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Variational characterization of the Knothe-Rosenblatt type rearrangements and its stochastic version

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1 Introduction.

The Knothe-Rosenblatt rearrangement plays a crucial role in many fields, e.g., the Brunn-Minkowski inequality and statistics (see [12], [13], [22] and the references therein).

Let $d \geq 1$ and let $\mathcal{M}_1(\mathbb{R}^d)$ denote the set of all Borel probability measures on $\mathbb{R}^d$ with a weak topology. For a distribution function $F$ on $\mathbb{R}$, let

\begin{align*}
F^{-1}(u) := \inf\{x \in \mathbb{R} | u \leq F(x)\} \quad (0 \leq u \leq 1) \quad (1.1)
\end{align*}

For $P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d)$, $x \in \mathbb{R}$, and $i = 0, 1$, let

\begin{align*}
F_{i,k}(x|x_{k-1}) := \begin{cases} 
    P_i((-\infty, x] \times \mathbb{R}^{d-1}) & (k = 1), \\
    P_i((-\infty, x] \times \mathbb{R}^{d-k}|x_{k-1}) & (1 < k < d), \\
    P_i((-\infty, x]|x_{d-1}) & (k = d),
\end{cases}
\end{align*}

\begin{align*}
\varphi_k(x_k) := F_{1,k}(\cdot|\varphi_1(x_1), \cdots, \varphi_{k-1}(x_{k-1}))^{-1}(F_{0,k}(x_k|x_{k-1}))(1 \leq k \leq d),
\end{align*}

where $x_k := (x_i)_{1 \leq i \leq k} \in \mathbb{R}^k$ for $x = (x_i)_{1 \leq i \leq d} \in \mathbb{R}^d$ and $P_i(\cdot|x_{k-1})$ denotes the regular conditional probability of $P_i$ given $x_{k-1}$.

Suppose that $F_{0,k}(\cdot|x_{k-1})$ is continuous for all $k = 1, \cdots, d$. Then $P_1$ is the image measure of $P_0$ by

\begin{align*}
T_{KR}(x_d) := (\varphi_1(x_1), \cdots, \varphi_d(x_d)).
\end{align*}

$T_{KR}$ is called the Knothe-Rosenblatt rearrangement. Suppose, in addition, that $F_{1,k}(\cdot|x_{k-1})$ is continuous for all $k = 1, \cdots, d$. Then $T_{KR}$ is invertible and the minimizer of the following weakly converges to $P_0(dx)\delta_{T_{KR}(x)}(dy)$ as $\epsilon \to 0$: for $p > 1$, 

\begin{align*}
\int |y - T_{KR}(x)|^p \, P_0(dx) \delta_{T_{KR}(x)}(dy)
\end{align*}
\[
\inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \sum_{k=1}^{d} \epsilon^{2(k-1)} |y_k - x_k|^p \mu(dxdy) \bigg| \mu(dx \times \mathbb{R}^d) = P_0(dx), \right.
\]
\[
\left. \mu(\mathbb{R}^d \times dy) = P_1(dy) \right\},
\]
(1.3)

provided \( \int_{\mathbb{R}^d} |x|^p (P_0(dx) + P_1(dx)) \) is finite (see [2]). Here \( \delta_x(dy) \) denotes the delta measure on \( \{x\} \).

For \( 1 \leq k \leq d \), \( x_{k-1} \in \mathbb{R}^{k-1} \), \( dF_{0,k}(x|x_{k-1})\delta_{\varphi_k(x_k)}(dy) \) is the unique minimizer of

\[
\inf \left\{ \int_{\mathbb{R} \times \mathbb{R}} |y - x|^p \mu(dxdy) \bigg| \mu(dx \times \mathbb{R}) = dF_{0,k}(x|x_{k-1}), \right.
\]
\[
\left. \mu(dx \times dy) = dF_{1,k}(y|\varphi_k(x_k)) \right\}
\]
(1.4)

(see e.g. [21], [24]). (1.4) also implies that \( P_0(dx_k \times \mathbb{R}^{d-k}) \delta_{(\varphi_1(x_1), \cdots, \varphi_k(x_k))}(dy_k) \) is the unique minimizer of

\[
\inf \left\{ \int_{\mathbb{R}^k \times \mathbb{R}^k} |y_k - x_k|^p \mu(dxdy) \bigg| \mu(dx \times \mathbb{R}^k) = P_0(dx \times \mathbb{R}^{d-k}), \right.
\]
\[
\left. \mu(dx \times dy) = P_1(dy \times \mathbb{R}^{d-k}), \quad y_i = \varphi_i(x_{i-1})(i = 1, \cdots, k-1), \mu-a.s. \right\}.
\]
(1.5)

We generalize (1.5) and call the minimizer the Knothe-Rosenblatt type rearrangement. We also prove the duality theorem, give the convergence result which generalizes (1.3) by the idea of [2] and consider the similar problems in the stochastic control setting.

2 Knothe-Rosenblatt type rearrangement.

Let \( d \geq 2, 1 \leq d_1 < d, c(x, y) : \mathbb{R}^{d-d_1} \times \mathbb{R}^{d-d_1} \mapsto [0, \infty) \) be Borel measurable and \( \nu \in \mathcal{M}_1(\mathbb{R}^{2d_1}) \). For \( P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d) \), let

\[
T(P_0, P_1|\nu) := \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_{d_1,d}, y_{d_1,d}) \mu(dxdy) \bigg| \right.
\]
\[
\left. \mu(dx_{d_1} \times \mathbb{R}^{d-d_1} \times dy_{d_1} \times \mathbb{R}^{d-d_1}) = \nu(dx_{d_1}dy_{d_1}), \mu(dx \times \mathbb{R}^d) = P_0(dx), \mu(\mathbb{R}^d \times dy) = P_1(dy) \right\},
\]
(2.1)
where $x_{i,j} := (x_{k})_{i+1 \leq k \leq j} \in \mathbb{R}^{j-i}$ for $x = (x_{k})_{1 \leq k \leq d} \in \mathbb{R}^{d}$. If the set over which the infimum is taken is empty, then we consider the infimum is equal to infinity. If there exists a Borel measurable function $\varphi : \mathbb{R}^{d_1} \rightarrow \mathbb{R}^{d_1}$ such that $y_{d_1} = \varphi(x_{d_1})$, $\nu$-a.s., then we write, for simplicity,

$$T(P_0, P_1|\varphi) := T(P_0, P_1|\nu).$$

We first show the existence of the Knothe-Rosenblatt type rearrangement.

**Proposition 2.1** Suppose that $c$ is lower semi-continuous. Then, for any $P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d)$, $T(P_0, P_1|\nu)$ has a minimizer, provided it is finite.

(Proof) Let $\{\mu_n\}_{n \geq 1}$ be a minimizing sequence of $T(P_0, P_1|\nu)$. Since $\mu_n(dx \times \mathbb{R}^d) = P_0(dx)$ and $\mu_n(\mathbb{R}^d \times dy) = P_1(dy)$, it has a weakly convergent subsequence which we denote by $\{\mu_{n(k)}\}_{k \geq 1}$. Let $\mu$ denote the limit. Then by Skorohod’s representation theorem, Fatou’s lemma and the lower semicontinuity of $c$,

$$T(P_0, P_1|\nu) = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_{d_1,d}, y_{d_1,d}) \mu_{n(k)}(dxdy) \geq \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_{d_1,d}, y_{d_1,d}) \mu(dxdy). \quad (2.2)$$

For any $f \in C(\mathbb{R}^{d_1} \times \mathbb{R}^{d_1})$,

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(x_{d_1}, y_{d_1}) \mu(dxdy) = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x_{d_1}, y_{d_1}) \mu_{n(k)}(dxdy) = \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_1}} f(x_{d_1}, y_{d_1}) \nu(dx_{d_1} dy_{d_1}). \quad (2.3)$$

In the same way, one can show that $\mu(dx \times \mathbb{R}^d) = P_0(dx)$ and $\mu(\mathbb{R}^d \times dy) = P_1(dy). \square$

### 2.1 Duality Theorem

It is easy to see that the following holds:

$$T(P_0, P_1|\varphi) = \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{c(x_{d_1,d}, y_{d_1,d})}{1_{\{\varphi(x_{d_1})\}}(y_{d_1})} \mu(dxdy) \mid \mu(dx \times \mathbb{R}^d) = P_0(dx), \mu(\mathbb{R}^d \times dy) = P_1(dy) \right\}, \quad (2.4)$$

where $1_A(x) := 1$ if $x \in A$ and $:= 0$ if $x \not\in A$ for the set $A$. This leads us to the duality theorem for $T(P_0, P_1|\varphi)$ which can be obtained from [11] (see also p. 76 in Vol. 1 of [21]).
Theorem 2.1 For any $P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d)$,

$$T(P_0, P_1|\varphi) = \sup \left\{ \int_{\mathbb{R}^d} f_1(y)P_1(dy) - \int_{\mathbb{R}^d} f_0(x)P_0(dx) \middle| f_0, f_1 \in C_b(\mathbb{R}^d), f_1(y) - f_0(x) \leq c(x_{d_1,d}, y_{d_1,d}) \right\}. \tag{2.5}$$

For $f \in C_b(\mathbb{R}^d)$ and $x = (x_{d_1}, x_{d_1,d}) \in \mathbb{R}^d$,

$$v(x; f|\varphi) := \sup \{ f(\varphi(x_{d_1}), y) - c(x_{d_1,d}, y) \middle| y \in \mathbb{R}^{d-d_1} \}. \tag{2.6}$$

Then, from (2.5),

$$T(P_0, P_1|\varphi) = \sup \left\{ \int_{\mathbb{R}^d} f(y)P_1(dy) - \int_{\mathbb{R}^d} v(x; f|\varphi)P_0(dx) \middle| f \in C_b(\mathbb{R}^d) \right\}. \tag{2.7}$$

We easily obtain the following (see e.g. (2.8)-(2.9) in [16]).

Proposition 2.2 Suppose that $\varphi \in C(\mathbb{R}^{d_1} : \mathbb{R}^{d_1})$, $c(x, y) \in C(\mathbb{R}^{d-d_1} \times \mathbb{R}^{d-d_1} : [0, \infty))$ and $\lim_{|y-x| \to \infty} c(x, y) = \infty$. Then for any $f \in C_b(\mathbb{R}^d)$, $v(\cdot; f|\varphi)$ is continuous.

We formally derive the Hamilton-Jacobi Equation (HJ Eqn for short) for $v(x; f|\varphi)$. Let

$$\Phi(t, x) := x + t(\varphi(x) - x),$$

$$b(t, x) := \varphi(\Phi(t, \cdot)^{-1}(x)) - \Phi(t, \cdot)^{-1}(x) \quad ((t, x) \in [0, 1] \times \mathbb{R}^{d_1}), \tag{2.8}$$

provided it exists. Then

$$\frac{d\Phi(t, x)}{dt} = \varphi(x) - x = b(t, \Phi(t, x)). \tag{2.9}$$

In case $c(x, y) = \ell(y-x)$ for a convex $\ell$, we consider the following HJ Eqn:

$$\frac{\partial v(t, x)}{\partial t} + \langle \nabla_{d_1}v(t, x), b(t, x_{d_1}) \rangle + h(\nabla_{d_1,d}v(t, x)) = 0 \quad ((t, x) \in (0,1) \times \mathbb{R}^d), \tag{2.10}$$

where $\nabla_{d_1} := (\partial/\partial x_i)_{i=1}^{d_1}$, $\nabla_{d_1,d} := (\partial/\partial x_i)_{i=d_1+1}^d$ and

$$h(z) := \sup\{ <u, z> - \ell(u) \middle| u \in \mathbb{R}^{d-d_1} \} \quad (z \in \mathbb{R}^{d-d_1}).$$

Then we have
Proposition 2.3 Suppose that \(c(x, y) = \ell(y - x)\) for a convex \(\ell\), that \(\Phi(t, \cdot)\) is injective for all \(t \in [0, 1]\), that the HJ Eqn (2.10) has a classical solution \(v\) and that the following ODE has an absolutely continuous solution: for any \(\phi_2(0) = x_{d_1,d}\)

\[
\frac{d\phi_2(t)}{dt} = \nabla h(\nabla_{x_{d_1,d}} v(t, \Phi(t, x_{d_1}), \phi_2(t))).
\]  

(2.11)

Then \(v(0, x) = v(x; v(1, \cdot)|\varphi)\).

(Proof) For any \(\phi_2 \in AC(\mathbb{R}^{d-d_1})\), from (2.9), we have

\[
\begin{align*}
v(1, \Phi(1, x_{d_1}), \phi_2(1)) - v(0, \Phi(0, x_{d_1}), \phi_2(0)) &= \int_0^1 \left\{ \frac{\partial v(t, \Phi(t, x_{d_1}), \phi_2(t))}{\partial t} + \langle \nabla_{d_1} v(t, \Phi(t, x_{d_1}), \phi_2(t)), b(t, \Phi(t, x_{d_1})) \rangle ight. \\
&\quad + \left. \langle \nabla_{d_1,d} v(t, \Phi(t, x_{d_1}), \phi_2(t)), \frac{d\phi_2(t)}{dt} \rangle \right\} dt \\
&\leq \int_0^1 \ell \left( \frac{d\phi_2(t)}{t} \right) dt,
\end{align*}
\]  

(2.12)

where the equality holds if (2.11) holds. By Jensen’s inequality,

\[
v(0, x) = \sup \left\{ v(1, \varphi(x_{d_1}), \phi_2(1)) - \int_0^1 \ell \left( \frac{d\phi_2(t)}{t} \right) dt \mid \phi_2(0) = x_{d_1,d} \right\} \\
= \sup \left\{ v(1, \varphi(x_{d_1}), \phi_2(1)) - \ell(\phi_2(1) - \phi_2(0)) \mid \phi_2(0) = x_{d_1,d} \right\} \\
= v(x; v(1, \cdot)|\varphi).\Box
\]  

(2.13)

Before we formulate the duality theorem in the framework of the theory of viscosity solutions, we give assumptions.

(A.1) \(b(t, x)\) is bounded and there exists \(K > 0\) such that

\[
|b(t, x) - b(t, y)| \leq K|x - y| \quad (t \in [0, 1], x, y \in \mathbb{R}^{d_1}).
\]

(A.2) There exists \(m \in C([0, 1] \times \mathbb{R}^{d_1} \times [0, 1] \times \mathbb{R}^{d_1} \times [0, \infty))\) such that \(m(t, x, s, y, 0) = 0\) and that

\[
|b(t, x) - b(s, y)| \leq m(t, x, s, y, |t - s| + |x - y|) \quad (t, s \in [0, 1], x, y \in \mathbb{R}^{d_1}).
\]
\( \ell : \mathbb{R}^{d-d_1} \mapsto [0, \infty) \) is convex and \( \lim \inf_{|v| \to \infty} \frac{\ell(v)}{|v|} = \infty \).

**Example 2.1** Suppose that \( d_1 = 1 \). Then (A.1)-(A.2) holds if \( 1 < d \varphi(x)/dx \leq K+1 \).

For \( (t, x) \in [0,1] \times \mathbb{R}^d \) and \( f \in C_b(\mathbb{R}^d) \),

\[
\nu(t, x; f|\varphi) \\
= \sup \left\{ f(\Phi(1, y_{d_1}), \phi_2(1)) - \int_t^1 \ell \left( \frac{d\phi_2(s)}{ds} \right) ds \middle| (\Phi(t, y_{d_1}), \phi_2(t)) = x \right\}. \tag{2.14}
\]

Then it is easy to see that the following holds:

\[
\nu(t, x; f|\varphi) \\
= \sup \left\{ f(x_{d_1} + (1-t)b(t, x_{d_1}), y) - (1-t)\ell \left( \frac{y-x_{d_1,d}}{1-t} \right) \middle| y \in \mathbb{R}^{d-d_1} \right\}. \tag{2.15}
\]

Then, from Theorem 10.1 in p. 95 of [7], under (A.1), \( \nu_n(t, x; f) \) is the unique bounded uniformly continuous viscosity solution of the following HJ Eqn: for \( (t, x) \in (0,1) \times \mathbb{R}^d \),

\[
\nu_n(t, x; f) := \sup \left\{ f(x_{d_1} + (1-t)b(t, x_{d_1}), \phi_2(1)) - \int_t^1 \ell \left( \frac{d\phi_2(s)}{ds} \right) ds \middle| \phi_2(t) = x_{d_1,d}, \left| \frac{d\phi_2(s)}{ds} \right| \leq n \right\}. \tag{2.17}
\]

**Corollary 2.1** Suppose that \( c(x, y) = \ell(y-x) \) and that (A.1)-(A.3) hold. Then for any Lipschitz continuous \( f : \mathbb{R}^d \mapsto \mathbb{R} \), \( \nu(t, x; f|\varphi) \) is a Lipschitz continuous viscosity solution of (2.10). In particular, for any \( P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d) \),

\[
T(P_0, P_1|\varphi) = \sup \left\{ \int_{\mathbb{R}^d} \nu(1, y)P_1(dy) - \int_{\mathbb{R}^d} \nu(0, x)P_0(dx) \middle| \nu(1, \cdot) \in C_b^\infty(\mathbb{R}^d) \right\}, \tag{2.16}
\]

where \( \nu(t, x) \) denotes a bounded uniformly continuous viscosity solution of (2.10).
\[
\frac{\partial v(t, x)}{\partial t} + <\nabla_{d_{1}}v(t, x), b(t, x_{d_{1}})> + h_{n}(\nabla_{d_{1},d}v(t, x)) = 0, \\
v(1, x) = f(x), \quad (2.18)
\]

where

\[
h_{n}(z) := \sup\{<u, z> - \ell(u)|u \in \mathbb{R}^{d-d_{1}}, |u| \leq n\}.
\]

Let \(\overline{v}\) be a bounded uniformly continuous viscosity solution of (2.10) with \(\overline{v}(1, x) = f(x)\). Then it is a bounded uniformly continuous viscosity supersolution of (2.18) with \(\overline{v}(1, x) = f(x)\) and

\[
v_{n}(t, x; f) \leq \overline{v}(t, x) \quad (2.19)
\]

from Theorem 9.1 in p. 86 of [7]. Let \(n \to \infty\) in (2.19). Then we obtain \(v(t, x; f|\varphi) \leq \overline{v}(t, x)\) from Theorem 9.1.

### 2.2 Convergence Theorem

Let \(2 \leq k \leq d\), \(0 = d_{0} < d_{1} < \cdots < d_{k} = d\) and

(A.4) \(c_{i} \in LSC(\mathbb{R}^{d_{i}-d_{i-1}} \times \mathbb{R}^{d_{i}-d_{i-1}} : [0, \infty)) (i = 1, \cdots, k)\).

For \(\epsilon \geq 0\), \(P_{0}, P_{1} \in \mathcal{M}_{1}(\mathbb{R}^{d})\),

\[
T^{\epsilon}(P_{0}, P_{1}) := \inf\left\{ \int_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \sum_{i=1}^{k} \epsilon^{i-1}c_{i}(x_{d_{i-1},d_{i}}, y_{d_{i-1},d_{i}})\mu(dx\,dy) \right\}
\]

\[
\mu(dx \times \mathbb{R}^{d}) = P_{0}(dx), \mu(\mathbb{R}^{d} \times dy) = P_{1}(dy) \quad (2.20)
\]

It is known that if \(c_{i}(x, y) = \ell_{i}(y - x)\) and \(\ell_{i}\) is strictly convex and superlinear \((i = 1, \cdots, k)\) and if \(P_{0}(dx)\) is absolutely continuous with respect to the Lebesgue measure \(dx\), then \(T^{\epsilon}(P_{0}, P_{1})\) has the unique minimizer, provided that it is finite (see e.g. [21], [24], [25]).

\[
T_{1}(P_{0,1}, P_{1,1}) := \inf\left\{ \int_{\mathbb{R}^{d_{1}} \times \mathbb{R}^{d_{1}}} c_{1}(x, y)\mu(dx\,dy) \right\}
\]

\[
\mu(dx \times \mathbb{R}^{d_{1}}) = P_{0,1}(dx), \mu(\mathbb{R}^{d_{1}} \times dy) = P_{1,1}(dy) \quad (2.21)
\]

where \(P_{t,i}(dx_{d_{i}}) := P_{t}(dx_{d_{i}} \times \mathbb{R}^{d-d_{i}}) (t = 0, 1)\). For \(i \geq 2\) and \(\nu_{i-1} \in \mathcal{M}_{1}(\mathbb{R}^{2d_{i-1}})\),
The following theorem can be proved in the same way as [2] (see also section 1) and is proved for the readers' convenience.

**Theorem 2.2** Let $P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d)$. Suppose that $k = 2$ and (A.4) holds and that $T_1(P_{0,1}, P_{1,i})$ and $T_2(P_0, P_1|\nu_1)$ have the unique minimizers $\nu_1$ and $\nu_2$, respectively. Then a minimizer of $T^\varepsilon(P_0, P_1)$ exists and weakly converges to $\nu_2$ as $\varepsilon \to 0$ and the following holds:

\[
\begin{align*}
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \times \mathbb{R}^d} c_1(x_{d_{1}}, y_{d_{1}}) \mu^\varepsilon(dx dy) &= T_1(P_{0,1}, P_{1,1}), \\
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \times \mathbb{R}^d} c_2(x_{d_{1},d}, y_{d_{1},d}) \mu^\varepsilon(dx dy) &= T_2(P_0, P_1|\nu_1).
\end{align*}
\]  

(2.23)  

(2.24)

(Proof). In the same way as in the proof of Proposition 2.1, by a standard method, one can show that $T^\varepsilon(P_0, P_1)$ has a minimizer $\mu^\varepsilon$, since

\[
T^\varepsilon(P_0, P_1) \leq T_1(P_{0,1}, P_{1,1}) + \varepsilon T_2(P_0, P_1|\nu_1) < +\infty. 
\]  

(2.25)

Since the set of $\mu$ for which $\mu(dx \times \mathbb{R}^d) = P_0(dx)$ and $\mu(\mathbb{R}^d \times dy) = P_1(dy)$ is compact, any sequence $\{\mu^{\varepsilon_n}\}_{n \geq 1}$ ($\varepsilon_n \to 0$ as $n \to \infty$) has a weakly convergent subsequence $\{\mu^{\varepsilon_n(\ell)}\}_{\ell \geq 1}$ and for the limit $\mu$,

\[
\mu_1(dx_{d_1}, dy_{d_1}) := \mu(dx_{d_1} \times \mathbb{R}^{d-d_{1}} \times dy_{d_1} \times \mathbb{R}^{d-d_{1}})
\]

is the minimizer of $T_1(P_{0,1}, P_{1,1})$ by the uniqueness of the minimizer and (2.23) holds. Indeed, from (2.25),

\[
\begin{align*}
T_1(P_{0,1}, P_{1,1}) &\leq \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_1}} c_1(x_{d_{1}}, d_{1}, y_{d_{1}}) \mu_1(dx dy) = \int_{\mathbb{R}^d \times \mathbb{R}^d} c_1(x_{d_{1}}, y_{d_{1}}) \mu(dx dy) \\
&\leq \liminf_{\ell \to \infty} T^{\varepsilon_n(\ell)}(P_0, P_1) \leq \limsup_{\ell \to \infty} T^{\varepsilon_n(\ell)}(P_0, P_1) \\
&\leq T_1(P_{0,1}, P_{1,1}).
\end{align*}
\]  

(2.26)
Since
\[ T_1(P_{0,1}, P_{1,1}) + \epsilon \int_{\mathbb{R}^d \times \mathbb{R}^d} c_2(x_{d_{i-1},d_{i}}, y_{d_{i-1},d_{i}})\mu^\epsilon(dx dy) \leq T^\epsilon(P_0, P_1), \] (2.27)
we also have, from (2.25) and (2.27),
\[ T_2(P_0, P_1|\nu_1) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} c_2(x_{d_{i-1},d_{i}}, y_{d_{i-1},d_{i}})\mu^\epsilon(dx dy) \leq \lim_{\ell \rightarrow \infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} c_2(x_{d_{i-1},d_{i}}, y_{d_{i-1},d_{i}})\mu^{\epsilon_{n(\ell)}}(dx dy) \leq T_2(P_0, P_1|\nu_1). \] (2.28)
The uniqueness of the minimizer of $T_2(P_0, P_1|\nu_1)$ completes the proof. \(\square\)

**Theorem 2.3** Let $P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d)$. Suppose that (A.4) holds, that $T_1(P_{0,1}, P_{1,1})$ and $T_i(P_{0,i}, P_{1,i}|\nu_{i-1})$ have the unique minimizers $\nu_1$ and $\nu_i$ ($i = 2, \cdots, k$), respectively and that $\nu \mapsto T_i(P_{0,i}, P_{1,i}|\nu)$ is continuous ($i = 3, \cdots, k$). Then a minimizer of $T^\epsilon(P_0, P_1)$ exists and weakly converges to $\nu_k$ as $\epsilon \rightarrow 0$ and the following holds:

\[ \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^d \times \mathbb{R}^d} c_1(x_{d_{i-1},d_{i}}, y_{d_{i-1},d_{i}})\mu^\epsilon(dx dy) = T_1(P_{0,1}, P_{1,1}), \] (2.29)
\[ \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^d \times \mathbb{R}^d} c_i(x_{d_{i-1},d_{i}}, y_{d_{i-1},d_{i}})\mu^\epsilon(dx dy) = T_i(P_{0,i}, P_{1,i}|\nu_{i-1})(i = 2, \cdots, k). \] (2.30)

(Proof). In the same way as in (2.25), one can show that $T^\epsilon(P_0, P_1)$ has a minimizer $\mu^\epsilon$ and that any subsequence $\{\mu^{\epsilon_n}\}_{n \geq 1}$ ($\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$) has a weakly convergent subsequence $\{\mu^{\epsilon_{n(\ell)}}\}_{\ell \geq 1}$. Let $\mu$ denote the weak limit of $\mu^{\epsilon_{n(\ell)}}$ as $\ell \rightarrow \infty$. We prove the theorem by induction. For $i = 2, \cdots, k$,

\[ T_{i-1}^\epsilon(P_{0,i-1}, P_{1,i-1}) := \inf \left\{ \int_{\mathbb{R}^{d_{i-1}} \times \mathbb{R}^{d_{i-1}}} \sum_{j=1}^{i-1} \epsilon^{j-1} c_j(x_{d_{j-1},d_{j}}, y_{d_{j-1},d_{j}})\nu(dx dy) \right\}, \] (2.31)

Let $\mu^\epsilon_{i-1}$ and $\nu^\epsilon_{i,j}$ denote a minimizer of $T_{i-1}^\epsilon(P_{0,i-1}, P_{1,i-1})$ and $T_j(P_{0,j}, P_{1,j}|\nu^\epsilon_{i,j-1})$ ($j = i, \cdots, k$), respectively, where $\nu^\epsilon_{i,i-1} := \mu^\epsilon_{i-1}$. Then
\begin{align}
T_{i-1}^\epsilon(P_{0_{i-1}}, P_{1,i-1}) + \int_{R^d \times R^d} \sum_{j=i}^{k} \epsilon^{j-1} c_j(x_{d_{j-1},d_{j}}, y_{d_{j-1},d_{j}}) \mu^\epsilon(dxdy) \\
\leq T^\epsilon(P_0, P_1) \\
\leq T_{i-1}^\epsilon(P_{0_{i-1}}, P_{1,i-1}) + \sum_{j=i}^{k} \epsilon^{j-1} T_j(P_{0,j}, P_{1,j}|\nu_{i,j-1}^\epsilon). \tag{2.32}
\end{align}

From Theorem 2.2, \( \mu_2^\epsilon \rightarrow \nu_2 \) as \( \epsilon \rightarrow 0 \) and (2.23)-(2.24) holds. Suppose that \( \mu_i^\epsilon \rightarrow \nu_i \) as \( \epsilon \rightarrow 0 \) for \( i \leq k - 1 \). In the same way as in Theorem 2.2, one can show that for \( j = 1, 2 \),

\[ \mu(dx_{d_j} \times R^{d-d_j} \times dy_{d_j} \times R^{d-d_j}) = \nu_j(dx_{d_j}dy_{d_j}). \tag{2.33} \]

Suppose that (2.33) holds for \( j = i - 1 \). Then, from (2.32) and the assumption of induction,

\begin{align}
T_i(P_{0,i}, P_{1,i}|\nu_{i-1}) &\leq \int_{R^d \times R^d} c_i(x_{d_{i-1},d_i}, y_{d_{i-1},d_i}) \mu(dx_{d_i}dy_{d_i}) \\
&\leq \lim\inf_{t \rightarrow \infty} \int_{R^d \times R^d} c_t(x_{d_{i-1},d_i}, y_{d_{i-1},d_i}) \mu^\epsilon_n(t)(dxdy) \\
&\leq \lim\sup_{t \rightarrow \infty} \int_{R^d \times R^d} c_t(x_{d_{i-1},d_i}, y_{d_{i-1},d_i}) \mu^\epsilon_n(t)(dxdy) \\
&\leq \lim_{t \rightarrow \infty} T_i(P_{0,i}, P_{1,i}|\nu_{i-1}^\epsilon) = T_i(P_0, P_1|\nu_{i-1}). \tag{2.34}
\end{align}

(2.34) implies (2.30) and the uniqueness of the minimizer \( \nu_i \) of \( T_i(P_{0,i}, P_{1,i}|\nu_{i-1}) \) implies that (2.33) holds for \( j = i \). \( \square \)

From (2.32), we also have

**Proposition 2.4** Suppose that the assumption in Theorem 2.3 holds. Then, for \( i = 1, \cdots, k - 1 \),

\[ 0 \leq \frac{\int_{R^d \times R^d} \sum_{j=1}^{i} \epsilon^{j-1} c_j(x_{d_{j-1},d_j}, y_{d_{j-1},d_j}) \mu^\epsilon(dxdy) - T^\epsilon_i(P_{0,i}, P_{1,i})}{\epsilon^i} \rightarrow 0 \quad (\epsilon \rightarrow 0). \tag{2.35} \]

We don’t know the real convergence rate of (2.35).

**Example 2.2** Let \( P_0, P_1 \in \mathcal{M}_1(R^d) \). Suppose that

(i) \( d_{i+1} = d_i + 1 \) (\( i = 1, \cdots, k - 1 \),
(ii) \( c_i(x, y) = \ell_i(y - x) \) and \( \ell_i : \mathbb{R}^{d_i} \mapsto [0, \infty) \) is strictly convex and superlinear \((i = 1, \ldots, k)\),

(iii) \( P_0 \) is absolutely continuous with respect to the Lebesgue measure \( dx \),

(iv) \( T_1(P_{0,1}, P_{1,1}) \) is finite.

Then \( T_1(P_{0,1}, P_{1,1}) \) has the unique minimizer \( \nu_1 \) which can be written as follows:

\[
\nu_1(dx_{d_1}dy_{d_1}) = P_{0,1}(dx_{d_1})\delta_{\phi_1(x_{d_1})}(dy_{d_1}),
\]

where \( \phi_1 \) is a Borel measurable function \((\text{see e.g. [21], [24])}\).

Suppose, in addition, that

(v) \( T_i(P_{0,i}, P_{1,i}|\nu_{i-1}) \) is finite for \( i = 2, \ldots, k \). \((\text{If } T_i(P_{0,i}, P_{1,i}|\nu_{i-1}) \text{ is finite, then it has a minimizer (see the proof of Prop. 2.1.).}\)

Then the following holds:

\[
\nu_i(dx_{d_i}dy_{d_i}) = P_{0,i}(dx_{d_i})\delta_{\phi_{\nu_0,\ldots,\nu_{i-1}}(x_{d_i})}(dy_{d_i}),
\]

where \( \phi_{\nu_0,\ldots,\nu_{i-1}}(x_{d_i}) := (\phi_{\nu_0}(x_{d_i}), \ldots, \phi_{\nu_{i-1}}(x_{d_i})) \), \( \phi_{\nu_0} := \phi_1 \) and

\[
\phi_{\nu_{i-1}}(x_{d_i}) := (F_{\nu_{i-1},1}(\cdot|x_{d_{i-1}}, \Phi_{\nu_0,\ldots,\nu_{i-2}}(x_{d_{i-1}})))^{-1}(F_{\nu_{i-1},0}(x_{d_i}|x_{d_{i-1}}, \Phi_{\nu_0,\ldots,\nu_{i-2}}(x_{d_{i-1}}))),
\]

\[
F_{\nu_{i-1},1}(x|x_{d_{i-1}}, \Phi_{\nu_0,\ldots,\nu_{i-2}}(x_{d_{i-1}})) := \nu_{i-1}(R \times (-\infty, x]|(x_{d_{i-1}}, \Phi_{\nu_0,\ldots,\nu_{i-2}}(x_{d_{i-1}}))),
\]

\[
F_{\nu_{i-1},0}(x_{d_i}|x_{d_{i-1}}, \Phi_{\nu_0,\ldots,\nu_{i-2}}(x_{d_{i-1}})) := \nu_{i-1}((-\infty, x] \times R|(x_{d_{i-1}}, \Phi_{\nu_0,\ldots,\nu_{i-2}}(x_{d_{i-1}}))).
\]

In particular, \( \phi_{\nu_{i-1}} \) is a minimizer of the following:

\[
\min\left\{ \int_{\mathbb{R}^{d_i}} \ell_i(\phi(x) - x_{d_i})P_{0,i}(dx)|P_{0,i}(\Phi_{\nu_0,\ldots,\nu_{i-2}}, \phi)^{-1} = P_{1,i} \right\} = T_i(P_{0,i}, P_{1,i}|\nu_{i-1}).
\]

3 Stochastic version of Knothe-Rosenblatt type rearrangement.

Let \( \mathcal{A} \) denote the set of all \( \mathbb{R}^d \)-valued, continuous semimartingales \( \{X(t)\}_{0 \leq t \leq 1} \) on a (possibly different) complete filtered probability space such that there exists a Borel measurable \( \beta_X : [0, 1] \times C([0, 1]) \mapsto \mathbb{R}^d \) for which

(i) \( \omega \mapsto \beta_X(t, \omega) \) is \( \mathcal{B}(C([0, 1]))_{\tau} \)-measurable for all \( t \in [0, 1] \),
(ii) \( X(t) = X(0) + \int_0^t \beta_X(s, X)ds + W_X(t) \) \( (0 \leq t \leq 1) \).
Here \( \mathcal{B}(C([0, t]))_{+} := \cap_{s>t} \mathcal{B}(C([0, s])) \), \( \mathcal{B}(C([0, t])) \) and \( W_X(t) \) denote the Borel \( \sigma \)-field of \( C([0, t]) \) and an \( (\mathcal{F}_t^X) \)-Brownian motion, respectively, and \( \mathcal{F}_t^X := \sigma[X(s) : 0 \leq s \leq t] \) (see e.g. [14]). Let \( d \geq 2 \) and \( 1 \leq d_1 < d \), and let \( b_1 : [0, 1] \times \mathbb{R}^{d_1} \rightarrow \mathbb{R}^{d_1} \) be a Borel measurable function such that the following SDE has a weak solution for a given initial distribution:

\[
\frac{dX_1(t)}{dt} = b_1(t, X_1(t)) \ dt + dW_{X_1}(t).
\]

(3.1)

Let \( L(t, x; u) : [0, 1] \times \mathbb{R}^d \times \mathbb{R}^{d-d_1} \rightarrow [0, \infty) \).

A minimizer of the following can be considered as the stochastic optimal control (SOC for short) version of the Knothe Rosenblatt type rearrangement: for \( P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d) \),

\[
V(P_0, P_1|b_1) := \inf \left\{ E \left[ \int_0^1 L(t, Y(t); \beta_{Y,2}(t, Y)) dt \right] \middle| Y \in \mathcal{A}, \beta_{Y,1}(t, Y) = b_1(t, Y_1(t)), PY(0)^{-1} = P_0, PY(1)^{-1} = P_1 \right\},
\]

(3.2)

where we write \( \beta_Y(t, Y) = (\beta_{Y,1}(t, Y), \beta_{Y,2}(t, Y)) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d-d_1} \).

**Example 3.1** For \( P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d) \), take \( T_{KR} \) in section 1 and, on a complete filtered probability space, consider

\[
Z(t) = Z(0) + \int_0^t T_{KR}(Z(0)) - Z(s) \frac{ds}{1-s} + W_Z(t).
\]

(3.3)

Then \( Z(1) = T_{KR}(Z(0)) \). In particular, \( PZ(1)^{-1} = P_1 \), provided \( PZ(0)^{-1} = P_0 \). Besides, \( \beta_{Z_i}(t, Z) = \beta_{Z_i}(t, Z_i) \) for all \( i = 1, \cdots, d \). Suppose that \( p \in [1, 2) \) and that \( \int_{\mathbb{R}^d} |x|^p (P_0(dx) + P_1(dx)) \) is finite. Then

\[
E \left[ \int_0^1 \left| \frac{T_{KR}(Z(0)) - Z(s)}{1-s} \right|^p ds \right] < \infty.
\]

(3.4)

Indeed, \( W_o(t) := Z(t) - Z(0) - (T_{KR}(Z(0)) - Z(0))t \) is a tided down brownian motion starting and arriving at \( o \), and

\[
\frac{T_{KR}(Z(0)) - Z(s)}{1-s} = T_{KR}(Z(0)) - Z(0) - \frac{W_o(s)}{1-s}.
\]

We describe our assumption in this section to show the existence of the stochastic analogue of the Knothe Rosenblatt type rearrangement.
(H.1). (i) $L \in C([0,1] \times \mathbb{R}^d \times \mathbb{R}^{d-d_1} : [0, \infty))$, (ii) $u \mapsto L(t, x; u)$ is strictly convex. (H.2). There exists $\gamma > 1$ such that
\[
\lim \inf_{|u| \rightarrow \infty} \inf \{L(t, x; u) : (t, x) \in [0,1] \times \mathbb{R}^d\} / |u|^\gamma > 0.
\]
(H.3).
\[
\Delta L(\varepsilon_1, \varepsilon_2) := \sup \frac{L(t, x; u) - L(s, y; u)}{1 + L(s, y; u)} \rightarrow 0 \quad \text{as} \quad \varepsilon_1, \varepsilon_2 \rightarrow 0,
\]
where the supremum is taken over all $(t, x)$ and $(s, y) \in [0,1] \times \mathbb{R}^d$ for which $|t - s| \leq \varepsilon_1$, $|x - y| < \varepsilon_2$ and over all $u \in \mathbb{R}^d$.

The following can be proved in the same way as Prop. 2.1 in [19], and the proof is omitted.

**Proposition 3.1** Suppose that (H.1)-(H.3) hold. Then for any $P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d)$, $V(P_0, P_1|b_1)$ has a minimizer, provided it is finite.

### 3.1 Duality Theorem

We consider the following HJB Equation:
\[
\frac{\partial v(t, x)}{\partial t} + \frac{1}{2} \triangle v(t, x) + <\nabla_{x_{d_1}} v(t, x), b_1(t, x_{d_1})>
+ H(t, x; \nabla_{x_{d_1}} v(t, x)) = 0,
\]
\((t, x) \in (0,1) \times \mathbb{R}^d\), where
\[
H(t, x; z) := \sup \{<u, z> - L(t, x; u)|u \in \mathbb{R}^{d-d_1}\} \quad (z \in \mathbb{R}^{d-d_1}).
\]
For $f \in C_b(\mathbb{R}^d)$,
\[
u(t, x; f|b_1)(x) := \sup \left\{ \mathbb{E} \left[ f(Y(1)) - \int_t^1 L(s, Y(s); \beta_{Y,2}(s, Y))ds \right] \right\}
Y(t) = x, \beta_{Y,1}(s, Y) = b_1(s, Y_1(s)), Y \in \mathcal{A} \right\}.
\]
(H.4). (i) $L(t, x; 0)$ is bounded; (ii) $\Delta L(0, \infty)$ is finite; (iii) $b_1 \in C^{1,2}([0,1] \times \mathbb{R}^d) \cap C_0^{0,1}([0,1] \times \mathbb{R}^d)$, $|D_x L(t, x; u)|/(1 + L(t, x; u))$ is bounded on $[0,1] \times \mathbb{R}^d \times \mathbb{R}^{d_1}$ and $D_u L(t, x; u)$ is bounded on $[0,1] \times \mathbb{R}^d \times B_R$ for all $R > 0$, where $B_R := \{x \in \mathbb{R}^{d_1}||x| \leq R\}$. 


The following can be proved in the same way as Theorem 11.1 in IV.11 of [7], and the proof is omitted.

**Proposition 3.2** Suppose that (H.1)-(H.2) and (H.4,i,iii) hold. Then, for any $f \in C^5(\mathbb{R}^d) \cap C_b^3(\mathbb{R}^d)$, $u(t, x; f|b_1) \in C^{1,2}([0,1] \times \mathbb{R}^d) \cap C^0_b([0,1] \times \mathbb{R}^d)$ and is the unique classical solution of the HJB Equation (3.7) with $v(1, x) = f(x)$.

It is easy to see that the following holds:

$$V(P_0, P_1|b_1) := \inf \left\{ E \left[ \int_0^1 \frac{L(t, Y(t); \beta_{Y,2}(t, Y))}{1_{\{b_1(t, Y(t))\}}(\beta_{Y,1}(t, Y))} dt \right] \right\} | Y \in \mathcal{A},$$

$$PY(0)^{-1} = P_0, PY(1)^{-1} = P_1 \right\},$$

which implies the duality theorem for $V(P_0, P_1|b_1)$.

**Theorem 3.1** Suppose that (H.1)-(H.4) hold. Then for any $P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d)$,

$$V(P_0, P_1|b_1) = \sup \left\{ \int_{\mathbb{R}^d} v(1, y)P_1(dy) - \int_{\mathbb{R}^d} v(0, x)P_0(dx) \right\},$$

where the supremum is taken over all classical solutions $v$ of (3.7) with $v(1, y) \in C^\infty_b(\mathbb{R}^d)$.

(Proof). Under (H.1)-(H.3) and (H.4,i,ii), (3.9) implies that $V(P_0, \cdot|b_1)$ is convex and lower-semicontinuous, which can be proved in the same way as in [19] and is not identically equal to infinity by considering the case where $\beta_{Y,2}(s, Y) = 0$ from (H.4,i). Hence, from Theorem 2.2.15 and Lemma 3.2.3 in [3],

$$V(P_0, P_1|b_1) = \sup \left\{ \int_{\mathbb{R}^d} f(y)P_1(dy) - V(P_0, \cdot|b_1)^*(f) \right\} f \in C_b(\mathbb{R}^d),$$

where

$$V(P_0, \cdot|b_1)^*(f) := \sup \left\{ \int_{\mathbb{R}^d} f(y)P(dy) - V(P_0, P|b_1) \right\} P \in \mathcal{M}_1(\mathbb{R}^d).$$

One can replace $C_b(\mathbb{R}^d)$ by $C_b^\infty(\mathbb{R}^d)$ in (3.11) in the same way as in the proof of Theorem 2.1 in [19]. For $f \in C_b^\infty(\mathbb{R}^d)$, from Proposition 3.2,
\[ V(P_0, b_1)^*(f) = \sup \left\{ E \left[ f(Y(1)) - \int_0^1 \frac{L(t, Y(t); \beta_{Y,2}(t, Y))}{1_{b_1(t, Y_1(t))}(\beta_{Y_1,1}(t, Y))} dt \right] \bigg| Y \in \mathcal{A}, P(Y(0))^{-1} = P_0 \right\} = \int_{\mathbb{R}^d} u(0, x; f|b_1) P_0(dx), \] 

where the optimal control is \( \beta_{Y,2}(t, Y) = \nabla_{d_1, d} u(t, Y(t); f|b_1). \square \)

As a corollary to Theorem 3.1, in the same way as [19], we easily obtain

**Corollary 3.1** Suppose that (H.1)-(H.4) hold. Then for any \( P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d) \) for which \( V(P_0, P_1|b_1) \) is finite, there exists a Borel measurable function \( b_2^o : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d-d_1} \) such that for a minimizer \( \{Y(t)\}_{0 \leq t \leq 1} \), \( \beta_{Y,2}(t, Y) = b_2^o(t, Y(t)). \)

We consider the following marginal problem:

\[ v(P_0, P_1|b_1) := \inf \int_0^1 dt \int_{\mathbb{R}^d} L(t, x; B_2(t, x)) Q_t(dx), \] 

where the infimum is taken over all \( \{Q_t(dx)\}_{0 \leq t \leq 1} \subset \mathcal{M}_1(\mathbb{R}^d) \) for which \( B_1 = b_1 \), \( Q_t = P_t \) \( (t = 0, 1) \) and

\[ \frac{\partial Q_t(dx)}{\partial t} = \frac{1}{2} \Delta Q_t(dx) - \text{div}(B(t, x)Q_t(dx)), \]

in a weak sense. Here we write \( B(t, x) = (B_1(t, x), B_2(t, x)) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d-d_1} \).

In the same way as [17], we have

**Theorem 3.2** Suppose that (H.1)-(H.4) hold. Then for any \( P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d) \),

\[ v(P_0, P_1|b_1) = \sup \left\{ \int_{\mathbb{R}^d} v(1, y) P_1(dy) - \int_{\mathbb{R}^d} v(0, x) P_0(dx) \right\}, \]

where the supremum is taken over all classical solutions \( v \) of (3.7) with \( v(1, y) \in C_0^\infty(\mathbb{R}^d) \). In particular, \( V(P_0, P_1|b_1) = v(P_0, P_1|b_1)(\in [0, \infty)) \).

We introduce an additional assumption to formulate the duality theorem in the framework of the theory of viscosity solutions.

(H.4)'. (i) \( \partial L(t, x; u)/\partial t \) and \( D_x L(t, x; u) \) is bounded on \( [0, 1] \times \mathbb{R}^d \times \mathbb{R}_R \) for all \( R > 0 \); (ii) \( \Delta L(0, \infty) \) is finite; (iii) \( b_1 \in C^1_b([0, 1] \times \mathbb{R}^d) \).

In the same way as in Lemma 4.5 in [17], one can prove
Proposition 3.3 Suppose that (H.1)-(H.3) and (H.4)' hold. Then for any \( f \in UC_b(\mathbb{R}^d) \), \( u(t,x;f|b_1) \) is a bounded continuous viscosity solution of (3.7) with \( v(1,x) = f(x) \) and for any \( Q \in M_1(\mathbb{R}^d) \) and \( t \in [0,1] \),

\[
\int_{\mathbb{R}^d} u(t,x;f|b_1)Q(dx) = \sup \left\{ E \left[ f(Y(1)) - \int_0^1 L(s,Y(s);\beta_Y(s,Y))ds \right] \mid PY^{-1}(t) = Q, \beta_{Y,1}(s,Y) = b_1(s,Y_1(s)), Y \in \mathcal{A} \right\}.
\]

(3.16)

In addition, for any bounded continuous viscosity solution \( u \) of (3.7) with \( u(1,x) = f(x) \), \( u(t,x) \geq u(t,x;f|b_1) \), that is, \( u(t,x;f|b_1) \) is minimal.

In the same was as in Theorem 3.1, from Prop. 3.3, we have

Theorem 3.3 Suppose that (H.1)-(H.3) and (H.4)' hold. Then for any \( P_0, P_1 \in M_1(\mathbb{R}^d) \),

\[
V(P_0, P_1|b_1) = \sup \left\{ \int_{\mathbb{R}^d} v(1,y)P_1(dy) - \int_{\mathbb{R}^d} v(0,x)P_0(dx) \mid v(t,x;f) \right\};
\]

(3.17)

where the supremum is taken over all bounded continuous viscosity solutions \( v(t,x;f) \) of (3.7) with \( v(1,x) \in C_b^\infty(\mathbb{R}^d) \).

Remark 3.1 (H.3) and (i) in (H.4)' implies (i) in (H.4).

3.2 Convergence Theorem

Let \( L_1 : [0,1] \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_1} \mapsto [0,\infty) \) and \( L_2 : [0,1] \times \mathbb{R}^d \times \mathbb{R}^{d-d_1} \mapsto [0,\infty) \). For \( \epsilon > 0 \), \( P_0, P_1 \in M_1(\mathbb{R}^d) \),

\[
V^\epsilon(P_0, P_1) := \inf \left\{ \int_0^1 L_i(t,Y_i(t);\beta_{Y,i}(t,Y))dt \mid PY(0)^{-1} = P_0, PY(1)^{-1} = P_1, Y \in \mathcal{A} \right\},
\]

(3.18)

where \( Y_1(t) := Y_1(t) \) and \( Y_2(t) := Y(t) \) for \( Y(t) = (Y_1(t), Y_2(t)) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d-d_1} \).

If (H.1)-(H.3) holds for \( L = L_i \) for all \( i = 1,2 \), then \( V^\epsilon(P_0, P_1) \) has a minimizer, provided it is finite (see Prop. 2.1 in [19]).

\[
V_1(P_{0,1}, P_{1,1}) := \inf \left\{ \int_0^1 L_1(t,Y(t);\beta_Y(t,Y))dt \mid Y \in \mathcal{A}_1 \right\},
\]

(3.19)

where \( \mathcal{A}_1 \) denotes \( \mathcal{A} \) with \( d = d_1 \).
Remark 3.2 If (H.1)-(H.4) with $L = L_i$ holds and that $V_i(P_{0,1}, P_{1,1})$ is finite. Then there exists a Borel measurable function $b : [0,1] \times \mathbb{R}^{d_1} \rightarrow \mathbb{R}^{d_1}$ such that for any minimizer $\{Y(t)\}_{0 \leq t \leq 1}$ of $V_i(P_{0,1}, P_{1,1})$, $\beta_Y(t, Y) = b(t, Y(t))$ (see [19]).

Let $b_1$ denote the drift vector of the minimizer of $V_i(P_{0,1}, P_{1,1})$, provided it exists and let $V_2(P_0, P_1|b_1)$ denote $V(P_0, P_1|b_1)$ with $L = L_2$. Then

Theorem 3.4 Let $P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d)$. Suppose that (H.1)-(H.3) with $L = L_i$ holds ($i = 1, 2$) and that $V_1(P_{0,1}, P_{1,1})$ and $V_2(P_0, P_1|b_1)$ is finite and have the unique minimizers $\{X_1(t)\}_{0 \leq t \leq 1}$ and $\{X(t)\}_{0 \leq t \leq 1}$, respectively. Then a minimizer $\{Y^\epsilon(t)\}_{0 \leq t \leq 1}$ of $V^\epsilon(P_0, P_1)$ exists and weakly converges to $\{X(t)\}_{0 \leq t \leq 1}$ as $\epsilon \rightarrow 0$. In particular,

$$\lim_{\epsilon \rightarrow 0} E \left[ \int_0^1 L_1(t, Y_1^\epsilon(t); \beta_{Y^\epsilon,1}(t, Y^\epsilon)) dt \right] = V_1(P_{0,1}, P_{1,1}),$$  

(3.20)

$$\lim_{\epsilon \rightarrow 0} E \left[ \int_0^1 L_2(t, Y^\epsilon(t); \beta_{Y^\epsilon,2}(t, Y^\epsilon)) dt \right] = V_2(P_0, P_1|b_1).$$  

(3.21)

(Proof) In the same way as Prop. 2.1 in [19], one can show that there exists a minimizer $Y^\epsilon(t)$ of $V^\epsilon(P_0, P_1)$ since

$$V^\epsilon(P_0, P_1) \leq V_1(P_{0,1}, P_{1,1}) + \epsilon V_2(P_0, P_1|b_1).$$  

(3.22)

In the same way as in Lemma 3.1 in [19], from (H.2), one can show that any sequence $\{Y^\epsilon_n(\cdot)\}_{n \geq 1}$ in $\mathcal{A}$ ($\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$) has a weakly convergent subsequence $\{Y^\epsilon_{n(k)}(\cdot)\}_{k \geq 1}$. Indeed,

$$E \left[ \int_0^1 L_1(t, Y_1^\epsilon(t); \beta_{Y^\epsilon,1}(t, Y^\epsilon)) dt \right] \leq V^\epsilon(P_0, P_1),$$  

(3.23)

$$E \left[ \int_0^1 L_2(t, Y^\epsilon(t); \beta_{Y^\epsilon,2}(t, Y^\epsilon)) dt \right] \leq V_2(P_0, P_1|b_1).$$  

(3.24)

We prove (3.24). In the same way as in Lemma 3.1 in [19], from (H.1,ii), by Jensen's inequality,

$$V_1(P_{0,1}, P_{1,1}) \leq E \left[ \int_0^1 L_1(t, Y_1^\epsilon(t); \beta_{Y_1^\epsilon}(t, Y^\epsilon)) dt \right] \leq E \left[ \int_0^1 L_1(t, Y_1^\epsilon(t); \beta_{Y_1^\epsilon,1}(t, Y^\epsilon)) dt \right].$$  

(3.25)

Indeed, $Y_1^\epsilon \in A_1$ with
\[ \beta_{Y_1^e}(t, Y_1^e) = E[\beta_{1,Y^\zeta}(t, Y^\epsilon)|Y_1^e(s), 0 \leq s \leq t] \]

(see e.g., p. 258 of [14]). (3.25) and (3.22) implies (3.24).

Let \( Y^0(t) \) denote the weak limit of \( \{Y^\epsilon_{n(k)}(\cdot)\}_{k \geq 1} \) as \( n \to \infty \). Then, again in the same way as in Lemma 3.1 in [19] and (3.25), from (H.1,ii) and (3.22)-(3.23), by Jensen's inequality,

\[
V_1(P_{0,1}, P_{1,1}) \leq E \left[ \int_0^1 L_1(t, Y_1^0(t); \beta_{Y_1^0}(t, Y_1^0)) dt \right] \\
\leq \liminf_{k \to \infty} V^\epsilon_{n(k)}(P_0, P_1) \leq \limsup_{k \to \infty} V^\epsilon_{n(k)}(P_0, P_1) \\
\leq V_1(P_{0,1}, P_{1,1}).
\]

(3.26)

\( \beta_{Y_0,1}(t, Y_0^0) = \beta_{Y_1^0}(t, Y_1^0) \) from the strict convexity of \( L_1 \) in \( u \), and \( Y_1^0 \) is equal to the minimizer \( X_1 \) of \( V_1(P_{0,1}, P_{1,1}) \) by the uniqueness of the minimizer of \( V_1(P_{0,1}, P_{1,1}) \) and we obtain (3.20). From (3.24), we also have

\[
V_2(P_0, P_1|b_1) \leq E \left[ \int_0^1 L_2(t, Y^0(t); \beta_{Y^0,2}(t, Y^0)) dt \right] \\
\leq \liminf_{k \to \infty} E \left[ \int_0^1 L_2(t, Y^\epsilon_{n(k)}(t); \beta_{Y^\epsilon_{n,2}(t, Y^\epsilon_{n(k)})}) dt \right] \\
\leq \limsup_{k \to \infty} E \left[ \int_0^1 L_2(t, Y^\epsilon_{n(k)}(t); \beta_{Y^\epsilon_{n,2}(t, Y^\epsilon_{n(k)})}) dt \right] \\
\leq V_2(P_0, P_1|b_1) .
\]

(3.27)

The uniqueness of the minimizer of \( V_2(P_0, P_1|b_1) \) completes the proof. □

One can easily prove

**Corollary 3.2** Let \( P_0, P_1 \in \mathcal{M}_1(\mathbb{R}^d) \). Suppose that (H.1)-(H.3) with \( L = L_i \) holds \((i = 1, 2)\), that \( \gamma = 2 \) in (H.2), and that \( V_1(P_{0,1}, P_{1,1}) \) and \( V_2(P_0, P_1|b_1) \) is finite. Then the minimizers \( \{X_1(t)\}_{0 \leq t \leq 1}, \{X(t)\}_{0 \leq t \leq 1} \) and \( \{Y^\epsilon(t)\}_{0 \leq t \leq 1} \) of \( V_1(P_{0,1}, P_{1,1}), V_2(P_0, P_1|b_1) \) and \( V^\epsilon(P_0, P_1) \) exist uniquely, respectively. In addition, \( \{Y^\epsilon(t)\}_{0 \leq t \leq 1} \) weakly converges to \( \{X(t)\}_{0 \leq t \leq 1} \) as \( \epsilon \to 0 \) and (3.20)-(3.21) holds.

From (3.21)-(3.22) and (3.25), we easily have
Proposition 3.4 Suppose that the assumption in Theorem 3.3 holds. Then for any minimizer \( \{Y^\epsilon\}_{0 \leq t \leq 1} \) of \( V^\epsilon(P_0, P_1) \),

\[
0 \leq \frac{E\left[\int_0^1 L_1(t, Y_1^\epsilon(t); \beta_{Y^\epsilon,1}(t, Y^\epsilon)) \, dt\right] - V_1(P_{0,1}, P_{1,1})}{\epsilon} \rightarrow 0 \quad (\epsilon \rightarrow 0).
\]

(3.28)

We don’t know the real convergence rate!

4 Discussion

In section 2, Theorem 2.3, we assumed that \( \nu \mapsto T_i(P_{0,i}, P_{1,i}|\nu) \) is continuous \((i = 3, \cdots, k)\). This continuity is known only in the case of the Knothe-Rosenblatt rearrangement where the representation of the minimizer is known. It is difficult to prove that \( \nu \mapsto T(P_0, P_1|\nu) \) is continuous, which is our future problem.

In section 3.2, we only considered the case where \( k = 2 \) because of the similar reason to above. The point is that we do not even know any example such as the Knothe-Rosenblatt rearrangement. This is also our future problem.

The Knothe-Rosenblatt rearrangement implies the Brunn-Minkowskii inequality. We would like to find, in future, the inequality which can be obtained by the result in section 3.

References


