i) In the supercritical case $\rho>\rho_{F}(N):=1+\frac{2}{N}$, the solution u behaves like $\theta_{0}G(t,x)$ as $t\rightarrow\infty$ for a suitable constant θ_{0} and the Gauss kernel $G(t,x)=(4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^{2}}{4t}}$, which is the fundamental solution of the corresponding linear parabolic equation

$$\phi_{t}-\Delta\phi=0.$$

(ii) In the critical case $\rho=\rho_{F}(N)$, the solution behaves like the approximate Gauss kernel $G(t,x)(\log t)^{-\frac{N}{2}}$.

(iii) In the subcritical case $\rho<\rho_{F}(N)$, the solution u behaves like the self-similar solution $w(t,x):=(t+1)^{\frac{\rho-1}{2}}f(|x|/\sqrt{t+1})$ of the corresponding semilinear parabolic equation

$$\phi_{t}-\Delta\phi+|\phi|^{\rho-1}\phi=0.$$

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In fact, several cases have been investigated, but we here devote ourselves to (1.1) and (1.2), not necessarily $\alpha = \beta = 0$. When α and β are small, the similar situations are expected to (i)-(iii) above. Actually we show the optimal or almost optimal decay properties of solutions in the supercritical and subcritical cases provided that $0 \le \alpha, \beta < 1$. In the special case we obtain the asymptotic profile. The proofs are mainly given by the L^2-energy method with suitable weights.

2 Time-dependent damping case

In this section we treat the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t + |u|^{\rho-1}u = 0, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ (u, u_t)(0,x) = (u_0, u_1)(x), & x \in \mathbf{R}^N, \end{cases} \tag{2.1}$$

where

$$b(t) = (t+1)^{-\beta}, \quad 0 \le \beta < 1 \tag{2.2}$$

and $(u_0, u_1) \in H^1 \times L^2$ are compactly supported. For the corresponding linear equation

$$v_{tt} - \Delta v + b(t)v_t = 0, \tag{2.3}$$

Wirth [11, 12] showed the followings by the Fourier transformation. When $-1 < \beta < 1$, the damping is effective and the solution of (2.3) decays as $t \to \infty$ with rate

$$\|v(t)\|_{L^p} = O(B(t)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}) = O(t^{-\frac{(1+\beta)N}{2}(\frac{1}{q}-\frac{1}{p})}), \quad B(t) = \int_0^t \frac{1}{b(\tau)}d\tau \tag{2.4}$$

for $1 \le q \le 2 \le p \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and the data in L^q with suitable regularity. When $-1/3 < \beta < 1$, the solution has, more precisely, the diffusion phenomena. When $\beta > 1$, the damping is non-effective, or the solution has the wave property. See also Yamazaki [13, 14] for the abstract setting.

Based on these results, we consider (2.1) with (2.2), when the diffusion phenomena is expected. The corresponding linear parabolic equation is

$$-\Delta\phi + b(t)\phi_t = 0 \quad \text{or} \quad \phi_t = \frac{1}{b(t)}\Delta\phi, \tag{2.5}$$

whose solution with $\phi(0,x) = \phi_0(x)$ is given by

$$\phi(t,x) = \int_{\mathbf{R}^N} (4\pi B(t))^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4B(t)}} \phi_0(y)dy. \tag{2.6}$$

Hence, if $\phi_0 \in L^q$, then the L^p-L^q estimate

$$\|\phi(t)\|_{L^p} \le C\|\phi_0\|_{L^q} t^{-\frac{(1+\beta)N}{2}(\frac{1}{q}-\frac{1}{p})} \tag{2.7}$$

holds for $1 \leq q \leq p \leq \infty$. On the other hand, the corresponding nonlinear parabolic equation is

$$b(t)\phi_t - \Delta\phi + |\phi|^{\rho-1}\phi = 0. \tag{2.8}$$

When $(1 <)\rho < 1 + 2/N$, (2.8) has the self-similar solution of the form

$$w_0(t,x) = (c+ct)^{-\frac{1+\beta}{\rho-1}} F\left(\frac{x}{(c+ct)^{\frac{1+\beta}{2}}}\right), \tag{2.9}$$

where $c^{1+\beta}(1+\beta) = 1$ and

$$\Delta F + \frac{c^{1+\beta}(1+\beta)}{2} y \cdot \nabla F + \frac{c^{1+\beta}(1+\beta)}{\rho - 1} F = |F|^{\rho-1}F, \quad \lim_{|x|\to\infty} |x|^{\frac{2}{\rho-1}} F(x) = 0. \tag{2.10}$$

Note that

$$\|w_0(t,\cdot)\|_{L^p} = O(t^{-(\frac{1}{\rho-1} - \frac{N}{2p})(1+\beta)}). \tag{2.11}$$

Compared (2.11) and (2.7) with $q = 1$, we can expect that the critical exponent is

$$\rho_F(N) := 1 + \frac{2}{N} \text{ (Fujita exponent)},$$

even in the time-dependent damping case. In fact, we have the following theorems.

Theorem 2.1 (Nishihara [6]) *Assume* $1 < \rho < \frac{N+2}{[N-2]_+} = \begin{cases} \infty & (N = 1,2) \\ \frac{N+2}{N-2} & (N \geq 3) \end{cases}$, *(2.2) and* $(u_0, u_1) \in H^1 \times L^2$ *whose supports are compact. Then the following assertions hold.*

(i) the weak solution $u \in C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$ *to (2.1) satisfies the decay properties*

$$(t+1)^{\frac{(1+\beta)(N+2)}{2} - \varepsilon} E(t) + (t+1)^{\frac{(1+\beta)N}{2} - \varepsilon} \int_{\mathbf{R}^N} e^{2\psi} u^2 \, dx \leq C I_0^2, \tag{2.12}$$

where $\psi(t,x) = \frac{(1+\beta)|x|^2}{4(2+\delta)(t+1)^{1+\beta}}$ $(0 < \forall\delta \ll 1)$, $\varepsilon = \varepsilon(\delta) > 0$ *satisfying* $\varepsilon(\delta) \to 0$ *as* $\delta \to 0$,

$$E(t) = \int_{\mathbf{R}^N} e^{2\psi}(u_t^2 + |\nabla u|^2 + |u|^{\rho+1}) \, dx \tag{2.13}$$

and

$$I_0^2 = \int_{\mathbf{R}^N} e^{2\psi(0,x)}(u_1^2 + |\nabla u_0|^2 + |u_0|^{\rho+1} + u_0^2) \, dx < \infty. \tag{2.14}$$

(ii) Moreover, assume $N = 1$, $\rho_F(1) = 3 < \rho < \infty$ *and* $(u_0, u_1) \in H^2 \times H^1$. *Then, it follows that, for* $p \geq 1$

$$\|u(t,\cdot) - \theta_0 G_B(t,\cdot)\|_{L^p} = o(t^{-\frac{1+\beta}{2}(1-\frac{1}{p})}), \tag{2.15}$$

where

$$\begin{aligned} \theta_0 = &\int_{\mathbf{R}^1} (u_1 + (1-\beta)u_0) \, dx \\ &+ \int_0^\infty \int_{\mathbf{R}^N} (\beta(1-\beta)(\tau+1)^{-(2-\beta)}u - (\tau+1)^\beta |u|^{\rho-1}u)(\tau,x) \, dx \, d\tau. \end{aligned} \tag{2.16}$$

Theorem 2.2 (Nishihara and Zhai [7]) *Assume* $1 < \rho < \frac{N+2}{[N-2]_+}$, *(2.2) and* $(u_0, u_1) \in H^1 \times L^2$ *whose supports are compact. Then the time-global solution* u *to (2.1) satisfies*

$$\int_{\mathbf{R}^N} e^{2\psi} u(t,x)^2 \, dx \le C I_0^2 (t+1)^{-(1+\beta)(\frac{2}{\rho-1}-\frac{N}{2})} \tag{2.17}$$

with $\psi(t,x) = \frac{a|x|^2}{(t+t_0)^{1+\beta}}$ $(0 < a \ll 1,\ t_0 \gg 1)$ *and (2.14).*

Both Theorem 2.1 and 2.2 are available for $1 < \rho < \frac{N+2}{[N-2]_+}$. But Theorem 2.1 is effective in the supercritical exponent. The decay rate of u in L^2 is almost same as that of $G_B(t,x)$ and almost optimal. When $N = 1$, $\theta_0 G_B(t,x)$ is an asymptotic profile. The asymptotic profile for $N \ge 2$ is not obtained yet. While Theorem 2.2 is effective in the subcritical exponent. The decay rate of u in L^2 is as same as the self-similar solution. Though the self-similar is expected to be an asymptotic profile, it remains open.

The proofs of Theorem 2.1 (i) and Theorem 2.2 are given by the L^2-energy mehtod with suitable weights, originally developed in Todorova and Yordanov [9]. For Theorem 2.1 (ii), $\int_{\mathbf{R}^N} u(t,x)\,dx$ heuristically tends to θ_0 as $t \to \infty$ and hence $\theta_0 G_B(t,x)$ is expected to be an asymptotic profile of the solution. By (2.6), the solution (2.1) is regarded as that of the integral equation

$$u(t,x) = \int_{\mathbf{R}^N} G_B(t, x-y) u_0(y)\,dy + \int_0^t \int_{\mathbf{R}^N} G_B(t-\tau, x-y) f(\tau, y; u)\,dy\,d\tau \tag{2.18}$$

with $f(t,x;u) = -\frac{1}{b(t)}(|u|^{\rho-1}u + u_{tt})(t,x)$. To show (2.15) with (2.16), we need the suitable decay estimate of u_{tt}. When $N = 1$, we get the L^∞-estimate of u by (2.12)-(2.13) and the Sobolev inequality, so that the estimates of higher derivatives of u are obtained by the energy method, and the proof of (2.15) will be done by the estimate of (2.18).

3 Space-dependent damping case

In this section we consider

$$\begin{cases} u_{tt} - \Delta u + \langle x \rangle^{-\alpha} u_t + |u|^{\rho-1} u = 0, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ (u, u_t)(0,x) = (u_0, u_1)(x), & x \in \mathbf{R}^N \end{cases} \tag{3.1}$$

with $\langle x \rangle = \sqrt{1+|x|^2}$. When $\alpha > 1$, Mochizuki [3] showed that the solution have the wave property. So, assume

$$0 \le \alpha < 1, \tag{3.2}$$

then the diffusion phenomena is expected. The energy method with suitable weight can still be applied and the decay estimates of the solution are obtained, similarly to the time-dependent damping case.

Theorem 3.1 (Nishihara [4]) *Assume* $1 < \rho < \frac{N+2}{[N-2]_+}$, *(2.2) and* $(u_0, u_1) \in H^1 \times L^2$ *whose supports are compact. Put*

$$\rho_c(N,\alpha) = 1 + \frac{2}{N-\alpha}, \quad \rho_{subc}(N,\alpha) = 1 + \frac{2\alpha}{N-\alpha}. \tag{3.3}$$

Then, the weak solution u *to (3.1) with (3.2) decays as* $t \to \infty$ *with its rates*

$$\|u(t)\|_{L^2} = \begin{cases} O(t^{-\frac{N-2\alpha}{2(2-\alpha)}+\varepsilon}) & \rho_c(N,\alpha) \le \rho < \frac{N+2}{[N-2]_+} \\ O(t^{-\frac{2}{2-\alpha}(\frac{1}{\rho-1}-\frac{N}{4})}) & \rho_{subc}(N,\alpha) < \rho \le \rho_c(N,\alpha) \\ O(t^{-\frac{2}{2-\alpha}(\frac{1}{\rho-1}-\frac{N}{4})}(\log t)^{\frac{1}{2}}) & \rho = \rho_{subc}(N,\alpha) \\ O(t^{-\frac{1}{\rho-1}+\frac{\alpha}{2(2-\alpha)}}) & 1 < \rho < \rho_{subc}(N,\alpha) \end{cases} \tag{3.4}$$

for any small $\varepsilon > 0$.

We believe that our decay rates (3.4) are optimal or almost optimal. But, we do not know how to obtain the asymptotic profile or the optimality. Because, in the space-dependent damping case we cannot apply the Fourier transformation method nor the explicit formula like (2.18). Hence we cannot say that $\rho_c(N,\alpha)$ and $\rho_{subc}(N,\alpha)$ in (3.3) are exactly critical. When $\alpha \to 0$,

$$\rho_c(N,\alpha) \to \rho_F(N) \quad \text{and} \quad \rho_{subc}(N,\alpha) \to 1.$$

Formally, put $\alpha = 0$ in (3.4), then the decay rates correspond to those of the solution to (3.1) with $\alpha = 0$. Also,

$$\phi_\alpha(t,x) = A(t+1)^{-\frac{N-\alpha}{2-\alpha}} e^{-\frac{|x|^{2-\alpha}}{(2-\alpha)^2(t+1)}}$$

is an exact solution to $-\Delta\phi + |x|^{-\alpha}\phi_t = 0$, and its L^2-norm decays with rate

$$\|\phi_\alpha(t)\|_{L^2} = O(t^{-\frac{N-2\alpha}{2(2-\alpha)}}),$$

which is almost same as the decay rate in the "supercritical" exponent in (3.4). These facts may be circumstantial evidences that $\rho_c(N,\alpha)$ and $\rho_{subc}(N,\alpha)$ are exactly critical.

The same problem was investigated by Todorova and Yordanov [10], in which there is a small misprint and their "critical" exponents are reduced to ours after a correction (Private communications with the authors).

Theorem 3.1 is shown as corollaries of the following two theorems in [4].

Theorem 3.2 *Assume* $1 < \rho < \frac{N+2}{[N-2]_+}$ *and* $(u_0, u_1) \in H^1 \times L^2$ *with compact support. Then, there exists a unique solution* $u \in C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$ *to (3.1), which satisfies*

$$\int_{\mathbf{R}^N} e^{2\psi}\langle x\rangle^{-\alpha} u^2(t,x)\,dx \le \begin{cases} CI_0^2(t+1)^{-2m_1}, & \rho > \rho_{subc}(N,\alpha) \\ CI_0^2(t+1)^{-2m_2}\log(t+2), & \rho = \rho_{subc}(N,\alpha) \\ CI_0^2(t+1)^{-2m_2}, & \rho < \rho_{subc}(N,\alpha), \end{cases} \tag{3.5}$$

$$E(t) \le \begin{cases} CI_0^2(t+1)^{-2m_1-1}, & \rho > \rho_{subc}(N,\alpha) \\ CI_0^2(t+1)^{-2m_2-1}\log(t+2), & \rho = \rho_{subc}(N,\alpha) \\ CI_0^2(t+1)^{-2m_2-1}, & \rho < \rho_{subc}(N,\alpha), \end{cases} \tag{3.6}$$

where $E(t)$ is given in (2.13) with $\psi(t,x) = \frac{\langle x\rangle^{2-\alpha}}{16(t+1)}$,

$$I_0^2 = 1 + \int_{\mathbf{R}^N} e^{2\psi(0,x)}\{\langle x\rangle^\alpha(u_1^2 + |\nabla u_0|^2 + |u_0|^{\rho+1})(x) + \langle x\rangle^{-\alpha}u_0^2(x)\}\,dx < \infty \tag{3.7}$$

and

$$m_1 = \frac{2}{2-\alpha}(\frac{1}{\rho-1} - \frac{N-\alpha}{4}), \quad m_2 = \frac{1}{\rho-1}. \tag{3.8}$$

Theorem 3.3 *Under the conditions same as in Theorem 3.2, the solution $u(t,x)$ to (3.1) satisfies*

$$\int_{\mathbf{R}^N} e^{2\psi}\langle x\rangle^{-\alpha}u^2(t,x)\,dx \le CI_0^2(t+1)^{-\frac{N-\alpha}{2-\alpha}+\varepsilon}, \tag{3.9}$$

$$\int_{\mathbf{R}^N} e^{2\psi}(u_t^2 + |\nabla u|^2 + |u|^{\rho+1})(t,x)\,dx \le CI_0^2(t+1)^{-\frac{N-\alpha}{2-\alpha}-1+\varepsilon} \tag{3.10}$$

with $\psi = \frac{1}{(2-\alpha-\delta)^2}\frac{\langle x\rangle^{2-\alpha}}{(t+1)}$ $(0 < \forall\delta \ll 1)$ and $\varepsilon = \frac{N-\alpha}{2-\alpha}(1 - (\frac{2-\alpha}{2-\alpha+\delta})^2) > 0$.

Since

$$\langle x\rangle^{-\alpha} = (\frac{\langle x\rangle^{2-\alpha}}{t+1})^{-\frac{\alpha}{2-\alpha}} \cdot (t+1)^{-\frac{\alpha}{2-\alpha}}$$

and $e^y y^{-\frac{\alpha}{2-\alpha}} \ge c\,(y > 0)$, both (3.5) and (3.9) yield the decay rates (3.4), after simple calculations.

4 Basic weighted energy estimates

For the proofs of theorems in Sections 2-3 the weighted energy method is used. But, we need many calculations which are simple but tedious. Since we treat the case that the solution of (2.1) or (3.1) may have the diffusion phenomena, the solution behaves like that of the corresponding linear parabolic equation in the supercritical exponent, while that of the corresponding nonlinear parabolic equation in the subcritical exponent. But, we cannot use the strong tool in the parabolic problems like the maximum principle etc. In particular, we don't know the useful methods for (3.1) except for the energy method. Therefore, if we just return back to the beginning, then we will face to the problems. These are whether we can get the suitable estimates of the solution to the linear parabolic equation

$$u_t - \Delta u = 0, \quad u(0,x) = u_0(x), \tag{4.1}$$

and the nonlinear parabolic equation

$$u_t - \Delta u + |u|^{\rho-1}u = 0, \quad u(0,x) = u_0(x), \tag{4.2}$$

using only the weighted energy method, not the Fourier transformation nor the Gauss kernel.

Thus, in this section we treat the simplist problems (4.1) and (4.2). Assuming

(A) when $|x| \to \infty$, $u(t,x)$ and $u_0(x)$ decay sufficiently fast,

we assert by the weighted energy method that

Craim I. the solution $u(t,x)$ to (4.1) satisfies

$$\|u(t)\|_{L^2} = O(t^{-\frac{N}{4}}) \quad \text{as} \quad t \to \infty, \tag{4.3}$$

Craim II. the solution $u(t,x)$ to (4.2) satisfies

$$\|u(t)\|_{L^2} = O(t^{-(\frac{1}{\rho-1}-\frac{N}{4})}) \quad \text{as} \quad t \to \infty. \tag{4.4}$$

Note that the assumption (A) is available in our problems (2.1) and (3.1) provided that the data are compactly supported. Dependent on the problems, the weight $\psi(t,x)$ will be chosen suitably, and similar process to the proofs of Craim I and II yields the proofs of Theorems, though the calculations are much more complicated. Details are referred to [4, 6, 7]. The problems for wave equations with time- or space-dependent damping are also investigated in [1, 2, 5, 8].

Proof of Craim I. To show (4.3) we derive the differential inequality

$$\frac{d}{dt}E(t) + \frac{N/2}{t+1}E(t) \leq 0 \tag{4.5}$$

for some $E(t) \geq 0$. Because, we easily have

$$E(t) \leq E(0)(t+1)^{-N/2} \quad \text{or} \quad E(t)^{1/2} = O(t^{-\frac{N}{4}})$$

by (4.5). We now multiply (4.1) by $2e^{2\psi}u$ to get

$$(e^{2\psi}u^2)_t - 2\nabla\cdot(e^{2\psi}u\nabla u) + 2\Big[e^{2\psi}(-\psi_t)u^2 + \underbrace{e^{2\psi}2\nabla\psi\cdot u\nabla u}_{(*)} + e^{2\psi}|\nabla u|^2\Big] = 0. \tag{4.6}$$

Here, choose $\psi = \frac{a|x|^2}{t+1}$ $(a > 0)$, then

$$-\psi_t = \frac{a|x|^2}{(t+1)^2}, \quad \nabla\psi = \frac{2ax}{t+1} \tag{4.7}$$

and hence

$$-\psi_t = \frac{1}{4a}|\nabla\psi|^2 \quad\text{and}\quad \Delta\psi = \frac{2aN}{t+1}. \tag{4.8}$$

Regarding as $E(t) = \int_{\mathbf{R}^N} e^{2\psi}u^2(t,x)\,dx$, if we simply change $(*)$ in (4.6) to

$$(*) = \nabla\cdot(e^{2\psi}u^2\nabla\psi) - e^{2\psi}2|\nabla\psi|^2u^2 - e^{2\psi}(\Delta\psi)u^2,$$

then the sign of the last two terms are not good. So, after changing $(*)$ to

$$(*) = e^{2\psi}4\nabla\psi\cdot u\nabla u \underbrace{-e^{2\psi}2\nabla\psi\cdot u\nabla u}_{(**)},$$

we change $(**)$ to

$$(**) = -\nabla\cdot(e^{2\psi}u^2\nabla\psi) + e^{2\psi}2|\nabla\psi|^2u^2 + e^{2\psi}(\Delta\psi)u^2.$$

Then, (4.6) becomes

$$\begin{aligned}&(e^{2\psi}u^2)_t - 2\nabla\cdot(e^{2\psi}u\nabla u + e^{2\psi}u^2\nabla\psi)\\ &+2e^{2\psi}\Big[\underbrace{(-\psi_t + 2|\nabla\psi|^2)}_{(\frac{1}{4a}+2)|\nabla\psi|^2}u^2 + 4u\nabla\psi\cdot\nabla u + |\nabla u|^2\Big] + e^{2\psi}\underbrace{(2\Delta\psi)}_{\frac{4aN}{t+1}}u^2 = 0.\end{aligned} \tag{4.9}$$

Taking $a = 1/8$, integrating (4.8) over $\mathbf{R}^N$ and using (A), we have

$$\frac{d}{dt}\int_{\mathbf{R}^N} e^{2\psi}u^2\,dx + 2\int_{\mathbf{R}^N} e^{2\psi}|2u\nabla\psi + \nabla u|^2dx + \frac{N/2}{t+1}\int_{\mathbf{R}^N} e^{2\psi}u^2\,dx = 0, \tag{4.10}$$

which implies (4.3) and Craim I.

Proof of Craim II. For Craim II we derive

$$\frac{d}{dt}E(t) + H(t) \le 0 \tag{4.11}$$

for $E(t), H(t) \ge 0$, and hence

$$\frac{d}{dt}(t+1)^kE(t) + (t+1)^k(H(t) - \frac{k}{t+1}E(t)) \le 0. \tag{4.12}$$

Then we show, for some $K > 0$

$$H(t) - \frac{k}{t+1}E(t) \ge -C(t+1)^{-K}. \tag{4.13}$$

If we have (4.11)-(4.13), then the choise of $k = K - 1 + \gamma\,(\forall\gamma > 0)$ yields

$$\frac{d}{dt}(t+1)^{K-1+\gamma}E(t) \le C(t+1)^{-1+\gamma} \quad\text{and}\quad E(t) \le C(t+1)^{-(K-1)}.$$

We now multiply (4.2) by $2e^{2\psi}u$ and use (4.7)-(4.8) to get

$$(e^{2\psi}u^2)_t - 2\nabla\cdot(e^{2\psi}u\nabla u)+2e^{2\psi}\Big[\underbrace{(-\psi_t)}_{\frac{1}{4a}|\nabla\psi|^2}u^2+\underbrace{2u\nabla u\cdot\nabla\psi}_{\geq -|\nabla u|^2-|\nabla\psi|^2u^2}+|\nabla u|^2+|u|^{\rho+1}\Big]=0.$$

Hence, taking $a \leq 1/16$, we have

$$\frac{d}{dt}\int e^{2\psi}u^2\,dx+\int e^{2\psi}\Big[|\nabla\psi|^2u^2+|u|^{\rho+1}\Big]\,dx\leq 0, \tag{4.14}$$

which is the form of (4.11). Multiplying (4.14) by $(t+1)^k$, we reach to

$$\frac{d}{dt}(t+1)^k\int_{\mathbf{R}^N}e^{2\psi}u^2\,dx+(t+1)^k\int_{\mathbf{R}^N}\underbrace{e^{2\psi}\Big[|\nabla\psi|^2u^2+|u|^{\rho+1}-\frac{k}{t+1}u^2\Big]}_{(\#)}\,dx\leq 0.$$

We decompose the integrand $\mathbf{R}^N$ to $\Omega := \{\frac{4a^2|x|^2}{t+1}\geq k\}$ and $\Omega^c=\{|x|\leq\sqrt{\frac{k(t+1)}{4a^2}}\}$, then clearly $\int_\Omega(\#)\,dx\geq 0$, because of $|\nabla\psi|^2=\frac{4a^2|x|^2}{(t+1)^2}$. Since $\frac{2}{\rho+1}+\frac{\rho-1}{\rho+1}=1$,

$$-\frac{k}{t+1}u^2\geq -|u|^{\rho+1}-C(t+1)^{-\frac{\rho+1}{\rho-1}}.$$

Hence,

$$\int_{\Omega^c}(\#)\,dx\geq -C\int_{\Omega^c}(t+1)^{-\frac{\rho+1}{\rho-1}}dx\geq -C(t+1)^{-\frac{\rho+1}{\rho-1}+\frac{N}{2}},$$

which means $K=\frac{\rho+1}{\rho-1}-\frac{N}{2}$. Thus we obtain

$$\int_{\mathbf{R}^N}e^{2\psi}u^2\,dx\leq C(t+1)^{-(\frac{\rho+1}{\rho-1}-\frac{N}{2}-1)}=C(t+1)^{-(\frac{2}{\rho-1}-\frac{N}{2})},$$

which implies (4.4) and Craim II.

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