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VARIATION FORMULAS FOR PRINCIPAL FUNCTIONS AND HARMONIC SPANS

SACHIKO HAMANO

ABSTRACT. The purpose of this article is to give a summary of the seminar lecture with title Variation formulas for principal functions and harmonic spans by the author in the conference held at RIMS, Kyoto, Japan, December 2009. The former half of this article is in Hamano [4] and the latter half is in the manuscript [6] which is a joint work with Maitani and Yamaguchi.

1. INTRODUCTION

Let $B = \{|t| < \rho\}$ and let $\tilde{\mathcal{R}}$ be an unramified (two-dimensional) Riemann domain sheeted over $B \times \mathbb{C}_z$. We write $\tilde{\mathcal{R}} = \bigcup_{t \in B}(t, \tilde{R}(t))$, where $\tilde{R}(t)$ is a fiber over each $t \in B$, i.e., $\tilde{R}(t) = \{z : (t, z) \in \tilde{\mathcal{R}}\}$, so that each $\tilde{R}(t)$ consists of unramified Riemann surfaces sheeted over $\mathbb{C}_z$. Consider a subdomain $\mathcal{R}$ in $\tilde{\mathcal{R}}$ such that, if we put $\mathcal{R} = \bigcup_{t \in B}(t, R(t)))$, then

1. $\tilde{R}(t) \supset R(t) \neq \emptyset$, $t \in B$;
2. the boundary $\partial \mathcal{R} = \bigcup_{t \in B}(t, \partial R(t))$ of $\mathcal{R}$ in $\tilde{\mathcal{R}}$ is $C^\omega$ smooth in $\tilde{\mathcal{R}}$;
3. each $R(t)$, $t \in B$ is a connected Riemann surface of genus $g \geq 0$, where $g$ is independent of $t \in B$;
4. each $\partial R(t)$, $t \in B$ in $\tilde{R}(t)$ consists of a finite number of $C^\omega$ smooth contours $C_j(t), j = 0, 1, \ldots, \nu$, where $\nu$ is independent of $t \in B$. We give the orientation of $C_j(t)$ such that $\partial R(t) = C_0(t) + C_1(t) + \cdots + C_\nu(t)$.

We usually regard two-dimensional Riemann domain $\mathcal{R}$ over $B \times \mathbb{C}_z$ as a $C^\omega$ smooth variation of Riemann surface $R(t)$ over $\mathbb{C}_z$ with $C^\omega$ smooth boundary $\partial R(t)$ with complex parameter $t \in B$:

$\mathcal{R} : t \in B \rightarrow R(t)$.

Variation formula of the Green function for $(R(t), 0)$ We assume that $\mathcal{R}$ contains $B \times \{0\}$, precisely, there exists at least one constant section $O$ of $\mathcal{R}$ over $B \times \{0\}$. We consider the Green function

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$g(t, z)$ with pole at $z = 0$ and the Robin constant $\lambda(t)$ for $(R(t), 0)$, so that

$$g(t, z) = \log \frac{1}{|z|} + \lambda(t) + h(t, z).$$

Here $h(t, z)$ is harmonic for $z$ in a neighborhood of $z = 0$ in $R(t)$ such that

$$h(t, 0) = 0 \quad \text{for } t \in B.$$ 

Let $\varphi(t, z)$ be a defining function of $\partial \mathcal{R}$ in $B \times \mathbb{C}_z$. For $(t, z) \in \partial \mathcal{R}$, we consider the following quantities:

$$k_1(t, z) = \frac{\partial \varphi}{\partial t} \left| \frac{\partial \varphi}{\partial z} \right|,$$

$$k_2(t, z) = \left( \frac{\partial^2 \varphi}{\partial t \partial \overline{t}} \left| \frac{\partial \varphi}{\partial z} \right|^2 - 2 \Re \left\{ \frac{\partial^2 \varphi}{\partial t \partial z} \frac{\partial \varphi}{\partial z} \frac{\partial \varphi}{\partial \overline{z}} \right\} + \left| \frac{\partial \varphi}{\partial t} \right|^2 \frac{\partial^2 \varphi}{\partial z \partial \overline{z}} \right) \left| \frac{\partial \varphi}{\partial z} \right|^3.$$

We note that they do not depend on the choice of defining functions $\varphi(t, z)$ of $\partial \mathcal{R}$. We denote by $ds_z$ the arc length element of $\partial R(t)$ at $z$. The function $k_2(t, z)$ on $\partial \mathcal{R}$ is due to Maitani-Yamaguchi in [8] which is based on [7], which is called the Levi curvature for $\partial \mathcal{R}$. Then the following variation formulas for the Robin constants are shown in Hadamard [3], Maitani-Yamaguchi [8].

**Fact.** It holds for $t \in B$ that

$$\frac{\partial \lambda(t)}{\partial t} = -\frac{1}{\pi} \int_{\partial \mathcal{R}(t)} k_1(t, z) \left| \frac{\partial g(t, z)}{\partial z} \right|^2 ds_z,$$

$$\frac{\partial^2 \lambda(t)}{\partial t \partial \overline{t}} = -\frac{1}{\pi} \int_{\partial \mathcal{R}(t)} k_2(t, z) \left| \frac{\partial g(t, z)}{\partial z} \right|^2 ds_z - \frac{4}{\pi} \int_{\mathcal{R}(t)} \left| \frac{\partial^2 g(t, z)}{\partial \overline{t} \partial z} \right|^2 dx dy.$$

2. **Variation Formula of $L_1$-Principal Function for $(R(t), 0, C_0(t))$**

This and the next sections are quoted from S. Hamano [4]. Under the same conditions for the unramified domain $\mathcal{R} = \bigcup_{t \in B}(t, R(t))$ in $\overline{\mathcal{R}}$ over $B \times \mathbb{C}_z$ and $\partial \mathcal{R}(t) = \sum_{j=0}^{\nu} C_j(t)$, we assume that the total space $\mathcal{R}$ contains $B \times \{0\}$. For each $t \in B$, we conventionally write $0$ for the point $O \cap R(t)$.

Let $t \in B$ be fixed. In the theory of one complex variable, it is known that there exists a unique real-valued function $u(t, z)$ on $R(t) \setminus \{0\}$ satisfying the following four conditions:

1. $u(t, z)$ is harmonic on $R(t) \setminus \{0\}$ and is continuous on $\overline{R(t)}$;
2. $u(t, z) - \log \frac{1}{|z|}$ is harmonic at $z = 0$;
3. $u(t, z) = 0$ on $C_0(t)$;
for each $i = 1, \ldots, \nu$, we have

(i) $u(t, z) = a_i(t)$: constant on $C_i(t)$;

(ii) \[
\int_{C_i(t)} d u(t, z) = 0.
\]

We note that $u(t, z)$ extends harmonically across $\partial R(t)$ as a harmonic function on $V(t)$ such that $\partial R(t) \subseteq V(t) \subseteq \tilde{R}(t)$. By (2), we find a neighborhood $U_0(t)$ of $z = 0$ such that

\[ u(t, z) = \log \frac{1}{|z|} + \gamma(t) + h(t, z) \]

on $U_0(t)$, where $\gamma(t)$ is the constant term and $h(t, z)$ is harmonic for $z$ on $U_0(t)$ such that $h(t, 0) = 0$, $t \in B$.

The function $u(t, z)$ is called the $L_1$-principal function on $R(t)$ with logarithmic pole at 0 with respect to $C_0(t)$, and $\gamma(t)$ is called the $L_1$-constant on $R(t)$ with logarithmic pole at 0 with respect to $C_0(t)$ (cf: \[1\]). In this article, we simply call $u(t, z)$ the $L_1$-principal function for $(R(t), 0, C_0(t))$, and $\gamma(t)$ the $L_1$-constant for $(R(t), 0, C_0(t))$. We note that $u(t, z) > 0$ in $R(t) \setminus \{0\}$ and $a_i(t) > 0$ ($i = 1, \ldots, \nu$).

Then we have the following variation formula for the $L_1$-constant $\gamma(t)$ for $(R(t), 0, C_0(t))$.

**Lemma 1.** It holds for $t \in B$ that

\[
\frac{\partial \gamma(t)}{\partial t} = -\frac{1}{\pi} \int_{\partial R(t)} k_1(t, z) \left| \frac{\partial u(t, z)}{\partial z} \right|^2 ds_z,
\]

\[
\frac{\partial^2 \gamma(t)}{\partial t \partial \overline{t}} = -\frac{1}{\pi} \int_{\partial R(t)} k_2(t, z) \left| \frac{\partial u(t, z)}{\partial z} \right|^2 ds_z - \frac{4}{\pi} \int_{R(t)} \left| \frac{\partial^2 u(t, z)}{\partial \overline{t} \partial z} \right|^2 dxdy.
\]

This variation formula is formally the same as that for the Robin constant $\lambda(t)$ in section 1. The essential difference of the proofs for $\gamma(t)$ and $\lambda(t)$ comes from the fact that $u(t, z)$ is not a defining function of $\partial \mathcal{R}$ contrary to the case of the Green function $g(t, z)$.

**Theorem 2.** Under the same conditions in Lemma 1, if $\mathcal{R}$ is pseu-doconvex over $B \times \mathbb{C}_z$, then $\gamma(t)$ is a $C^\omega$ superharmonic function on $B$.

**Remark 1.** For Lemma 1, we assumed that $\mathcal{R}$ is unramified over $B \times \mathbb{C}_z$. However, even if each $R(t), t \in B$ has a finite number of branch points $\zeta_k(t)$ ($k = 1, \ldots, m$) for $t \in B$ such that $\zeta_k(t)$ is a holomorphic function on $B$ with $\zeta_k(t) \neq \zeta_l(t)$ ($k \neq l$), $t \in B$, then Lemma 1 and hence Theorem 2 hold. For, this case can be reduced to Lemma 1 by the standard method by use of Y. Nishimura’s theorem \[10\].
In the special case when $R(t)$ is a planar Riemann surface, the $L_{1}$-principal function $u(t, z)$ induces a circular slit mapping $f(t, z)$. That is, if we choose a branch $u^{*}(t, z)$ of harmonic conjugate function of $u(t, z)$ on $R(t)$, $t \in B$ such that

$$f(t, z) = e^{\gamma(t) - (u(t, z) + iu^{*}(t, z))}$$

is of the form

$$w = f(t, z) = z + \sum_{j=2}^{\infty} b_{j}(t)z^{j} \quad \text{on} \quad U_{0}(t),$$

then $f(t, z)$ conformally maps $R(t)$ onto a circular slit domain $\{|w| < e^{\gamma(t)}\} \setminus \bigcup_{i=1}^{\nu} \ell_{i}$, where $\ell_{i}(t) = f(t, C_{i}(t))$ (an arc of the circle $\{|w| = e^{\gamma(t) - a_{i}(t)}\}$). If $\mathcal{R}$ is pseudoconvex over $B \times \mathbb{C}_{z}$, then $e^{\gamma(t)}$ is logarithmic superharmonic on $B$, so that the total space $\bigcup_{t \in B} \{|w| < e^{\gamma(t)}\}$ is a Hartogs pseudoconvex domain in $B \times \mathbb{C}_{w}$.

**Remark 2.** In the theory of one complex variable, the circular slit mapping and the radial slit mapping have good correspondence. But the same result for the corresponding radius of the radial slit mapping does not hold. In fact, we have the following counterexamples (i) and (ii) of pseudoconvex domains $\mathcal{R}$ in $B \times \mathbb{C}_{z}$ such that the radii of radial slit mappings are not logarithmic superharmonic or not logarithmic subharmonic on $B$:

(i) The radius of radial slit mapping is not logarithmic superharmonic on $B$: Let

$$\mathcal{R} = \{ |t| < \frac{1}{2} \} \times \{ |z| < 1 \} \setminus \{ (t, z) : |z - \frac{1}{2}| \leq |t| < \frac{1}{2} \},$$

$$B = \{ |t| < \frac{1}{2} \}, \quad R(t) = \{ |z| < 1 \} \setminus \{ |z - \frac{1}{2}| \leq |t| \},$$

so that $\partial R(t) = C_{0}(t) + C_{1}(t)$ where $C_{0}(t) = \{ |z| = 1 \}$ and $C_{1}(t) = \{ |z - 1/2| = |t| \}$.

(ii) The radius of radial slit mapping is not logarithmic subharmonic on $B$: Let

$$\mathcal{R} = \bigcup_{t \in B} \{ |z| < r(t) \} \setminus B \times (C_{1} \cup C_{2}),$$

where $C_{1} = [\frac{1}{2}, \frac{3}{2}]$, $C_{2} = [\frac{1}{2}, \frac{2i}{3}]$ in $\mathbb{C}_{z}$, $r(t) > 1$ and $\log r(t)$ is superharmonic on $B$. Thus $\partial R(t) = C_{0}(t) + C_{1}(t) + C_{2}(t)$ where $C_{0}(t) = \{ |z| = r(t) \}$, $C_{i}(t) = C_{i}$, $i = 1, 2$.

3. **Variation formula of $L_{1}$-principal function for $(R(t), 0, \xi(t))$**

Under the same conditions for the unramified domain $\mathcal{R} = \bigcup_{t \in B} (t, R(t))$ in $\tilde{\mathcal{R}}$ over $B \times \mathbb{C}_{z}$ and $\partial R(t) = \sum_{j=0}^{\nu} C_{j}(t)$, we assume that there exist
two holomorphic sections:

$$\Xi_0 : z = 0 \quad \text{and} \quad \Xi_1 : z = \xi(t)$$

of $\mathcal{R}$ over $B$ such that $\Xi_0 \cap \Xi_1 = \emptyset$. Let $t \in B$ be fixed. In the theory of one complex variable, there exists a unique real-valued function $p(t, z)$ on $R(t) \setminus \{0, \xi(t)\}$ satisfying the following four conditions:

1. $p(t, z)$ is harmonic on $R(t) \setminus \{0, \xi(t)\}$ and continuous on $\overline{R(t)}$;
2. $p(t, z) - \log 1/|z|$ is harmonic at $z = 0$ and
   \[
   \lim_{z \to 0} \left( p(t, z) - \log \frac{1}{|z|} \right) = 0;
   \]
3. $p(t, z) - \log |z - \xi(t)|$ is harmonic at $z = \xi(t)$;
4. for each $j = 0, 1, \ldots, \nu$, we have
   (i) $p(t, z) = a_j(t)$ : constant on $C_j(t)$;
   (ii) $\int_{C_j(t)} *dp(t, z) = 0$.

We note that $p(t, z)$ extends harmonically across $\partial R(t)$ as a harmonic function on $V(t)$ such that $\partial R(t) \subseteq V(t) \subseteq \tilde{R}(t)$, $-\infty < p(t, z) < +\infty$, and $-\infty < a_j(t) < +\infty$.

By (2), we find a neighborhood $U_0(t)$ of $z = 0$ such that

$$p(t, z) = \log \frac{1}{|z|} + h_0(t, z) \quad \text{on} \quad U_0(t),$$

where $h_0(t, z)$ is harmonic for $z$ on $U_0(t)$ and

$$h_0(t, 0) = 0, \quad t \in B.$$

By (3), we find a neighborhood $U_\xi(t)$ of $z = \xi(t)$ such that

$$p(t, z) = \log |z - \xi(t)| + \alpha(t) + h_\xi(t, z) \quad \text{on} \quad U_\xi(t),$$

where $\alpha(t)$ is a real constant and $h_\xi(t, z)$ is harmonic for $z$ on $U_\xi(t)$ and

$$h_\xi(t, \xi(t)) = 0, \quad t \in B.$$

In this article, we simply call $p(t, z)$ the $L_1$-principal function for $(R(t), 0, \xi(t))$, and $\alpha(t)$ the $L_1$-constant for $(R(t), 0, \xi(t))$.

Under these situations, we have

**Lemma 3.** It holds for $t \in B$ that

\[
\frac{\partial \alpha(t)}{\partial t} = \frac{1}{\pi} \int_{\partial R(t)} k_1(t, z) \left| \frac{\partial p(t, z)}{\partial z} \right|^2 ds_z + 2 \frac{\partial h_\xi}{\partial z} \bigg|_{(t, \xi(t))} \cdot \xi'(t),
\]

\[
\frac{\partial^2 \alpha(t)}{\partial t \partial \overline{t}} = \frac{1}{\pi} \int_{\partial R(t)} k_2(t, z) \left| \frac{\partial p(t, z)}{\partial z} \right|^2 ds_z + \frac{4}{\pi} \int_{R(t)} \left| \frac{\partial^2 p(t, z)}{\partial t \partial z} \right|^2 dxdy
\]
If $\mathcal{R}$ is pseudoconvex in $\tilde{\mathcal{R}}$, then $k_2(t, z) \geq 0$ on $\partial \mathcal{R}$ (Levi condition), and the converse is also true. We then apply Lemma 1 to obtain the following

**Theorem 4.** Under the same conditions in Lemma 3, if $\mathcal{R}$ is pseudoconvex over $B \times \mathbb{C}_z$, then $a(t)$ is a $C^\omega$ subharmonic function on $B$. This is also true under the same condition for $\mathcal{R}$ as in Remark 1.

**Application of Theorem 4** As an application of Theorem 4, we show that the following fact. Let $B$ be a simply connected domain in $\mathbb{C}_t$. Let $\pi : \mathcal{S} \to B$ be a holomorphic family of compact Riemann surfaces $S(t) = \pi^{-1}(t)$ over $B$ such that each fiber $S(t)$ is of genus $\geq 2$ and non-singular in $\mathcal{S}$. For a fixed $t \in B$, we consider the Schottky covering $\tilde{S}(t)$ of each $S(t)$ (cf: Sec. 101 in Ford [2], and 19F in p.241 in Ahlfors-Sario [1]). We denote by $\tilde{S}$ the total space of the variation: $t \in B \to \tilde{S}(t)$, namely, $\tilde{S} = \bigcup_{t \in B} (t, \tilde{S}(t))$. Then we have:

**Theorem 5.** The total space $\tilde{S}$ consisting of the Schottky covering $\tilde{S}(t)$ of compact Riemann surfaces $S(t)$ with complex parameter $t \in B$ is holomorphically uniformized to a univalent domain on $B \times \mathbb{P}^1$.

In [8], Maitani and Yamaguchi proved that, if $\mathcal{R} = \bigcup_{t \in B} (t, R(t))$ is an unramified pseudoconvex domain over $B \times \mathbb{C}_z$ such that each $R(t)$, $t \in B$ is planar and parabolic, then $\mathcal{R}$ is holomorphically uniformizable to a domain in $B \times \mathbb{P}^1$. Since the Schottky covering $\tilde{S}(t)$ of a compact Riemann surface $S(t)$ of genus $g \geq 2$ is planar but not parabolic, their theorem and method cannot be applicable to our case. In [12], Yamaguchi wrote a resume about Theorem 5 with a rough sketch of the proof. However his sketch had a "gap". Then I bridge the gap by establishing the variation formula for $L_1$-principal function (Lemma 3), and obtain Theorem 5.

4. **VARiATION FORMULA OF $L_0$-PRINCIPAL FUNCTION FOR**

$$(R(t), 0, \xi(t))$$

This section is quoted from S. Hamano, F. Maitani and H. Yamaguchi [6]. Under the same conditions for the unramified domain $\mathcal{R} = \bigcup_{t \in B} (t, R(t))$ in $\tilde{\mathcal{R}}$ over $B \times \mathbb{C}_z$ and $\partial R(t) = \sum_{j=0}^{\nu} C_j(t)$, we assume that $\mathcal{R}$ has two holomorphic section over $B$:

$$\Xi_0 : z = 0 \quad \text{and} \quad \Xi_1 : z = \xi(t)$$

such that $\Xi_0 \cap \Xi_1 = \emptyset$. Let $t \in B$ be fixed. Then it is known (cf: Sario-Nakai [9]) that $R(t)$ carries the following real-valued function $q(t, z)$.

1. $q(t, z)$ is harmonic on $R(t) \setminus \{0, \xi(t)\}$ and is continuous on $\overline{R(t)}$;
2. $q(t, z) - \log 1/|z|$ is harmonic at $z = 0$ and

$$\lim_{z \to 0} (q(t, z) - \log 1/|z|) = 0;$$
(3) $q(t, z) - \log|z - \xi(t)|$ is harmonic at $z = \xi(t)$;
(4) $\frac{\partial q(t, z)}{\partial n_z} = 0$ on $\partial R(t)$.

We call the function $q(t, z)$ the \textit{L$_0$-principal function} for $(R(t), 0, \xi(t))$.

Note that $q(t, z)$ extends harmonically across $\partial R(t)$ as a harmonic function on $V(t)$ such that $\partial R(t) \subseteq V(t) \subseteq \tilde{R}(t)$.

By (2) for $q(t, z)$, we find a neighborhood $U_0(t)$ of $z = 0$ such that

$$ q(t, z) = \log \frac{1}{|z|} + h_0(t, z) \quad \text{on} \quad U_0(t), $$

where $h_0(t, z)$ is harmonic for $z$ on $U_0(t)$ and

$$ h_0(t, 0) = 0, \quad t \in B. $$

By (3) for $q(t, z)$, we find a neighborhood $U_\xi(t)$ of $z = \xi(t)$ such that

$$ q(t, z) = \log|z - \xi(t)| + \beta(t) + \mathfrak{h}_\xi(t, z) \quad \text{on} \quad U_\xi(t), $$

where $\beta(t)$ is a constant and $\mathfrak{h}_\xi(t, z)$ is harmonic for $z$ on $U_\xi(t)$ and

$$ \mathfrak{h}_\xi(t, \xi(t)) = 0, \quad t \in B. $$

We call $\beta(t)$ the \textit{L$_0$-constant} for $(R(t), 0, \xi(t))$.

We shall give the variational formulas for $L_0$-constant $\beta(t)$. In order to prove the formula for $\beta(t)$, we have to add a new idea to the proof for $\alpha(t)$. In fact, the formulas for $\alpha(t)$ do not concern to the genus of $R(t)$ but the variation formula of the second order for $\beta(t)$ does concern to the genus of $R(t)$. It seems to be curious that of the first order does not concern to the genus as below. In case when $R(t)$ is of positive genus $g \geq 1$, we take $\{A_i(t), B_i(t)\}_{1 \leq i \leq g}$ be usual $A, B$ cycles on $R(t)$ with intersection number condition: for $k, l = 1, \ldots, g$,

$$ A_k(t) \times B_l(t) = \delta_{k,l}, \quad A_k(t) \times A_l(t) = 0, \quad B_k(t) \times B_l(t) = 0. $$

Here $\delta_{k,l}$ is Kronecker's delta; $A_k(t) \times B_l(t)$ means that $A_k(t)$ crosses $B_l(t)$ from the left-side to the right-side of the direction $B_l(t)$; and each $A_k(t)(B_k(t))$, $k = 1, \ldots, g$ varies continuously with parameter $t \in B$ such that $A_k(t), B_k(t)$ do not pass through $\{0, \xi(t)\}$. On each $R(t), t \in B$ we denote by $\ast dq(t, z)$ the conjugate differential of $dq(t, z)$.

Then we have

\textbf{Lemma 6.} It holds for $t \in B$ that

$$ \frac{\partial \beta(t)}{\partial t} = -\frac{1}{\pi} \int_{\partial R(t)} k_1(t, z) \left| \frac{\partial q(t, z)}{\partial z} \right|^2 ds_z + 2 \frac{\partial \mathfrak{h}_\xi}{\partial z}(t, \xi(t)) \cdot \xi'(t), $$

$$ \frac{\partial^2 \beta(t)}{\partial t \partial \overline{t}} = -\frac{1}{\pi} \int_{\partial R(t)} k_2(t, z) \left| \frac{\partial q(t, z)}{\partial z} \right|^2 ds_z - \frac{4}{\pi} \int_{R(t)} \left| \frac{\partial^2 q(t, z)}{\partial t \partial z} \right|^2 dxdy $$

$$ - \frac{2}{\pi} \sum_{k=1}^{g} \left( \frac{\partial}{\partial t} \int_{A_k(t)} \ast dq(t, z) \right) \cdot \left( \frac{\partial}{\partial \overline{t}} \int_{B_k(t)} \ast dq(t, z) \right). $$
Lemma 3 implied that if \( \mathcal{R} \) is pseudoconvex in \( \tilde{\mathcal{R}} \), then \( L_1 \)-constant \( \alpha(t) \) for \( (R(t), 0, \xi(t)) \) is a \( C^\omega \) subharmonic function on \( B \). On the other hand, Lemma 6 implied the following:

**Theorem 7.** Under the same conditions in Lemma 6, if \( \mathcal{R} \) is pseudoconvex in \( \tilde{\mathcal{R}} \) and each \( R(t), t \in B \) is planar, then the \( L_0 \)-constant \( \beta(t) \) for \( (R(t), 0, \xi(t)) \) is a \( C^\omega \) superharmonic function on \( B \).

The contrast between the subharmonicity of \( \alpha(t) \) and the superharmonicity of \( \beta(t) \) are unified with the notion of the harmonic span \( s(t) \) for \( (R(t), 0, \xi(t)) \) due to M. Schiffer [11].

**Variations formula for the harmonic spans.** We recall some notions studied in the theory of one complex variable. Let \( R \) be a domain in \( \mathbb{C}_z \) bounded by a finite number of closed curves \( C_j, j = 0, 1, \ldots, \nu \). For simplicity we assume \( 0 \in R \). For a point \( \xi \neq 0 \), we consider the \( L_1 \)- and \( L_0 \)-function \( p(z) \) and \( q(z) \) for \( (R, 0, \xi) \) and the \( L_1 \) - and \( L_0 \)-constant \( \alpha \) and \( \beta \) for \( (R, 0, \xi) \). In the function theory of one complex variable, it is known (cf: [1] and [9])

\[
s(R) = \frac{\pi}{2} (\alpha - \beta)
\]
as the harmonic span \( s(R) \) for \( (R, 0, \xi) \).

We return to the variation of Riemann surfaces. Let \( \mathcal{R} : t \in B \to R(t) \) satisfy the conditions in the beginning of Sections 3 and 4. For a fixed \( t \in B \), we denote by \( p(t, z) \) \( (q(t, z)) \) the \( L_1 \)\-(\( L_0 \))-principal function, by \( \alpha(t) \) \((\beta(t))\) the \( L_1 \)-\( (L_0 \))-constant and by \( s(t) \) the harmonic span for \( (R(t), 0, \xi(t)) \). Then combining Lemmas 3 and 6, we immediately have the following:

**Lemma 8.** Assume that \( R(t), t \in B \) is planar. Then it holds that

\[
\frac{\partial s(t)}{\partial t} = \frac{1}{2} \int_{\partial R(t)} k_2(t, z) \left( \left| \frac{\partial p(t, z)}{\partial z} \right|^2 + \left| \frac{\partial q(t, z)}{\partial z} \right|^2 \right) ds_z
\]

\[
+ 2 \int_{R(t)} \left( \left| \frac{\partial^2 p(t, z)}{\partial \overline{t} \partial z} \right|^2 + \left| \frac{\partial^2 q(t, z)}{\partial \overline{t} \partial z} \right|^2 \right) dx dy
\]

Lemma 8 implied the following:

**Theorem 9.** Under the same conditions in Lemma 6, if \( \mathcal{R} \) is pseudoconvex in \( \tilde{\mathcal{R}} \) and each \( R(t), t \in B \) is planar, then the harmonic span \( s(t) \) for \( (R(t), 0, \xi(t)) \) is a \( C^\omega \) subharmonic function on \( B \).

5. **Examples**

We begin with a simple example of our general result shown in this article. Let \( B = \{ |t| < \rho \} \) be a disk in \( \mathbb{C}_t \). For each \( t \in B \), let \( R(t) \) be a disk \( \{ |z| < r(t) \} \in \mathbb{C}_z \), where \( \log r(t) \) is a superharmonic function.
on $B$. If we set the Hartogs domain of disks $\mathcal{R} = \bigcup_{t \in B}(t, R(t))$, then $\mathcal{R}$ is a pseudoconvex domain in $B \times \mathbb{C}_z$. Assume that there exists a holomorphic section $\xi : t \to \xi(t)(\neq 0) \in R(t)$.

**[Example of Theorem 4.]** We consider the following function:

$$f(t, z) = -\frac{1}{\xi(t)} \cdot \frac{r(t)^2(z - \xi(t))}{z(r(t)^2 - \bar{\xi}(t)z)} \quad \text{on } R(t).$$

Then $f$ is a circular slit mapping on $R(t)$ with zero at $z = \xi(t)$ and pole at $z = 0$. The $L_1$-constant $\alpha(t)$ on $B$ is written into

$$\alpha(t) = \log \left| \frac{\partial f}{\partial z}(t, \xi(t)) \right| = \log \left| -\frac{1}{\xi(t)^2} \cdot \frac{r(t)^2}{r(t)^2 - |\xi(t)|^2} \right|.$$

Since $\xi(t)$ is holomorphic on $B$ and since $\log r(t)$ is superharmonic on $B$, $\log \frac{|\xi(t)|}{r(t)}$ is subharmonic on $B$, so is the second term in the right-hand side. Hence, $\alpha(t)$ is a subharmonic function on $B$.

**[Example of Theorem 7.]** We put $\theta(t) = \arg \xi(t)$. Then

$$F(t, z) = \frac{1}{2} \left( \frac{z}{r(t)e^{i\theta(t)}} + \frac{r(t)e^{i\theta(t)}}{z} \right) - \frac{1}{2} \left( \frac{|\xi(t)|}{r(t)} + \frac{r(t)}{|\xi(t)|} \right)$$

is the radial slit mapping on $R(t)$ with zero at $z = \xi(t)$ and pole at $z = 0$. The $L_0$-constant $\beta(t)$ is written into

$$\beta(t) = \log \left| \frac{\partial Q}{\partial z}(t, \xi(t)) \right| = -2\log |\xi(t)| + \log \left[ 1 - \left( \frac{|\xi(t)|}{r(t)} \right)^2 \right],$$

which certainly is superharmonic on $B$.

**[Example of Theorem 9.]** We also see that the harmonic span $s(t) = \frac{\pi}{2}(\alpha(t) - \beta(t))$ for $(R(t), 0, \xi(t))$ is

$$s(t) = \log \frac{1}{1 - \left( \frac{|\xi(t)|}{r(t)} \right)^2},$$

which is subharmonic on $B$.

**REFERENCES**


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