数理解析研究所講究録
第 1694 巻 2010 年 70-88

RELATIVE CLASSES OF HARMONIC FUNCTIONS ON RIEMANN SURFACES

MITSURU NAKAI
Nagoya Institute of Thechnology (Professor Emeritus)

1. Introduction. We follow the traditional notation in the classification theory of Riemann surfaces (cf. e.g. [1], [2], [14], [15], [17], etc.) in the sequel in this treatise. We denote by $H(R)$ the linear space of harmonic functions on an open Riemann surface R (cf. e.g. [1]). In this article we are mainly concerned with the linear subspace $HD(R)$ of $H(R)$ consisting of $u \in H(R)$ with finite Dirichlet integral

$$D(u; R) := \int_R du \wedge *du.$$

For two functions u and v in $HD(R)$, the mutual Dirichlet integral

$$D(u, v; R) := \int_R du \wedge *dv$$

of u and v can be considered so that $D(u; R) = D(u, u; R)$. Then the space $HD(R)$ with the possible inner product $D(\cdot, \cdot; R)$ almost forms a Hilbert space except for one small point: one of the conditions of the norm property that $D(u; R) = 0$ implies $u = 0$ fails to hold since $D(u; R) = 0$ is only equivalent to $u \in \mathbb{R}$, the real number field. To save this crisis, often the following normalizations are adopted: to replace $HD(R)$ by $HD(R; a) := \{u \in HD(R) : u(a) = 0\}$ for a fixed reference point $a \in R$, or by the quotient space $HD(R)/\mathbb{R}$, or to consider the space $dHD(R) := \{du : u \in HD(R)\}$ of square integrable exact harmonic 1-forms on R in place of $HD(R)$. However these devices have several more or less seirious drawbacks. Firstly, the linear dimension $\dim HD(R)$ of $HD(R)$ is not preserved in general by considering any one of $HD(R; a)$, $HD(R)/\mathbb{R}$, or $dHD(R)$ so that the linear structure is not preserved but not too worse. Secondly, not only the linear structure but also, even more gravely, the important structure of $HD(R)$ that $HD(R)$ forms a Riesz space (i.e. vector lattice) is lost by considering any one of $HD(R; a)$, $HD(R)/\mathbb{R}$, or $dHD(R)$, where lattice operations the join $u \vee v$ and the meet $u \wedge v$ of u and v in $HD(R)$ are given by the least harmonic majorant of u and v and by the greatest harmonic minorant of u an v. Thirdly, and lastly, we mention the following , which is the most serious. We denote by $L^{1,2}(R)$ the Dirichlet space on R which is the linear space of functions $u \in W^{1,2}_{loc}(R)$, the local Sobolev space, such that $D(u; R) < +\infty$. The Dirichlet null space $L^{1,2}_0(R)$ is a linear subspace of the Dirichlet space $L^{1,2}(R)$ consisting of $u \in L^{1,2}(R)$ for which there is a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset C^\infty_0(R)$ such that $\lim_{n\to\infty} \varphi_n = u$ a.e. on R and simultaneously $\lim_{n\to\infty} D(u - \varphi_n; R) = 0$ (cf. e.g. [4]). Any member of the Dirichlet null space $L^{1,2}_0(R)$ is referred to as a Dirichlet potential. The Weyl-Royden-Brelot theorem (cf. e.g. [15]) says that

$$L^{1,2}(R) = HD(R) \oplus L^{1,2}_0(R), \tag{1,1}$$

i.e. any $f \in L^{1,2}(R)$ has a unique decomposition

$$f = u + \varphi \quad (u \in HD(R),\ \varphi \in L^{1,2}_0(R))$$

with the orthogonal relation

$$D(f; R) = D(u; R) + D(\varphi; R).$$

It can happen that $1 \in L^{1,2}_0(R)$, or equivalently $\mathbb{R} \subset L^{1,2}_0(R)$, when and only when R is parabolic (i.e. nonhyperbolic, $R \in \mathcal{O}_G$ in notation) in the sense that there is no harmonic Green function on R (cf. e.g. [15]). It is reasonable, or rather it should be, that we make a convention that $HD(R) = \{0\}$ if and only if $R \in \mathcal{O}_G$. We denote by $\mathcal{O}_{HD}$ the family of open Riemann surfaces R with $HD(R) \subset \mathbb{R}$. We know (cf. e.g. [1], [15], etc.) that

$$\mathcal{O}_G < \mathcal{O}_{HD} \quad \text{(strict inclusion)}. \tag{1.2}$$

If $R \in \mathcal{O}_G$ ($R \in \mathcal{O}_{HD} \setminus \mathcal{O}_G$, resp.), then $HD(R) = \{0\}$ ($HD(R) = \mathbb{R}$, resp.) but $HD(R; a) = \{0\}$ ($HD(R; a) = \{0\}$, resp.), for example. The fact mentioned above shows that the important structural information of R whether it is in $\mathcal{O}_G$ or not in terms of the space $HD(R)$ is completely lost by considering the normalized space $HD(R; a)$, for example.

In view of the above observations, we need some other normalization of the class $HD(R)$ which does not destroy the Riesz space strucure of $HD(R)$ and also the coherent relation between the space $HD(R)$ and the base space R. It is the relative class $HD(W; \partial W)$ described below that entirely meet our requirment. It is then a genuine Hilbert space carrying the reproducing kernel, which we call the Bergman kernel. We will describe some new feature of the structure of the class $HD(W; \partial W)$ by using its Bergman kernel.

2. The relative class. We will replace a tiny set $\{a\}$ consisting of a single point

$a \in R$ in the normalized space $HD(R;a)$ by a larger set A which is the closure of a regular subregion (i.e. a relatively compact subregion whose relative boundary consists of a finite number of mutually disjoin smooth Jordan curves) of R. An open subset W of R is referred to as an *end* of R if $R \setminus \overline{W}$ is a regular subregion of R. In most occasions it is enough to consider only the case $R \setminus \overline{W}$ is a parametric disc but still for the sake of generality we allow the case W is disconnected. The *relative class* $H(W;\partial W)$ of the absolute class $H(R)$ is, by definition,

$$H(W;\partial W) := \{u \in H(W) \cap C(R) : u|R \setminus W = 0\} \tag{2.1}$$

(cf. [1], [2] , [13], [15], [17]). To relate the relative class $H(W;\partial W)$ with the original absolute class $H(R)$ we first define an operator $D : C(\partial W) \to H(W) \cap C(\overline{W})$, which is referred to as the outer Dirichlet operator relative to W. Let (Ω) be the exhaustion of R consisting of regular subregions Ω of R directed by inclusion. Take any $\varphi \in C(\partial W)$. For each $\Omega \supset R \setminus W$ we consider the $u_\Omega \in H(W \cap \Omega) \cap C(\overline{W \cap \Omega})$ with $u_\Omega|\partial W = \varphi$ and $u_\Omega|\partial\Omega = 0$. Then it can be easily seen that

$$D\varphi := \lim_{\Omega \uparrow R} u_\Omega \in H(W) \cap C(\overline{W})$$

exists. Then we restrict D to $H(R)|\partial W$ and finally, by using the same notation D, we define the operator $D : H(R) \to H(W) \cap C(R)$ by $Du := D(u|\partial W)$ on W and $Du = u$ on $R \setminus W$. The operator $D : H(R) \to H(W) \cap C(R)$ is order preserving, linear, and bounded in the sense that

$$\|Du; R\|_\infty := \sup_R |Du| = \sup_{\partial W} |u| =: \|u; \partial W\|_\infty \quad (u \in H(R)).$$

Lastly, we define one more operator $E : H(R) \to H(W;\partial W)$ by

$$Eu := u - Du \quad (u \in H(R)). \tag{2.2}$$

It is an order preserving and linear operator. If E is bijective, then we say that $H(R)$ and $H(W;\partial W)$ are *canonically isomorphic*,

$$H(R) \cong H(W;\partial W) \tag{2.3}$$

in notation. It is hence important to know when this is the case. We mention (cf. [14], [13], [12], etc.):

THEOREM 2.4. *The absolute class $H(R)$ and the relative class $H(W;\partial W)$ are canonically isomorphic if and only if R is hyperbolic: $R \notin \mathcal{O}_G$.*

PROOF: Suppose first that $R \in \mathcal{O}_G$. As the compensation of the nonexistence of the Green function on R, $R \in \mathcal{O}_G$ is characterized by the existence of an *Evans function* $e(\cdot, a)$ on R with its negative pole at $a \in R$, where $e(\cdot, a) \in H(R \setminus \{a\})$, $e(\cdot, a)$ has a negative logarithmic singularity at a, and $\lim_{z \to \infty_R} e(z, a) = +\infty$ with ∞_R the Alexandroff point of R ([9], cf. also [16], [15]). Let $v := e(\cdot, a) - D(e(\cdot, a)|\partial W)$ on W and $v := 0$ on $R \setminus W$ so that $v \in H(W; \partial W)$. Contrary to the assertion, assume E is bijective and in particular surjective. Then there is a $u \in H(R)$ such that $Eu = v$. The function $u - v = u - Eu = Du$ is bounded on $\overline{W}$ and a fortiori

$$\lim_{z \to \infty_R} u(z) = \lim_{z \to \infty_R} v(z) = \lim_{z \to \infty_R} e(z, a) = +\infty$$

so that $u \equiv +\infty$ on R by the maximum principle, a contradiction.

The proof is complete if we show that E is bijective if $R \notin \mathcal{O}_G$. Here we use another characterization for R not being in the class $\mathcal{O}_G$ that $D1 \not\equiv 1$. First we assert that E is injective, i.e. $Eu = 0$ implies $u = 0$ for $u \in H(R)$. Then

$$\sup_W |u| = \sup_W |Du| = \sup_{\partial W} |u|$$

is a cosequence of $Eu = 0$ or $u = Du$ and on the other hand, trivially, $\sup_{R \setminus \overline{W}} |u| = \sup_{\partial W} |u|$. Therefore $\sup_R |u| = \sup_{\partial W} |u|$, which implies the constancy of u on R by the maximum principle so that $u = Du$ implies that $u \equiv 0$ on R. The essential part of the proof is thus the surjectivity of E under the assumption $R \notin \mathcal{O}_G$. We take a regular subregion $B \supset R \setminus W$ and set $\beta := \partial B$. As usual we denote by H_φ^B the function in $H(B) \cap C(\overline{B})$ with $H_\varphi^B|\beta = \varphi$ for $\varphi \in C(\beta)$. Consider the linear operator T from the Banach space $C(\beta)$ with the supremum norm $\|\varphi\|_\infty = \|\varphi; \beta\|_\infty = \sup_\beta |\varphi|$ to itself given by

$$T\varphi := D(H_\varphi^B|\partial W)|\beta \tag{2.5}$$

It is bounded, i.e. the operator norm

$$\|T\| := \sup_{\varphi \in C(\beta), \|\varphi\|_\infty = 1} \|\varphi\|_\infty < +\infty.$$

Since T is positive and linear, we can easily see that $\|T\| \leqq 1$ but $\|T\| = 1$ can happen if W is disconnected. Again we use the characterization $D1 \not\equiv 1$ of $R \notin \mathcal{O}_G$. For any $\varphi \in C(\beta)$, we have

$$|T\varphi| \leqq T|\varphi| \leqq T\|\varphi\|_\infty = (T1)\|\varphi\|_\infty.$$

Observe that $T1 = D(H_1^B|\partial W)|\beta$. Here $0 < D1 \leqq 1$ and $D1|\beta \not\equiv 1$ so that $H_{D1}^B|\partial W < 1$ and $D(H_{D1}^B|\partial W)|\beta \in (0,1)$. Thus

$$T^2 1 = T(T1) = T(D1|\beta) = D(H_{D1|\beta}^B|\partial W)|\beta \in (0,1)$$

and a fortiori

$$k := \sup_{\beta} T^2 1 \in (0,1).$$

Therefore, again by the fact that T^2 is also positive and linear, we see that

$$\|T^2\| = \sup_{\varphi\in C(\beta), \|\varphi\|_\infty=1} \|T^2\varphi\|_\infty \leqq k \in (0,1) \tag{2.6}$$

as a conseqeunce of

$$\|T^2\varphi\|_\infty \leqq \|T^2 1\|_\infty \|\varphi\|_\infty.$$

In view of the above (2.6) we have that

$$\begin{aligned}\left\|\sum_{n=0}^{\infty} T^n\right\| &= \left\|\sum_{m=0}^{\infty}(T^2)^m + \sum_{m=0}^{\infty} T(T^2)^m\right\| \\ &\leqq 2\sum_{m=0}^{\infty} \|T^2\|^m \leqq 2\sum_{m=0}^{\infty} k^m = \frac{2}{1-k}\end{aligned}$$

so that we see that the operator $(I-T)^{-1} = \sum_{n=0}^{\infty} T^n$ is a bounded linear operator of $C(\beta)$ to itself, where I is the identity operator of $C(\beta)$ to itself:

$$\|(I-T)^{-1}\| = \left\|\sum_{n=0}^{\infty} T^n\right\| \leqq \frac{2}{1-k}. \tag{2.7}$$

We are to show that there is a $u \in H(R)$ such that $Eu = u - Du = v$ for an arbitrarily given $v \in H(W;\partial W)$ in advance. We set $s := v|\beta \in C(\beta)$ and consider an abstract Fredholm equation

$$(I-T)\varphi = s. \tag{2.8}$$

By virtue of (2.7) we see that the equation (2.8) has a unique solution $\varphi \in C(\beta)$ given by the C. Neumann series $\varphi = \sum_{n=0}^{\infty} T^n s$. Let $p := H_\varphi^B$ defined on $\overline{B}$ and $q := v + D(p|\partial W)$ defined on W. On β, we see that

$$\begin{aligned}p|\beta &= \varphi = s + T\varphi = s + D(H_\varphi^B|\partial W)|\beta \\ &= s + D(p|\partial W) = (v + D(p|\partial W))|\beta = q|\beta,\end{aligned}$$

i.e. the harmonic function p on $\overline{B}$ and the harmonic function q on $\overline{W}$ coincide with each other on β. On ∂W, we observe that

$$\begin{aligned}p|\partial W &= H_\varphi^B|\partial W = 0 + D(p|\partial W)|\partial W \\ &= v|\partial W + D(p|\partial W)|\partial W = (v + D(p|\partial W))|\partial W = q|\partial W,\end{aligned}$$

i.e. the harmonic function p on $\overline{B}$ and the harmonic function q on $\overline{W}$ coincide with each other on ∂W. This shows that two harmonic functions p and q on $\overline{B \cap W}$

coincide with each other on $\partial(B \cap W) = \beta \cup \partial W$ so that $p \equiv q$ on $B \cap W$. Then the function u on R given by p on B and by q on W is a well defined harmonic function on R: $u \in H(R)$. Then finally we see that

$$\begin{aligned} Eu &= u - Du = u|W - Du|W \\ &= q - D(p|\partial W) = (v + D(p|\partial W)) - D(p|\partial W) = v, \end{aligned}$$

which was to be shown. This completes the proof. □

REMARK 2.9. The present Theorem 2.4 has never been in any literature and thus is new in the point that the connectedness of W is not postulated.

3. The Hilbert space $HD(W; \partial W)$. As we mentioned in the introduction 1, the motivation of considering the relative class $H(W; \partial W)$ instead of other normalization such as $H(R; a)$ is that $H(W; \partial W)$ inherits most of important structure of $H(R)$. We suppose that (2.3) holds so that $R \notin \mathcal{O}_G$. We denote by S either the operator $E : H(R) \to H(W; \partial W)$ or the inverse operator $E^{-1} : H(W; \partial W) \to H(R)$. It is clear that

FACT 3.1. *The operator S is positive and linear.*

Here the positiveness of S means that if $u \geqq 0$ on R, then $Su \geqq 0$ on R. Hence we can also say that E preserves the order in the sense that for any u and v in $H(R)$ we have $Eu \geqq Ev$ if and only if $u \geqq v$ on R. As a consequence we can say that

FACT 3.2. *The operator S preserves the lattice operations.*

This means that if $u \vee v$ ($u \wedge v$, resp.) can be defined for u and v in $H(R)$, then $(Eu) \vee (Ev)$ ($(Eu) \wedge (Ev)$, resp.) can be defined and

$$E(u \vee v) = (Eu) \vee (Ev) \quad (E(u \wedge v) = (Eu) \wedge (Ev), \text{resp.})$$

and the same is true for E^{-1}.

FACT 3.3. *The operator S preserves the supremum norm.*

Therefore $u \in H(R)$ is bounded if and only if $Eu \in H(W; \partial W)$ is bounded and

$$\|u; R\|_\infty = \|Eu; W\|_\infty$$

for every $u \in H(R)$.

FACT 3.4. *The operator S preserves the finiteness of Dirichlet integrals.*

Take an arbitrary $u \in H(R)$. Let (Ω) be the exhaustion of R consisting of regular subregions Ω directed by inclusion. For each $\Omega \supset R \backslash W$ let $u_\Omega \in H(W \cap \Omega) \cap C(R)$ with $u_\Omega = u$ on $R \setminus W$ and $u_\Omega = 0$ on $R \setminus \Omega$. By the Stokes formula, for $\overline{\Omega} \subset \Omega'$, we have

$$D(u_{\Omega'} - u_\Omega, u_{\Omega'}; R) = \int_{\partial(W \cap \Omega')} (u_{\Omega'} - u_\Omega) * du_{\Omega'} = 0$$

and thus $D(u_\Omega, u_{\Omega'}; R) = D(u_{\Omega'}; R)$ so that

$$D(u_\Omega - u_{\Omega'}; R) = D(u_\Omega; R) - D(u_{\Omega'}; R).$$

Since $\lim_{\Omega \uparrow R} u_\Omega = Du$, the above displayed relation implies that $D(u_{\Omega'}; R) \leqq D(u_\Omega; R)$ and

$$\lim_{\Omega \to R} D(Du - u_\Omega; R) = 0.$$

Thus we can in particular conclude that

$$D(Du; R) < +\infty \quad (u \in H(R)). \tag{3.5}$$

We now consider the relative class $HD(W; \partial W)$ corresponding to the absolut class $HD(R)$ so that

$$HD(W; \partial W) := \{v \in H(W; \partial W) : D(v; W) = D(v; R) < +\infty\}.$$

Hence, if $u \in HD(R)$, then

$$D(Eu; R)^{1/2} = D(u - Du; R)^{1/2} \leqq D(u; R)^{1/2} + D(Du; R)^{1/2} < +\infty.$$

Viewing $Eu \in L^{1,2}(R)$ and applying (1.1) to Eu, we have

$$D(Eu; R) = D(u; R) + D(Du; R)$$

and thus we conclude that

$$D(u; R) \leqq D(Eu; R) < +\infty \quad (u \in HD(R)). \tag{3.6}$$

Conversely, let $v \in HD(W; \partial W)$ and set $u = E^{-1}v$ so that $v = Eu = u - Du$ is the decomposition of $v \in L^{1,2}(R)$ in (1.1) with u the harmonic part of v and Du the potential part of v. Hence $D(v; R) = D(u; R) + D(Du; R) < +\infty$ and

$$D(E^{-1}v; R) \leqq D(v; R) < +\infty \quad (v \in HD(W; \partial W)). \tag{3.7}$$

A somewhat detailed account of Fact 3.4 is thus (3.6) and (3.7).

When $R \notin \mathcal{O}_G$, the above observation with Theorem 2.4 thus assures that $HD(R) \cong HD(W; \partial W) = HD(W) \cap H(W; \partial W)$. Even in the case $R \in \mathcal{O}_G$, since $HD(W; \partial W) = \{0\}$ and $HD(R) = \{0\}$ by our convention based upon the equivalence of $\mathbb{R} \subset L_0^{1,2}(R)$ and $R \in \mathcal{O}_G$, $HD(R) = HD(W; \partial W) = \{0\}$ and trivially $HD(R) \cong HD(W;\partial W)$. Hence we can conclude:

THEOREM 3.8. *The absolute class $HD(R)$ is canonically isomorphic to the relative class $HD(W;\partial W)$ for any R regardless of whether $R \in \mathcal{O}_G$ or not:*

$$HD(R) \cong HD(W;\partial W). \tag{3.9}$$

Thus by handling the relative class $HD(W;\partial W)$ in place of the absolute class $HD(R)$ we are not loosing any important property possessed by the absolute class $HD(R)$. For example, as is well known $HD(R)$ forms a Riesz space (i.e. vector lattice) and the same is true for $HD(W;\partial W)$. Actually we are moreover gaining a fabulous reword by adopting the relative class $HD(W;\partial W)$: the linear space $HD(W;\partial W)$ with the inner product $D(\cdot,\cdot;W)$ forms a Hilbert space, which was not the case of $HD(R)$ with $D(\cdot,\cdot;R)$. A good point having HD as a Hilbert space is that it carries the Bergman kernel. We will study certain properties of $HD(W;\partial W)$ from the view point that it is a Hilbert space with the Bergman kernel.

4. Royden compactification. An essential, important, and convenient tool for the class $HD(R)$ and also $HD(W;\partial W)$ is the theory of Royden compactification R^* of R, which is the compactification of R, i.e. a compact Hausdorff space containing R as its open and dense subset, such that every function $f \in L^{1,2}(R) \cap C(R)$ is extended to R^* as a $[-\infty,\infty]$-valued continuous function and thus extended class $L^{1,2}(R) \cap C(R)$ separates points in R^*, i.e. for any two distinct points ξ and η in R^* there is an $f \in L^{1,2}(R) \cap C(R)$ with $f(\xi) \neq f(\eta)$. We call the set $\gamma = \gamma R := R^* \setminus R$ the Royden boundary of R. The set $\delta = \delta R$ of regular points $\zeta \in \gamma$ in the sense of the standard PWB (i.e. Perron-Wiener-Brelot) procedure of solving harmonic Dirichlet problem on R with boundary data on γ is referred to as the *Royden harmonic boundary.* The following characterization of δ is remarkable:

$$\delta = \bigcap_{f \in L_0^{1,2}(R) \cap C(R)} f^{-1}(0), \tag{4.1}$$

so that δ is a compact subset of the compact subset of the compact subset γ of R^*. Thus three conditions $R \in \mathcal{O}_G$, $\mathbb{R} \subset L_0^{1,2}(R)$, and $\delta = \emptyset$ are equivalent by pairs (cf. e.g. [2], [15], [7], etc.).

The following tow kinds of measurements for compact subsets $K \subset \gamma$ are also innevitable tools in the theory of Royden compactifications: the capacity $\mathrm{cap}(K)$ of K and the harmonic measure $\mathrm{hm}(K)$ of K. Take any end W of R, which is bounded by γ and ∂W. For any compact subset $K \subset \gamma$ we define the *capacity*, or more precisely the variational 2 capacity, $\mathrm{cap}(K)$, of K relative to W by

$$\mathrm{cap}(K) := \inf_f D(f; R), \tag{4.2}$$

where f runs over the class of functions $f \in L^{1,2}(R) \cap C(R)$ such that $f|K \geqq 1$ and $f|R \setminus W \leqq 0$. We denote by H_f^R for any $f \in C(\gamma)$ the unique harmonic function on R such that $H_f^R \in C(R^*)$ and $H_f^R|\delta = f$. Take a reference point $a \in R$. Then there is a unque Borel measure ω on γ supported by δ such that

$$H_f^R(a) = \int_\gamma f\,d\omega \quad (f \in C(\gamma)). \tag{4.3}$$

Then the *harmonic measure* $\mathrm{hm}(K)$ of K reative to $a \in R$ is given by

$$\mathrm{hm}(K) = \int_K d\omega. \tag{4.4}$$

It can be seen easily that

$$\mathrm{cap}(\gamma \setminus \delta) = 0 \quad \text{and} \quad \mathrm{hm}(\gamma \setminus \delta) = 0$$

and therefore we consider $\mathrm{cap}(K)$ and $\mathrm{hm}(K) = \omega(K)$ only for compact subsets $K \subset \delta$. Starting from capacities (measues, resp.) for compact subsets of δ we can consider outer capacities (outer measurs, resp.) which gives rise to the notion of capacitability (measurability, resp.). Borel subsets of δ are capacitable (measurable, rep.). Subsets of δ of outer capacity zero (of outer measure zero) are capacitable (measurable, rep.) and thus of capacity zero (measure zero, resp.). A property concerning δ is said to hold on δ *quasieverywhere*, abbreviated as q.e., (*almost everywhere*, abbreviated as a.e., resp.) if it holds on δ except for its subset of capacity zero (of measure zero, resp.)

To understand $\mathrm{cap}(K)$ more precisely we need to consider the extention of the conjugate differential $*du$ of u harmonic and Dirichlet finite on an ideal boundary neighborhood of γ, i.e. a complement in R of a compact subset A of R, to δ. We say that for $u \in HD(R \setminus A)$ $*du$ can be defined on δ as a signed Radon measure on δ if for any end W of R with $W \subset R \setminus A$

$$D(v, u; W) = \int_\delta v * du \tag{4.5}$$

for every $v \in HD(W; \partial W)$. The measure $*du$ on δ, if it exists, is uniquely determined. Moreover, if $*du$ is determined by (4.5) for one admissible W, then the same $*du$ satisfies (4.5) for every admissible W. In short, W is immaterial in the definition of $*du$ as far as W satisfies $W \subset R \setminus A$.

Returning to the definition (4.2) of the capacity $c(K)$ of a compact subset $K \subset \delta$, it is readily seen that the family of competing functions for the variation (4.2) can be reduced to the subfamily $\mathcal{F}_K$ of $HD(W : \partial W)$ consisting of functions f with $0 \leqq f \leqq 1$ on R and $f|K = 1$. Then we can conclude the unique existence of the extremal function c_K in the closure $\overline{\mathcal{F}}_K$ of $\mathcal{F}_K$ in $HD(W; \partial W)$ for the variation (4.2): $\text{cap}(K) = D(c_K; R)$ with $0 \leqq c_K \leqq 1$ on R. The function c_K is referred to as the *capacitary function* of K. The following characterization of the capacitary function c_K is important and also usefull ([11]): first of all, $c_K \in HD(W; \partial W)$; secondly, $*dc_K$ exists on δ and $*dc_K \geqq 0$ there; thirdly, $*dc_K = 0$ on $\delta \setminus K$; fourthly and lastly, $c_K = 1$ q.e. on K. Conversely, if a function f on R satisfies the above four conditions, then $f = c_K$ on R. The measure $*dc_K$ on δ is a Borel measure μ_K on δ such that

$$*dc_K = d\mu_K. \tag{4.6}$$

The measure μ_K is referred to as the *capacitary measure* of K. In these terms, $\text{cap}(K)$ is expressed as follows:

$$\text{cap}(K) = D(c_K; W) = \int_\delta *dc_K = \mu_K(\delta) = \mu_K(K). \tag{4.7}$$

Between capacities and harmonic measures we have the following relations ([11]):

$$\text{hm}(K) \leqq \kappa \cdot \text{cap}(K)^{1/2} \tag{4.8}$$

for any compact subset $K \subset \delta$, where κ is a finite positive constant depending only upon W and $a \in R$.

5. Bergman kernel. The point evaluation $u \mapsto u(a)$ on $HD(W; \partial W)$ for a fixed poiny $a \in R$ is a bounded functional on $HD(W; \partial W)$. There are many proofs for this fact, some of which are simple and elementary. By virtue of the fact that $HD(W; \partial W)$ forms a Hilbert space, the boundedness of point evaluation at $a \in R$ is equivalent to the existence of the reproducing kernel $B(\cdot, a) \in HD(W; \partial W)$ characterlized by

$$u(a) = D(u, B(\cdot, a); R)$$

for every $u \in HD(W; \partial W)$. The function $B(\cdot, \cdot) = B(\cdot, \cdot; W)$ on $W \times W$ or even on $R \times R$ is called the *Bergman kernel* on $W \times W$, or simply on W. It is also called, more precisely, the Dirichlet finite harmonic Bergman kernel on W. Recall that the Green function $G(\cdot, \zeta) = G(\cdot, \zeta; W)$ on W with its pole at $\zeta \in W$ is the unique solution of the Poisson equation

$$-\Delta G(\cdot, \zeta) = 2\pi Dirac_\zeta \tag{5.1}$$

with the boundary data

$$G(\cdot,\zeta)|\partial W = 0 \quad \text{and} \quad G(\cdot,\zeta)|\delta = 0,$$

where $Dirac_\zeta$ is the Dirac measure on R supported at $\zeta \in W$. This gives the Green kernel $G(\cdot,\cdot)$ on $W \times W$, or simply on W. Similarly, the Neumann function $N(\cdot,\zeta) = N(\cdot,\zeta;W)$ on W with its pole at $\zeta \in W$ is the unique solution of the same Poisson equation as (5.1)

$$-\Delta N(\cdot,\zeta) = 2\pi Dirac_\zeta \tag{5.2}$$

on W with the boundary data

$$N(\cdot,\zeta)|\partial W = 0 \quad \text{and} \quad *dN(\cdot,\zeta)|\delta = 0.$$

This gives the Neumann kernel $N(\cdot,\cdot)$ on $W \times W$, or simply on W. By using the standard exhaustion method with the Stokes formula, we can establish the following more explit representation of the Bergman function or kernel:

$$B(\cdot,\zeta) = N(\cdot,\zeta) - G(\cdot,\zeta) \geqq 0. \tag{5.3}$$

As a basic general property of the reproducing kernel, the Bergman kernel is symmetric: $B(a,b) = B(b,a)$ for $(a,b) \in W \times W$. In view of (5.3) this also follows from those of the Neumann kernel $N(\cdot,\cdot)$ and the Green kernel $G(\cdot,\cdot)$. Since $HD(W;\partial W) = \{0\}$ is equivalent to $R \in \mathcal{O}_G$, we see that $B(\cdot,\zeta) \equiv 0$ if and only if $R \in \mathcal{O}_G$. With (5.3) this also follows from the fact that $N(\cdot,\zeta) \equiv G(\cdot,\zeta)$ if and only $R \in \mathcal{O}_G$. Hence we can say based upon (5.3) that

$$B(\cdot,\zeta) > 0 \quad \text{if and only if} \quad R \notin \mathcal{O}_G$$

for every $\zeta \in W$. Thus by applying the Harnack inequality we can see that

$$B(\cdot,\cdot) \in C(W \times W)$$

and therefore $B(\cdot,\cdot)$ is not only separately harmonic but also harmonic on $W \times W$. The most economical way to derive the above, though not too elementary, is just to appeal to the harmonic version of the Hartogs theorem due to Lelong [6] that the separate harmonicity implies the joint harmonicity. However, we cannot unfortunately make any efficient use of the harmonicity of $B(\cdot,\cdot)$ as a function of two variables at present. One step further we need to investigate the continuity of B on the Royden compactification R^* of R. For the purpose we can also use (5.3) since the continuity problem of N and G on R^* is relatively easier than that of B itself. However the continuity of N and G can only be obtained in the sense of $[0, +\infty]$-valued continuity. Hence we have to be very careful in the application of (5.3) when $\infty - \infty$ occurs. Keeping this in mind we start by introducing the notation

$$N \cap \alpha = \min\{N, \alpha\} \quad \text{and} \quad G \cap \alpha = \min\{G, \alpha\}$$

for every $\alpha \in \mathbb{R}^+$. The basic relations for the study of continuity of B on R^* are the following (5.4) and (5.5). First we maintain that

$$D(N(\cdot, \zeta) \cap \alpha; W) = D(G(\cdot, \zeta) \cap \alpha; W) = 2\pi\alpha \tag{5.4}$$

for every $\alpha \in \mathbb{R}^+$ and $\zeta \in W$ (cf. [2], [15]). The proof of (5.4) is almost trivial for α so large as to have the compact $\{z \in W : N(z, \zeta) \geqq \alpha\}$ and compact $\{z \in W : G(z, \zeta) \geqq \alpha\}$ in W. The proof for small $\alpha \in \mathbb{R}^+$ except for the case $\alpha = 0$ such that the above two sets are not compact in W is far form simple and easy especially for the part of N. For a complete proof we need a couple of pages at least. Anyhow once (5.4) is established we see that $N(\cdot, \zeta) \cap \alpha$ and $G(\cdot, \zeta) \cap \alpha$ belong to $L^{1,2}(R) \cap C(R)$ for every α in $\mathbb{R}^+$ so that these are continuous on $(\partial W) \cup W \cup \gamma$ and hence on R^* by setting $N(z, \zeta) = G(z, \zeta) = 0$ when z or ζ is in $R \setminus W$. Thus $N(\cdot, \zeta) = N(\zeta, \cdot)$ and $G(\cdot, \zeta) = G(\zeta, \cdot)$ are $[0, \infty]$-valued continious on R^* and finitely continuous on $R^* \setminus \{\zeta\}$. Next, applying the Fatou lemma to (5.4) as the interior point $\zeta \in R$ goes to a point $\zeta \in \gamma$, we deduce

$$D(N(\cdot, \zeta) \cap \alpha; W) \leqq 2\pi\alpha, \qquad D(G(\cdot, \zeta) \cap \alpha; W) \leqq 2\pi\alpha \tag{5.5}$$

for every $\zeta \in \gamma$ and hence for every $\zeta \in R^*$ (cf. [2], [15]). Therefore $N(\cdot, \zeta) = N(\zeta, \cdot)$ and $G(\cdot, \zeta) = G(\zeta, \cdot)$ are $[0, \infty]$-valued continuous on R^*. Thus $N(\cdot, \cdot)$ and $G(\cdot, \cdot)$ can be defined on $R^* \times R^*$ by their separate continuity and the extended ones are also symmetric: $N(a, b) = N(b, a)$ and $G(a, b) = G(b, a)$ for every $(a, b) \in R^* \times R^*$. As we saw above, $N(\cdot, \zeta) = N(\zeta, \cdot)$ and $G(\cdot, \zeta) = G(\zeta, \cdot)$ are $[0, \infty]$-valued continuous on R^* for any fixed $\zeta \in R^*$ and in particular for any $\zeta \in R \cup \delta$. At this point it is fatally important that

$$G(\cdot, \zeta) = G(\zeta, \cdot) \equiv 0 \tag{5.6}$$

on R and hence on R^* for any $\zeta \in \delta$. We can thus conclude that $B(\cdot, \zeta) = B(\zeta, \cdot)$ can be defined by $N(\cdot, \zeta) - G(\cdot, \zeta) = N(\zeta, \cdot) - G(\zeta, \cdot) = N(\cdot, \zeta) = N(\zeta, \cdot)$ for any $\zeta \in \delta$. In short we have seen that the Bergman kernel $B(\cdot, \cdot)$ can be extended to $(R \cup \delta) \times (R \cup \delta)$ as a symmetric kernel which is separately $[0, \infty]$-valued continuous there. We stress that

$$B(z, \zeta) = N(z, \zeta) = N(\zeta, z) = B(\zeta, z) \tag{5.7}$$

for $(z, \zeta) \in R^* \times \delta$. As for the continuity of B we once more state the following:

PROPOSITION 5.8 *The Bergman kernel $B(\cdot,\cdot)$ is finitely continuous on $R \times R$; the function $B(\cdot,\zeta) = B(\zeta,\cdot)$ is finitely continuous on R^* for any fixed $\zeta \in R$; the function $B(\cdot,\zeta) = B(\zeta,\cdot)$ is $[0,\infty]$-valued continuous on R^* for any fixed $\zeta \in \delta$; the function $(z,\zeta) \mapsto B(z,\zeta) = B(\zeta,z)$ of two variables $(z,\zeta) \in R^* \times \delta$ is lower semicontinuous on $R^* \times \delta$.*

PROOF: All assertions except for the last have already been explained. In view of (5.7), the function $(z,\zeta) \mapsto B(z,\zeta)$ is identical with the function $(z,\zeta) \mapsto N(z,\zeta)$ for $(z,\zeta) \in R^* \times \delta$. By the very definition of the Kuramochi compactification of R , $N(z,\zeta) = N(\zeta,z)$ is lower semicontinuous on the product space of Kuramochi compactification of R (cf. [2]). Observe that the Kuramochi compactification of R is a quotient space of R^*. Hence $N(z,\zeta)$ is lower semicontinuous on $R^* \times R^*$ and in particular on $R^* \times \delta$. Hence we have deduced the last assertion. □

6. Bergman integrals. Fix a point $a \in W$. Since

$$u \mapsto u(a) = D(u, B(\cdot,a); W) : HD(W;\partial W) \to \mathbb{R}$$

is a positive linear functional and

$$u \mapsto u|\delta : HD(W;\partial W) \to HD(W;\partial W)|\delta$$

is a bijective order preserving linear isomorphism and the linear subspace $HD(W;\partial W)|\delta$ of $C(\delta)$ is densely and isometrically embedded in $C(\delta)$ with respect to the supremum norm $\|\cdot;\delta\|_\infty$, we can conclude the unique existence of a Borel measure ν on δ such that

$$u(a) = D(u, B(\cdot,a); W) = \int_\delta u d\nu$$

for every $u \in HD(W;\partial W)$, which assures the existence of $*dB(\cdot,a)$ and $*dB(\cdot,a) = d\nu$ on δ. Thus we have obtained the following identity:

$$(I-D)H_\varphi^R(a) = \int_\delta \varphi * dB(\cdot,a) \tag{6.1}$$

for every $\varphi \in C(\delta)$. This shows that $*dB(\cdot,a)$ and the harmonic measure $d\omega = d\mathrm{hm}$ are mutually absolutely continuous and

$$*dB(\cdot,a) = b d\mathrm{hm}, \tag{6.2}$$

where b is a Borel function on δ with $k^{-1} \leqq b \leqq k$ for some finite positive constant k. In short, $*dB(\cdot,a)$ is essentially the harmonic measure on δ.

For any (signed) Radon measure μ on δ the function

$$B\mu := \int_\delta B(\cdot,\zeta)d\mu(\zeta) \tag{6.3}$$

on R is referred to as a *Bergman integral* of μ. In the special case of $d\mu = fd\omega$, we denote $B\mu$ simply by Bf:

$$Bf := \int_\delta B(\cdot,\zeta)f(\zeta)d\omega(\zeta),$$

which we call the Bergman integral of f. The Bergman integral plays an important role in the Neumann problem like the Poisson integral plays an important role in the Dirichlet problem. The Neumann problem (or rather the Robin problem) we consider here is the following: given a Radon measure μ on δ and we are to find a $u \in HD(W;\partial W)$ such that

$$*du = d\mu \tag{6.4}$$

on δ. If the solution $u \in HD(W;\partial W)$ with (6.4) is found, then it has the following Bergmann integral expression:

$$u = B\mu. \tag{6.5}$$

In fact, for any point $z \in R$, we have

$$\begin{aligned} u(z) &= D(u, B(\cdot,z);W) = D(B(z,\cdot),u;W) \\ &= \int_\delta B(z,\zeta) * du(\zeta) = \int_\delta B(z,\zeta)d\mu(\zeta) = B\mu(z), \end{aligned}$$

i.e. $u = B\mu$, which was to be shown. We define the mutual energy of two Radon measures μ and ν on δ by

$$B[\mu,\nu] := \int_\delta \left(\int_\delta B(z,\zeta)d\mu(z)\right) d\nu(\zeta)$$

and the energy of μ on δ by $B[\mu] := B[\mu,\mu]$ if these can be defined. Then for the solvability of (6.4) the measure μ must satisfy the condition

$$B[\mu] < +\infty. \tag{6.6}$$

In fact, since we have

$$\begin{aligned} D(B\mu;W) &= \int_\delta B\mu * dB\mu = \int_\delta B\mu d\mu \\ &= \int_\delta \left(\int_\delta B(\zeta,\xi)d\mu(\xi)\right) d\mu(\zeta) = B[\mu] \end{aligned}$$

and $D(B\mu;W) < \infty$, we must conclude that (6.6) is valid. We have thus seen the following:

PROPOSITION 6.7. *If the Neumann problem* (6.4) *is solved, then the solution is the Bergman integral* $B\mu$ *of* μ *and* μ *is of finite energy:* $B[\mu] = D(B\mu; W) < +\infty$.

Therefore the problem is to settle the following expectation:

CONJECTURE 6.8. *If* μ *is a Radon measure on* δ *with* $B[\mu] = D(B\mu; W) < +\infty$, *then* $*dB\mu = d\mu$ *on* δ, *i.e.* (6.4) *is solvable.*

PSEUDO-PROOF: By the Fubini theorem, for every $v \in HD(W; \partial W)$, we have

$$D(v, B\mu; W) = D(v, \int_\delta B(\cdot, \zeta) d\mu(\zeta); W)$$

$$= \int_\delta D(v, B(\cdot, \zeta); W) d\mu(\zeta) = \int_\delta v(\zeta) d\mu(\zeta),$$

which shows that $*dB\mu$ exists on δ and $*dB\mu = d\mu$. □

Obviously, one sees at once that the argument above is a fake but still it sounds considerably plausible, which is a reason we suspect the conjecture might be true. A few positive results known thus far for Radon measure μ on δ with $B[\mu] = D(B\mu; W) < +\infty$ are: if μ is ω absolutely continuous with the density $d\mu/d\omega \in L^2(\delta, d\omega)$, then $*dB\mu = d\mu$ (Maeda [7]); if μ is the lift up of a Radon measure on the Kuramochi boundary of R, then $*dB\mu = d\mu$ (Constantinescu and Cornea [2]).

7. Capacitary functions, revisited. We denote by $\mathcal{C}$ the family of capacitary functions of compact subsets of the harmonic boundary δ of R:

$$\mathcal{C} := \{c_K = B\mu_K : K \subset \delta, \text{ compact}\}. \tag{7.1}$$

A subset Z of a Banach space X (or more generally a locally convex linear topological space X) is said to be a *fundamental set* (cf. e.g. [3]) if the closed linear span, i.e. the closure of the set of all finite linear combinations of elements in Z, $\mathrm{cls}(Z)$ in notation, coincides with the total space X. If the set Z consists of easily handled elements possessing some characteristic properties in X, then the set Z should be helpful to investigate the space X. In this sense the following result contributes to clarifying the structure of the Hilbert space $HD(W; \partial W)$.

THEOREM 7.2. *The family* $\mathcal{C}$ *given by* (7.1) *is a fundamental set of the Hilbert space* $HD(W; \partial W)$:

$$HD(W; \partial W) = \mathrm{cls}(\mathcal{C}). \tag{7.3}$$

PROOF: We denote by $\mathcal{C}^\perp$ the set of members in $HD(W;\partial W)$ perpendicular to each element in $\mathcal{C}$:

$$\mathcal{C}^\perp := \{u \in HD(W;\partial W) : D(u, c_K; W) = 0 \text{ for every } c_K \in \mathcal{C}\}.$$

Then, since we have the orthogonal decomposition

$$HD(W;\partial W) = \mathrm{cls}(\mathcal{C}) \oplus \mathcal{C}^\perp,$$

the required assertion is equivalent to $\mathcal{C}^\perp = \{0\}$ which now we derive. Contrary to the assertion assume the existence of a $u \in \mathcal{C} \setminus \{0\}$. If $u|\delta$ is a constant k, then $k \in \mathbb{R} \setminus \{0\}$ and $u = kc_\delta \in \mathrm{cls}(\mathcal{C})$. Originally

$$u \in \mathcal{C}^\perp \equiv (\mathrm{cls}(\mathcal{C}))^\perp.$$

This is clearly a contradiction. Thus $u|\delta$ is not constant. Any nonempty open subset of δ is of positive harmonic measure and thus, by (4.8), any compact subset with nonempty interior in δ is of positive capacity. In view of this observation, we can find two values

$$-\infty < \alpha < \beta < +\infty$$

and two disjoint compact subsets K_α and K_β of δ which are the closures of open subsets of δ so that their capacities are strictly positive such that

$$\sup_{K_\alpha} u \leqq \alpha < \beta \leqq \inf_{K_\beta} u.$$

Let $d\mu_\alpha$ ($d\mu_\beta$, resp.) be the unit positive Borel measure on δ given by

$$(1/\mathrm{cap}(K_\alpha)) * dc_{K_\alpha} \quad ((1/\mathrm{cap}(K_\beta)) * dc_{K_\beta}, \quad \text{resp.}).$$

Then, since

$$B_{\mu_\alpha} = (1/\mathrm{cap}(K_\alpha))c_{K_\alpha} \quad (B_{\mu_\beta} = (1/\mathrm{cap}(K_\beta))c_{K_\beta}, \quad \text{resp.}),$$

we see that the function $B_{\mu_\alpha} - B_{\mu_\beta}$ belongs to the class $\mathrm{cls}(\mathcal{C})$ so that

$$D(u, B_{\mu_\alpha} - B_{\mu_\beta}; W) = 0. \tag{7.4}$$

On the other hand we see that

$$\begin{aligned} D(u, B_{\mu_\alpha} - B_{\mu_\beta}; W) &= \int_\delta u\,d\mu_\alpha - \int_\delta u\,d\mu_\beta \\ &= \int_{K_\alpha} u\,d\mu_\alpha - \int_{K_\beta} u\,d\mu_\beta \leqq \alpha - \beta. \end{aligned}$$

This estimate $D(u, B_{\mu_\alpha} - B_{\mu_\beta}; W) \leqq \alpha - \beta$ with (7.4) yields $\alpha \geqq \beta$, which contradicts the standing assumption $\alpha < \beta$. $\square$

COROLLARY 7.5. *The set of $u \in HD(W; \partial W)$ for which $*du$ exists on δ is dense in the Hilbert space $HD(W; \partial W)$.*

PROOF: The set in question described in the statement of the above assertion contains as its subset the linear span of $\mathcal{C}$, which is dense in $HD(W; \partial W)$ by Theoem 7.2 and all the more the set in question is dense in $HD(W; \partial W)$. $\square$

REMARK 7.6. Concerning two fundamental sets firstly in the convex space of essentially positive harmonic functions (i.e., expressible as differences of two positive harmonic functions) $HP(R)$ with the topology of the locall uniform convergence and secondly in the Banach space $HB(R)$ of bounded harmonic functions with the supremum norm, we recall the following two known results, one is long established extremely well known theory of Martin [8] and the other [10] is poorly publicized meager and tiny result.

The set $\mathcal{K} := \{K(\cdot, \zeta) : \zeta \in \Delta_1\}$ of minimal Martin kernels $K(\cdot, \zeta)$ with its pole ζ in the minimal Martin boundary Δ_1 constitutes a fundamental set in the convex space $HP(R)$ with the topology of local uniform convergence. An important point here is that $\mathcal{K}$ is not only a fundamental set in $HP(R)$ but also we have a definite way of approximating each member u of $HP(R)$ using the set $\mathcal{K}$: the Martin integral representation. For any $u \in HP(R)$, there exists a unique Radon measure μ on Δ_1 such that

$$u = \int_{\Delta_1} K(\cdot, \zeta) d\mu(\zeta) \tag{7.7}$$

on R ([8], also cf. [2], [5]). These are of course concrete examples of the famous theorem of Krein-Milman in functional analysis and, as its precision, the Choquet integral representation theorem (cf. e.g. [18]).

The set $\mathcal{W} := \{w :$ harmonic measure function on $R\}$, where a harmonic function w on R is said to be a harmonic measure function if it satisfies the condition $w \wedge (1 - w) \equiv 0$ on R so that it is bounded on R, is a fundamental set in the Banach space $HB(R)$ of bounded harmonic functions on R with the supremum norm $\|\cdot; R\|_\infty$. A one parameter family $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ is referred to as a resolution of unity of finite type if the following 4 conditions are satisfied: $e_\lambda \in \mathcal{W}$ for every $\lambda \in \mathbb{R}$; $e_\lambda \leqq e_\mu$ for $\lambda \leqq \mu$; $e_{\lambda+0} = e_\lambda$ for every $\lambda \in \mathbb{R}$ in the order sense, i.e. $e_{\lambda+0} := \inf_{\mu \geqq \lambda} e_\mu$; there are two finite numbers $-\infty < \underline{\lambda} \leqq \overline{\lambda} < +\infty$ such that $e_\lambda = 0$ for all $\lambda < \underline{\lambda}$ and $e_\lambda = 1$ for all $\lambda \geqq \overline{\lambda}$. Again an important point here about the family $\mathcal{W}$ is that it is not only a fundamental set in the Banach space

$HB(R)$ with the supremum norm but also we have a canonical way of approximating each function u in $HB(R)$ by using the set $\mathcal{W}$: the spectral resolution theorem. There is a bijective correspondence $u \leftrightarrow \{e_\lambda\}_{\lambda\in\mathbb{R}}$ between $HB(R)$ and the family of resolutions of unity of finite type such that

$$u = \int_{-\infty}^{\infty} \lambda de_\lambda \quad \text{(Riemann-Stieltjes integral)} \tag{7.8}$$

([10]). The above integral representation of u is referred to as the spectral resolution of the function u and $\{e_\lambda\}_{\lambda\in\mathbb{R}}$ appearing in the above integral as the resolution of unity of the function u.

Standing upon the above view points backing up the expressions like (7.7) and (7.8), it is an intresting and probably very important theme to seek and establish some canonical way to express each function u in $HD(W;\partial W)$ by using a certain standard way of selecting functions from $\mathcal{C}$ and forming their linear combintions by e.g. integration or the like.

References

[1] L. V. AHLFORS AND L. SARIO: *Riemann Surfaces*, Princeton Mathematical Series, No. **26**, Princeton Univ. Press, 1960.

[2] C. CONSTANTINESCU UND A. CORNEA: *Ideale Ränder Riemannscher Flächen*, Ergebnisse der Mathematik und ihre Grenzgebiete, Band **32**, Springer-Verlag, 1963.

[3] N. DUNFORD AND J. T. SCHWARTZ: *Linear Operators, (Pasrt I: General Theory)*, Pure and Applied Mathematics, Vol. **7**, Interscience Publishers, 1967.

[4] J. HEINONEN, T. KILPELÄINEN AND O. MARTIO: *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, Clarendon Press, 1993.

[5] L. L. HELMS: *Introduction to Potential Theory*, Pure and Applied Mathematics, Vol. **22**, Wiley-Interscience, 1969.

[6] P. LELONG: *Fonctions plurisousharmoniques et fonctions analytiques de variables réelles*, Ann. Inst. Fourier (Grenoble), **11**(1961), 515-562.

[7] F.-Y. MAEDA: *Dirichlet Integrals on Harmonic Spaces*, Lecture Notes in Mathematics, **803**, Springer-Verlag, 1980.

[8] R. S. MARTIN: *Minimal positive harmonic functions*, Trans. American Math. Soc., **49**(1941), 137-172.

[9] M. NAKAI: *On Evans potential*, Proc. Japan Acad., **38**(1962), 624-629.

[10] M. NAKAI: *Spectral resolutions of bounded harmonic functions*, Proceedings of the 2008 Workshop on Potential Theory in Akita, 81-104.

[11] M. NAKAI: *Extremal functions for capacities*, J. Math. Soc. Japan, **61**(2009), 345-361.

[12] M. NAKAI, S. SEGAWA AND T. TADA: *Surfaces carrying no singular functions*, Proc. Japan Acad., **85**, Ser. A(2009), 163-166.

[13] M. NAKAI AND T. TADA: *Monotoneity and homogeneity of Picard dimensions for signed radial densities*, Hokkaido Math. J., **26**(1997), 253-296.

[14] B. RODIN AND L. SARIO: *Principal Functions*, University Series in Higher Mathematics, Van Nostrand, 1968.

[15] L. SARIO AND M. NAKAI: *Classification Theory of Riemann Surfaces*, Grundlehren der mathematischen Wissenschaften in Einzelldarstellungen, Band **164**, Springer-Verlag, 1970.

[16] L. SARIO AND K. NOSHIRO: *Value Distribution Theory*, University Series in Higher Mathematics, D. Van Nostrand, 1966.

[17] L. SARIO AND K. OIKAWA: *Capacity Functions*, Grundlehren der mathematischen Wissenschaften in Einzelldarstellungen, Band **149**, Springer-Verlag, 1969.

[18] K. YOSIDA: *Functional Analysis*, Grundlehren der matematischen Wissenschaften in Einzeldarstellungen, Band **123**, Springer-Verlag, 1965.