<table>
<thead>
<tr>
<th>Title</th>
<th>Variational Inequalities with Gradient Constraint and Applications to Optimal Dividend Payments (Viscosity Solutions of Differential Equations and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Morimoto, Hiroaki</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1695: 101-111</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2010-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/141633">http://hdl.handle.net/2433/141633</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Variational Inequalities with Gradient Constraint and Applications to Optimal Dividend Payments

Hiroaki Morimoto

Department of Mathematics, Ehime University, Japan

1 Variational inequalities arisen from dividend payments

We consider the variational inequality of the form:

\[(a)\quad w'(x) \geq 1, \quad x > 0, \quad w'(0+) > 1,\]
\[(b)\quad -\alpha w + \frac{1}{2} \sigma^2 w'' + \mu w' \leq 0, \quad x > 0,\]
\[(c)\quad (-\alpha w + \frac{1}{2} \sigma^2 w'' + \mu w')(w' - 1)^+ = 0, \quad x > 0,\]
\[(d)\quad w(0) = 0, \quad \mu, \sigma > 0 : \text{constants.}\]

Define

\[w(x) = \begin{cases} w_0(x), & x \leq m, \\ x - m + w_0(m), & x > m, \end{cases}\]

where \(w_0\) is the solution of

\[\mathcal{A} w_0 := -\alpha w_0 + \frac{1}{2} w_0'' + \mu w_0 = 0, \quad x \leq m,\]

and \(m > 0\) is chosen as \(w'_0(m) = 1\).

Theorem 1.1 \(w \in C^2(0, \infty) \cap C[0, \infty)\) is a concave solution of the variational inequality \((a)-(d)\).

The variational inequality \((a)-(d)\) is closely related to optimal dividend payments. The reserve \(R_t\) of an insurance company at time \(t \geq 0\) is assumed to be governed by

\[R_t = \mu t + \sigma B_t - L_t, \quad R_0 = x - L_0 \geq 0,\]
where $B_t$ is a standard Brownian motion, $\mu, \sigma > 0$ constants, $x \geq 0$ the initial position of reserve and $L_t$ the rate of dividend payment at time $t$ (0 acts absorbing barrier for $R_t$). Note that $R_0 = x - L_0$ means that if there is a pay-out of dividends at time 0, then $R_t$ instantaneously decreases from $x$ to $x - L_0$. The dividend process \{L_t\} is called admissible if

$$L_t : \mathcal{F}_t := \sigma(B_s, s \leq t)\text{-measurable}, \quad x - L_0 \geq 0,$$

$L_t$ is nonnegative, nondecreasing, continuous,

and we denote by $\mathcal{L}$ the class of all admissible dividend processes \{L_t\}.

The objective is to find an optimal dividend payment \{L_t^*\} $\in \mathcal{L}$ so as to maximize the expected total pay-out of dividend

$$J_x(L) = E[\int_0^\tau e^{-\alpha t} dL_t], \quad L \in \mathcal{L},$$

where $\alpha > 0$ is the discount rate and $\tau$ the absorption time, $\tau = \inf\{t \geq 0 : R_t = 0\}$.

**Theorem 1.2** We have

$$J_x(L) \leq w(x).$$

Define

$$R_t^* = x + \mu t + \sigma B_t - L_t^*, \quad R_0^* = x - L_0^* \geq 0,$$

$$L_t^* = \max_{s \leq t}(x + \mu s + \sigma B_s - m)^+. $$

**Theorem 1.3** We assume that the initial position $x \leq m$. Then \{L_t^*\} is optimal.

**Remark 1.4** Instead of the variational inequality, we consider the Black-Scholes Model:

(a) $w'(x) \geq 1, \quad x > 0, \quad w'(0+) > 1,$

(b) $-\alpha w + \frac{1}{2} \sigma^2 x^2 w'' + \mu x w' \leq 0, \quad x > 0,$

(c) $(-\alpha w + \frac{1}{2} \sigma^2 x^2 w'' + \mu x w')(w' - 1)^+ = 0, \quad x > 0,$

(d) $w(0) = 0,$
where $\mu, \sigma > 0$ constants. Then $w(x) = x$ and (a) fails if $\alpha > \mu$.

**Remark 1.5** Consider the following variational inequality:

(a) $w'(x) \geq 1, \; x > 0, \; w'(0+) > 1,$

(b) $-\alpha w + \frac{1}{2} \sigma^2 x^2 w'' + \mu w' \leq 0, \; x > 0,$

(c) $(-\alpha w + \frac{1}{2} \sigma^2 x^2 w'' + \mu w')(w' - 1)^+ = 0, \; x > 0,$

(d) $w(0) = 0.$

Then this variational inequality seems to have no solution.

## 2 Variational inequalities in the Stochastic Ramsey problem

From now on, we consider the variational inequality associated with optimal dividends for the stochastic Ramsey model. We define the following quantities:

- $K_t = \text{capital stock of a firm at time } t,$
- $K^\gamma = \text{the Cobb-Douglas function for the amount of capital stock } K, \quad 0 < \gamma < 1,$
- $B_t = \text{1-dim. Brownain motion},$
- $\mathcal{F}_t = \sigma(B_s, s \leq t),$
- $\sigma = \text{diffusion constant, } \sigma > 0$
- $x = \text{initial position, } x > 0.$

Dividends are paid from the profit of the firm for shareholders and the remainder accumulates in capital stock. We assume that the flow of dividend payments at time $t$ can be written as $K_t dD_t,$ where $dD_t$ denotes the per capital stock dividend payments. Let $\mathcal{A}$ be the class of all nonnegative, nondecreasing, continuous, $\{\mathcal{F}_t\}$-adapted stochastic processes $D = \{D_t\}$ such that $x_D := x - D_0 > 0.$ Given a policy $D \in \mathcal{A},$ the capital stock process $\{K_t\}$ evolves according to

$$dK_t = K_t^\gamma dt + \sigma K_t dB_t - K_t dD_t, \quad K_0 = x - D_0 > 0.$$
Our objective is to find an optimal policy \( D^* = \{D_t^*\} \) so as to maximize the expected total pay-out functional with discount factor \( \alpha > 0 \):

\[
J(D) = E\left[ \int_0^\infty e^{-\alpha t} K_t dD_t \right], \quad \forall D \in A.
\]

The associated variational inequality is given by

\[
\begin{align*}
& v'(x) \geq 1, \quad x > 0, \quad v'(0+) > 1, \\
& -\alpha v + \frac{1}{2} \sigma^2 x^2 v'' + x^{\gamma} v' \leq 0, \quad x > 0, \\
& (-\alpha v + \frac{1}{2} \sigma^2 x^2 v'' + x^{\gamma} v')(v' - 1)^+ = 0, \quad x > 0.
\end{align*}
\]

For the existence of \( K_t \), we have the following.

**Proposition 2.1** For each \( D \in A \), there exists uniquely a positive solution \( \{K_t\} \) of

\[
dK_t = K_t^{\gamma} dt + \sigma K_t dB_t - K_t dD_t, \quad K_0 = x_D = x - D_0 > 0.
\]

such that

\[
E[K_t] \leq 2^\beta (x_D + t^\beta),
\]

\[
E[K_t^2] \leq 2^{2\beta} e^{\sigma^2 t} (x_D^2 + t^{2\gamma\beta}/\sigma^2),
\]

where \( \beta = 1/(1 - \gamma) \).

**Outline of the proof.** We set \( k_t = K_t^{1-\gamma} \). Then, by Ito's formula

\[
dk_t = (1 - \gamma)K_t^{-\gamma} dK_t + \frac{\sigma^2}{2} K_t^2 (1 - \gamma)(-\gamma) K_t^{-\gamma-1} dt
\]

\[
= (1 - \gamma) dt + \sigma K_t^{1-\gamma} dB_t - K_t^{1-\gamma} dD_t \\
+ \frac{\sigma^2}{2} (1 - \gamma)(-\gamma) K_t^{1-\gamma} dt
\]

\[
= (1 - \gamma) \{(1 - \frac{\sigma^2}{2} \gamma k_t) dt + \sigma k_t dB_t - k_t dD_t\},
\]

\[k_0 = x_D^{1-\gamma}.
\]

By linearity, there exists a unique positive solution \( \{k_t\} \).
Proposition 2.2 Assume $\sigma = 0$. Then there exists a concave solution $v_0 \in C^2(0, \infty)$ of (VI).

Outline of the proof. We solve the equation $-\alpha h + x^\gamma h' = 0$ to have

$$h(x) = Q \exp\{\alpha x^{1-\gamma}/(1-\gamma)\}.$$  

Define

$$v_0(x) = \begin{cases} h(x) & \text{if } x \leq x_*, \\ x - x_* + h(x_*) & \text{if } x_* < x, \end{cases}$$

Choose $x_* = (\gamma/\alpha)^{1/(1-\gamma)}$, $Q > 0$ such that $h'(x_*) = 1$. Then we have

$$h''(x_*) = 0,$$

and

$$-\alpha v_0 + x^\gamma v'_0 = -\alpha\{x - x_* + h(x_*)\} + x^\gamma \leq 0 \text{ for } x > x_*.$$

3 Probabilistic solution of the penalty equation

We consider the penalty equation

$$(p) \quad -\alpha u + \frac{1}{2}\sigma^2 x^2 u'' + x^\gamma u' + \frac{x}{\epsilon} (u' - 1)^- = 0, \quad x > 0,$$

which can be rewritten as

$$-\alpha u + \frac{1}{2}\sigma^2 x^2 u'' + x^\gamma u' + \frac{x}{\epsilon} \max_{\frac{u}{\epsilon} \leq c \leq 1} (1 - u') c = 0, \quad x > 0.$$ 

Let $C$ be the class of all $\{\mathcal{F}_t\}$-progressively measurable processes $c = \{c_t\}$ such that $0 \leq c_t \leq 1$, a.s. for all $t \geq 0$. For any $c \in C$, let $\{X_t\}$ be the solution of

$$dX_t = X_t^\gamma dt + \sigma X_t dB_t - \frac{1}{\epsilon} c_t X_t dt, \quad X_0 = x > 0.$$ 

Define

$$u(x) = \sup_{c \in C} E\left[ \int_0^\infty e^{-\alpha t} \frac{1}{\epsilon} c_t X_t dt \right],$$

where the supremum is taken over all systems $(\Omega, \mathcal{F}, \mathbb{P}, \{c_t\}, \{B_t\})$. Then we observe that the penalty equation $(p)$ is a Hamilton-Jacobi-Bellman equation.
Theorem 3.1 We have

\[ 0 \leq u(x) \leq v_0(x) \leq C(1 + x), \quad x > 0, \]

for some constant \( C > 0 \).

Theorem 3.2 For any \( \rho > 0 \), there exists \( C_{\rho, \epsilon} > 0 \) such that

\[ |u(x) - u(y)| \leq C_{\rho, \epsilon}|x - y| + \rho(1 + x + y), \quad x, y > 0. \]

Theorem 3.3 \( u \) is concave on \((0, \infty)\).

4 Solution of the penalty equation

In this section, we show that the probabilistic solution \( u \) is a classical solution of the penalty equation \((p)\).

Definition 4.1 Let \( w \in C(0, \infty) \). Then \( w \) is called a viscosity solution of \((p)\) if

(a) \( w \) is a viscosity subsolution of \((p)\), that is, for any \( \phi \in C^2(0, \infty) \) and any local maximum point \( z > 0 \) of \( w - \phi \),

\[ -\alpha w + \frac{1}{2} \sigma^2 x^2 \phi'' + x^\gamma \phi' + \frac{x}{\epsilon} (\phi' - 1)^{-} \bigg|_{x=z} \geq 0, \]

and (b) \( w \) is a viscosity supersolution of \((p)\), that is, for any \( \phi \in C^2(0, \infty) \) and any local minimum point \( \bar{z} > 0 \) of \( w - \phi \),

\[ -\alpha w + \frac{1}{2} \sigma^2 x^2 \phi'' + x^\gamma \phi' + \frac{x}{\epsilon} (\phi' - 1)^{-} \bigg|_{x=\bar{z}} \leq 0. \]

By Theorems 3.1 and 3.2, we can show that the dynamic programming principle holds for \( u \), i.e.,

\[ u(x) = \sup_{c \in \mathcal{C}} E\left[ \int_0^s e^{-\alpha t} \frac{1}{\epsilon} c_t X_t dt + e^{-\alpha s} u(X_s) \right] \]

for any \( s \geq 0 \). By the theory of viscosity solutions, taking into account Proposition 2.1, we have the viscosity property of \( u \). For details, we refer to [9].
Theorem 4.2 \( u \) is a viscosity solution of \((p)\).

Theorem 4.3 We have
\[
u \in C^2(0, \infty).
\]

5 Solution of the variational inequality

In this section, we study the convergence of \( u = u_\epsilon \) to a viscosity solution \( v \) of the variational inequality \((VI)\) as \( \epsilon \to 0 \).

5.1 Limit of the penalized problem

Definition 5.1 Let \( w \in C(0, \infty) \). Then \( w \) is called a viscosity solution of \((VI)\), if the following assertions are satisfied:

\begin{enumerate}
\item For any \( \phi \in C^2 \) and any local minimum point \( \bar{z} > 0 \) of \( w - \phi \),
\[
\phi'(\bar{z}) \geq 1, \quad -\alpha w + \frac{1}{2} \sigma^2 x^2 \phi'' + x^\gamma \phi' \bigg|_{x=\bar{z}} \leq 0,
\]
\item For any \( \phi \in C^2 \) and any local maximum point \( z > 0 \) of \( w - \phi \),
\[
(-\alpha w + \frac{1}{2} \sigma^2 x^2 \phi'' + x^\gamma \phi') (\phi' - 1)^+ \bigg|_{x=z} \geq 0.
\]
\end{enumerate}

By concavity and Theorem 3.1, we get
\[
0 \leq u_\epsilon'(x)x \leq u_\epsilon(x) - u_\epsilon(0) \leq v_0(x), \quad x > 0.
\]

Hence, for any \( 0 < a < b \),
\[
\sup_{\epsilon} \| u_\epsilon' \|_{C[a,b]} < \infty.
\]

By the Ascoli-Arzelà theorem and Theorem 4.2, we have the following.

Theorem 5.2 There exists a subsequence \( \{ u_{\epsilon_n} \} \) such that
\[
u_{\epsilon_n} \to v \in C(0, \infty) \quad \text{locally uniformly in } (0, \infty) \text{ as } \epsilon_n \to 0.
\]

Furthermore, \( v \) is a viscosity solution of \((VI)\).
5.2 Regularity

In this subsection, we study the regularity of the viscosity solution $v$ of (VI). By concavity, we can show that

$$u_{\epsilon_n}' \geq 1 \text{ on } [a, b].$$

We rewrite the penalty equation as

$$-u''_{\epsilon} = \frac{2}{\sigma^2 x^2} \{-\alpha u_{\epsilon} + x^\gamma u_{\epsilon}' + \frac{x}{\epsilon} (u_{\epsilon}' - 1)^-\}.$$

Thus we have:

**Theorem 5.3** For any $0 < a < b$, we have

$$\sup_{n \geq 1} \|u_{\epsilon_n}''\|_{C[a,b]} < \infty.$$

By Theorem 5.3, extracting a subsequence, we have

$$u_{\epsilon_n}' \rightarrow v' \text{ locally uniformly in } (0, \infty) \text{ as } n \rightarrow \infty,$$

and $v'$ is locally Lipschitz on $(0, \infty)$.

**Theorem 5.4** We have

$$v \in C_{loc}^{1,1}(0, \infty), \text{ piecewise } C^2, \quad v' \geq 1 \text{ on } (0, \infty).$$

Furthermore, by using Proposition 2.2, we can state the following.

**Theorem 5.5** We have

$$v'(0+) > 1,$$

and there exists $x^* > 0$ such that

$$x^* = \inf\{x > 0 : v'(x) = 1\}.$$
6 Optimal dividend payments

In this section, we give a synthesis of the optimal policy $D^* \in \mathcal{A}$ of the maximization problem.

Consider the SDE with reflecting barrier conditions:

(a) $dK_t^* = (K_t^*)^\gamma dt + \sigma K_t^* dB_t - K_t^* dD_t^*$, \quad $K_0^* = x - D_0^* > 0$,
(b) $D_t^* = (x - x^*)^+ + \int_0^t 1_{\{K_s^* = x^*\}} dD_s^*$,
(c) $D_t^*$ is continuous a.s.,
(d) $K_t^* \in \mathcal{R}$, \quad $\forall t \geq 0$, a.s.,
(e) $\int_0^t 1_{\{K_s^* = x^*\}} ds = 0$, \quad $\forall t \geq 0$, a.s.,

where $\mathcal{R} := (0, x^*]$ for $x^* = \inf\{x > 0 : v'(x) = 1\}$.

**Theorem 6.1** We assume that the initial position $x \leq x^*$, (by making $D_0 = x - x^*$ if $x > x^*$). Then the optimal policy $D^* = \{D_t^*\}$ is given by (a) - (e).

**Lemma 6.2** There exists a unique solution $\{(K_t^*), \{D_t^*\}\}$ of (a) - (e).

Proof. There exists a unique solution $\{(M_t, \Delta_t)\}$ of the SDE with reflecting barrier conditions:

- $dM_t = (1 - \gamma)(dt - \sigma^2 \gamma M_t dt + \sigma M_t dB_t) - d\Delta_t$, \quad $M_0 = x^{1-\gamma} - \Delta_0 > 0$,
- $\Delta_t = (x^{1-\gamma} - (x^*)^{1-\gamma})^+ + \int_0^t 1_{\{M_s \in \partial S\}} d\Delta_s$,
- $\Delta_t$ is continuous a.s.,
- $M_t \in \mathcal{S}$, \quad $\forall t \geq 0$, a.s.,
- $\int_0^t 1_{\{M_s \in \partial S\}} ds = 0$, \quad $\forall t \geq 0$, a.s.,

where $\mathcal{S} = [0, (x^*)^{1-\gamma}]$ and $\{\Delta_t\}$ is a bounded variation process. Define

$$K_t^* = M_t^\beta, \quad D_t^* = \Delta_t^\beta + \int_0^t \beta M_s^{-1} 1_{\{M_s > 0\}} d\Delta_s, \quad \beta := 1/(1 - \gamma).$$

Then, Ito’s formula completes the proof.
Proof of Theorem 6.1. Let $D \in \mathcal{A}$ be arbitrary. By the variational inequality and the continuity of $\{D_t\}$, we can apply the generalized Ito formula to $\{K_t\}$ for convex functions (cf. [5]). Then

$$e^{-\alpha s}v(K_s) - v(x_D) = \int_0^s e^{-\alpha t}\{ -\alpha v + \frac{1}{2} \sigma^2 x^2 v'' + x^\gamma v'\}|_{x=K_t} dt$$

$$+ \int_0^s e^{-\alpha t}v'(K_t)\sigma K_t dB_t - \int_0^s e^{-\alpha t}v'(K_t)K_t dD_t$$

$$\leq \int_0^s e^{-\alpha t}v'(K_t)\sigma K_t dB_t - \int_0^s e^{-\alpha t}v'(K_t)K_t dD_t, \quad a.s. \ s \geq 0.$$

Hence

$$E[\int_0^{\tau_R} e^{-\alpha t}K_t dD_t] \leq v(x_D) \leq v(x).$$

where $\tau_R := R \wedge \inf\{t \geq 0 : K_t \geq R \text{ or } K_t \leq 1/R\}$ for $R > 0$. Letting $R \to \infty$,

$$J(D) = E[\int_0^\infty e^{-\alpha t}K_t dD_t] \leq v(x).$$

By the same argument as above, we get

$$v(x) = E\left[\int_0^\infty e^{-\alpha t}v'(K^*_t)K^*_t dD^*_t\right].$$

Since $D^*_t$ increases only when $K^*_t = x^*$ and $v'(x^*) = 1$,

$$v(x) = E[\int_0^\infty e^{-\alpha t}v'(K^*_t)\mathbb{1}_{K_t=x^*} K_t^* dD_t] = E[\int_0^\infty e^{-\alpha t}K_t^* dD_t] = J(D^*),$$

which completes the proof.

References


