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A random walk model for nonlinear diffusion

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Abstract

In the present paper, we discuss the asymptotic behaviors of solutions for a couple of nonlinear parabolic equations associated with nonlinear Laplace operators and make an attempt to explain the mechanism of their behaviors by using a macroscopic random walk model.

1 Introduction

The nonlinear generalization of the usual linear Laplace operator $\Delta = \sum_{i=1}^{N} D_{ii}^2$ would be one of stimulus subjects in nonlinear analysis. Particularly, parabolic equations involving such nonlinear Laplace operators appear in the study of nonlinear diffusion. For instance, the p-Laplace operator Δ_p is defined by

$$\begin{aligned} \Delta_p u &:= \operatorname{div}(|Du|^{p-2}Du) \\ &= (p-2)|Du|^{p-4}\langle D^2uDu, Du\rangle + |Du|^{p-2}\Delta u \end{aligned}$$

for $1 < p < \infty$. Degenerate parabolic equations associated with p-Laplace operators for $p > 2$ such as

$$u_t = \Delta_p u \tag{1}$$

are known to describe the motion of non-Newtonian fluids, some critical-state model for type-II superconductors, an approximate model for sandpile growth and so on. Equation (1) has been vigorously studied from various points of view by many authors (see, e.g., [12], [23], [24] and references therein).

The infinity-Laplace operator Δ_∞ defined by

$$\Delta_\infty u = \langle D^2uDu, Du\rangle$$

was introduced by G. Aronsson [5] to derive an Euler equation for a variational problem in L^∞ related to some Lipschitz extension problem into a domain Ω of $\mathbb{R}^N$ for functions defined on the boundary $\partial\Omega$. More precisely, a function $\phi \in W^{1,\infty}(\Omega)\cap C(\overline{\Omega})$ is called an *absolutely minimizing Lipschitz extension* (AMLE for short) of a function $\varphi : \partial\Omega \to \mathbb{R}$ into Ω, if $\phi = \varphi$ on $\partial\Omega$ and

$$|D\phi|_{L^\infty(U)} \leq |Dw|_{L^\infty(U)} \tag{2}$$

for every open subset U of Ω and $w \in W^{1,\infty}(U) \cap C(\overline{U})$ satisfying $w = \phi$ on ∂U. Then (2) is regarded as a variational problem in L^∞. Aronsson [5] proposed the following Dirichlet problem:

$$\Delta_\infty \phi = 0 \quad \text{in} \quad \Omega, \tag{3}$$

$$\phi = \varphi \quad \text{on} \quad \partial\Omega \tag{4}$$

as an Euler equation for the variational problem. He also proved the equivalence between smooth AMLEs of φ into Ω and classical solutions of (3), (4). Moreover, Jensen [15] imported the notion of viscosity solution to this subject and proved the equivalence of general AMLEs of φ and viscosity solutions to (3), (4). Furthermore, there are a vast amount of contributions to the elliptic problem (3) (see the survey papers [6], [9]).

On the other hand, there are fewer contributions to parabolic problems associated with the infinity-Laplace operator. Juutine-Kawohl [17] studied the well-posedness in the viscosity sense of the Cauchy/Cauchy-Dirichlet problem for

$$u_t = \frac{\Delta_\infty u}{|Du|^2}, \tag{5}$$

and moreover, Akagi-Suzuki [2] also proved that for

$$u_t = \Delta_\infty u \tag{6}$$

(see also an earlier work due to Crandall-Wang [11]). Furthermore, the asymptotic behaviors of viscosity solutions for the Cauchy/Cauchy-Dirichlet problem for (6) were investigated by Akagi-Juutinen-Kajikiya [1]. The asymptotic behavior of solutions for (5) was also treated by Juutinen [16] with the homogeneous Dirichlet boundary condition.

The main purposes of the present paper are to compare the asymptotic behaviors of solutions for parabolic equations associated with the usual Laplace operator and nonlinear Laplace operators and to make an attempt to explain the mechanism of the behaviors of solutions by deriving such nonlinear parabolic equations from a macroscopic random walk model.

Several papers also treated the formulations of fully nonlinear parabolic equations from microscopic view points. Cheridito et al [8] provided an approach using backward stochastic differential equations, and Kohn-Serfaty [19] gave a deterministic-control-based approach (see also [18], [20]). Moreover, Peres et al [22] proposed a derivation of the infinity-Laplace equation as well as a singular parabolic equation involving the infinity-Laplacian in terms of a class of zero-sum two-player stochastic games called tug-of-war (see also [21]). Furthermore, nonlocal evolution equations were also exploited to model nonlinear diffusion processes (see, e.g., [7], [3], [4]). Our formulation would be simpler than those and could provide an intuitive interpretation to the behaviors of solutions by sacrificing mathematical rigor.

In Section 2, we briefly review the asymptotic behavior of solutions for linear and nonlinear parabolic equations involving (1), (5) and (6). We particularly provide the optimal decay rate of viscosity solutions of the Cauchy problem for (5) with a proof. In Section 3, we first recall a usual random walk model for linear diffusion and discuss its formal generalizations for nonlinear diffusion. In the final section, we also make an attempt to explain the mechanism of asymptotic behaviors of solutions for the nonlinear diffusion equations (1), (5) and (6).

2 Asymptotic behaviors of solutions

2.1 Usual linear diffusion equation

Let us first consider the usual linear diffusion equation,

$$u_t = \Delta u, \quad x \in \mathbb{R}^N, \ t > 0. \tag{7}$$

Then the Gauss kernel $G(x,t)$ is a well-known self-similar solution of (7) in $\mathbb{R}^N$ given by

$$G(x,t) = \frac{1}{(4\pi t)^{N/2}} \exp\left(-\frac{|x|^2}{4t}\right) \quad \text{for } \ x \in \mathbb{R}^N, \ t > 0.$$

Moreover, for any $u_0 \in C_0(\mathbb{R}^N)$, the solution of (7) satisfying $u(0,\cdot) = u_0$ is explicitly written as follows.

$$u(x,t) = \int_{\mathbb{R}^N} G(x-y,t)u_0(y)\,dy \quad \text{for } \ x \in \mathbb{R}^N, \ t > 0.$$

Then we observe that

$$\operatorname{supp} u(\cdot,t) = \mathbb{R}^N \quad \text{for } \ t > 0$$

and

$$\sup_{x\in\mathbb{R}^N} |u(x,t)| \le \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} |u_0(y)|\,dy.$$

Hence the support of $u(\cdot,t)$ expands at an infinite speed, and solutions decay at the rate of $O(t^{-N/2})$ as $t \to \infty$.

2.2 p-Laplace parabolic equations

We next consider p-Laplace parabolic equations of the form

$$u_t = \Delta_p u, \quad x \in \mathbb{R}^N, \ t > 0 \tag{8}$$

with a p-Laplace operator Δ_p for $p > 2$. Equation (8) belongs to the class of degenerate parabolic equations. As for self-similar solutions of (8) in $\mathbb{R}^N$, a *Barenblatt-type solution* $U(x,t)$ is given by

$$U(x,t) = t^{-\alpha_N}\left(C - k\,|x|^{\frac{p}{p-1}}\,t^{-\frac{p}{p-1}\frac{\alpha_N}{N}}\right)_+^{\frac{p-1}{p-2}} \tag{9}$$

with

$$\alpha_N = \frac{N}{N(p-2)+p}, \quad k = \frac{p-2}{p}\left(\frac{\alpha_N}{N}\right)^{\frac{1}{p-1}}, \quad C > 0.$$

Then

$$\operatorname{supp} U(\cdot,t) = \left\{x \in \mathbb{R}^N;\ |x| \le \left(\frac{C}{k}\right)^{\frac{p-1}{p}} t^{\frac{\alpha_N}{N}}\right\} \quad \text{and} \quad U(0,t) = C^{\frac{p-1}{p-2}}t^{-\alpha_N}.$$

Let $u = u(x,t)$ be a solution of the Cauchy problem for (8) with an initial data $u_0 \in C_0(\mathbb{R}^N)$. Then by virtue of the comparison principle, the decay rate of $u = u(x,t)$ is $O(t^{-\alpha_N})$. Moreover, the support of $u(\cdot,t)$ is bounded for all $t > 0$ and extends in all directions (see [12], [23] for more details).

2.3 Infinity-Laplace parabolic equation: degenerate type

In this subsection we discuss the optimal decay rate of viscosity solutions of the Cauchy problem for

$$u_t = \Delta_\infty u, \quad x \in \mathbb{R}^N, \ t > 0, \tag{10}$$

where Δ_∞ denotes the infinity-Laplace operator, with an initial data u_0 whose support is compact.

Theorem 2.1 (Akagi-Juutinen-Kajikiya [1]). *For initial data $u_0 \in C_0(\mathbb{R}^N)$, the unique viscosity solution $u = u(x,t)$ of the Cauchy problem for* (10) *satisfies*

$$|u(\cdot,t)|_{L^\infty(\mathbb{R}^N)} \leq C(t+1)^{-1/6} \quad \textit{for} \ \ t > 0 \tag{11}$$

with some $C > 0$ independent of x and t. In addition, if $u_0 \geq 0$ in $\mathbb{R}^N$ and $u_0 \not\equiv 0$, then there is $c > 0$ independent of x and t such that

$$c(t+1)^{-1/6} \leq |u(\cdot,t)|_{L^\infty(\mathbb{R}^N)} \quad \textit{for} \ \ t > 0. \tag{12}$$

Therefore $(t+1)^{-1/6}$ is the optimal decay rate of solutions in $\Omega = \mathbb{R}^N$. Moreover, the support of $u(\cdot,t)$ is bounded in $\mathbb{R}^N$ for any $t \geq 0$.

We refer the readers to [1] for its proof. Furthermore, we can also prove that the support of $u(\cdot,t)$ expands in all directions by slightly modifying the argument of proof used in [1].

2.4 Infinity-Laplace parabolic equation: singular type

This subsection is devoted to the Cauchy problem for

$$u_t = \frac{\Delta_\infty u}{|Du|^2}, \quad x \in \mathbb{R}^N, \ t > 0. \tag{13}$$

There seems to be no contribution for the asymptotic behavior of solutions to (13) (c.f., see [17] for the well-posedness). As in [1], we can prove:

Theorem 2.2. *For initial data $u_0 \in C_0(\mathbb{R}^N)$, the unique viscosity solution u of the Cauchy problem for* (13) *satisfies*

$$|u(\cdot,t)|_{L^\infty(\mathbb{R}^N)} \leq C(t+1)^{-1/2} \quad \textit{for} \ \ t > 0 \tag{14}$$

with some $C > 0$ independent of x and t. In addition, if $u_0 \geq 0$ in $\mathbb{R}^N$ and $u_0 \not\equiv 0$, then there is $c > 0$ independent of x and t such that

$$c(t+1)^{-1/2} \leq |u(\cdot,t)|_{L^\infty(\mathbb{R}^N)} \quad \textit{for} \ \ t > 0. \tag{15}$$

Therefore $(t+1)^{-1/2}$ is the optimal decay rate of solutions in $\Omega = \mathbb{R}^N$. Moreover, the support of $u(\cdot,t)$ coincides with $\mathbb{R}^N$ for any $t > 0$.

Here we recall the definition of viscosity solutions for (13). We use the following notation:

$$\Lambda(X) := \max_{\xi\in\mathbb{R}^N,|\xi|=1} \langle X\xi,\xi\rangle, \quad \lambda(X) := \min_{\xi\in\mathbb{R}^N,|\xi|=1} \langle X\xi,\xi\rangle$$

for an $N\times N$ matrix X. Moreover, let us denote by $\mathcal{P}^{2,\pm}u(x,t)$ the parabolic super- and subjets of a function u at (x,t) (see §8 of [10] for more details).

Definition 2.3. *Let $Q := \mathbb{R}^N \times (0,\infty)$. An upper semicontinuous function $u : Q \to \mathbb{R}$ is said to be a viscosity subsolution of (13), if it holds that*

$$s \le \langle Xp,p\rangle/|p|^2 \quad \text{if} \quad p \neq 0, \quad \text{and} \quad s \le \Lambda(X) \quad \text{if} \quad p = 0$$

for all $(x,t)\in Q$ and $(p,X)\in\mathcal{P}^{2,+}u(x,t)$.

A lower semicontinuous function $u : Q \to \mathbb{R}$ is said to be a viscosity supersolution of (13), if it holds that

$$s \ge \langle Xp,p\rangle/|p|^2 \quad \text{if} \quad p \neq 0, \quad \text{and} \quad s \ge \lambda(X) \quad \text{if} \quad p = 0$$

for all $(x,t)\in Q$ and $(p,X)\in\mathcal{P}^{2,-}u(x,t)$.

To prove Theorem 2.2, we first construct a family of exact solutions for (13). Define $\rho_\varepsilon \in C_0^\infty(\mathbb{R})$ by

$$\rho_\varepsilon(\zeta) := \varepsilon\rho\left(\frac{\zeta}{\varepsilon}\right) \qquad \text{for} \quad \zeta\in\mathbb{R}, \ \varepsilon > 0,$$

where $\rho \in C_0^\infty(\mathbb{R})$ is given by

$$\rho(\zeta) := \begin{cases} \exp\left(\dfrac{\zeta^2}{\zeta^2-1}\right) & \text{if} \quad |\zeta| \le 1, \\ 0 & \text{if} \quad |\zeta| > 1. \end{cases}$$

It then follows that ρ_ε is even, and

$$\begin{aligned} &\rho_\varepsilon \ge 0 \ \text{ in } \mathbb{R}, \quad \max_{\mathbb{R}} \rho_\varepsilon = \varepsilon, \quad \text{supp } \rho_\varepsilon = [-\varepsilon,\varepsilon], \quad \rho_\varepsilon\left(\frac{\varepsilon}{2}\right) = \varepsilon\rho\left(\frac{1}{2}\right) > 0, \\ &\rho_\varepsilon' > 0 \ \text{ in } (-\varepsilon,0), \quad \rho_\varepsilon' < 0 \ \text{ in } (0,\varepsilon), \quad \rho_\varepsilon'(\zeta) = 0 \ \text{ if } \ |\zeta|\ge\varepsilon \ \text{ or } \ \zeta = 0, \\ &\rho_\varepsilon''(0) < 0. \end{aligned}$$

Furthermore, we set

$$G(\zeta,t) := (4\pi t)^{-1/2}\exp\left(-\frac{\zeta^2}{4t}\right) \qquad \text{for } \zeta\in\mathbb{R}, \ t>0,$$

and

$$g_\varepsilon(\xi,t) := \left(G(\cdot,t) * \rho_\varepsilon\right)(\xi) = \int_{-\infty}^{\infty} G(\xi-\eta,t)\rho_\varepsilon(\eta)d\eta \quad \text{for } \xi\in\mathbb{R}, \ t>0.$$

Then $g_\varepsilon \in C^\infty(\mathbb{R} \times (0,\infty))$, and it solves

$$\begin{cases} u_t = u_{\xi\xi} & \text{in } \mathbb{R} \times (0,\infty), \\ u(\cdot,0) = \rho_\varepsilon(|\cdot|) & \text{in } \mathbb{R} \end{cases}$$

in the classical sense.

Now, we set

$$V_\varepsilon(x,t) := g_\varepsilon(|x|,t) \quad \text{for } x \in \mathbb{R}^N \text{ and } t > 0.$$

Then we have

Lemma 2.4. (i) supp $V_\varepsilon(\cdot,t) = \mathbb{R}^N$ *for any* $t > 0$.

(ii) $D_i V_\varepsilon(x,t) = \partial_\xi g_\varepsilon(|x|,t)\dfrac{x_i}{|x|}$ *for all* $x \in \mathbb{R}^N \setminus \{0\}$ *and* $t > 0$.

(iii) $D^2_{ij} V_\varepsilon(x,t) = \partial^2_{\xi\xi} g_\varepsilon(|x|,t)\dfrac{x_i x_j}{|x|^2} + \partial_\xi g_\varepsilon(|x|,t)\dfrac{\delta_{ij}|x|^2 - x_i x_j}{|x|^3}$ *for all* $x \in \mathbb{R}^N \setminus \{0\}$ *and* $t > 0$.

(iv) *For* $t > 0$, *it holds that* $|DV_\varepsilon(x,t)| = 0$ *if and only if* $x = 0$.

(v) V_ε *belongs to* $C^2(\mathbb{R}^N \times \mathbb{R}^+)$.

(vi) $D^2_{ij} V_\varepsilon(0,t) = \partial^2_\xi g_\varepsilon(0,t)\delta_{ij}$ *for all* $t > 0$.

Proof. Since supp $g_\varepsilon(\cdot,t) = \mathbb{R}$ for $t > 0$, we get (i). Moreover, it is obvious that V_ε belongs to $C^2((\mathbb{R}^N \setminus \{0\}) \times \mathbb{R}^+)$, and therefore (ii) and (iii) follow immediately from the definition of V_ε. We next claim that

$$\partial_\xi g_\varepsilon(\xi,t) = 0 \text{ if and only if } \xi = 0$$

for $t > 0$. Indeed, we see

$$\begin{aligned} \partial_\xi g_\varepsilon(\xi,t) &= \int_{-\infty}^{\infty} \partial_\xi G(\xi-\eta,t)\rho_\varepsilon(\eta)d\eta \\ &= \int_{-\infty}^{\infty} -\partial_\eta G(\xi-\eta,t)\rho_\varepsilon(\eta)d\eta \\ &= -\int_{-\varepsilon}^{\varepsilon} \partial_\eta G(\xi-\eta,t)\rho_\varepsilon(\eta)d\eta \\ &= -\left[G(\xi-\eta,t)\rho_\varepsilon(\eta)\right]_{\eta=-\varepsilon}^{\eta=\varepsilon} + \int_{-\varepsilon}^{\varepsilon} G(\xi-\eta,t)\rho'_\varepsilon(\eta)d\eta. \end{aligned}$$

Using the fact that $\rho_\varepsilon(\pm\varepsilon) = 0$ and changing the variable by $\sigma := \pm\rho_\varepsilon(\eta)$, we obtain

$$\partial_\xi g_\varepsilon(\xi,t) = \int_0^\varepsilon G(\xi-\eta(\sigma),t)d\sigma - \int_0^\varepsilon G(\xi+\eta(\sigma),t)d\sigma.$$

Hence since G is even, it follows that $\partial_\xi g_\varepsilon(\xi,t) = 0$ if and only if $\xi = 0$. Moreover, it also implies that $V_\varepsilon \in C^1(\mathbb{R}^N \times \mathbb{R}^+)$, and thus (iv) follows.

Finally, we prove (v) and (vi). By virtue of Lagrange's mean value theorem, for $i, j = 1, 2, \dots, N$ and $h \in \mathbb{R}$, there exists $\theta \in (0,1)$ depending on h, i, j such that

$$D_i V_\varepsilon(he_j, t) - D_i V_\varepsilon(0,t) = D^2_{ij} V_\varepsilon(\theta h e_j, t)h = \partial^2_{\xi\xi} g_\varepsilon(\theta|h|, t)\delta_{ij} h,$$

where e_j denotes the j-th unit basis vector in $\mathbb{R}^N$. Hence we find V_ε belongs to $C^2(\mathbb{R}^N \times \mathbb{R}^+)$, and moreover,

$$D^2_{ij} V_\varepsilon(0,t) = \partial^2_{\xi\xi} g_\varepsilon(0,t)\delta_{ij} \quad \text{for all } \; i, j = 1, 2, \dots, N \; \text{ and } \; t > 0.$$

This completes our proof. □

Moreover, V_ε becomes a radially symmetric viscosity solution for (13) in $\mathbb{R}^N \times \mathbb{R}^+$. Indeed, we have:

Theorem 2.5. *For each $\varepsilon > 0$, the function V_ε solves (13) in $\mathbb{R}^N \times \mathbb{R}^+$ in the viscosity sense. Moreover, $V_\varepsilon(\cdot, t) \to \rho_\varepsilon(|\cdot|)$ uniformly in $\mathbb{R}^N$ as $t \to +0$.*

Proof. We have, for $x \neq 0$ and $t > 0$,

$$|DV_\varepsilon(x,t)| = |\partial_\xi g_\varepsilon(|x|, t)| \quad \text{ and } \quad \Delta_\infty V_\varepsilon(x,t) = \partial^2_{\xi\xi} g_\varepsilon(|x|,t)\, |\partial_\xi g_\varepsilon(|x|,t)|^2,$$

and for all $x \in \mathbb{R}^N$ and $t > 0$,

$$\partial_t V_\varepsilon(x,t) = \partial_t g_\varepsilon(|x|, t).$$

Here, by (iv) of Lemma 2.4, we note that $|DV_\varepsilon(x,t)| > 0$ for all $x \in \mathbb{R}^N \setminus \{0\}$ and $t > 0$. Thus V_ε becomes a classical solution of (13) in $(\mathbb{R}^N \setminus \{0\}) \times \mathbb{R}^+$.

As for $x = 0$, by Lemma 2.4, we have $DV_\varepsilon(0,t) = 0$ and $D^2_{ij} V_\varepsilon(0,t) = \partial^2_\xi g_\varepsilon(0,t)\delta_{ij}$ for all $t > 0$, which yields

$$\lambda\left(D^2 V(0,t)\right) = \Lambda\left(D^2 V(0,t)\right) = \partial^2_{\xi\xi} g_\varepsilon(0,t).$$

Let $t > 0$ and let $(s, p, X) \in \mathcal{P}^{2,+} V_\varepsilon(0,t)$ be fixed. Then, since V_ε belongs to $C^{2,1}(\mathbb{R}^N \times \mathbb{R}^+)$, we obtain

$$s = \partial_t V_\varepsilon(0,t) = \partial_t g_\varepsilon(0,t), \quad p = DV_\varepsilon(0,t) = 0, \quad X \geq D^2 V_\varepsilon(0,t.)$$

Hence we can deduce that

$$s - \lambda(X) \leq s - \lambda(D^2 V_\varepsilon(0,t)) = \partial_t g_\varepsilon(0,t) - \partial^2_{\xi\xi} g_\varepsilon(0,t) = 0,$$

and also that $s \geq \Lambda(X)$ for $(s,p,X) \in \mathcal{P}^{2,-} V_\varepsilon(0,t)$. Consequently, V_ε becomes a viscosity solution of (13) in $\mathbb{R}^N \times \mathbb{R}^+$. Finally, since $g_\varepsilon(\cdot, t) \to \rho_\varepsilon$ uniformly in $\mathbb{R}$ as $t \to 0$, we deduce that $V_\varepsilon(\cdot, t) \to \rho_\varepsilon(|\cdot|)$ uniformly in $\mathbb{R}^N$ as $t \to 0$. □

Now, let us establish an estimate from below for $u(x,t) \geq 0$ by using $V_\varepsilon(x,t)$ as a barrier function.

Lemma 2.6. *Let $u_0 \in C(\mathbb{R}^N)$ be such that $u_0 \geq 0$ and $u_0 \not\equiv 0$. Let u be a unique viscosity solution of the Cauchy problem for* (13) *in $\mathbb{R}^N \times \mathbb{R}^+$. Then*

$$V_\varepsilon(x - x_0, t) \leq u(x,t) \quad \textit{for all} \;\; x \in \mathbb{R}^N \;\; \textit{and} \;\; t > 0$$

for some $\varepsilon > 0$ and $x_0 \in \mathbb{R}^N$. In particular, the support of $u(\cdot, t)$ coincides with $\mathbb{R}^N$ for $t > 0$.

Proof. We can assume that $u_0(0) > 0$ without any loss of generality by an appropriate translation. Then we can take $\varepsilon > 0$ such that

$$u_0(x) \geq \frac{u_0(0)}{2} \;\; \text{if} \;\; |x| < \varepsilon \quad \text{and} \quad \varepsilon < \frac{u_0(0)}{2}.$$

Then ρ_ε satisfies

$$\rho_\varepsilon(|x|) \leq u_0(x) \;\; \text{for} \;\; x \in \mathbb{R}^N.$$

Exploiting the comparison principle, we have

$$V_\varepsilon(x,t) \leq u(x,t) \;\; \text{for all} \;\; x \in \mathbb{R}^N \;\; \text{and} \;\; t > 0.$$

Moreover, by (i) of Lemma 2.4, the support of u instantly coincides with $\mathbb{R}^N$. □

Finally, we give an estimate from above for $u(x,t)$ to complete our proof of Theorem 2.2.

Lemma 2.7. *Let $u_0 \in C_0(\mathbb{R}^N)$ and let u be a unique viscosity solution of the Cauchy problem for* (13) *in $\mathbb{R}^N \times \mathbb{R}^+$. Then there exists a constant $C > 0$ independent of x and t such that*

$$|u(\cdot, t)|_\infty \leq C(t+1)^{-1/2} \quad \textit{for all} \;\; t > 0.$$

Moreover, by virtue of Lemma 2.6, the decay rate $O(t^{-1/2})$ is optimal.

Proof. Since u_0 has a compact support, we can take $R > 0$ so large that

$$\text{supp}\; u_0 \subset B(0, R/2) \quad \text{and} \quad |u_0|_\infty \leq R\rho(1/2).$$

Then we observe that

$$-\rho_R(|x|) \leq u_0(x) \leq \rho_R(|x|) \;\; \text{for} \;\; x \in \mathbb{R}^N.$$

Hence the comparison principle ensures that

$$-V_\varepsilon(x,t) \leq u(x,t) \leq V_\varepsilon(x,t) \;\; \text{for all} \;\; x \in \mathbb{R}^N \;\; \text{and} \;\; t > 0,$$

which implies

$$|u(\cdot, t)|_\infty \leq C(t+1)^{-1/2} \;\; \text{for all}\; t > 0.$$

This completes our proof. □

3 Random walk model

In this section, we first recall the well-known derivation of the usual linear diffusion equation in view of a macroscopic random walk. Our purpose of this section is to derive nonlinear parabolic equations involving nonlinear Laplace operators by formally modifying the probability density function for random steps. The probability density functions proposed here will be interpreted in the final section to give an explanation to the asymptotic behavior of solutions shown in the last section.

Let $u(x,t)$ denote the density for randomly moving particles to be at x at time t. Let τ be a (short) duration of each step and let $y \in \mathbb{R}^N$ be a random step. Then we recall a macroscopic random walk model given by

$$u(x,t+\tau) = \int_{\mathbb{R}^N} u(x-y,t)p_\tau(y)dy \quad \text{for} \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \tag{16}$$

where $p_\tau(y)$ denotes a probability density of choosing a random step $y \in \mathbb{R}^N$ for $\tau > 0$ (see §4 of a celebrated paper [13] due to A. Einstein).

3.1 Derivation of a linear diffusion equation

Let us recall the well-known derivation of a usual linear diffusion equation by setting $p_\tau = \mathcal{N}_N(0,\Sigma)$, i.e., the N-dimensional normal distribution with zero mean given by

$$p_\tau(y) = \frac{1}{(2\pi)^{N/2}(\det\Sigma)^{1/2}} \exp\left(-\frac{1}{2}\left\langle y, \Sigma^{-1}y\right\rangle\right), \tag{17}$$

with the covariance matrix $\Sigma = 2\tau I$ (here, I is an $N \times N$ unit matrix). Hence

$$\int_{\mathbb{R}^N} p_\tau(y)dy = 1, \quad \int_{\mathbb{R}^N} y_i p_\tau(y)dy = 0, \quad \int_{\mathbb{R}^N} y_i y_j p_\tau(y)dy = 2\tau\delta_{ij}. \tag{18}$$

Fix $(x,t) \in \mathbb{R}^N \times (0,\infty)$ and expand $u(x,t+\tau)$ as a Taylor series in t and $u(x-y,t)$ in x, that is,

$$u(x,t+\tau) = u(x,t) + u_t(x,t)\tau + O(\tau^2)$$

and

$$u(x-y,t) = u(x,t) - D_i u(x,t)y_i + \frac{1}{2}D_{ij}^2 u(x,t)y_i y_j + O(|y|^3).$$

It then follows that

$$\begin{aligned} &u(x,t) + u_t(x,t)\tau + O(\tau^2) \\ &= u(x,t)\int_{\mathbb{R}^N} p_\tau(y)dy - D_i u(x,t)\int_{\mathbb{R}^N} y_i p_\tau(y)dy \\ &\quad + \frac{1}{2}D_{ij}^2 u(x,t)\int_{\mathbb{R}^N} y_i y_j p_\tau(y)dy + \int_{\mathbb{R}^N} O(|y|^3)p_\tau(y)dy. \end{aligned}$$

Dividing both sides by τ and letting $\tau \to 0$, we obtain a linear diffusion equation,

$$u_t(x,t) = \Delta u(x,t). \tag{19}$$

3.2 Derivation of infinity-Laplace parabolic equations

Now, let us discuss modifications on the derivation of linear diffusion equations for non-linear parabolic equations. We first derive the infinity-Laplace parabolic equations from (16) by specifying a probability density p_τ for random steps.

Let $(x,t) \in \mathbb{R}^N \times (0,\infty)$ be fixed and set

$$p_\tau(y) = \int_{\mathbb{R}} \rho_\tau(\epsilon)\delta^N(y - \epsilon \mathbf{v}(x,t))d\epsilon \tag{20}$$

with a vector $\mathbf{v}(x,t) \in \mathbb{R}^N$ and $\rho_\tau = \mathcal{N}_1(0,\sigma^2)$, a one-dimensional normal distribution,

$$\rho_\tau(\epsilon) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{\epsilon^2}{2\sigma^2}\right),$$

with the variance $\sigma^2 = 2\tau$. Here we note that

$$\int_{\mathbb{R}} \rho_\tau(\epsilon)d\epsilon = 1, \quad \int_{\mathbb{R}} \epsilon\rho_\tau(\epsilon)d\epsilon = 0, \quad \int_{\mathbb{R}} \epsilon^2\rho_\tau(\epsilon)d\epsilon = 2\tau,$$

which implies

$$\int_{\mathbb{R}^N} p_\tau(y)dy = 1, \quad \int_{\mathbb{R}^N} y_i p_\tau(y)dy = 0$$

and

$$\int_{\mathbb{R}^N} y_i y_j p_\tau(y)dy = 2\tau v_i(x,t)v_j(x,t).$$

Now, we formally derive from (16) that

$$\begin{aligned}
u(x,t+\tau) &= \int_{\mathbb{R}^N} u(x-y,t)p_\tau(y)dy \\
&= \int_{\mathbb{R}^N} u(x-y,t)\left(\int_{\mathbb{R}} \rho_\tau(\epsilon)\delta^N(y-\epsilon\mathbf{v}(x,t))d\epsilon\right)dy \\
&= \int_{\mathbb{R}} \rho_\tau(\epsilon)\left(\int_{\mathbb{R}^N} u(x-y,t)\delta^N(y-\epsilon\mathbf{v}(x,t))dy\right)d\epsilon \\
&= \int_{\mathbb{R}} u(x-\epsilon\mathbf{v}(x,t),t)\rho_\tau(\epsilon)d\epsilon,
\end{aligned}$$

which leads us to

$$u(x,t+\tau) = \int_{\mathbb{R}} u(x-\epsilon\mathbf{v}(x,t),t)\rho_\tau(\epsilon)d\epsilon.$$

Here by performing the Taylor expansion, we deduce that

$$u(x,t+\tau) = u(x,t) + u_t(x,t)\tau + O(\tau^2)$$

and

$$\begin{aligned}
&\int_{\mathbb{R}} u(x-\epsilon\mathbf{v}(x,t),t)\rho_\tau(\epsilon)d\epsilon \\
&= \int_{\mathbb{R}}\Big(u(x,t)-\epsilon\langle Du(x,t),\mathbf{v}(x,t)\rangle \\
&\quad +\frac{\epsilon^2}{2}\langle D^2u(x,t)\mathbf{v}(x,t),\mathbf{v}(x,t)\rangle + O(\epsilon^3)\Big)\rho_\tau(\epsilon)d\epsilon \\
&= u(x,t)\int_{\mathbb{R}}\rho_\tau(\epsilon)d\epsilon - \langle Du(x,t),\mathbf{v}(x,t)\rangle\int_{\mathbb{R}}\epsilon\rho_\tau(\epsilon)d\epsilon \\
&\quad +\frac{1}{2}\langle D^2u(x,t)\mathbf{v}(x,t),\mathbf{v}(x,t)\rangle\int_{\mathbb{R}}\epsilon^2\rho_\tau(\epsilon)d\epsilon + \int_{\mathbb{R}}O(\epsilon^3)\rho_\tau(\epsilon)d\epsilon \\
&= u(x,t)+\tau\langle D^2u(x,t)\mathbf{v}(x,t),\mathbf{v}(x,t)\rangle + O(\tau^4).
\end{aligned}$$

Hence

$$u_t(x,t)\tau = \tau\langle D^2u(x,t)\mathbf{v}(x,t),\mathbf{v}(x,t)\rangle + O(\tau^2).$$

Divide both sides by τ and pass to the limit as $\tau\to 0$. Then

$$u_t(x,t) = \langle D^2u(x,t)\mathbf{v}(x,t),\mathbf{v}(x,t)\rangle.$$

In particular, put $\mathbf{v}(x,t)=Du(x,t)$. Then

$$u_t(x,t) = \langle D^2u(x,t)Du(x,t),Du(x,t)\rangle = \Delta_\infty u(x,t).$$

Therefore the infinity-Laplace parabolic equation is derived from the limit as $\tau\to 0$ of the relation,

$$u(x,t+\tau) = \int_{\mathbb{R}} u(x-\epsilon\mathbf{v}(x,t),t)\rho_\tau(\epsilon)d\epsilon \quad \text{for } (x,t)\in\mathbb{R}^N\times(0,\infty)$$

with

$$\mathbf{v}(x,t) = Du(x,t),$$

which is obtained from (16) with the probability density for random steps in the form:

$$p_\tau(y) = \int_{\mathbb{R}}\rho_\tau(\epsilon)\delta^N(y-\epsilon Du(x,t))d\epsilon. \tag{21}$$

We can also derive the singular infinity-Laplace parabolic equation,

$$u_t = \frac{\Delta_\infty u}{|Du|^2},$$

from (16) together with the probability density,

$$p_\tau(y) = \int_{\mathbb{R}}\rho_\tau(\epsilon)\delta^N\left(y-\epsilon\frac{Du(x,t)}{|Du(x,t)|}\right)d\epsilon. \tag{22}$$

3.3 Derivation of p-Laplace parabolic equations

Let us next propose a probability density function for random steps to derive p-Laplace parabolic equations. Fix $(x,t) \in \mathbb{R}^N \times (0,\infty)$ and set

$$p_\tau(y) = \frac{1}{2}\int_{\mathbb{R}} \rho_\tau(\epsilon)\delta^N(y-\epsilon\mathbf{v}(x,t))d\epsilon + \frac{1}{2}\int_{\mathbb{R}^N} q_\tau(\varepsilon)\delta^N(y-\varepsilon c(x,t))d\varepsilon$$

with a vector $\mathbf{v}(x,t) \in \mathbb{R}^N$ and a scalar $c(x,t) \in \mathbb{R}$. Moreover, put

$$\rho_\tau = \mathcal{N}_1(0,2\tau) \quad \text{and} \quad q_\tau = \mathcal{N}_N(0,2\tau I).$$

Then

$$\int_{\mathbb{R}} \rho_\tau(\epsilon)d\epsilon = 1, \quad \int_{\mathbb{R}} \epsilon\rho_\tau(\epsilon)d\epsilon = 0, \quad \int_{\mathbb{R}} \epsilon^2\rho_\tau(\epsilon)d\epsilon = 2\tau$$

and

$$\int_{\mathbb{R}^N} q_\tau(\varepsilon)d\varepsilon = 1, \quad \int_{\mathbb{R}^N} \varepsilon_i q_\tau(\varepsilon)d\varepsilon = 0, \quad \int_{\mathbb{R}^N} \varepsilon_i\varepsilon_j q_\tau(\varepsilon)d\varepsilon = 2\tau\delta_{ij}.$$

Moreover, we formally obtain

$$\int_{\mathbb{R}^N} u(x-y,t)\left(\int_{\mathbb{R}} \rho_\tau(\epsilon)\delta^N(y-\epsilon\mathbf{v}(x,t))d\epsilon\right)dy = \int_{\mathbb{R}} u(x-\epsilon\mathbf{v}(x,t),t)\rho_\tau(\epsilon)d\epsilon$$

and

$$\int_{\mathbb{R}^N} u(x-y,t)\left(\int_{\mathbb{R}^N} q_\tau(\varepsilon)\delta^N(y-\varepsilon c(x,t))d\varepsilon\right)dy = \int_{\mathbb{R}^N} u(x-\varepsilon c(x,t),t)q_\tau(\varepsilon)d\varepsilon.$$

Hence combining those with (16), we have

$$u(x,t+\tau) = \frac{1}{2}\int_{\mathbb{R}} u(x-\epsilon\mathbf{v}(x,t),t)\rho_\tau(\epsilon)d\epsilon + \frac{1}{2}\int_{\mathbb{R}^N} u(x-\varepsilon c(x,t),t)q_\tau(\varepsilon)d\varepsilon.$$

We have already known that

$$\begin{aligned}
&\int_{\mathbb{R}} u(x-\epsilon\mathbf{v}(x,t),t)\rho_\tau(\epsilon)d\epsilon \\
&\quad = u(x,t) + \tau\langle D^2u(x,t)\mathbf{v}(x,t),\mathbf{v}(x,t)\rangle + O(\tau^4),
\end{aligned}$$

and it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} u(x-\varepsilon c(x,t),t)q_\tau(\varepsilon)d\varepsilon. \\
&= u(x,t) + \frac{1}{2}c(x,t)^2 D^2_{ij}u(x,t)\int_{\mathbb{R}^N} \varepsilon_i\varepsilon_j q_\tau(\varepsilon)d\varepsilon + \int_{\mathbb{R}} O(|\varepsilon|^3)q_\tau(\varepsilon)d\varepsilon \\
&= u(x,t) + \tau c(x,t)^2\Delta u(x,t) + O(\tau^4).
\end{aligned}$$

Thus by letting $\tau \to 0$,

$$u_t(x,t) = \frac{1}{2}\langle D^2u(x,t)\mathbf{v}(x,t), \mathbf{v}(x,t)\rangle + \frac{1}{2}c(x,t)^2\Delta u(x,t).$$

In particular, set

$$\mathbf{v}(x,t) = \sqrt{2(p-2)}|Du(x,t)|^{(p-4)/2}Du(x,t)$$

and

$$c(x,t) = \sqrt{2}|Du(x,t)|^{(p-2)/2}.$$

Then

$$\begin{aligned}
u_t(x,t) &= (p-2)|Du(x,t)|^{p-4}\langle D^2u(x,t)Du(x,t), Du(x,t)\rangle \\
&\quad +|Du(x,t)|^{p-2}\Delta u(x,t) \\
&= \Delta_p u(x,t).
\end{aligned}$$

Hence p-Laplace parabolic equations are derived from a limit as $\tau \to 0$ of the relation,

$$\begin{aligned}
u(x,t+\tau) &= \frac{1}{2}\int_{\mathbb{R}} u(x-\epsilon\mathbf{v}(x,t),t)\rho_\tau(\epsilon)d\epsilon \\
&\quad +\frac{1}{2}\int_{\mathbb{R}^N} u(x-\varepsilon|Du(x,t)|^{p-2})q_\tau(\varepsilon)d\varepsilon \\
&\quad \text{for } (x,t) \in \mathbb{R}^N \times (0,\infty)
\end{aligned}$$

with

$$\begin{aligned}
\mathbf{v}(x,t) &= \sqrt{2(p-2)}|Du(x,t)|^{(p-4)/2}Du(x,t), \\
c(x,t) &= \sqrt{2}|Du(x,t)|^{(p-2)/2},
\end{aligned}$$

which is obtained by (16) coupled with the probability density for random steps,

$$p_\tau(y) = \frac{1}{2}\int_{\mathbb{R}} \rho_\tau(\epsilon)\delta^N(y-\epsilon\mathbf{v}(x,t))d\epsilon + \frac{1}{2}\int_{\mathbb{R}^N} q_\tau(\varepsilon)\delta^N(y-\varepsilon c(x,t))d\varepsilon \tag{23}$$

with $\mathbf{v}(x,t)$ and $c(x,t)$ given above.

4 Conclusion

In §2, we particularly observe that

(i) In the case of linear diffusion equations (7) and p-Laplace parabolic equations (8), the decay rates of solutions depend on the space dimension N. In the case of infinity-Laplace parabolic equations (10) and (13), the decay rates of solutions are independent of N.

(ii) Let u_0 be a continuous initial data with compact support in $\mathbb{R}^N$. In the case of (8) with $p > 2$ and (10), the support of each solution is bounded in $\mathbb{R}^N$ for all $t > 0$. On the other hand, in the case of (7) and (13), the support of every solution instantly coincides with $\mathbb{R}^N$ (hence the speed of propagation is infinite).

(iii) In all examples, for initial data $u_0 \in C_0(\mathbb{R}^N)$, the support of each solution expands in every direction.

Let us first discuss why the decay rate of solutions for infinity-Laplace equations do not depend on N from the view point of the macroscopic random walk model (16). In the derivation of (10), due to the probability density (21) of random steps, particles move only in the direction parallel to $Du(x,t)$. Furthermore, we can also observe that the probability density (21) is quite similar to that for the one-dimensional p-Laplace parabolic equation (8) with $p=4$. Indeed, the decay rate of solutions for (10) is $O(t^{-1/6})$, and moreover, it coincides with the decay rate, $O(t^{-\alpha_N})$, of solutions for the p-Laplace parabolic equation with $p=4$ and $N=1$ (see §2.2). Hence solutions of (10) behave like a one-dimensional diffusion given by (8) with $N=1$ and $p=4$ in the direction parallel to $Du(x,t)$ at each (x,t). In the derivation of (13), recalling the probability density (22) of random steps, we observe that particles also move only in the direction parallel to $Du(x,t)/|Du(x,t)|$. We can also deduce that solutions for (13) behave like a one-dimensional linear diffusion described by (7) with $N=1$ in the direction parallel to $Du(x,t)/|Du(x,t)|$ at each (x,t). This observation could support the facts that the decay rate of solutions for (13) coincides with that for (7) with $N=1$.

The one-dimensional diffusion phenomena described by solutions for infinity-Laplace parabolic equations are easily observed in the radial symmetric case. Indeed, substitute $u(x,t)=\phi(r,t)$ with $r=|x|$ into (10) (resp., (13)). Then the function ϕ solves the one-dimensional p-Laplace parabolic equation with $p=4$ (resp., linear diffusion equation), even if $N>1$. Hence the decay rates of radially symmetric solutions $u(x,t)=\phi(|x|,t)$ are independent of N.

The supports of solutions for (10) and (13) expand in all directions, although the diffusion occurs one-dimensionally. We can also explain this fact from the view point of the random walk model. For a smooth initial data u_0, the family of gradients $Du_0(x)$ for $x \in [u_0=0] := \{x \in \mathbb{R}^N; u_0(x)=0\}$ covers all directions of $\mathbb{R}^N$. Hence the support of $u(\cdot,t)$ could expand in all directions.

We next focus on the variance of stride length for random steps. Let λ_i be the i-th eigenvalue of the covariance matrix for p_τ and let $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\ldots,\lambda_N)$. We call $\lambda := |\boldsymbol{\lambda}|$ the variance of stride length for random steps. In the case of (21) and (23), the variance of stride length depends on $|Du(x,t)|$. In particular, if $Du(x,t)$ vanishes, then λ also vanishes, and hence, particles will not move at (x,t). On the other hand, in the case of (17) and (22), the variance of stride length are always constant at every $(x,t) \in \mathbb{R}^N \times (0,\infty)$. Such a difference seems to cause the difference of the speed of propagation.

Finally, we give a remark on the anisotropy of nonlinear diffusion described by p-Laplace parabolic equations. Recalling the probability density (23), $p_\tau(y)$ takes a larger value in the direction parallel to $Du(x,t)$ than other directions. Indeed, the second term of p_τ in (23) provides the same weight in all directions; however, the first term of p_τ increases the weight only in the direction parallel to $Du(x,t)$. Hence the diffusion phenomena described by solutions for (8) could be anisotropic.

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