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An integral formula for powers of the Bergman kernel on representative bounded homogeneous domains

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Abstract. The representative domain gives a nice realization for a bounded homogeneous domain. For the classical domain, its representative domain is a constant multiple of the standard realization. We show that the integral of the negative power K^{-s} of the normalized Bergman kernel K of the domain equals the reciprocal of a polynomial of s, called the Hua polynomial, whose roots are negative rational numbers determined explicitly from structure of the holomorphic automorphism group of the domain.

Introduction.

In [5], Hua proved fascinating formulas about harmonic analysis on classical domains. For instance, if we write $R_I(m,n)$ $(1 \le n \le m)$ for the classical domain $\{ Z \in \mathrm{Mat}(m,n;\mathbb{C})\,;\, I - ZZ^* \text{ is positive definite}\}$ of type I, we find the following integral evaluation in [5, p. 40]:

$$\int_{R_I(m,n)} \det(I - ZZ^*)^\lambda \, dV(Z) = \pi^{mn} \cdot \frac{\prod_{j=1}^n \Gamma(\lambda + j) \prod_{k=1}^m \Gamma(\lambda + k)}{\prod_{l=1}^{m+n} \Gamma(\lambda + l)} \qquad (\lambda > -1), \tag{1}$$

where dV denotes the Lebesgue measure with respect to the natural complex coordinate. In particular, we get the volume $\mathrm{Vol}(R_I(m,n))$ of the domain $R_I(m,n)$ by putting $\lambda = 0$. Furthermore, Hua showed similar integral formulas for the other classical domains, where the results are always expressed as quotients of products of the Gamma functions. Now we observe that the right-hand side of (1) is rewritten as

$$\pi^{mn} \prod_{j=1}^n \frac{\Gamma(\lambda + j)}{\Gamma(\lambda + m + n + 1 - j)} = \frac{\pi^{mn}}{\prod_{j=1}^n (\lambda + j)_{m+n+1-2j}},$$

where $(a)_p$ denotes the Pochhammer polynomial: $(a)_p = a(a+1)\cdots(a+p-1)$. Note that the denominator is a polynomial of λ with the degree being $\sum_{j=1}^n (m +$

$n+1-2j) = mn = \dim_{\mathbb{C}} R_I(m,n)$. This observation is valid for each classical domain. Indeed, using theory of Jordan triple system, Yin, Lu and Roos [13] generalized Hua's result to bounded symmetric domains as follows. Let $\mathcal{S}$ be the Harish-Chandra realization of an irreducible bounded symmetric domain of dimension N, and $\mathcal{N}(Z,W)$ be the associated generic minimal polynomial (if $\mathcal{S} = R_I(m,n)$, then $\mathcal{N}(Z,W) = \det(I - ZW^*)$). Then it is shown [13, (2.5)] that

$$\int_{\mathcal{S}} \mathcal{N}(Z,Z)^{\lambda}\, dV(Z) = \frac{p(0)}{p(\lambda)} \mathrm{Vol}(\mathcal{D}) \qquad (\Re\lambda > -1),$$

where $p(\lambda)$ is a polynomial of degree N, called *the Hua polynomial*, whose roots are negative half integers determined explicitly.

In this article, we shall consider further generalization of Hua's result to a bounded homogeneous domain (BHD) $\mathcal{U}$. Since there is no Jordan triple system corresponding to a non-symmetric BHD, it is a non-trivial question what the generalization should be. We recall that, for the symmetric case $\mathcal{U} = \mathcal{S}$, the Bergman kernel $K_{\mathcal{S}}(Z,W)$ equals $\mathrm{Vol}(\mathcal{S})^{-1}\mathcal{N}(Z,W)^{-\gamma_{\mathcal{S}}}$ where $\gamma_{\mathcal{S}}$ is a certain positive integer. Thus, for a general BHD $\mathcal{U}$, we substitute the reciprocal $\{\mathrm{Vol}(\mathcal{U})K_{\mathcal{U}}(Z,W)\}^{-1}$ of the normalized Bergman kernel for the generic minimal polynomial $\mathcal{N}(Z,W)$. On the other hand, results in [6] suggest that the representative domain can be regarded as a standard realization of BHD like the Harish-Shandra realization of bounded symmetric domain. Eventually, we obtain the following result: Let $\mathcal{U}$ be a representative BHD of dimension N. Then we can determine rational numbers $a_1, a_2, \ldots, a_N$ so that

$$\int_{\mathcal{U}} \{\mathrm{Vol}(\mathcal{U})K_{\mathcal{U}}(\zeta,\zeta)\}^{-s}\, dV(\zeta) = \frac{\mathrm{Vol}(\mathcal{U})}{F(s)} \qquad (\Re s > -\min a_i), \tag{2}$$

$$\text{where} \quad F(s) := \prod_{i=1}^{N} (1 + \frac{s}{a_i}). \tag{3}$$

Let $\mathcal{D}$ be a (not necessarily bounded) domain biholomorphic to the representative BHD $\mathcal{U}$. Thanks to a canonical nature of the Bergman kernel $K_{\mathcal{U}}$ (Theorem 1), the formula (2) is equivalent to

$$\int_{\mathcal{D}} |F(s)K_{\mathcal{D}}(z,w)^{s+1}|^2 K_{\mathcal{D}}(z,z)^{-s} dV(z) = F(s)K_{\mathcal{D}}(w,w)^{s+1} \qquad (w \in \mathcal{D},\ \Re s > -\min a_i), \tag{4}$$

which implies that the weighted Bergman space $L^2_a(\mathcal{D}, K_{\mathcal{D}}(z,z)^{-s}dV(z))$ has the reproducing kernel given by $F(s)K_{\mathcal{D}}(z,w)^{s+1}$. We should notice that the statement in this form is already known essentially in [4] (see also [10]) where $\mathcal{D}$ is a homogeneous Siegel domain, and $F(s)$ is expressed as a quotient of products of the Gamma functions (see Section 3). Nevertheless, we think that the formulation (2) in terms of the representative domain as well as the expression of $F(s)$ as a polynomial is worth claiming to be new.

§1. Preliminaries.

1.1. Let $\mathcal{D} \subset \mathbb{C}^N$ be a bounded complex domain, and $K_{\mathcal{D}}$ the Bergman kernel of $\mathcal{D}$. If $K_{\mathcal{D}}(z,w) \neq 0$ for $z, w \in \mathcal{D}$, we set

$$T_{\mathcal{D}}(z,w) := \Big(\frac{\partial^2}{\partial z_i \partial \bar{w}_j} \log K_{\mathcal{D}}(z,w)\Big)_{i,j} \in \mathrm{Mat}(N, \mathbb{C}).$$

Take $p \in \mathcal{D}$ and assume that $K_{\mathcal{D}}(z,p) \neq 0$ for all $z \in \mathcal{D}$. Then we define *the Bergman mapping* $\sigma_p : \mathcal{D} \to \mathbb{C}^N$ by

$$\sigma_p(z) := T_{\mathcal{D}}(p,p)^{-1/2} \mathrm{grad}_{\bar{w}} \log \frac{K_{\mathcal{D}}(z,w)}{K_{\mathcal{D}}(p,w)}\Big|_{w=p} \qquad (z \in \mathcal{D}),$$

where $\mathrm{grad}_{\bar{w}} f(w) := {}^t(\frac{\partial f}{\partial \bar{w}_1}, \frac{\partial f}{\partial \bar{w}_2}, \dots, \frac{\partial f}{\partial \bar{w}_n})$ for an anti-holomorphic function f on $\mathcal{D}$. A domain $\mathcal{U}$ is called a *representative domain* if it is the image $\sigma_p(\mathcal{D})$ of some Bergman mapping $\sigma_p : \mathcal{D} \to \mathbb{C}^N$.

1.2. In what follows, we assume that a bounded domain $\mathcal{D}$ is *homogeneous*, that is, the holomorphic automorphism group $\mathrm{Aut}(\mathcal{D})$ acts on $\mathcal{D}$ transitively. The notion of the representative domain works very well for such BHDs. Since $K_{\mathcal{D}}(z,p) \neq 0$ for any z, $p \in \mathcal{D}$ in this case, the Bergman mapping $\sigma_p : \mathcal{D} \to \mathbb{C}^N$ is always well-defined. It is shown in [12, Theorem 4.7] and [6, Theorem 3.3] that $\sigma_p(D)$ is a bounded domain and σ_p gives a biholomorphism from $\mathcal{D}$ onto $\sigma_p(D)$. Thus, any BHD $\mathcal{D}$ is realized as a representative BHD $\mathcal{U}$, which is unique up to unitary linear transform by [6, Proposition 2.1, Lemma 3.2]. A representative BHD $\mathcal{U}$ is characterized by the following properties: (U1) $0 \in \mathcal{U}$, and (U2) $T_{\mathcal{U}}(\zeta, 0) = I_N$ $(\forall \zeta \in \mathcal{U})$. For example, $\sqrt{2}\Delta = \{ z \in \mathbb{C};\, |z| < \sqrt{2} \}$ is a representative domain. In general, the Harish-Chandra realization of an irreducible bounded symmetric domain (e.g. a classical domain) coincides with a constant multiple of the representative domain.

1.3. For a representative BHD $\mathcal{U}$, we see from [6, Proposition 3.8] that

$$K(\zeta, 0) = \frac{1}{\mathrm{Vol}(\mathcal{U})} \qquad (\forall \zeta \in \mathcal{U}), \tag{5}$$

which is equivalent to the mean value property

$$f(0) = \frac{1}{\mathrm{Vol}(\mathcal{U})} \int_{\mathcal{U}} f(\zeta) dV(\zeta) \qquad (f \in L^2_a(\mathcal{U})).$$

From this observation, we can deduce the following general formula.

Theorem 1. *For a (not necessarily bounded) domain $\mathcal{D}$ biholomorphic to a representative BHD $\mathcal{U}$ and a biholomorphism $\Phi : \mathcal{D} \to \mathcal{U}$, putting $a := \Phi^{-1}(0) \in \mathcal{D}$, one has*

$$K_{\mathcal{U}}(\Phi(z), \Phi(w)) = \frac{1}{\mathrm{Vol}(\mathcal{U})} \frac{K_{\mathcal{D}}(z, w) K_{\mathcal{D}}(a, a)}{K_{\mathcal{D}}(z, a) K_{\mathcal{D}}(a, w)} \qquad (z,\, w \in \mathcal{D}). \tag{6}$$

Proof. By the transformation rule of the Bergman kernel, we have

$$K_{\mathcal{D}}(z, w) = K_{\mathcal{U}}(\Phi(z), \Phi(w)) \det J(\Phi, z) \overline{\det J(\Phi, w)}.$$

In particular, putting $w = a$, we have by (5)

$$K_{\mathcal{D}}(z, a) = \frac{\det J(\Phi, z) \overline{\det J(\Phi, a)}}{\mathrm{Vol}(\mathcal{U})}.$$

Similarly, we see that

$$K_{\mathcal{D}}(a, w) = \frac{\det J(\Phi, a) \overline{\det J(\Phi, w)}}{\mathrm{Vol}(\mathcal{U})}.$$

Furthermore, for the case $z = w = a$, we have

$$K_{\mathcal{D}}(a, a) = \frac{|\det J(\Phi, a)|^2}{\mathrm{Vol}(\mathcal{U})}.$$

Substituting these equalities, we obtain (6). □

§2. Main result.

For a representative BHD $\mathcal{U}$, structure of the holomorphic automorphism group $\mathrm{Aut}(\mathcal{U})$ is rather complicated in general, while the Lie algebra $\mathfrak{b}$ of the Iwasawa subgroup (maximal connected split solvable Lie subgroup) $B \subset \mathrm{Hol}(\mathcal{U})$ has a specific root space decomposition (Theorem 2). The subgroup B is unique up to inner automorphisms in $\mathrm{Aut}(\mathcal{U})$, so that the structure of B and $\mathfrak{b}$ are canonically determined from the BHD $\mathcal{U}$. Our main result is stated in terms of the dimensions of the root subspaces of $\mathfrak{b}$.

2.1. Since the group B acts on the domain $\mathcal{U}$ simply transitively ([11]), we have the linear isomorphism $\iota : \mathfrak{b} \ni Y \mapsto Y \cdot 0 \in T_0\mathcal{U} \equiv \mathbb{C}^N$. Let us transfer the complex structure and the Bergman metric $(ds^2_{\mathcal{U}})_0$ on $T_0\mathcal{U}$ to $\mathfrak{b}$ by means of ι. Let $j : \mathfrak{b} \to \mathfrak{b}$ be a linear map defined in such a way that $\iota(jY) = \sqrt{-1}\iota(Y)$ $(Y \in \mathfrak{b})$, and $(\cdot\,|\,\cdot)_{\mathfrak{b}}$ an inner product on $\mathfrak{b}$ given by $(Y_1|Y_2)_{\mathfrak{b}} := ds^2_{\mathcal{U}}(\iota(Y_1), \iota(Y_2))_0$ $(Y_1, Y_2 \in \mathfrak{b})$. Let $\mathfrak{a}$ be the orthogonal complement of the subspace $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{b}$ with respect to $(\cdot\,|\,\cdot)_{\mathfrak{b}}$. Then $\mathfrak{a}$ is a commutative Cartan subalgebra of the solvable Lie algebra $\mathfrak{b}$. For $\alpha \in \mathfrak{a}^*$, we denote by $\mathfrak{b}_\alpha$ the root subspace $\mathfrak{b}_\alpha := \{\, Y \in \mathfrak{b}\,;\, [C, Y] = \alpha(C)Y \ (\forall C \in \mathfrak{a}) \,\}$. The number $r := \dim \mathfrak{a}$ is called the *rank* of $\mathfrak{b}$.

Theorem 2 ([9, Chapter 2, Section 3]). *There exists a basis $\{\alpha_1, \ldots, \alpha_r\}$ of $\mathfrak{a}^*$ such that $\mathfrak{b} = \mathfrak{b}(1) \oplus \mathfrak{b}(1/2) \oplus \mathfrak{b}(0)$,*

$$\mathfrak{b}(0) = \mathfrak{a} \oplus \sum_{1\le k<m\le r}^{\oplus} \mathfrak{b}_{(\alpha_m-\alpha_k)/2}, \quad \mathfrak{b}(1/2) = \sum_{1\le k\le r}^{\oplus} \mathfrak{b}_{\alpha_k/2},$$
$$\mathfrak{b}(1) = \sum_{1\le k\le r}^{\oplus} \mathfrak{b}_{\alpha_k} \oplus \sum_{1\le k<m\le r}^{\oplus} \mathfrak{b}_{(\alpha_m+\alpha_k)/2}.$$

Let $\{A_1, \cdots, A_r\}$ be the basis of $\mathfrak{a}$ dual to $\{\alpha_1, \ldots, \alpha_r\}$, and put $E_k := -jA_k$ $(k = 1, \ldots, r)$. Then $\mathfrak{b}_{\alpha_k} = \mathbb{R}E_k$. One has $j\mathfrak{b}(0) = \mathfrak{b}(1)$, $j\mathfrak{b}(1/2) = \mathfrak{b}(1/2)$ and

$$[\mathfrak{b}(p), \mathfrak{b}(q)] \subset \mathfrak{b}(p+q) \quad (\textit{if } p > 1, \textit{ then } \mathfrak{b}(p) := \{0\}). \tag{7}$$

for $p, q = 0,\ 1/2,\ 1$.

We note that some root spaces $\mathfrak{b}_{(\alpha_m\pm\alpha_k)/2}$ or $\mathfrak{b}_{\alpha_k/2}$ may be zero.

2.2. For $k = 1, \ldots, r$, we set

$$p_k := \sum_{i<k} \dim \mathfrak{b}_{(\alpha_k-\alpha_i)/2}, \quad q_k := \sum_{m>k} \dim \mathfrak{b}_{(\alpha_m-\alpha_k)/2}, \quad b_k := (\dim \mathfrak{b}_{\alpha_k/2})/2.$$

Then we state our main result as follows.

Theorem 3. *Putting*

$$P(s) := \prod_{k=1}^{r} \bigl(s(2 + p_k + q_k + b_k) + 1 + q_k/2\bigr)_{1+p_k+b_k}, \tag{8}$$

one has

$$\int_{\mathcal{U}} \{\mathrm{Vol}(\mathcal{U}) K_{\mathcal{U}}(\zeta, \zeta)\}^s \, dV(\zeta) = \mathrm{Vol}(\mathcal{U}) \frac{P(0)}{P(s)}, \tag{9}$$

where s is a complex number for which the real part of every factor of $P(s)$ is positive.

The polynomial $F(s)$ in (2) is $P(s)/P(0)$. Indeed, the degree of $P(s)$ is $\sum_{k=1}^{r}(1+p_k+q_k) = \dim \mathfrak{b}(0) + (\dim \mathfrak{b}(1/2))/2 = (\dim \mathfrak{b})/2$, which is nothing but $N = \dim_{\mathbb{C}} \mathcal{U}$. For the case $\mathcal{U}$ is (a constant multiple of) $R_I(m,n)$, we have $p_k = 2(k-1)$, $q_k = 2(n-k)$ and $b_k = m-n$, so that Theorem 3 is compatible with (1).

§3. Evaluation of integrals on a homogeneous Siegel domain.

The solvable group B acts on the representative BHD $\mathcal{U}$ simply transitively, while we shall see that the same B acts on a certain Siegel domain $\mathcal{D}$ as an affine transformation group. The domain $\mathcal{D}$ is biholomorphic to $\mathcal{U}$. This is a generalization of the relation between the upper half plane and the unit disc in the complex plane $\mathbb{C}$. In this section, making use of Theorem 1, we reduce the integral (9) over $\mathcal{U}$ to integrals over the Siegel domain $\mathcal{D}$, whose evaluation is essentially due to Gindikin [3] and [4].

3.1. Thanks to (7), we see that $\mathfrak{b}(0)$ and $\mathfrak{b}(1)$ are a subalgebra and a commutative ideal of $\mathfrak{b}$ respectively, and that the group $B(0) := \exp \mathfrak{b}(0)$ of B acts on $\mathfrak{b}(1)$ by the adjoint representation. Putting $E := E_1 + \cdots + E_r \in \mathfrak{b}(1)$, we set $\Omega := B(0) \cdot E \subset \mathfrak{b}(1)$. Then Ω is a regular open convex cone in $\mathfrak{b}(1)$, on which the group $B(0)$ acts simply transitively. The linear map $j|_{\mathfrak{b}(1/2)}$ gives a complex structure on the space $\mathfrak{b}(1/2)$. We definite the Hermitian map $Q : \mathfrak{b}(1/2) \times \mathfrak{b}(1/2) \to \mathfrak{b}(1)_{\mathbb{C}}$ on the complex vector space $(\mathfrak{b}(1/2), j)$ by $Q(u,u') := ([ju,u'] + i[u,u'])/4$. Let us consider the Siegel domain $\mathcal{D} \subset \mathfrak{b}(1)_{\mathbb{C}} \times (\mathfrak{b}(1/2), j)$ given by

$$\mathcal{D} := \{ Z = (z,u) \in \mathfrak{b}(1)_{\mathbb{C}} \times (\mathfrak{b}(1/2), j) ; \Im z - Q(u,u) \in \Omega \}.$$

An action of the solvable group B on $\mathcal{D}$ is defined by

$$b_0 \cdot (z,u) := (h_0 \cdot z + x_0 + iQ(h_0 \cdot u, u_0) + iQ(u_0,u_0)/2,\ h_0 \cdot u + u_0) \qquad ((z,u) \in \mathcal{D})$$

for $b_0 = \exp(x_0 + u_0)h_0 \in B$ ($x_0 \in \mathfrak{b}(1)$, $u_0 \in \mathfrak{b}(1/2)$, $h_0 \in B(0)$). It is easy to check that the point $a_0 := (iE, 0)$ belongs to $\mathcal{D}$. Then we can describe the Bergman mapping $\mathcal{C} := \sigma_{a_0} : \mathcal{D} \xrightarrow{\sim} \mathcal{U}$ concretely ([6], [8]).

Noting that $\mathfrak{b}(0) = \mathfrak{a} \oplus [\mathfrak{b}(0), \mathfrak{b}(0)]$, we define a one-dimensional representation $\chi_{\underline{\sigma}} : B(0) \to \mathbb{C}^{\times}$ for $\underline{\sigma} = (\sigma_1, \ldots, \sigma_r) \in \mathbb{C}^r$ by $\chi_{\underline{\sigma}}(\exp C) := e^{\sum \sigma_i \alpha_i(C)}$ $(C \in \mathfrak{a})$. Let $\Delta_{\underline{\sigma}}$ be a smooth function on the cone Ω given by $\Delta_{\underline{\sigma}}(h \cdot E) := \chi_{\underline{\sigma}}(h)$ $(h \in B(0))$. This $\Delta_{\underline{\sigma}}$ can be expressed as a product of powers of rational functions, and it can be extended as a holomorphic function on the complex domain $\Omega + i\mathfrak{b}(1)$. Define

$\underline{d} = (d_1, \ldots, d_r)$ by $d_k := 1 + (p_k + q_k)/2$ $(k = 1, \ldots, r)$. Then $\Delta_{-\underline{d}}(x)\,dx$ is an invariant measure on Ω with respect to the action of $B(0)$.

Proposition 4 ([3, Lemma 5.1]). *The Bergman kernel $K_{\mathcal{D}}$ of the homogeneous Siegel domain $\mathcal{D}$ is given by*

$$K_{\mathcal{D}}(Z, Z') = C_{\mathcal{D}} \Delta_{-(2\underline{d}+\underline{b})}\left(\frac{z - \bar{z}'}{2i} - Q(u, u')\right) \qquad (Z = (z, u),\ Z' = (z', u') \in \mathcal{D}),$$

where $C_{\mathcal{D}}$ is a constant independent of Z and Z'.

3.2. Let $E^* \in \mathfrak{b}(1)^*$ be the linear form on $\mathfrak{b}(1)$ given by $\langle x, E^* \rangle = \sum_{k=1}^r x_{kk}$ for elements $x = \sum_{k=1}^r x_{kk} E_k + \sum_{1 \le k < m \le r} X_{mk} \in \mathfrak{b}(1)$ $(x_{kk} \in \mathbb{R},\ X_{mk} \in \mathfrak{b}_{(\alpha_m + \alpha_k)/2})$. Then E^* belongs to the dual cone $\Omega^* := \{\xi \in \mathfrak{b}(1)^* \,;\, \langle x, \xi \rangle > 0\ (\forall x \in \overline{\Omega} \setminus \{0\})\}$ of Ω. Moreover, for any $\xi \in \Omega^*$, there exists a unique $h \in B(0)$ for which $\xi = E^* \circ h$. Therefore, we can define a function $\delta_{\underline{\sigma}}$ by $\delta_{\underline{\sigma}}(E^* \circ h) := \chi_{\underline{\sigma}}(h)$ $(h \in B(0))$.

Proposition 5 ([3, Theorem 2.1, Proposition 2.3]). (i) *For a parameter $\underline{\sigma} = (\sigma_1, \ldots, \sigma_r) \in \mathbb{C}^r$, the integral $\Gamma_\Omega(\underline{\sigma}) := \int_\Omega e^{-\langle x, E^* \rangle} \Delta_{\underline{\sigma}-\underline{d}}(x)\,dx$ converges if and only if $\Re\sigma_k > p_k/2$ $(k = 1, \ldots, r)$. In this case, one has $\Gamma_\Omega(\underline{\sigma}) = C_\Gamma \prod_{k=1}^r \Gamma(\sigma_k - p_k/2)$, where C_Γ is a constant independent of $\underline{\sigma}$. Moreover, one has*

$$\delta_{-\underline{\sigma}}(\xi) = \frac{1}{\Gamma_\Omega(\underline{\sigma})} \int_\Omega e^{-\langle x, \xi \rangle} \Delta_{\underline{\sigma}-\underline{d}}(x)\,dx \qquad (\xi \in \Omega^*). \tag{10}$$

(ii) *The integral $\gamma_{\Omega^*}(\underline{\sigma}) := \int_{\Omega^*} e^{-\langle E, \xi \rangle} \delta_{\underline{\sigma}-\underline{d}}(\xi)\,dx$ converges if and only if $\Re\sigma_k > q_k/2$ $(k = 1, \ldots, r)$, and in this case, $\gamma_{\Omega^*}(\underline{\sigma}) = \Gamma_\Omega(\underline{\sigma} + (\underline{p} - \underline{q})/2) = C_\Gamma \prod_{k=1}^r \Gamma(\sigma_k - q_k/2)$. Moreover, one has*

$$\Delta_{-\underline{\sigma}}(z) = \frac{1}{\gamma_{\Omega^*}(\underline{\sigma})} \int_\Omega e^{-\langle z, \xi \rangle} \delta_{\underline{\sigma}-\underline{d}}(\xi)\,d\xi \qquad (z \in \Omega + i\mathfrak{b}(1)). \tag{11}$$

(iii) *For $\xi \in \Omega^*$, one has*

$$\int_{\mathfrak{b}(1/2)} e^{-\langle Q(u,u), \xi \rangle}\,dV(u) = C_Q \delta_{-\underline{b}}(\xi), \tag{12}$$

where C_Q is a constant independent of ξ.

3.3. By the transformation rule of the Bergman kernels, we have $K_{\mathcal{U}}(\zeta, \zeta)\,dV(\zeta) = K_{\mathcal{D}}(Z, Z)\,dV(Z)$ for the change of variable $\zeta = \mathcal{C}(Z)$ $(Z \in \mathcal{D})$. This together with Theorem 1 tells us that the left-hand side of (9) equals

$$\frac{\mathrm{Vol}(\mathcal{U})}{K_{\mathcal{D}}(a_0, a_0)^{s+1}} \int_{\mathcal{D}} |K_{\mathcal{D}}(Z, a_0)^{s+1}|^2 K_{\mathcal{D}}(Z, Z)^{-s}\,dV(Z),$$

which is rewritten as

$$C_{\mathcal{D}}\,\mathrm{Vol}(\mathcal{U})\int_{\mathcal{D}}|\Delta_{-(s+1)(2\underline{d}+\underline{b})}(\frac{z+iE}{2i})|^2\Delta_{s(2\underline{d}+\underline{b})}(\frac{z-\bar{z}}{2i}-Q(u,u))dV(Z)$$

owing to Proposition 4. In order to evaluate this integral, we consider the change of variable

$$Z=(x+iy+iQ(u,u),\,u)\in\mathcal{D}\qquad(x\in\mathfrak{b}(1),\ y\in\Omega,\ u\in\mathfrak{b}(1/2)).$$

For simplicity, we assume that the real part of s are large enough for the convergene of the integrals in Proposition 5. First of all, by (11) and the Plancherel formula, we have

$$\begin{aligned}&\int_{\mathfrak{b}(1)}|\Delta_{-(s+1)(2\underline{d}+\underline{b})}(\frac{z+iE}{2i})|^2dx\\&=\frac{(4\pi)^{N_1}}{\gamma_{\Omega^*}((s+1)(2\underline{d}+\underline{b}))^2}\int_{\Omega^*}e^{-\langle E+y+Q(u,u),\xi\rangle}\delta_{2(s+1)(2\underline{d}+\underline{b})-2\underline{d}}(\xi)\,d\xi,\end{aligned}$$

where $N_1:=\dim\mathfrak{b}(1)$. Next, by (12) we have

$$\begin{aligned}&\int_{\mathfrak{b}(1/2)}\int_{\mathfrak{b}(1)}|\Delta_{-(s+1)(2\underline{d}+\underline{b})}(\frac{z+iE}{2i})|^2dx\,dV(u)\\&=\frac{(4\pi)^{N_1}C_Q}{\gamma_{\Omega^*}((s+1)(2\underline{d}+\underline{b}))^2}\int_{\Omega^*}e^{-\langle E+y,\xi\rangle}\delta_{(2s+1)(2\underline{d}+\underline{b})}(\xi)\,d\xi.\end{aligned}$$

Furthermore, we see from (10) that

$$\begin{aligned}&\int_{\Omega}\int_{\mathfrak{b}(1/2)}\int_{\mathfrak{b}(1)}|\Delta_{-(s+1)(2\underline{d}+\underline{b})}(\frac{z+iE}{2i})|^2\Delta_{s(2\underline{d}+\underline{b})}(y)\,dx\,dV(u)\,dy\\&=\frac{(4\pi)^{N_1}C_Q\Gamma_\Omega(s(2\underline{d}+\underline{b})+\underline{d})}{\gamma_{\Omega^*}((s+1)(2\underline{d}+\underline{b}))^2}\int_{\Omega^*}e^{-\langle E,\xi\rangle}\delta_{(s+1)(2\underline{d}+\underline{b})-\underline{d}}(\xi)\,d\xi\\&=\frac{(4\pi)^{N_1}C_Q\Gamma_\Omega(s(2\underline{d}+\underline{b})+\underline{d})}{\gamma_{\Omega^*}((s+1)(2\underline{d}+\underline{b}))},\end{aligned}$$

where we use Proposition 5 (ii) for the second equality. Therefore, the left-hand side of (9) is equal to

$$\frac{\Gamma_\Omega(s(2\underline{d}+\underline{b})+\underline{d})}{\gamma_{\Omega^*}((s+1)(2\underline{d}+\underline{b}))}$$

up to a constant multiple, and this is nothing but the reciprocal of $P(s)$ in (8) thanks to Proposition 5 (i) and (ii). Hence we obtain Theorem 3.

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