<table>
<thead>
<tr>
<th>Title</th>
<th>Variational characterization of the Knothe-Rosenblatt type rearrangements and its stochastic version (Viscosity Solutions of Differential Equations and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Mikami, Toshio</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1695: 35-55</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2010-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/141639">http://hdl.handle.net/2433/141639</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
Variational characterization of the Knothe-Rosenblatt type rearrangements and its stochastic version

広島大学・工学研究科 三上 敏夫 (Toshio Mikami)
Department of Applied Mathematics
Hiroshima University

1 Introduction.

The Knothe-Rosenblatt rearrangement plays a crucial role in many fields, e.g., the Brunn-Minkowski inequality and statistics (see [12], [13], [22] and the references therein).

Let $d \geq 1$ and let $\mathcal{M}_1(\mathbb{R}^d)$ denote the set of all Borel probability measures on $\mathbb{R}^d$ with a weak topology. For a distribution function $F$ on $\mathbb{R}$, let

$$F^{-1}(u) := \inf\{x \in \mathbb{R}|u \leq F(x)\} \quad (0 \leq u \leq 1). \quad (1.1)$$

For $P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d)$, $x \in \mathbb{R}$, and $i = 0, 1$, let

$$F_{i,k}(x|x_{k-1}) := \begin{cases} P_i((-\infty, x] \times \mathbb{R}^{d-1}) & (k = 1), \\ P_i((-\infty, x] \times \mathbb{R}^{d-k}|x_{k-1}) & (1 < k < d), \\ P_i((-\infty, x]|x_{d-1}) & (k = d), \end{cases}$$

$$\varphi_k(x_k) := F_{1,k}(\cdot|\varphi_1(x_1), \cdots, \varphi_{k-1}(x_{k-1}))^{-1}(F_{0,k}(x_k|x_{k-1}))(1 \leq k \leq d), \quad (1.2)$$

where $x_k := (x_i)_{1 \leq i \leq k} \in \mathbb{R}^k$ for $x = (x_i)_{1 \leq i \leq d} \in \mathbb{R}^d$ and $P_i(\cdot|x_{k-1})$ denotes the regular conditional probability of $P_i$ given $x_{k-1}$.

Suppose that $F_{0,k}(\cdot|x_{k-1})$ is continuous for all $k = 1, \cdots, d$. Then $P_1$ is the image measure of $P_0$ by

$$T_{KR}(x_d) := (\varphi_1(x_1), \cdots, \varphi_d(x_d)).$$

$T_{KR}$ is called the **Knothe-Rosenblatt rearrangement**. Suppose, in addition, that $F_{1,k}(\cdot|x_{k-1})$ is continuous for all $k = 1, \cdots, d$. Then $T_{KR}$ is invertible and the minimizer of the following weakly converges to $P_0(dx)\delta_{T_{KR}(x)}(dy)$ as $\epsilon \to 0$: for $p > 1$,
\[
\inf\left\{ \int_{R^d \times R^d} \sum_{k=1}^{d} \varepsilon^{2(k-1)} |y_k - x_k|^{p} \mu(dxdy) \bigg| \mu(dx \times R^d) = P_0(dx), \mu(R^d \times dy) = P_1(dy) \right\},
\]

(1.3)

provided \( \int_{R^d} |x|^p (P_0(dx) + P_1(dx)) \) is finite (see [2]). Here \( \delta_x(dy) \) denotes the delta measure on \( \{x\} \).

For \( 1 \leq k \leq d \), \( x_{k-1} \in R^{k-1} \), \( dF_{0,k}(x|x_{k-1})\delta_{\varphi_{k}(x_{k})}(dy) \) is the unique minimizer of

\[
\inf\left\{ \int_{R \times R} |y - x|^p \mu(dxdy) \bigg| \mu(dx \times R) = dF_{0,k}(x|x_{k-1}), \mu(R \times dy) = dF_{1,k}(y|\varphi_{1}(x_{1}), \cdots, \varphi_{k-1}(x_{k-1})) \right\}
\]

(1.4)

(see e.g. [21], [24]). (1.4) also implies that \( P_0(dx_k \times R^{d-k})\delta_{(\varphi_{1}(x_{1}), \cdots, \varphi_{k}(x_{k}))}(dy_k) \) is the unique minimizer of

\[
\inf\left\{ \int_{R^k \times R^k} |y_k - x_k|^p \mu(dxdy) \bigg| \mu(dx \times R^k) = P_0(dx \times R^{d-k}), \mu(R^k \times dy) = P_1(dy \times R^{d-k}), y_i = \varphi_{i-1}(x_{i-1})(i = 1, \cdots, k - 1), \mu - a.s. \right\}.
\]

(1.5)

We generalize (1.5) and call the minimizer the Knothe-Rosenblatt type rearrangement. We also prove the duality theorem, give the convergence result which generalizes (1.3) by the idea of [2] and consider the similar problems in the stochastic control setting.

2 Knothe-Rosenblatt type rearrangement.

Let \( d \geq 2 \), \( 1 \leq d_1 < d \), \( c(x, y) : R^{d-d_1} \times R^{d-d_1} \mapsto [0, \infty) \) be Borel measurable and \( \nu \in M_1(R^{2d_1}) \). For \( P_0, P_1 \in M_1(R^d) \), let

\[
T(P_0, P_1|\nu) := \inf\left\{ \int_{R^d \times R^d} c(x_{d_1,d}, y_{d_1,d}) \mu(dxdy) \bigg| \mu(dx_{d_1} \times R^{d-d_1} \times dy_{d_1} \times R^{d-d_1}) = \nu(dx_{d_1}dy_{d_1}), \mu(dx \times R^d) = P_0(dx), \mu(R^d \times dy) = P_1(dy) \right\},
\]

(2.1)
$x_{i,j} := (x_k)_{i+1 \leq k \leq j} \in R^{j-i}$ for $x = (x_k)_{1 \leq k \leq d} \in R^d$. If the set over which the infimum is taken is empty, then we consider the infimum is equal to infinity. If there exists a Borel measurable function $\varphi : R^{d_1} \mapsto R^{d_1}$ such that $y_{d_1} = \varphi(x_{d_1})$, $\nu$-a.s., then we write, for simplicity,

$$T(P_0, P_1|\varphi) := T(P_0, P_1|\nu).$$

We first show the existence of the Knothe-Rosenblatt type rearrangement.

**Proposition 2.1** Suppose that $c$ is lower semi-continuous. Then, for any $P_0, P_1 \in M_1(R^d), T(P_0, P_1|\nu)$ has a minimizer, provided it is finite.

(Proof) Let $\{\mu_n\}_{n \geq 1}$ be a minimizing sequence of $T(P_0, P_1|\nu)$. Since $\mu_n(dx \times R^d) = P_0(dx)$ and $\mu_n(R^d \times dy) = P_1(dy)$, it has a weakly convergent subsequence which we denote by $\{\mu_{n(k)}\}_{k \geq 1}$. Let $\mu$ denote the limit. Then by Skorohod's representation theorem, Fatou's lemma and the lower semicontinuity of $c$,

$$T(P_0, P_1|\nu) = \lim_{k \to \infty} \int_{R^d \times R^d} c(x_{d_1,d}, y_{d_1,d}) \mu_{n(k)}(dx dy) \geq \int_{R^d \times R^d} c(x_{d_1,d}, y_{d_1,d}) \mu(dx dy). \tag{2.2}$$

For any $f \in C(R^{d_1} \times R^{d_1})$,

$$\int_{R^d \times R^d} f(x_{d_1,d}, y_{d_1,d}) \mu(dx dy) = \lim_{k \to \infty} \int_{R^d \times R^d} f(x_{d_1,d}, y_{d_1,d}) \mu_{n(k)}(dx dy) = \int_{R^{d_1} \times R^{d_1}} f(x_{d_1,d}, y_{d_1,d}) \nu(dx_{d_1,d} dy_{d_1,d}). \tag{2.3}$$

In the same way, one can show that $\mu(dx \times R^d) = P_0(dx)$ and $\mu(R^d \times dy) = P_1(dy)$. \square

### 2.1 Duality Theorem

It is easy to see that the following holds:

$$T(P_0, P_1|\varphi) = \inf \left\{ \int_{R^d \times R^d} \frac{c(x_{d_1,d}, y_{d_1,d})}{1_{\{\varphi(x_{d_1})\}}(y_{d_1})} \mu(dx dy) \right\}$$

$$\mu(dx \times R^d) = P_0(dx), \mu(R^d \times dy) = P_1(dy), \tag{2.4}$$

where $1_A(x) := 1$ if $x \in A$ and $:= 0$ if $x \not\in A$ for the set $A$. This leads us to the duality theorem for $T(P_0, P_1|\varphi)$ which can be obtained from [11] (see also p. 76 in Vol. 1 of [21]).
Theorem 2.1 For any $P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d)$,

$$T(P_0, P_1|\varphi) = \sup\left\{ \int_{\mathbb{R}^d} f_1(y)P_1(dy) - \int_{\mathbb{R}^d} f_0(x)P_0(dx) \middle| f_0, f_1 \in C_b(\mathbb{R}^d), f_1(y) - f_0(x) \leq \frac{c(x_{d_1,d}, y_{d_1,d})}{1_{\{\varphi(x_{d_1})\}}(y_{d_1})} \right\}. \quad (2.5)$$

For $f \in C_b(\mathbb{R}^d)$ and $x = (x_{d_1}, x_{d_1,d}) \in \mathbb{R}^d$,

$$v(x; f|\varphi) := \sup\{f(\varphi(x_{d_1}), y) - c(x_{d_1,d}, y) | y \in \mathbb{R}^{d-d_1}\}. \quad (2.6)$$

Then, from (2.5),

$$T(P_0, P_1|\varphi) = \sup\left\{ \int_{\mathbb{R}^d} f(y)P_1(dy) - \int_{\mathbb{R}^d} v(x; f|\varphi)P_0(dx) \middle| f \in C_b(\mathbb{R}^d) \right\}. \quad (2.7)$$

We easily obtain the following (see e.g. (2.8)-(2.9) in [16]).

Proposition 2.2 Suppose that $\varphi \in C(\mathbb{R}^{d_1} \to \mathbb{R}^{d_1}), c(x, y) \in C(\mathbb{R}^{d-d_1} \times \mathbb{R}^{d-d_1} : [0, \infty))$ and $\lim_{|y-x|\to\infty} c(x, y) = \infty$. Then for any $f \in C_b(\mathbb{R}^d)$, $v(\cdot; f|\varphi)$ is continuous.

We formally derive the Hamilton-Jacobi Equation (HJ Eqn for short) for $v(x; f|\varphi)$. Let

$$\Phi(t, x) := x + t(\varphi(x) - x), \quad b(t, x) := \varphi(\Phi(t, \cdot)^{-1}(x)) - \Phi(t, \cdot)^{-1}(x) \quad ((t, x) \in [0, 1] \times \mathbb{R}^{d_1}), \quad (2.8)$$

provided it exists. Then

$$\frac{d\Phi(t, x)}{dt} = \varphi(x) - x = b(t, \Phi(t, x)). \quad (2.9)$$

In case $c(x, y) = \ell(y-x)$ for a convex $\ell$, we consider the following HJ Eqn:

$$\frac{\partial v(t, x)}{\partial t} + \left< \nabla_{d_1} v(t, x), b(t, x_{d_1}) \right> + h(\nabla_{d_1,d} v(t, x)) = 0 \quad ((t, x) \in (0, 1) \times \mathbb{R}^d), \quad (2.10)$$

where $\nabla_{d_1} := (\partial/\partial x_i)_{i=1}^{d_1}$, $\nabla_{d_1,d} := (\partial/\partial x_i)_{i=d_1+1}^d$ and

$$h(z) := \sup\{ <u, z> - \ell(u) | u \in \mathbb{R}^{d-d_1}\} \quad (z \in \mathbb{R}^{d-d_1}).$$

Then we have
Proposition 2.3 Suppose that \( c(x, y) = \ell(y - x) \) for a convex \( \ell \), that \( \Phi(t, \cdot) \) is injective for all \( t \in [0, 1] \), that the HJ Eqn (2.10) has a classical solution \( v \) and that the following ODE has an absolutely continuous solution: for any \( \phi_2(0) = x_{d_1,d} \)

\[
\frac{d\phi_2(t)}{dt} = \nabla h(\nabla_{x_{d_1,d}} v(t, \Phi(t, x_{d_1}), \phi_2(t))).
\] (2.11)

Then \( v(0, x) = v(x; v(1, \cdot) | \varphi) \).

(Proof) For any \( \phi_2 \in AC(\mathbb{R}^{d-d_1}) \), from (2.9), we have

\[
v(1, \Phi(1, x_{d_1}), \phi_2(1)) - v(0, \Phi(0, x_{d_1}), \phi_2(0)) = \int_0^1 \left\{ \frac{\partial v(t, \Phi(t, x_{d_1}), \phi_2(t))}{\partial t} + \left< \nabla_{d_1} v(t, \Phi(t, x_{d_1}), \phi_2(t)), \frac{d\phi_2(t)}{dt} \right> \right\} dt
\]

\[
= \int_0^1 \left\{ -h(\nabla_{d_1,d} v(t, \Phi(t, x_{d_1}), \phi_2(t))) + \left< \nabla_{d_1,d} v(t, \Phi(t, x_{d_1}), \phi_2(t)), \frac{d\phi_2(t)}{dt} \right> \right\} dt
\]

\[
\leq \int_0^1 \ell\left( \frac{d\phi_2(t)}{t} \right) dt,
\] (2.12)

where the equality holds if (2.11) holds. By Jensen's inequality,

\[
v(0, x) = \sup \left\{ v(1, \varphi(x_{d_1}), \phi_2(1)) - \int_0^1 \ell\left( \frac{d\phi_2(t)}{t} \right) dt \mid \phi_2(0) = x_{d_1,d} \right\}
\]

\[
= \sup \left\{ v(1, \varphi(x_{d_1}), \phi_2(1)) - \ell(\phi_2(1) - \phi_2(0)) \mid \phi_2(0) = x_{d_1,d} \right\}
\]

\[
= v(x; v(1, \cdot) | \varphi). \square
\] (2.13)

Before we formulate the duality theorem in the framework of the theory of viscosity solutions, we give assumptions.

(A.1) \( b(t, x) \) is bounded and there exists \( K > 0 \) such that

\[
|b(t, x) - b(t, y)| \leq K|x - y| \quad (t \in [0, 1], x, y \in \mathbb{R}^{d_1}).
\]

(A.2) There exists \( m \in C([0, 1] \times \mathbb{R}^{d_1} \times [0, 1] \times \mathbb{R}^{d_1} \times [0, \infty)) \) such that \( m(t, x, s, y, 0) = 0 \) and that

\[
|b(t, x) - b(s, y)| \leq m(t, x, s, y, |t - s| + |x - y|) \quad (t, s \in [0, 1], x, y \in \mathbb{R}^{d_1}).
\]
(A.3). $\ell : \mathbb{R}^{d-d_{1}} \mapsto [0, \infty)$ is convex and $\liminf_{|v| \to \infty} \frac{\ell(v)}{|v|} = \infty$.

**Example 2.1** Suppose that $d_{1} = 1$. Then (A.1)-(A.2) holds if $1 < d\varphi(x)/dx \leq K + 1$.

For $(t, x) \in [0, 1] \times \mathbb{R}^{d}$ and $f \in C_{b}(\mathbb{R}^{d})$,

$$v(t, x; f|\varphi) := \sup \left\{ f(\Phi(1, y_{d_{1}}), \phi_{2}(1)) - \int_{t}^{1} \ell \left( \frac{d\phi_{2}(s)}{ds} \right) ds \bigg| (\Phi(t, y_{d_{1}}), \phi_{2}(t)) = x \right\}. \quad (2.14)$$

Then it is easy to see that the following holds:

$$v(t, x; f|\varphi) = \sup \left\{ f(x_{d_{1}} + (1-t)b(t, x_{d_{1}}), y) - (1-t)\ell \left( \frac{y-x_{d_{1},d}}{1-t} \right) \bigg| y \in \mathbb{R}^{d-d_{1}} \right\}. \quad (2.15)$$

(see (2.8)). We also have

**Corollary 2.1** Suppose that $c(x, y) = \ell(y - x)$ and that (A.1)-(A.3) hold. Then for any Lipschitz continuous $f : \mathbb{R}^{d} \mapsto \mathbb{R}$, $v(t, x; f|\varphi)$ is a Lipschitz continuous viscosity solution of (2.10). In particular, for any $P_{0}, P_{1} \in \mathcal{M}_{1}(\mathbb{R}^{d})$,

$$T(P_{0}, P_{1}|\varphi) = \sup \left\{ \int_{\mathbb{R}^{d}} v(1, y)P_{1}(dy) - \int_{\mathbb{R}^{d}} v(0, x)P_{0}(dx) \bigg| v(1, \cdot) \in C_{b}^{\infty}(\mathbb{R}^{d}) \right\}, \quad (2.16)$$

where $v(t, x)$ denotes a bounded uniformly continuous viscosity solution of (2.10).

(Proof) In the same way as in p. 127 in [4], by (A.1) and (A.3), one can prove that $v(\cdot, \cdot; f|\varphi)$ is Lipschitz continuous for Lipschitz continuous $f : \mathbb{R}^{d} \mapsto \mathbb{R}$. In addition, from Chap. II.16 of [7], under (A.1)-(A.3), $v(t, x; f|\varphi)$ is a bounded, uniformly continuous viscosity solution of (2.10). It is easy to see that the supremum in (2.7) can be taken only over all $f \in C_{b}^{\infty}(\mathbb{R}^{d})$. For $n \geq 1$, $f \in C_{b}^{\infty}(\mathbb{R}^{d})$ and $(t, x) \in [0, 1] \times \mathbb{R}^{d}$,

$$v_{n}(t, x; f) := \sup \left\{ f(x_{d_{1}} + (1-t)b(t, x_{d_{1}}), \phi_{2}(1)) - \int_{t}^{1} \ell \left( \frac{d\phi_{2}(s)}{ds} \right) ds \bigg| \phi_{2}(t) = x_{d_{1},d}, \left| \frac{d\phi_{2}(s)}{ds} \right| \leq n \right\}. \quad (2.17)$$

Then, from Theorem 10.1 in p. 95 of [7], under (A.1), $v_{n}(t, x; f)$ is the unique bounded uniformly continuous viscosity solution of the following HJ Eqn: for $(t, x) \in (0, 1) \times \mathbb{R}^{d}$,
\[ \frac{\partial v(t, x)}{\partial t} + \langle \nabla_{d_1} v(t, x), b(t, x_{d_1}) \rangle + h_n(\nabla_{d_1, d} v(t, x)) = 0, \]
\[ v(1, x) = f(x), \quad (2.18) \]
where
\[ h_n(z) := \sup \{ \langle u, z \rangle - \ell(u) | u \in \mathbb{R}^{d-d_1}, |u| \leq n \}. \]

Let \( \overline{v} \) be a bounded uniformly continuous viscosity solution of (2.10) with \( \overline{v}(1, x) = f(x) \). Then it is a bounded uniformly continuous viscosity supersolution of (2.18) with \( \overline{v}(1, x) = f(x) \) and
\[ v_n(t, x; f) \leq \overline{v}(t, x) \quad (2.19) \]
from Theorem 9.1 in p. 86 of [7]. Let \( n \to \infty \) in (2.19). Then we obtain \( v(t, x; f|\phi) \leq \overline{v}(t, x) \).

### 2.2 Convergence Theorem

Let \( 2 \leq k \leq d, \ 0 = d_0 < d_1 < \cdots < d_k = d \) and (A.4) \( c_i \in \text{LSC}(\mathbb{R}^{d_i-d_{i-1}} \times \mathbb{R}^{d_i-d_{i-1}} : [0, \infty)) \) \( (i = 1, \cdots, k) \).

For \( \epsilon \geq 0, \ P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d), \)
\[ T^\epsilon(P_0, P_1) := \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \sum_{i=1}^{k} \epsilon^{i-1} c_i(x_{d_{i-1},d_i}, y_{d_{i-1},d_i}) \mu(dx dy) \right\}, \quad (2.20) \]

It is known that if \( c_i(x, y) = \ell_i(y - x) \) and \( \ell_i \) is strictly convex and superlinear \( (i = 1, \cdots, k) \) and if \( P_0(dx) \) is absolutely continuous with respect to the Lebesgue measure \( dx \), then \( T^\epsilon(P_0, P_1) \) has the unique minimizer, provided that it is finite (see e.g. [21], [24], [25]).

\[ T_1(P_{0,1}, P_{1,1}) := \inf \left\{ \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_1}} c_1(x, y) \mu(dx dy) \right\}, \quad (2.21) \]

where \( P_t(dx_{d_1}) := P_t(dx_{d_1} \times \mathbb{R}^{d-d_1}) \ (t = 0, 1) \). For \( i \geq 2 \) and \( \nu_{i-1} \in \mathcal{M}_1(\mathbb{R}^{2d_i-1}), \)
\[ T_i(P_{0,i}, P_{1,i} | \nu_{i-1}) := \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} c_i(x_{d_{i-1}, d_{i}}, y_{d_{i-1}, d_{i}}) \mu(dxdy) \right\} \]

\[
\begin{align*}
\mu(dx_{d_{i-1}} \times \mathbb{R}^{d_{i}-d_{i-1}} \times dy_{d_{i-1}} \times \mathbb{R}^{d_{i}-d_{i-1}}) &= \nu_{i-1}(dx_{d_{i-1}} dy_{d_{i-1}}), \\
\mu(dx \times \mathbb{R}^{d_{i}}) &= P_{0,i}(dx), \\
\mu(\mathbb{R}^{d_{i}} \times dy) &= P_{1,i}(dy) \}
\end{align*}
\]

(2.22)

The following theorem can be proved in the same way as \cite{2} (see also section 1) and is proved for the readers' convenience.

**Theorem 2.2** Let \( P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d) \). Suppose that \( k = 2 \) and (A.4) holds and that \( T_1(P_{0,1}, P_{1,1}) \) and \( T_2(P_0, P_1 | \nu_1) \) have the unique minimizers \( \nu_1 \) and \( \nu_2 \), respectively. Then a minimizer of \( T^\epsilon(P_0, P_1) \) exists and weakly converges to \( \nu_2 \) as \( \epsilon \to 0 \) and the following holds:

\[
\begin{align*}
\lim_{\epsilon \to 0} \int_{\mathbb{R}^d \times \mathbb{R}^d} c_1(x_{d_{i}, d_{i}}) \mu^\epsilon(dxdy) &= T_1(P_{0,1}, P_{1,1}), \\
\lim_{\epsilon \to 0} \int_{\mathbb{R}^d \times \mathbb{R}^d} c_2(x_{d_{i}, d_{i}}, y_{d_{i}, d_{i}}) \mu^\epsilon(dxdy) &= T_2(P_0, P_1 | \nu_1).
\end{align*}
\]

(2.23) (2.24)

(Proof). In the same way as in the proof of Proposition 2.1, by a standard method, one can show that \( T^\epsilon(P_0, P_1) \) has a minimizer \( \mu^\epsilon \), since

\[ T^\epsilon(P_0, P_1) \leq T_1(P_{0,1}, P_{1,1}) + \epsilon T_2(P_0, P_1 | \nu_1) < +\infty. \]  

(2.25)

Since the set of \( \mu(dxdy) \) is compact, any sequence \( \{\mu^\epsilon_n\}_{n \geq 1} \) (\( \epsilon_n \to 0 \) as \( n \to \infty \)) has a weakly convergent subsequence \( \{\mu^\epsilon_{n(\ell)}\}_{\ell \geq 1} \) and for the limit \( \mu \),

\[ \mu_1(dx_{d_{i}} dy_{d_{i}}) := \mu(dx_{d_{i}} \times \mathbb{R}^{d-d_{i}} \times dy_{d_{i}} \times \mathbb{R}^{d-d_{i}}) \]

is the minimizer of \( T_1(P_{0,1}, P_{1,1}) \) by the uniqueness of the minimizer and (2.23) holds. Indeed, from (2.25),

\[
\begin{align*}
T_1(P_{0,1}, P_{1,1}) &\leq \int_{\mathbb{R}^{d_{i}} \times \mathbb{R}^{d_{i}}} c_1(x, y) \mu_1(dxdy) = \int_{\mathbb{R}^d \times \mathbb{R}^d} c_1(x_{d_{i}}, y_{d_{i}}) \mu(dxdy) \\
&\leq \liminf_{\ell \to \infty} T_{\epsilon_{n(\ell)}}^\epsilon(P_0, P_1) \leq \limsup_{\ell \to \infty} T_{\epsilon_{n(\ell)}}^\epsilon(P_0, P_1) \\
&\leq T_1(P_{0,1}, P_{1,1}).
\end{align*}
\]

(2.26)
Since
\[ T_1(P_{0,1}, P_{1,1}) + \epsilon \int_{\mathbb{R}^d \times \mathbb{R}^d} c_2(x_{d_1,d}, y_{d_1,d}) \mu^\epsilon(dx dy) \leq T^\epsilon(P_0, P_1), \] (2.27)
we also have, from (2.25) and (2.27),
\begin{align*}
T_2(P_0, P_1 | \nu_1) \leq & \int_{\mathbb{R}^d \times \mathbb{R}^d} c_2(x_{d_1,d}, y_{d_1,d}) \mu(dx dy) \\
\leq & \liminf_{\ell \to \infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} c_2(x_{d_1,d}, y_{d_1,d}) \mu^{\epsilon_n(\ell)}(dx dy) \\
\leq & \limsup_{\ell \to \infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} c_2(x_{d_1,d}, y_{d_1,d}) \mu^{\epsilon_n(\ell)}(dx dy) \\
\leq & T_2(P_0, P_1 | \nu_1). \tag{2.28}
\end{align*}

The uniqueness of the minimizer of $T_2(P_0, P_1 | \nu_1)$ completes the proof. □

**Theorem 2.3** Let $P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d)$. Suppose that (A.4) holds, that $T_1(P_{0,1}, P_{1,1})$ and $T_i(P_{0,i}, P_{1,i} | \nu_{i-1})$ have the unique minimizers $\nu_1$ and $\nu_i$ ($i = 2, \cdots, k$), respectively and that $\nu \mapsto T_i(P_{0,i}, P_{1,i} | \nu)$ is continuous ($i = 3, \cdots, k$). Then a minimizer of $T^\epsilon(P_0, P_1)$ exists and weakly converges to $\nu_k$ as $\epsilon \to 0$ and the following holds:

\begin{align*}
\lim_{\epsilon \to 0} \int_{\mathbb{R}^d \times \mathbb{R}^d} c_1(x_{d_1}, y_{d_1}) \mu^\epsilon(dx dy) & = T_1(P_{0,1}, P_{1,1}), \tag{2.29} \\
\lim_{\epsilon \to 0} \int_{\mathbb{R}^d \times \mathbb{R}^d} c_i(x_{d_{i-1},d_i}, y_{d_{i-1},d_i}) \mu^\epsilon(dx dy) & = T_i(P_{0,i}, P_{1,i} | \nu_{i-1})(i = 2, \cdots, k). \tag{2.30}
\end{align*}

(Proof). In the same way as in (2.25), one can show that $T^\epsilon(P_0, P_1)$ has a minimizer $\mu^\epsilon$ and that any subsequence $\{\mu^{\epsilon_n}\}_{n \geq 1}$ ($\epsilon_n \to 0$ as $n \to \infty$) has a weakly convergent subsequence $\{\mu^{\epsilon_n(\ell)}\}_{\ell \geq 1}$. Let $\mu$ denote the weak limit of $\mu^{\epsilon_n(\ell)}$ as $\ell \to \infty$. We prove the theorem by induction. For $i = 2, \cdots, k$,

\begin{align*}
T_{i-1}^\epsilon(P_{0,i-1}, P_{1,i-1}) := & \inf \left\{ \int_{\mathbb{R}^{d_{i-1}} \times \mathbb{R}^{d_{i-1}}} \sum_{j=1}^{i-1} \epsilon^{j-1} c_j(x_{d_{j-1},d_j}, y_{d_{j-1},d_j}) \nu(dx dy) \right. \\
& \left. \nu(dx \times \mathbb{R}^{d_{i-1}}) = P_{0,i-1}(dx), \nu(\mathbb{R}^{d_{i-1}} \times dy) = P_{1,i-1}(dy) \right\}. \tag{2.31}
\end{align*}

Let $\mu_{i-1}^\epsilon$ and $\nu_{i,j}^\epsilon$ denote a minimizer of $T_{i-1}^\epsilon(P_{0,i-1}, P_{1,i-1})$ and $T_j(P_{0,j}, P_{1,j} | \nu_{i,j-1}^\epsilon)$ ($j = i, \cdots, k$), respectively, where $\nu_{i,i-1}^\epsilon := \mu_{i-1}^\epsilon$. Then
$$T_{i}^{\epsilon}(P_{0,i-1}, P_{1,i-1}) + \int_{\mathbb{R}^d \times \mathbb{R}^d} \sum_{j=i}^{k} \epsilon^{j-1} c_{j}(x_{d_{j-1},d_{j}}, y_{d_{j-1},d_{j}}) \mu^{\epsilon}(dxdy) \leq T^{\epsilon}(P_{0}, P_{1})$$

$$\leq T_{i-1}^{\epsilon}(P_{0_{i-1}}, P_{1,i-1}) + \sum_{j=i}^{k} \epsilon^{j-1} T_{j}(P_{0,j}, P_{1,j}|\nu_{i,j-1}^{\epsilon}). \quad (2.32)$$

From Theorem 2.2, $\mu_{i}^{\epsilon} \rightarrow \nu_{i}$ as $\epsilon \rightarrow 0$ and (2.23)-(2.24) holds. Suppose that $\mu_{i}^{\epsilon} \rightarrow \nu_{i}$ as $\epsilon \rightarrow 0$ for $i \leq k - 1$. In the same way as in Theorem 2.2, one can show that for $j = 1, 2,$

$$\mu(dx_{d_{j}} \times \mathbb{R}^{d-d_{j}} \times dy_{d_{j}} \times \mathbb{R}^{d-d_{j}}) = \nu_{j}(dx_{d_{j}}dy_{d_{j}}). \quad (2.33)$$

Suppose that (2.33) holds for $j = i - 1$. Then, from (2.32) and the assumption of induction,

$$T_{i}(P_{0,i}, P_{1,i}|\nu_{i-1}) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} c_{i}(x_{d_{i-1},d_{i}}, y_{d_{i-1},d_{i}}) \mu(dx_{d_{i}}, dy_{d_{i}})$$

$$\leq \lim_{\ell \rightarrow \infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} c_{i}(x_{d_{i-1},d_{i}}, y_{d_{i-1},d_{i}}) \mu_{\mu_{i-1}^{\ell}}(dx_{d_{i}}, dy_{d_{i}})$$

$$\leq \lim_{\ell \rightarrow \infty} T_{i}(P_{0,i}, P_{1,i}|\mu_{i-1}^{\ell}) = T_{i}(P_{0}, P_{1}|\nu_{i-1}). \quad (2.34)$$

(2.34) implies (2.30) and the uniqueness of the minimizer $\nu_{i}$ of $T_{i}(P_{0,i}, P_{1,i}|\nu_{i-1})$ implies that (2.33) holds for $j = i.$ \(\square\)

From (2.32), we also have

**Proposition 2.4** Suppose that the assumption in Theorem 2.3 holds. Then, for $i = 1, \cdots, k - 1$,

$$0 \leq \frac{\int_{\mathbb{R}^d \times \mathbb{R}^d} \sum_{j=1}^{i} \epsilon^{j-1} c_{j}(x_{d_{j-1},d_{j}}, y_{d_{j-1},d_{j}}) \mu^{\epsilon}(dxdy) - T_{i}^{\epsilon}(P_{0,i}, P_{1,i})}{\epsilon^{i}} \rightarrow 0 \quad (\epsilon \rightarrow 0). \quad (2.35)$$

We don't know the real convergence rate of (2.35).

**Example 2.2** Let $P_{0}, P_{1} \in \mathcal{M}_{1}(\mathbb{R}^d)$. Suppose that

(i) $d_{i+1} = d_{i} + 1 \ (i = 1, \cdots, k - 1),$
(ii) $c_i(x, y) = \ell_i(y - x)$ and $\ell_i : \mathbb{R}^{d_i} \to [0, \infty)$ is strictly convex and superlinear ($i = 1, \ldots, k$).

(iii) $P_0$ is absolutely continuous with respect to the Lebesgue measure $dx$.

(iv) $T_1(P_{0,1}, P_{1,1})$ is finite.

Then $T_1(P_{0,1}, P_{1,1})$ has the unique minimizer $\nu_1$ which can be written as follows:

$$\nu_1(dx_{d_1}dy_{d_1}) = P_{0,1}(dx_{d_1})\delta_{\phi_1(x_{d_1})}(dy_{d_1}),$$

where $\phi_1$ is a Borel measurable function (see e.g. [21], [24]).

Suppose, in addition, that

(v) $T_i(P_{0,i}, P_{1,i}|\nu_{i-1})$ is finite for $i = 2, \ldots, k$. (If $T_i(P_{0,i}, P_{1,i}|\nu_{i-1})$ is finite, then it has a minimizer (see the proof of Prop. 2.1).)

Then the following holds:

$$\nu_i(dx_{d_i}dy_{d_i}) = P_{0,i}(dx_{d_i})\delta_{\Phi_{\nu_0,\ldots,\nu_{i-1}}(x_{d_i})}(dy_{d_i}),$$

where $\Phi_{\nu_0,\ldots,\nu_{i-1}}(x_{d_i}) := (\phi_{\nu_0}(x_{d_1}), \ldots, \phi_{\nu_{i-1}}(x_{d_i}))$, $\phi_{\nu_0} := \phi_1$ and

$$\phi_{\nu_{i-1}}(x_{d_i}) := \left( F_{\nu_{i-1},1}(x|x_{d_{i-1}}, \Phi_{\nu_0,\ldots,\nu_{i-2}}(x_{d_{i-1}}))^{-1}(F_{\nu_{i-1},0}(x_{d_i}|x_{d_{i-1}}, \Phi_{\nu_0,\ldots,\nu_{i-2}}(x_{d_{i-1}}))), \quad F_{\nu_{i-1},0}(x_{d_i}|x_{d_{i-1}}, \Phi_{\nu_0,\ldots,\nu_{i-2}}(x_{d_{i-1}})) \right).$$

In particular, $\phi_{\nu_{i-1}}$ is a minimizer of the following:

$$\min \left\{ \int_{\mathbb{R}^{d_i}} \ell_i(\phi(x) - x_{d_i})P_{0,i}(dx) \bigg| P_{0,i}(\Phi_{\nu_0,\ldots,\nu_{i-2}}, \phi)^{-1} = P_{1,i} \right\} = T_i(P_{0,i}, P_{1,i}|\nu_{i-1}).$$

### 3 Stochastic version of Knothe-Rosenblatt type rearrangement.

Let $\mathcal{A}$ denote the set of all $\mathbb{R}^d$-valued, continuous semimartingales $\{X(t)\}_{0 \leq t \leq 1}$ on a (possibly different) complete filtered probability space such that there exists a Borel measurable $\beta_X : [0, 1] \times C([0, 1]) \mapsto \mathbb{R}^d$ for which

(i) $\omega \mapsto \beta_X(t, \omega)$ is $\mathcal{B}(C([0, 1]))$-$\mathcal{L}$-measurable for all $t \in [0, 1]$,
(ii) \( X(t) = X(0) + \int_0^t \beta_X(s, X) \, ds + W_X(t) \) (0 \( \leq t \leq 1 \)).

Here \( \mathcal{B}(C([0, t]))_+ := \cap_{s > t} \mathcal{B}(C([0, s])) \), \( \mathcal{B}(C([0, t])) \) and \( W_X \) denote the Borel \( \sigma \)-field of \( C([0, t]) \) and an \( (\mathcal{F}_t^X) \)-Brownian motion, respectively, and \( \mathcal{F}_t^X := \sigma[X(s) : 0 \leq s \leq t] \) (see e.g. [14]). Let \( d \geq 2 \) and \( 1 \leq d_1 < d \), and let \( b_1 : [0,1] \times \mathbb{R}^{d_1} \mapsto \mathbb{R}^{d_1} \) be a Borel measurable function such that the following SDE has a weak solution for a given initial distribution:

\[
dX(t) = b_1(t, X(t)) \, dt + dW_{X_1}(t).
\]

(3.1)

Let \( L(t, x; u) : [0,1] \times \mathbb{R} \times \mathbb{R}^{d-d_1} \mapsto [0, \infty) \).

A minimizer of the following can be considered as the stochastic optimal control (SOC for short) version of the Knothe Rosenblatt type rearrangement: for \( P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d) \),

\[
V(P_0, P_1|b_1) := \inf \left\{ E\left[ \int_0^1 L(t, Y(t); \beta_{Y,2}(t, Y)) \, dt \right] \middle| Y \in \mathcal{A}, \beta_{Y,1}(t, Y) = b_1(t, Y_1(t)), PY(0)^{-1} = P_0, PY(1)^{-1} = P_1 \right\}, \tag{3.2}
\]

where we write \( \beta_{Y}(t, Y) = (\beta_{Y,1}(t, Y), \beta_{Y,2}(t, Y)) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d-d_1} \).

**Example 3.1** For \( P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d) \), take \( T_{KR} \) in section 1 and, on a complete filtered probability space, consider

\[
Z(t) = Z(0) + \int_0^t \frac{T_{KR}(Z(0)) - Z(s)}{1 - s} \, ds + W_{Z}(t). \tag{3.3}
\]

Then \( Z(1) = T_{KR}(Z(0)) \). In particular, \( PZ(1)^{-1} = P_1 \), provided \( PZ(0)^{-1} = P_0 \). Besides, \( \beta_{Z,i}(t, Z) = \beta_{Z_i}(t, Z_i) \) for all \( i = 1, \cdots, d \). Suppose that \( p \in [1,2) \) and that \( \int_{\mathbb{R}^d} |x|^p (P_0(dx) + P_1(dx)) \) is finite. Then

\[
E\left[ \int_0^1 \frac{\left| T_{KR}(Z(0)) - Z(s) \right|^p}{1 - s} \, ds \right] < \infty. \tag{3.4}
\]

Indeed, \( W_o(t) := Z(t) - Z(0) - (T_{KR}(Z(0)) - Z(0))t \) is a tided down Brownian motion starting and arriving at 0, and

\[
\frac{T_{KR}(Z(0)) - Z(s)}{1 - s} = T_{KR}(Z(0)) - Z(0) - \frac{W_o(s)}{1 - s}.
\]

We describe our assumption in this section to show the existence of the stochastic analogue of the Knothe Rosenblatt type rearrangement.
(H.1). (i) $L \in C([0,1] \times \mathbb{R}^d \times \mathbb{R}^{d-d_1} : [0,\infty))$, (ii) $u \mapsto L(t,x,u)$ is strictly convex.

(H.2). There exists $\gamma > 1$ such that
\[
\liminf_{|u| \to \infty} \frac{\inf\{L(t,x;u) : (t,x) \in [0,1] \times \mathbb{R}^d\}}{|u|^\gamma} > 0. \tag{3.5}
\]

(H.3).
\[
\Delta L(\epsilon_1, \epsilon_2) := \sup \frac{L(t,x;u) - L(s,y;u)}{1 + L(s,y;u)} \to 0 \quad \text{as } \epsilon_1, \epsilon_2 \to 0, \tag{3.6}
\]
where the supremum is taken over all $(t,x)$ and $(s,y) \in [0,1] \times \mathbb{R}^d$ for which $|t-s| \leq \epsilon_1$, $|x-y| < \epsilon_2$ and over all $u \in \mathbb{R}^d$.

The following can be proved in the same way as Prop. 2.1 in [19], and the proof is omitted.

**Proposition 3.1** Suppose that (H.1)-(H.3) hold. Then for any $P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d)$, $V(P_0, P_1|b_1)$ has a minimizer, provided it is finite.

### 3.1 Duality Theorem

We consider the following HJB Equation:
\[
\frac{\partial v(t,x)}{\partial t} + \frac{1}{2} \Delta v(t,x) + \langle \nabla_{x_{d_1}} v(t,x), b_1(t,x_{d_1}) \rangle + H(t,x; \nabla v(t,x)) = 0, \tag{3.7}
\]
$((t,x) \in (0,1) \times \mathbb{R}^d)$, where
\[
H(t,x;z) := \sup \{<u,z>-L(t,x;u) : u \in \mathbb{R}^{d-d_1}\} \quad (z \in \mathbb{R}^{d-d_1}).
\]

For $f \in C_b(\mathbb{R}^d),$
\[
u(t,x;f|b_1)(x) := \sup \left\{ E\left[f(Y(1)) - \int_t^1 L(s,Y(s);\beta_{Y,2}(s,Y))ds\right] \right\}
Y(t) = x, \beta_{Y,1}(s,Y) = b_1(s,Y_{1}(s)), Y \in \mathcal{A}. \tag{3.8}
\]

(H.4). (i) $L(t,x;0)$ is bounded; (ii) $\Delta L(0,\infty)$ is finite; (iii) $b_1 \in C^{1,2}([0,1] \times \mathbb{R}^d) \cap C^0_\beta([0,1] \times \mathbb{R}^d)$, $|D_{x}L(t,x;u)|/(1 + L(t,x;u))$ is bounded on $[0,1] \times \mathbb{R}^d \times \mathbb{R}^{d_1}$ and $D_uL(t,x;u)$ is bounded on $[0,1] \times \mathbb{R}^d \times B_R$ for all $R > 0$, where $B_R := \{x \in \mathbb{R}^{d_1}||x| \leq R\}$.  

The following can be proved in the same way as Theorem 11.1 in IV.11 of [7], and
the proof is omitted.

**Proposition 3.2** Suppose that (H.1)-(H.2) and (H.4,i,iii) hold. Then, for any \( f \in C^5(\mathbb{R}^d) \cap C_b^3(\mathbb{R}^d) \), \( u(t, x; f|b_1) \in C^{1,2}([0,1] \times \mathbb{R}^d) \cap C_b^{0,1}([0,1] \times \mathbb{R}^d) \) and is the unique classical solution of the HJB Equation (3.7) with \( v(1, x) = f(x) \).

It is easy to see that the following holds:

\[
V(P_0, P_1|b_1) := \inf \left\{ E \left[ \int_0^1 \frac{L(t, Y(t); \beta_{Y,2}(t, Y))}{1_{\{b_1(t, Y(t))\}}(\beta_{Y,1}(t, Y))} dt \right] | Y \in \mathcal{A}, \\
PY(0)^{-1} = P_0, PY(1)^{-1} = P_1 \right\}, \tag{3.9}
\]

which implies the duality theorem for \( V(P_0, P_1|b_1) \).

**Theorem 3.1** Suppose that (H.1)-(H.4) hold. Then for any \( P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d) \),

\[
V(P_0, P_1|b_1) = \sup \left\{ \int_{\mathbb{R}^d} v(1, y) P_1(dy) - \int_{\mathbb{R}^d} v(0, x) P_0(dx) \right\}, \tag{3.10}
\]

where the supremum is taken over all classical solutions \( v \) of (3.7) with \( v(1, y) \in C_b^\infty(\mathbb{R}^d) \).

(Proof). Under (H.1)-(H.3) and (H.4,i,ii), (3.9) implies that \( V(P_0, \cdot|b_1) \) is convex and lower-semicontinuous, which can be proved in the same way as in [19] and is not identically equal to infinity by considering the case where \( \beta_{Y,2}(s, Y) = 0 \) from (H.4,i). Hence, from Theorem 2.2.15 and Lemma 3.2.3 in [3],

\[
V(P_0, P_1|b_1) = \sup \left\{ \int_{\mathbb{R}^d} f(y) P_1(dy) - V(P_0, \cdot|b_1)^*(f) \right\} \bigg| f \in C_b(\mathbb{R}^d) \right\}, \tag{3.11}
\]

where

\[
V(P_0, \cdot|b_1)^*(f) := \sup \left\{ \int_{\mathbb{R}^d} f(y) P(dy) - V(P_0, P|b_1) \right\} \bigg| P \in \mathcal{M}_1(\mathbb{R}^d) \right\}. \tag{3.12}
\]

One can replace \( C_b(\mathbb{R}^d) \) by \( C_b^\infty(\mathbb{R}^d) \) in (3.11) in the same way as in the proof of Theorem 2.1 in [19]. For \( f \in C_b^\infty(\mathbb{R}^d) \), from Proposition 3.2,
$V(P_0, |b_1)^*(f)$

$= \sup \left\{ E \left[ f(Y(1)) - \int_0^1 \frac{L(t,Y(t); \beta_{Y,2}(t,Y))}{1_{b_1(t,Y(t))} \beta_{Y_1}(t,Y)} \right] \left| Y \in \mathcal{A}, P(Y(0))^{-1} = P_0 \right. \right\}$

$= \int_{\mathbb{R}^d} u(0, x; f|b_1) P_0(dx), \quad (3.13)$

where the optimal control is $\beta_{Y,2}(t,Y) = \nabla_{d_1,d} u(t,Y(t);f|b_1). \square$

As a corollary to Theorem 3.1, in the same way as [19], we easily obtain

**Corollary 3.1** Suppose that (H.1)-(H.4) hold. Then for any $P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d)$ for which $V(P_0, P_1|b_1)$ is finite, there exists a Borel measurable function $b_2^o : [0,1] \times \mathbb{R}^d \to \mathbb{R}^{d-d_1}$ such that for a minimizer $\{Y(t)\}_{0 \leq t \leq 1}$, $\beta_{Y,2}(t,Y) = b_2^o(t,Y(t))$.

We consider the following marginal problem:

$v(P_0, P_1|b_1) := \inf \int_0^1 dt \int_{\mathbb{R}^d} L(t, x; B_2(t, x)) Q_t(dx), \quad (3.14)$

where the infimum is taken over all $(Q_t(dx))_{0 \leq t \leq 1} \subset \mathcal{M}_1(\mathbb{R}^d)$ for which $B_1 = b_1$, $Q_t = P_t (t = 0, 1)$ and

$$\frac{\partial Q_t(dx)}{\partial t} = \frac{1}{2} \triangle Q_t(dx) - div(B(t,x)Q_t(dx),$$

in a weak sense. Here we write $B(t,x) = (B_1(t,x), B_2(t,x)) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d-d_1}$.

In the same way as [17], we have

**Theorem 3.2** Suppose that (H.1)-(H.4) hold. Then for any $P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d)$,

$v(P_0, P_1|b_1) = \sup \left\{ \int_{\mathbb{R}^d} v(1, y) P_1(dy) - \int_{\mathbb{R}^d} v(0, x) P_0(dx) \right\}, \quad (3.15)$

where the supremum is taken over all classical solutions $v$ of (3.7) with $v(1, y) \in C^\infty_b(\mathbb{R}^d)$. In particular, $V(P_0, P_1|b_1) = v(P_0, P_1|b_1)(\in [0, \infty))$.

We introduce an additional assumption to formulate the duality theorem in the framework of the theory of viscosity solutions.

(H.4)$'$. (i) $\partial L(t,x;u)/\partial t$ and $D_x L(t,x;u)$ is bounded on $[0,1] \times \mathbb{R}^d \times B_R$ for all $R > 0$; (ii) $\Delta L(0,\infty)$ is finite; (iii) $b_1 \in C^\infty(\mathbb{R}^{d})$.

In the same way as in Lemma 4.5 in [17], one can prove
Proposition 3.3 Suppose that (H.1)-(H.3) and (H.4)' hold. Then for any $f \in UC_b(R^d)$, $u(t, x; f|b_1)$ is a bounded continuous viscosity solution of (3.7) with $v(1, x) = f(x)$ and for any $Q \in \mathcal{M}_1(R^d)$ and $t \in [0, 1],$

$$
\int_{R^d} u(t, x; f|b_1)Q(dx) = \sup \left\{ E\left[f(Y(1)) - \int_{t}^{1} L(s, Y(s); \beta_{Y,2}(s, Y))ds \right] \right\}
$$

$$
PY^{-1}(t) = Q, \beta_{Y,1}(s, Y) = b_1(s, Y_1(s)), Y \in \mathcal{A}. \tag{3.16}
$$

In addition, for any bounded continuous viscosity solution $u$ of (3.7) with $u(1, x) = f(x)$, $u(t, x) \geq u(t, x; f|b_1)$, that is, $u(t, x; f|b_1)$ is minimal.

In the same way as in Theorem 3.1, from Prop. 3.3, we have

Theorem 3.3 Suppose that (H.1)-(H.3) and (H.4)' hold. Then for any $P_0, P_1 \in \mathcal{M}_1(R^d),$

$$
V(P_0, P_1|b_1) = \sup \left\{ \int_{R^d} v(1, y)P_1(dy) - \int_{R^d} v(0, x)P_0(dx) \right\}, \tag{3.17}
$$

where the supremum is taken over all bounded continuous viscosity solutions $v(t, x; f)$ of (3.7) with $v(1, x) \in C^\infty_b(R^d)$.

Remark 3.1 (H.3) and (i) in (H.4)' implies (i) in (H.4).

3.2 Convergence Theorem

Let $L_1 : [0, 1] \times R^{d_1} \times R^{d_1} \mapsto [0, \infty)$ and $L_2 : [0, 1] \times R^d \times R^{d-d_1} \mapsto [0, \infty)$. For $\epsilon > 0$, $P_0, P_1 \in \mathcal{M}_1(R^d),$

$$
V^\epsilon(P_0, P_1) := \inf \left\{ E\left[\sum_{i=1}^{2} \epsilon^{i-1} \int_{0}^{1} L_i(t, Y_i(t); \beta_{Y,i}(t, Y))dt \right] \right\}
$$

$$
PY(0)^{-1} = P_0, PY(1)^{-1} = P_1, Y \in \mathcal{A}, \tag{3.18}
$$

where $Y_1(t) := Y_1(t)$ and $Y_2(t) := Y(t)$ for $Y(t) = (Y_1(t), Y_2(t)) \in R^{d_1} \times R^{d-d_1}$.

If (H.1)-(H.3) holds for $L = L_i$ for all $i = 1, 2$, then $V^\epsilon(P_0, P_1)$ has a minimizer, provided it is finite (see Prop. 2.1 in [19]).

$$
V_1(P_{0,1}, P_{1,1}) := \inf \left\{ E\left[\int_{0}^{1} L_1(t, Y(t); \beta_{Y}(t, Y))dt \right] \right\} Y \in \mathcal{A}_1,
$$

$$
PY(0)^{-1} = P_{0,1}, PY(1)^{-1} = P_{1,1}, \tag{3.19}
$$

where $\mathcal{A}_1$ denotes $\mathcal{A}$ with $d = d_1$. 

**Remark 3.2** If (H.1)-(H.4) with $L = L_1$ holds and that $V_1(P_{0,1}, P_{1,1})$ is finite. Then there exists a Borel measurable function $b : [0, 1] \times \mathbb{R}^{d_1} \to \mathbb{R}^{d_1}$ such that for any minimizer $\{Y(t)\}_{0 \leq t \leq 1}$ of $V_1(P_{0,1}, P_{1,1})$, $\beta_Y(t, Y) = b(t, Y(t))$ (see [19]).

Let $b_1$ denote the drift vector of the minimizer of $V_1(P_{0,1}, P_{1,1})$, provided it exists and let $V_2(P_0, P_1|b_1)$ denote $V(P_0, P_1|b_1)$ with $L = L_2$. Then

**Theorem 3.4** Let $P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d)$. Suppose that (H.1)-(H.3) with $L = L_i$ holds ($i = 1, 2$) and that $V_1(P_{0,1}, P_{1,1})$ and $V_2(P_0, P_1|b_1)$ is finite and have the unique minimizers $\{X_1(t)\}_{0 \leq t \leq 1}$ and $\{X(t)\}_{0 \leq t \leq 1}$, respectively. Then a minimizer $\{Y^\epsilon(t)\}_{0 \leq t \leq 1}$ of $V^\epsilon(P_0, P_1)$ exists and weakly converges to $\{X(t)\}_{0 \leq t \leq 1}$ as $\epsilon \to 0$. In particular,

\[
\lim_{\epsilon \to 0} E\left[\int_0^1 L_1(t, Y_1^\epsilon(t); \beta_{Y^\epsilon,1}(t, Y^\epsilon)) dt\right] = V_1(P_{0,1}, P_{1,1}), \quad (3.20)
\]
\[
\lim_{\epsilon \to 0} E\left[\int_0^1 L_2(t, Y^\epsilon(t); \beta_{Y^\epsilon,2}(t, Y^\epsilon)) dt\right] = V_2(P_0, P_1|b_1). \quad (3.21)
\]

(Proof) In the same way as Prop. 2.1 in [19], one can show that there exists a minimizer $Y^\epsilon(t)$ of $V^\epsilon(P_0, P_1)$ since

\[
V^\epsilon(P_0, P_1) \leq V_1(P_{0,1}, P_{1,1}) + \epsilon V_2(P_0, P_1|b_1). \quad (3.22)
\]

In the same way as in Lemma 3.1 in [19], from (H.2), one can show that any sequence $\{Y_{\epsilon_n}(\cdot)\}_{n \geq 1}$ in $\mathcal{A}$ ($\epsilon_n \to 0$ as $n \to \infty$) has a weakly convergent subsequence $\{Y_{\epsilon_{n(k)}}(\cdot)\}_{k \geq 1}$. Indeed,

\[
E\left[\int_0^1 L_1(t, Y_1^\epsilon(t); \beta_{Y_1^\epsilon}(t, Y_1^\epsilon)) dt\right] \leq V^\epsilon(P_0, P_1), \quad (3.23)
\]
\[
E\left[\int_0^1 L_2(t, Y^\epsilon(t); \beta_{Y^\epsilon,2}(t, Y^\epsilon)) dt\right] \leq V_2(P_0, P_1|b_1). \quad (3.24)
\]

We prove (3.24). In the same way as in Lemma 3.1 in [19], from (H.1,ii), by Jensen's inequality,

\[
V_1(P_{0,1}, P_{1,1}) \leq E\left[\int_0^1 L_1(t, Y_1^\epsilon(t); \beta_{Y_1^\epsilon}(t, Y_1^\epsilon)) dt\right] \leq E\left[\int_0^1 L_1(t, Y_1^\epsilon(t); \beta_{Y^\epsilon,1}(t, Y^\epsilon)) dt\right]. \quad (3.25)
\]

Indeed, $Y_1^\epsilon \in A_1$ with
\[
\beta_{Y_{1}^{e}}(t, Y_{1}^{e}) = E[\beta_{1,Y^\epsilon}(t, Y^\epsilon)|Y_{1}^\epsilon(s), 0 \leq s \leq t]
\]
(see e.g., p. 258 of [14]). (3.25) and (3.22) implies (3.24).

Let \( Y^0(t) \) denote the weak limit of \( \{Y^{\epsilon_n(k)}(\cdot)\}_{k \geq 1} \) as \( n \to \infty \). Then, again in the same way as in Lemma 3.1 in [19] and (3.25), from (H.1,ii) and (3.22)-(3.23), by Jensen’s inequality,

\[
V_1(P_{0,1}, P_{1,1}) \leq E\left[ \int_0^1 L_1(t, Y_{1}^{0}(t); \beta_{Y_{1}^{0}}(t, Y_{1}^{0}))dt \right]
\leq \lim_{k \to \infty} \inf E\left[ \int_0^1 L_1(t, Y_{1}^{0}(t); \beta_{Y_{1}^{0},1}(t, Y^{0}))dt \right]
\leq \lim \sup_{k \to \infty} E\left[ \int_0^1 L_1(t, Y_{1}^{0}(t); \beta_{Y_{1}^{0},1}(t, Y^{0}))dt \right]
\leq V_1(P_{0,1}, P_{1,1}). \tag{3.26}
\]

\( \beta_{Y_{1}^{0},1}(t, Y^{0}) = \beta_{Y_{1}^{0}}(t, Y_{1}^{0}) \) from the strict convexity of \( L_1 \) in \( u \), and \( Y_{1}^{0} \) is equal to the minimizer \( X_{1} \) of \( V_1(P_{0,1}, P_{1,1}) \) by the uniqueness of the minimizer of \( V_1(P_{0,1}, P_{1,1}) \) and we obtain (3.20). From (3.24), we also have

\[
V_2(P_{0}, P_{1}|b_{1}) \leq E\left[ \int_0^1 L_2(t, Y_{1}^{0}(t); \beta_{Y_{1}^{0},2}(t, Y^{0}))dt \right]
\leq \lim_{k \to \infty} \inf E\left[ \int_0^1 L_2(t, Y_{1}^{\epsilon_n(k)}(t); \beta_{Y_{1}^{\epsilon_n(k)},2}(t, Y_{1}^{\epsilon_n(k)}))dt \right]
\leq \lim \sup_{k \to \infty} E\left[ \int_0^1 L_2(t, Y_{1}^{\epsilon_n(k)}(t); \beta_{Y_{1}^{\epsilon_n,k},2}(t, Y_{1}^{\epsilon_n(k)}))dt \right]
\leq V_2(P_{0}, P_{1}|b_{1}). \tag{3.27}
\]

The uniqueness of the minimizer of \( V_2(P_{0}, P_{1}|b_{1}) \) completes the proof. \( \square \)

One can easily prove

**Corollary 3.2** Let \( P_{0}, P_{1} \in \mathcal{M}_1(\mathbb{R}^d) \). Suppose that (H.1)-(H.3) with \( L = L_i \) holds \( (i = 1, 2) \), that \( \gamma = 2 \) in (H.2), and that \( V_1(P_{0,1}, P_{1,1}) \) and \( V_2(P_{0}, P_{1}|b_{1}) \) is finite. Then the minimizers \( \{X_{1}(t)\}_{0 \leq t \leq 1}, \{X(t)\}_{0 \leq t \leq 1}, \{Y_{1}^{\epsilon}(t)\}_{0 \leq t \leq 1} \) of \( V_1(P_{0,1}, P_{1,1}) \), \( V_2(P_{0}, P_{1}|b_{1}) \) and \( V_{\epsilon}(P_0, P_1) \) exist uniquely, respectively. In addition, \( \{Y_{1}^{\epsilon}(t)\}_{0 \leq t \leq 1} \) weakly converges to \( \{X(t)\}_{0 \leq t \leq 1} \) as \( \epsilon \to 0 \) and (3.20)-(3.21) holds.

From (3.21)-(3.22) and (3.25), we easily have
Proposition 3.4 Suppose that the assumption in Theorem 3.3 holds. Then for any minimizer \( \{Y^\varepsilon\}_{0 \leq t \leq 1} \) of \( V^\varepsilon(P_0, P_1) \),

\[
0 \leq \frac{E\left[\int_0^1 L_1(t, Y_1^\varepsilon(t); \beta_{Y^\varepsilon,1}(t, Y^\varepsilon))dt\right] - V_1(P_{0,1}, P_{1,1})}{\varepsilon} \to 0 \quad (\varepsilon \to 0).
\] (3.28)

We don’t know the real convergence rate!

4 Discussion

In section 2, Theorem 2.3, we assumed that \( \nu \mapsto T_i(P_0, P_1|\nu) \) is continuous \((i = 3, \ldots, k)\). This continuity is known only in the case of the Knothe-Rosenblatt rearrangement where the representation of the minimizer is known. It is difficult to prove that \( \nu \mapsto T(P_0, P_1|\nu) \) is continuous, which is our future problem.

In section 3.2, we only considered the case where \( k = 2 \) because of the similar reason to above. The point is that we do not even know any example such as the Knothe-Rosenblatt rearrangement. This is also our future problem.

The Knothe-Rosenblatt rearrangement implies the Brunn-Minkowskii inequality. We would like to find, in future, the inequality which can be obtained by the result in section 3.

References


