数理解析研究所講究録
第1693巻 2010年 168-179

Interface evolution by tristable Allen–Cahn type equation with collision free condition

明治大学先端数理科学インスティチュート　大塚 岳 (Takeshi Ohtsuka)
Meiji Institute for advanced study of Mathematical Sciences,
Meiji University

1. Introduction

In this paper we discuss on the singular limit of the tristable Allen-Cahn type equation of the form

$$u_t - \Delta u + \frac{f_0(u) + \varepsilon f_1(u)}{\varepsilon^2} = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, T), \tag{1.1}$$

$$u|_{t=0} = g \quad \text{on} \quad \mathbb{R}^N. \tag{1.2}$$

For $f_0, f_1 \in C^2(\mathbb{R})$ and $g \in BUC(\mathbb{R}^N)$ we assume that

(F1) either $f(u) = f(-u)$ or $f(u+1) = f(u)$ holds for $u \in (-1, 0)$,

(F2) there exist $a_0 \in (-1, 0)$ and $a_1 \in (0, 1)$ such that $f_0(-1) = f_0(a_0) = f_0(0) = f_0(a_1) = f_0(1) = 0$,

(F3) there exists $R > 1$ such that $f_0 > 0$ in $(-1, a_0) \cup (0, a_1) \cup (1, R)$ and $f_0 < 0$ in $(-R, -1) \cup (a_0, 0) \cup (a_1, 1)$,

(F4) $f_0'(k) > 0$ for $k = -1, 0, 1$, and $f_0'(a_k) < 0$ for $k = 0, 1$,

(F5) $\int_{-1}^0 f_0(u)du = \int_0^1 f_0(u)du = 0$,

(F6) $\int_{-1}^0 f_1(u)du \le \int_0^1 f_1(u)du$.

(G1) $\inf_{\mathbb{R}^N} g < b_0$, $\sup_{\mathbb{R}^N} g > b_1$,

(G2) there exists $\bar{\delta} > 0$ such that $\lambda_k(\delta) = \sup\{g(x);\ d_0^k(x) < -\delta\}$ and $\Lambda_k(\delta) = \inf\{g(x);\ d_0^k(x) > \delta\}$ are monotone decreasing and increasing for $\delta \in (0, \bar{\delta})$ and $k = 0, 1$, respectively, where d_0^k is the signed distance function of $\Gamma_0^k := \{x;\ g(x) = b_k\}$ defined as

$$d_0^k(x) := \begin{cases} \operatorname{dist}(x, \Gamma_0^k) & \text{if } x \in \{y \in \mathbb{R}^N;\ g(y) \ge b_k\}, \\ -\operatorname{dist}(x, \Gamma_0^k) & \text{if } x \in \{y \in \mathbb{R}^N;\ g(y) < b_k\}, \end{cases} \tag{1.3}$$

and $\operatorname{dist}(x, U) := \inf\{|x - y|;\ y \in U\}$ for $U \subset \mathbb{R}^N$.

The typical example of f_0 is

$$f_0(u) = \frac{d}{du} u^2(u-1)^2(u+1)^2 = 2u(u-1)(u+1)(3u^2-1),$$
$$f_0(u) = \frac{d}{du}\frac{1-\cos(2\pi u)}{2\pi} = \sin(2\pi u).$$

The equation (1.1) is the L^2 gradient flow of the following energy form

$$\mathcal{E}(u) = \int_{\mathbb{R}^N} \left[\frac{|\nabla u|^2}{2} + \frac{F_0(u) + \varepsilon F_1(u)}{\varepsilon^2}\right] du,$$

where $F_0 = \int f_0$ and $F_1 = \int f_1$. The assumptions (F2)–(F4) imply that $F_\varepsilon(u) := F_0(u) + \varepsilon F_1(u)$ has three local minima at $\alpha_k = k + O(\varepsilon)$ for $k = -1, 0, 1$, and two local maxima at $\beta_k = b_k + O(\varepsilon)$ for $k = 0, 1$ as $\varepsilon \to 0$, respectively. Thus, from analogy to the Allen–Cahn equation and (G1), one can find three stable equilibria expressed by the region satisfying $u \approx k$ for $k = -1, 0, 1$, and two evolving internal transition layers around $\{(x,t) \in \mathbb{R}^N \times (0,T);\ u(x,t) = \beta_k\}$ for $k = 0, 1$. By formal asymptotic analysis as in [10] or [11] the layers approximate the motion of interfaces evolving by

$$V = -H + A_k, \tag{1.4}$$

where V is the normal velocity of the interface, H is its mean curvature defined with the opposite normal vector for V, and A_k is the constant as

$$A_k = -C\int_{k-1}^{k} f_1(u)du,$$

where C is the numerical constant determined only on f_0, which is in particular independent of k. See also [9] for the details of asymptotic analysis.

Our aim is to give a rigorous convergence result of internal transition layers to the interfaces evolving by (1.4), in particular when A_k are different but satisfy (F6), i.e.,

$$A_0 \geq A_1. \tag{1.5}$$

The dynamics of internal transition layers for Allen–Cahn equation, which is the L^2 gradient flow of E with bistable potential F_ε, is studied by [10] with formal asymptotic analysis. Rigorous convergence results of layers to interfaces evolving by (1.4) are shown by [3, 2, 1]. Asymptotic analysis for Allen–Cahn type equation with multiple-well potential is given by [11] and [9]. The rigorous convergence result for Allen–Cahn equation with multiple-well potential is obtained by [8] when f_0 and f_1 is periodic with same periods.

Our problem is its extension with removing assumptions of periodicity for f_1. The crucial difference is that the driving forces A_k for internal transition layers depend on k, and then interfaces or internal transition layers may collide with each other. However, we are now assume (1.5). The important property obtained from above is that evolving interfaces Γ_t^k by (1.4) do not collide with each other if Γ_t^0 is on the outside of Γ_t^1. In this case we can investigate the motion of layers similarly as [8]. However we need each interfaces evolving (1.4) with $k = 0, 1$ to know the motion of layers. It is one of differences from the result of [8].

In the next section we consider the motion of interfaces by (1.4) in level set sense. Our situation includes the situation that interfaces are not compact. Thus we have to treat viscosity solutions to level set equation for (1.4) carefully. In the third section we shall sketch the proof of the convergence result. Throughout this paper we simply write

$$\{x \in \mathbb{R}^N;\ v(x,t) = \gamma\} = \{v(\cdot,t) = \gamma\},$$
$$\{(x,t) \in \mathbb{R}^N \times (0,\infty);\ v(x,t) = \gamma\} = \{v(\cdot,\cdot) = \gamma\} \quad (\text{or } \{v = \gamma\})$$

for $v\colon \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ and $\gamma \in \mathbb{R}$ for the simplicity of notations. Similarly we express $\{v(\cdot,t) \geq 0\}$, $\{v \geq 0\}$ and other inequalities. The second set on the above does not include $t = 0$, and accordingly we especially write as

$$\{v = \gamma\} \subset \mathbb{R}^N \times [0,\infty) \quad \text{or} \quad \{v = \gamma\} \subset \mathbb{R}^N \times (0,T)$$

if we have to clarify the time interval of the sets. We also denote the internal transition layer evolving by (1.1) or interfaces evolving by (1.4) just by layer or interface for simplicity.

2. Level set equations

In this section we construct target interfaces evolving by (1.4) with level set method for the convergence of internal transition layers by a solution of (1.1). We also prepare some properties of the interfaces.

In the level set method we describe the evolving interfaces Γ_t^k by (1.4) as

$$\Gamma_t^k := \{w^k(\cdot,t) = 0\} \tag{2.1}$$

with an at least continuous function $w^k\colon \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$. Here we give the direction of the motion by

$$\vec{\mathbf{n}}_k = -\frac{\nabla w^k}{|\nabla w^k|}.$$

Then, the level set equation of (1.4) is of the form

$$w_t^k - |\nabla w^k| \left\{ \operatorname{div} \frac{\nabla w^k}{|\nabla w^k|} + A_k \right\} = 0 \quad \text{in } \mathbb{R}^N \times (0,T). \tag{2.2}$$

(See [4] for the details.) In this paper we intend to prove that internal transition layers in (1.1)–(1.2) approximate the evolving interfaces Γ_t^k with initial interfaces

$$\Gamma_0^k := \{g = b_k\}.$$

Here we do not assume any compactness for initial interfaces Γ_0^k. Thus we have to consider the comparison principle for viscosity solutions for the problem in unbounded domain, which is key lemma to estimate the solution to (2.2) or (1.1). We now recall simple version of Theorem 2.1 in [5] adjusting to our problems.

Lemma 2.1. **([5, Theorem 2.1])** *Let u and v be, respectively, viscosity sub- and supersolution of* (2.2) *in $\mathbb{R}^N \times (0,T)$. Assume that*

(A1) $u(x,t) \le K(|x|+1)$, $v(x,t) \ge -K(|x|+1)$ for some $K > 0$ independent of $(x,t) \in \mathbb{R}^N \times (0,T)$;

(A2) there is a modulus m_T such that

$$u^*(x,0) - v_*(y,0) \le m_T(|x-y|) \quad \textit{for } (x,y) \in \mathbb{R}^{2N};$$

(A3) $u^*(x,0) - v_*(y,0) \le K(|x-y|+1)$ on $\mathbb{R}^{2N}$ for some $K > 0$ independent of $(x,y) \in \mathbb{R}^{2N}$.

Then there is a modulus m such that

$$u^*(x,t) - v_*(y,t) \le m(|x-y|) \quad \textit{for } (x,y,t) \in \mathbb{R}^{2N} \times (0,T].$$

In particular $u^* \le v_*$ on $\mathbb{R}^N \times (0,T]$.

The difference between the above and the usual comparison principle is the additional conditions (A1) and (A3). For not only the uniqueness of solutions but also the properties of interfaces evolving (1.4) with $k=0$ and $k=1$, we now sketch the construction of a viscosity solution w^k to (2.2) satisfying (A1) and (A3).

Let us choose an initial data for w^k as

$$w^k|_{t=0} = d_0^k \quad \text{on } \mathbb{R}^N, \tag{2.3}$$

where d_0^k is defined as (1.3). The basic strategy of the construction is by Perron's method due to H. Ishii. (See [6].) In the method the solution is given by

$$w^k(x,t) = \sup\left\{ z(x,t);\ \begin{array}{l} z \text{ is a viscosity subsolution to (2.2),} \\ \phi(x,t) \leq z(x,t) \leq \psi(x,t) \end{array} \right\}$$

with a viscosity sub- and super-solution ϕ and ψ satisfying $\phi(\cdot,0) = \psi(\cdot,0) = d_0^k$, respectively.

Here we construct only ψ since the construction of ϕ is similar. Note that d_0^k satisfies

$$|d_0^k(x) - d_0^k(y)| \leq |x-y| \leq \mu + \frac{1}{4\mu}|x-y|^2 \quad \text{for } (x,y) \in \mathbb{R}^{2N},\ \mu > 0.$$

Then, we now introduce

$$\begin{aligned} \tilde{v}^+_{y,\mu}(x,t) &= \frac{1}{2\mu}(N - 1 + 4|A_k|) + \frac{1}{4\mu}|x-y|^2 + \mu, \\ \bar{v}^+_{y,\mu}(x,t) &= \frac{1}{2\mu}(N - 1 + 4|A_k|) + |x-y| + \mu + \frac{1}{\mu} - 2. \end{aligned}$$

All the coefficients in the above functions are chosen by technical reason to satisfy all the following properties;

(i) $\tilde{v}^+_{y,\mu}$ is a viscosity supersolution to (2.2) for $(x,t) \in B_4(y) \times (0,T]$, where $B_r(y) := \{x \in \mathbb{R}^N;\ |x-y| < r\}$,

(ii) $\bar{v}^+_{y,\mu}$ is a viscosity supersolution to (2.2) for $(x,t) \in (\mathbb{R}^N \backslash \overline{B_{1/2}(y)}) \times (0,T]$ provided that $\mu < 1/4$,

(iii) $\tilde{v}^+_{y,\mu} < \bar{v}^+_{y,\mu}$ on $B_2(y) \times [0,T]$,

(iv) $\tilde{v}^+_{y,\mu} > \bar{v}^+_{y,\mu}$ on $(\mathbb{R}^N \setminus \overline{B_2(y)}) \times [0,T]$.

We now introduce

$$v_{y,\mu}(x,t) := \begin{cases} \tilde{v}_{y,\mu}(x,t) & \text{on } B_1(y) \times [0,T], \\ \min\{\tilde{v}_{y,\mu}(x,t), \bar{v}_{y,\mu}(x,t)\} & \text{on } (B_3(y) \setminus B_1(y)) \times [0,T], \\ \bar{v}_{y,\mu}(x,t) & \text{otherwise.} \end{cases}$$

Then $v_{y,\mu}$ is a viscosity supersolution of (2.2) in $\mathbb{R}^N \times (0,T]$ by stability of viscosity solutions provided that $\mu \in (0,1/4)$. Consequently, the function

$$\psi(x,t) := \inf\{v_{y,\mu}(x,t) + d_0^k(y);\ y \in \mathbb{R}^N,\ \mu \in (0,1/4)\}$$

is still a viscosity supersolution of (2.2) satisfying $\psi(x,0) = d_0^k(x)$ for $x \in \mathbb{R}^N$. The viscosity subsolution ϕ is constructed by $\tilde{v}_{y,\mu}^-(x,t) := -\tilde{v}_{y,\mu}^+(x,t)$, $\bar{v}_{y,\mu}^-(x,t) := -\bar{v}_{y,\mu}^+(x,t)$, and their supremum.

From the definition of ϕ and ψ we find

$$\phi(x,t) \geq -\left(|x| + Lt + \mu + \frac{1}{\mu} - 2\right),$$
$$\psi(x,t) \leq |x| + Lt + \mu + \frac{1}{\mu} - 2$$

for $(x,t) \in \mathbb{R}^N \times [0,T]$, where $L = (N - 1 + 4|A_k|)/\mu$. This implies (A1) and (A3). Moreover, from the Lipschitz continuity and [5, Corollary 2.11] we have

$$|w^k(x,t) - w^k(y,t)| \leq |x - y| \quad \text{for } x, y \in \mathbb{R}^N \text{ and } k = 0, 1.$$

The above and (F6) yield that interfaces $\Gamma_t^k := \{x \in \mathbb{R}^N;\ w^k(x,t) = 0\}$ do not collide with each other for $t > 0$.

Lemma 2.2. **([9, Lemma 3.1])** *Assume that* $A_0 \geq A_1$. *Let* w^k *be a viscosity solution to* (2.2) *with* (2.3). *Let*

$$U_t^k := \{w^k(\cdot,t) > 0\}.$$

Then $U_t^1 \subset U_t^0$ *for* $t \in [0,T]$. *Moreover,*

$$\operatorname{dist}(\Gamma_t^0, \Gamma_t^1) \geq \operatorname{dist}(\Gamma_0^0, \Gamma_0^1) > 0 \quad \textit{for } t \in [0,T],$$

where Γ_t^k *is given by* (2.1) *for* $k = 0, 1$.

Here we omit the proof of Lemma 2.2. See [9] for the detail of the proof.

3. Convergence

We intend to prove the convergence of internal transition layers in a solution to (1.1)–(1.2) to the interfaces evolving (1.4) with level set formulation.

In [3] the authors show the convergence results for the usual Allen–Cahn equation in $\mathbb{R}^N \times [0,\infty)$ with a special initial datum which is constructed by a standing wave and the signed distance from initial interface. However, it is not clear to find such an initial datum in our problem. Thus we shall prove the convergence under general initial data.

To state our main result we now prepare some notations. Let w^k be a viscosity solution to (2.2)–(2.3). We now denote "interfaces", "insides" and "outsides" of interfaces by Γ, I and O, respectively which are defined as

$$\Gamma_t^k := \{w^k(\cdot,t)=0\}, \qquad \Gamma^k := \bigcup_{t>0} \Gamma_t^k \times \{t\},$$
$$I_t^k := \{w^k(\cdot,t)>0\}, \qquad I^k := \bigcup_{t>0} I_t^k \times \{t\},$$
$$O_t^k := \{w^k(\cdot,t)<0\}, \qquad O^k := \bigcup_{t>0} O_t^k \times \{t\}.$$

Theorem 3.1. *Assume that (F1)–(F6), (G1)–(G2) hold. Let* u *be a solution to* (1.1)–(1.2). *Then,*

$$u \longrightarrow \left\{ \begin{array}{ccc} -1 & in & O^0 \cap O^1 \\ 0 & in & I^0 \cap O^1 \\ +1 & in & I^0 \cap I^1 \end{array} \right\} \text{ locally uniformly as } \varepsilon \to 0.$$

The our main result has a kind of advantages against to [3]'s one such that the convergence result holds for general initial data. However, the convergence does not hold at $t = 0$ (see the definitions of I^k and O^k.) It is because of general initial data.

We now sketch the proof. The strategy of the proof is similar to [8]. More precisely, it is in two steps expressed by the following lemmas. The first step is to show the behavior of a solution u to (1.1)–(1.2) that traveling fronts appear in very short time.

Lemma 3.2. (Generation of fronts.) *Let* u *be a solution of* (1.1)–(1.2). *Assume that (F1)–(F4). Then, for any* $\mu > 0$ *and* $m > 0$, *there exists* $\bar{\varepsilon} = \bar{\varepsilon}(\mu, m)$ *and* $\tau_0 = \tau_0(\mu)$ *such that*

$$\begin{array}{ll} u(x,\tau_0\varepsilon^2|\log\varepsilon|) \geq \alpha_{-1} - \mu\varepsilon & \text{for } x \in \mathbb{R}^N, \\ u(x,\tau_0\varepsilon^2|\log\varepsilon|) \leq \alpha_1 + \mu\varepsilon & \text{for } x \in \mathbb{R}^N, \\ u(x,\tau_0\varepsilon^2|\log\varepsilon|) \leq \alpha_{k-1} + \mu\varepsilon & \text{for } x \in \{y \in \mathbb{R}^N;\ g(y) \leq b_k - m\}, \\ u(x,\tau_0\varepsilon^2|\log\varepsilon|) \geq \alpha_k - \mu\varepsilon & \text{for } x \in \{y \in \mathbb{R}^N;\ g(y) \geq b_k + m\} \end{array}$$

provided that $\varepsilon \in (0, \bar{\varepsilon})$, *where* $\hat{\alpha} = \max\{|\alpha_{-1}|, |\alpha_1|\}$.

We are now assume (G1) without periodicity like as [8] so that we cannot apply [8, Theorem 3.1]. However, if $g \in [-1,0]$ or $g \in [0,1]$ in $\mathbb{R}^N$, then we obtain the above estimate by applying the method as in [2, §3] or [8, §3] with a little adjustment. Thus we modify their method to adjust to our problem. More precisely, we give the modification $\bar{f}_\varepsilon$ of f_ε as in [2] or [8] in the both domain $(-\infty,0]$ and $(0,\infty)$. To apply the similar argument as in [8, §3] independently in $\{g \le b_0 - m\}$ and $\{g \le b_1 - m\}$, then we obtain Lemma 3.2.

The second step is to construct a supersolution stated the following lemma for the estimate of the convergence.

Lemma 3.3. **(Large wave solution.)** *Assume that (F1)-(F6) and (G1) hold. Then, there exist $K_k > 0$ for $k = -1, 0, 1$ which is independent of ε such that for any $\delta > 0$ there exists a viscosity supersolution $\psi^{\varepsilon,\delta}$ to (1.1) satisfying*

$$\begin{aligned}\psi^{\varepsilon,\delta}(x,0) \ge &(\alpha_{-1} + \varepsilon K_{-1})\chi_{\{d_0^0 \le 2\delta\}}(x) \\ &+ (\alpha_0 + \varepsilon K_0)\chi_{\{d_0^1 \le 2\delta\}\setminus\{d_0^0 \le 2\delta\}}(x) \\ &+ (\alpha_1 + \varepsilon K_1)\chi_{\{d_0^1 > 2\delta\}}(x),\end{aligned} \tag{3.1}$$

$$\overline{\lim_{\varepsilon\to 0}}\, \psi^{\varepsilon,\delta}(x,t) \le \left\{ \begin{array}{ll} -1 & \text{in } \{d^0(\cdot,t) \le 0\} \\ 0 & \text{in } \{d^1(\cdot,t) \le 0\} \\ +1 & \text{otherwise} \end{array} \right\} \text{ for } t \ge 0,$$

where $d^k(\cdot,t)$ is the signed distance function of $\Gamma_t^k \subset \mathbb{R}^N$ with same sign as $w^k(\cdot,t)$ for $k = 0,1$, and $\chi_U\colon \mathbb{R}^N \to \mathbb{R}$ is the characteristic function defined as

$$\chi_U(x) = \left\{ \begin{array}{ll} 1 & \text{if } x \in U, \\ 0 & \text{otherwise} \end{array} \right.$$

for $U \subset \mathbb{R}^N$.

Note that $\{d_0^0 \le 2\delta\} \subset \{d_0^1 \le 2\delta\}$ and thus the right hand side of (3.1) takes only the three values $\alpha_k + \varepsilon K_k$ for $k = -1,0,1$. The strategy of the proof is to modify the method as in [8]. First, we construct a viscosity supersolution with a traveling wave solution and truncated distance function as in [3]. A traveling wave solution is of the form $u(x,t) = q_k(x \cdot e - ct)$ with a pair of a function and a constant (q_k, c) for some $e \in \mathbb{S}^{N-1}$, and thus q_k and c satisfy

$$\begin{aligned}-q_k'' - cq_k' + f_\varepsilon(q_k) = 0 \quad \text{in } \mathbb{R}, \\ q_k(-\infty) = \alpha_{k-1},\ q_k(0) = \beta_k,\ q_k(\infty) = \alpha_k.\end{aligned} \tag{3.2}$$

We now use $d^k(x,t)$ to construct a traveling wave solution related to Γ_t^k. By similar argument as in [3, §2] we obtain

$$d_t^k - \Delta d^k - A_k|\nabla d^k| \geq 0 \quad \text{in } \{d^k > 0\} \subset \mathbb{R}^N \times (0, T^*]$$

in viscosity sense, where T^* is the extinction time of $\{w^k(\cdot,t) = 0\}$ (see [3, §2] for the details of the extinction time). However, there is no such a good estimate in $\{d^k(\cdot,\cdot) \leq 0\}$, thus we also introduce a truncated distance function as in [3, §3]. Let $\eta \in C^\infty(\mathbb{R})$ be a cut-off function satisfying

$$\eta(s) = \begin{cases} -\delta & \text{for } s \in (-\infty, \delta/4), \\ s - \delta & \text{for } s \in (\delta/2, \infty), \end{cases}$$
$$0 \leq \eta'(s) \leq C_\eta, \quad |\eta''(s)| \leq C_\eta/\delta \quad \text{for } s \in \mathbb{R}$$

for $\delta > 0$, where C_η is a positive constant. Then, by the similar argument as in [8], for $\delta > 0$ there exist positive constants $K_{1,k}$ and $K_{2,k}$ which are independent of $\varepsilon > 0$ such that

$$\psi^k(x,t) := q_k\left(\frac{\eta(d^k(x,t)) + K_{1,k}t}{\varepsilon}\right) + \varepsilon K_{2,k}$$

is a viscosity supersolution of (1.1) for sufficiently small $\varepsilon > 0$. See [8] how to choose $K_{1,k}$ and $K_{2,k}$. The important properties are such that

$$\eta(d^k(x,t)) + K_1 t < -\frac{\delta}{2} \quad \text{for } (x,t) \in \{d^k \leq 0\} \subset \mathbb{R}^N \times [0,\infty),$$
$$\eta(d^k(x,t)) + K_1 t > \delta \quad \text{for } (x,t) \in \{d^k \geq 2\delta\} \subset \mathbb{R}^N \times [0,\infty).$$

The characteristic difficulty to prove the convergence with multiple-well potential is that each ψ^k is not useful to estimate a solution because of (3.2). In particular ψ^0 crosses to u and thus the comparison principle does not hold between ψ^0 and u. One attempt to consider

$$q(-\infty) = \alpha_{-1}, \quad q(\infty) = \alpha_1$$

instead of (3.2). However, the author remarked in [8] that there is no such a solution in general. To overcome this difficulty we pile up solutions ψ^k like as [8]. Note that we can choose $K_{1,0} = K_{1,1}$, and $K_{2,k} = 2^{-k}K_2$ for some positive constant K_2 which is independent of $k = 0, 1$. We now define

$$\psi(x,t) := \begin{cases} \min\{\psi^0(x,t), \psi^1(x,t)\} & \text{for } (x,t) \in \{d^0 \leq 2\delta\} \subset \mathbb{R}^N \times [0,T^*] \\ \psi^1(x,t) & \text{otherwise} \end{cases}$$

for sufficiently small $\delta > 0$. From Lemma 2.2 ψ is well-defined for sufficiently small $\delta > 0$. Moreover, from the properties of q, d^k and η we find ψ is a desired viscosity supersolution in Lemma 3.3.

The crucial difference between our problem and [8] is the way to construct ψ^k, especially we use each distance function d^k from Γ_t^k. If f_ε is periodic as in [8], we can choose $q_0(s) + 2$ instead of q_1, and $d^0(x,t) - \gamma$ for some $\gamma > 0$ instead of $d^1(x,t)$. However, it does not work well in our problem since A_k are depend on k. Thus we have to introduce an each distance function of interfaces and a traveling wave solution.

Finally we present a sketch of the proof of Theorem 3.1. It is similar to that of [8]. However there is a little difference in particular how to use Lemma 3.2. It is because of the difference of initial data for a solution to the level set equation.

Sketch of the proof of Theorem 3.1. We now present a sketch of the estimate of u from above since the estimate from below is similar.

Fix $\delta \in (0, \bar{\delta})$. Then there exists $m > 0$ such that $\{d_0^k \leq -\delta\} \subset \{g < b_k - m\}$. Thus, to apply Lemma 3.2 with $\mu = K_2/4$ we obtain

$$\begin{aligned} u(x, \tau_0\varepsilon^2|\log\varepsilon|) \leq &(\alpha_{-1} + \varepsilon K_2/4)\chi_{\{d_0^0 \leq -\delta\}}(x) \\ &+ (\alpha_0 + \varepsilon K_2/4)\chi_{\{d_0^1 \leq -\delta\}\setminus\{d_0^0\leq -\delta\}}(x) \\ &+ (\alpha_1 + \varepsilon K_2/4)\chi_{\{d_0^1 > -\delta\}}(x) \\ =:& v^\delta(x). \end{aligned}$$

We now consider

$$\Gamma_t^{k,\delta} := \{x \in \mathbb{R}^N;\ w^k(x,t) = -3\delta\}.$$

Then, since $w^k(x,t) + 3\delta$ is still a viscosity solution to (2.2), we find

$$\psi^{k,\delta}(x,t) := q_k\left(\frac{\eta(d^{k,\delta}(x,t)) + K_1 t}{\varepsilon}\right) + 2^{-k}\varepsilon K_2$$

is still a viscosity supersolution to (1.1) for $k = 0, 1$ and sufficiently small $\varepsilon > 0$, where $d^{k,\delta}(\cdot, t)$ is a signed distance function of $\Gamma_t^{k,\delta}$ with same sign as $w^k(\cdot,t) + 3\delta$ for $t \in [0, T_\delta^*]$, and T_δ^* is the extinction time of $\Gamma_t^{k,\delta}$. From definition of $d^{k,\delta}$ or d^k we have

$$\{d^{k,\delta}(\cdot, 0) \geq 2\delta\} \supset \{d_0^k(\cdot) \geq -\delta\}. \tag{3.3}$$

This implies that

$$\psi^{k,\delta}(x,0) \geq \alpha_k + \varepsilon K_2/4 \quad \text{for } x \in \{d_0^k \geq -\delta\}$$

for sufficiently small ε, since the convergence $\lim_{s\to\infty} q_k(s) = \alpha_k$ is exponentially fast (see [8]). Thus, from (3.3) we obtain

$$v^\delta(x) \le \psi^{k,\delta}(x,0) \quad \text{for } x \in \mathbb{R}^N.$$

From the comparison principle we have

$$u(x, t + \tau_0\varepsilon^2|\log\varepsilon|) \le \psi^{k,\delta}(x,t) \quad \text{for } (x,t) \in \mathbb{R}^N \times [0, T_\delta^*].$$

Thus we obtain

$$\overline{\lim_{\varepsilon\to 0}}\, u(x,t) \le k \quad \text{for } (x,t) \in \{\, w^k \le -3\delta\} \subset \mathbb{R}^N \times (0, T_\delta^*].$$

Since $O^k = \bigcup_{\delta>0}\{w^k \le -3\delta\}$ we obtain the estimate of the convergence in Theorem 3.1 from above. □

References

[1] G. Barles, H. M. Soner and P. E. Souganidis, Front propagation and phase field theory, SIAM J. Cont. Opt. **31**(1993), 439–469.

[2] X. Chen, Generation and propagation of interface in reaction–diffusion equations, J. Differential Equations **96**(1992), 116–141.

[3] L. C. Evans, H. M. Soner and P. E. Souganidis, Phase transitions and generalized motion by mean curvature, Comm. Pure Appl. Math. **45**(1992), 1097–1123.

[4] Y. Giga, *Surface Evolution Equations – A Level Set Approach*, Birkhäuser, 2006.

[5] Y. Giga, S. Goto, H. Ishii and H.-M. Sato, Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains, Indiana University Mathematics Journal **40**(1991), 443–470.

[6] H. Ishii, Perron's method for Hamilton–Jacobi Equations, Duke Math. J. **55**(1987), 369–384.

[7] M. A. Katsourakis, G. Kossioris and F. Reitech, Generalized motion by mean curvature with Neumann conditions and the Allen–Cahn model for phase transitions, J. Geom. Anal. **5**(1995), 255–279.

[8] T. Ohtsuka, Motion of interfaces by Allen–Cahn type equation with multiple-well potentials, Asymptotic analysis **56**(2008), 87–123.

[9] T. Ohtsuka, The singular limit of an Allen–Cahn type equation with unbalanced multiple-well potential, Proceedings of International Conference for the 25th Anniversary of Viscosity Solutions, GAKUTO International Series, Mathematical Sciences and Applications **30**(2008), 165–174.

[10] J. Rubinstein, P. Sternberg and J. B. Keller, Fast reaction, slow diffusion and curve shortening, SIAM J. Appl. Math. **49**(1989), 116–133.

[11] J. Rubinstein, P. Sternberg and J. B. Keller, Front interaction and nonhomogeneous equilibria for tristable reaction-diffusion equations, SIAM J. Appl. Math. **53**(1993), 1669–1685.