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Hamilton-Jacobi equations and Euclidean Sobolev inequality

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1 Introduction

The result of this note is a special case of [3], and the readers should refer to it for more detailed results and their proofs.

Let \( \Omega \) be a bounded and Lebesgue measurable set in \( \mathbb{R}^n \). Let \( 0 < \alpha < \beta < \infty \). Then, as is well-known, the following inequality holds:

\[
\left| \Omega \right|^{-1/\alpha} \| f \|_{\alpha, \Omega} \leq \left| \Omega \right|^{-1/\beta} \| f \|_{\beta, \Omega} \leq \| f \|_{\infty, \Omega}, \quad f \in L^\infty(\Omega)
\]

where \( |\Omega| \) is the Lebesgue measure of \( \Omega \) and \( \| \cdot \|_{\beta, \Omega} \) is the \( L^\beta(\Omega) \)-norm \((0 < \beta < \infty)\) with respect to the Lebesgue measure in \( \mathbb{R}^n \). Furthermore, this inequality is optimal in the sense that all inequalities in (1.1) are reduced to equalities when \( f \) is a constant function on \( \Omega \). This inequality show a norm-monotone property of \( \{ |\Omega|^{-1/\beta} \| f \|_{\beta, \Omega} \}_{0 < \alpha < \infty} \).

However, as far as we know, there is no inequality corresponding to (1.1) when a bounded and Lebesgue measurable set \( \Omega \) in \( \mathbb{R}^n \) is replaced by the whole domain \( \mathbb{R}^n \). A reason for it is that when \( \Omega = \mathbb{R}^n \), we have \( |\Omega|^{-1/\beta} = 0 \) for all \( 0 < \beta < \infty \).

The goal of this note is to provide an inequality corresponding to (1.1) when a bounded and Lebesgue measurable set \( \Omega \) in \( \mathbb{R}^n \) is replaced by the whole domain \( \mathbb{R}^n \). This inequality is obtained by using the Euclidean logarithmic Sobolev inequality and Hamilton-Jacobi equations. We use the inequalities obtained by [4, 5], and minimize this inequality with respect to some parameter, and finally get the desired inequality by letting another parameter tend to \( \infty \).

2 Preliminaries

In this section, we collect some results of [4, 5]. For \( p \geq 1 \), we denote by \( W^{1,p}(\mathbb{R}^n) \) the space of all weakly differentiable functions \( f \) on \( \mathbb{R}^n \) such that \( f \) and \( |Df| \) are in \( L^p(\mathbb{R}^n) \).

Throughout this note, the integral without its domain is understood as the one over \( \mathbb{R}^n \).

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*Department of Mathematics, University of Toyama, Toyama 930-8555, Japan. Supported in part by Grant-in-Aid for Scientific Research, No. 21540168, 18204009, JSPS.
Lemma 2.1 Let $p \geq 1$. Then, we have the following Euclidean logarithmic Sobolev inequality:

$$\int |f|^p \log |f|^p \, dx \leq \frac{n}{p} \log \left( L_p \int |Df|^p \, dx \right)$$

for $f \in W^{1,p}(\mathbb{R}^n)$ with $\int |f|^p \, dx = 1$.

Here,

$$L_p = \frac{p}{n} \left( \frac{p-1}{e} \right)^{p-1} \pi^{-p/2} \left( \frac{\Gamma(\frac{p}{2} + 1)}{\Gamma(n \frac{p-1}{2} + 1)} \right)^{p/n},$$

and this is the best possible constant satisfying (2.1).

We denote by $\| \cdot \|_\alpha$ the $L^\alpha(\mathbb{R}^n)$-norm with respect to the Lebesgue measure in $\mathbb{R}^n$.

Lemma 2.2 Let $p > 1$. For $f \in \text{Lip}(\mathbb{R}^n)$, let $u \in \text{Lip}(\mathbb{R}^n \times [0, \infty))$ be a viscosity subsolution of the Hamilton-Jacobi equation

$$u_t(x, t) + \frac{1}{p} |Du(x, t)|^p = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, \infty), \quad u = f \quad \text{on} \quad \mathbb{R}^n \times \{0\}.$$

If there is a constant $\alpha > 0$ such that $e^f \in L^\alpha(\mathbb{R}^n)$, then $e^{u(\cdot, t)} \in L^\beta(\mathbb{R}^n)$ for any $\beta \in (\alpha, \infty)$ and $t \in (0, \infty)$. Furthermore, we have

$$\|e^{u(\cdot, t)}\|_\beta \leq \|e^f\|_\alpha \left( \frac{n L_p e^{p-1} (\beta - \alpha)}{p^\delta t} \right)^{\frac{n-\alpha}{p} \cdot \frac{n-\beta}{p}} \frac{\alpha^{\frac{\beta-\alpha}{p}}}{\beta^{\frac{n-\alpha}{p} \cdot \frac{\beta-\alpha}{p}}}, \quad t > 0,$$

where $q > 1$ is the exponent conjugate of $p$, i.e., $(1/p) + (1/q) = 1$.

3 A result

Let $\theta > 0$. For $\alpha > 0$, we set

$$\mathcal{L}_{\alpha, \theta} = \left\{ f \in \text{Lip}(\mathbb{R}^n) : \text{Lip}(f) \leq \theta, \ e^f \in L^\alpha(\mathbb{R}^n) \right\},$$

where Lip$(f)$ is the Lipschitz constant of $f$, i.e., Lip$(f) = \sup_{x \neq y} |f(x) - f(y)|/|x - y|$. Let us denote by $\omega_{n-1}$ the surface area of the unit ball in $\mathbb{R}^n$. We set

$$k_n = \left( \frac{1}{\omega_{n-1} (n-1)!} \right)^{1/n}.$$

Now, we state our result of this note and give a sketch of its proof.

Theorem 3.1 Let $\alpha, \theta > 0$. For $f \in \mathcal{L}_{\alpha, \theta}$, we have the following inequality:

$$\|e^f\|_\infty \leq \|e^f\|_\beta (k_n \theta \beta)^{n/\beta} \leq \|e^f\|_\alpha (k_n \theta \alpha)^{n/\alpha}, \quad \alpha \leq \beta \leq \infty.$$

Inequality (3.3) is optimal in the sense that equality holds when $f(x) = C - \theta |x|$ for some constant $C \in \mathbb{R}$.
Remark. Note that $\lim_{\beta \to \infty} (k_n \theta)^{n/\beta} = 1$. Hence, the family 
\[ \{ \| e^f \| _\beta \ (k_n \theta)^{n/\beta} \}_{\alpha < \beta < \infty} \]
interpolates continuously and monotonically between 
\[ \| e^f \| _\alpha \ (k_n \theta)^{n/\alpha} \] \text{ and } \| e^f \| _\infty.

Sketch of Proof. Let $f \in L_{\alpha, \theta}$. Then, the function $v(x, t) = f(x) \ - \ (\theta^p t / p)$ is a subsolution of (2.3), so that $v \leq u$ on $\mathbb{R}^n \times [0, \infty)$ by [7]. By Lemma 2.2, we have, for any $\beta \in (\alpha, \infty)$ and $t \in (0, \infty)$,
\[
\| e^f \| _\beta \leq \| e^f \| _\alpha \ e^{\theta^p t / p \ t - \frac{n \theta^p}{p \ \theta^p} \ \theta^p} \ e^{\frac{nL_p e^{p-1}(\beta - \alpha)}{p^p}} \ \alpha^{\frac{n}{\alpha \beta} (\frac{\alpha}{p} + \frac{\beta}{q})} \ \beta^{\frac{n}{\alpha \beta} (\frac{\beta}{p} + \frac{\alpha}{q})}, \quad t > 0,
\]
where $q > 1$ is the exponent conjugate of $p$, i.e., $(1/p) + (1/q) = 1$. By minimizing the right-hand side of (3.4) with respect to the $t$-variable, we have
\[
\| e^f \| _\beta \leq \| e^f \| _\alpha \ \left( \frac{eL_p^{1/p}}{p} \right)^{\frac{n}{\alpha \beta} (\frac{\alpha}{p} + \frac{\beta}{q})} \ \alpha^{\frac{n}{\alpha \beta} (\frac{\alpha}{p} + \frac{\beta}{q})} \ \beta^{\frac{n}{\alpha \beta} (\frac{\beta}{p} + \frac{\alpha}{q})}.
\]
Hence, we obtain
\[
\| e^f \| _\beta \ (k_p^{(n)} \theta^\beta)^{n/\beta} \leq \| e^f \| _\alpha \ (k_p^{(n)} \theta^\alpha)^{n/\alpha},
\]
where
\[
k_p^{(n)} = \frac{eL_p^{1/p}}{p} = \left( \frac{n}{eq} \right)^{1/q} \left[ \Gamma \left( \frac{n}{q} + 1 \right) \right]^{-1/n} \frac{e}{n\sqrt{\pi}} \left[ \Gamma \left( \frac{n}{2} + 1 \right) \right]^{1/n}.
\]
Now, letting $p$ tend to $\infty$ in (3.7), i.e., letting $q$ tend to $1$ in (3.7), we conclude that
\[
\lim_{p \to \infty} k_p^{(n)} = \lim_{q \to 1} \left( \frac{n}{eq} \right)^{1/q} \left[ \Gamma \left( \frac{n}{q} + 1 \right) \right]^{-1/n} \frac{e}{n\sqrt{\pi}} \left[ \Gamma \left( \frac{n}{2} + 1 \right) \right]^{1/n} = \frac{1}{\sqrt{\pi}} \left( \frac{n}{\omega_{n-1}(n-1)!} \right)^{1/n} = k_n.
\]
The proof is completed. \[ \square \]

References


