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Spectral properties of Schrödinger operators with singular magnetic fields supported by a circle in $\mathbb{R}^3$

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1 Introduction

In 1959, Aharonov-Bohm [AB] asserted that an electrically shielded solenoid can affect the phase of an electron moving outside the solenoid; this phenomenon is called the Aharonov-Bohm effect. Since then, numerous experimental attempts to demonstrate the Aharonov-Bohm effect were performed. However, as far as they used a solenoid of finite length, they could not avoid the criticism that their experimental result is caused by the leaking magnetic field from the ends of the solenoid. To avoid this criticism, Tonomura et al. [To] made a decisive experiment using a toroidal magnetic field enclosed by a superconductive material in 1986. Historical reviews in these subjects are found in e.g. Peshkin-Tonomura [PT] and Afanasiev [A].

After the experiment of Tonomura et al., several authors studied the Schrödinger operators with toroidal magnetic field. Afanasiev [A] gives a numerical calculation for the scattering amplitude by the toroidal solenoids. Ballesteros-Weder [BW] consider magnetic fields supported on handle bodies $K$ (the boundary sum of several tori), and study the inverse scattering problem by means of the high-velocity limit for the Schrödinger operators defined on the exterior region $\mathbb{R}^3 \setminus K$ with Dirichlet boundary conditions.

We consider the Schrödinger operators $H_\epsilon$ in $\mathbb{R}^3$ with magnetic fields supported in a torus of thickness $\epsilon$, and consider the singular limit $\epsilon \to 0$. The result of this type is obtained particularly in the two-dimensional case; see e.g. Albeverio et al. [AGHH] for the scalar potential case, and Tamura [Ta] for the magnetic case.

We have shown in [IMS] that, if we choose the magnetic field and the vector potential appropriately, then $H_\epsilon$ converges in the norm resolvent sense to some operator $H_0$, which is the Schrödinger operator with a singular magnetic field supported on a circle.

We like to present some improvements of our result of [IMS]: First we show the choice of the three dimensional magnetic fields $B_\epsilon$ is arbitrary in the sense that, if $B_\epsilon$ are of the form given in the condition (A2) below, the two dimensional magnetic field $b$ is arbitrary
as far as it satisfies the assumption (A1) below. Second we show "the norm resolvent convergence" of the Schrödinger operators defined on the exterior region $\mathbb{R}^3 \setminus \mathcal{T}_\epsilon$ with Dirichlet boundary conditions as $\epsilon$ tends to zero, where $\mathcal{T}_\epsilon$ is the torus of thickness $\epsilon$ around a fixed circle and where we should interpret the meaning of the norm resolvent convergence appropriately since the operators considered are defined on different domains (see Theorem 1.2 below). The proof is very similar to that of [IMS] with some refinement of the argument. In the last section we give some result concerning spectral and scattering properties of the operator $H_0$ with singular magnetic fields: see Theorems 9.1 and 9.3 below.

Now let us explain the rigorous mathematical setting. We consider magnetic Schrödinger operators on $\mathbb{R}^3$

$$\mathcal{L}_\epsilon = (D - A_\epsilon)^2 = \sum_{j=1}^{3} (D_j - A_{\epsilon,j})^2,$$

where $0 \leq \epsilon < \epsilon_0$ ($\epsilon_0$ is some positive constant), $D_j = \frac{1}{i} \partial_j$, $\partial_j = \frac{\partial}{\partial x_j}$, $D = t(D_1, D_2, D_3)$ and $A_\epsilon = t(A_{\epsilon,1}, A_{\epsilon,2}, A_{\epsilon,3})$. The magnetic field $B = t(B_1, B_2, B_3)$ corresponding to a vector potential $A = t(A_1, A_2, A_3)$ is given by

$$B = \nabla \times A = \begin{pmatrix} \partial_2 A_3 - \partial_3 A_2 \\ \partial_3 A_1 - \partial_1 A_3 \\ \partial_1 A_2 - \partial_2 A_1 \end{pmatrix}.$$

We denote

$$B_\epsilon = \nabla \times A_\epsilon. \quad (1.1)$$

We shall define our magnetic fields as follows. Let $a > \epsilon_0$ be a constant. We introduce a local coordinate

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} (a + \tau \cos \phi) \cos \theta \\ (a + \tau \cos \phi) \sin \theta \\ \tau \sin \phi \end{pmatrix} \quad (1.2)$$

where $0 \leq \tau < a$, $\phi \in \mathbb{R}/2\pi \mathbb{Z}$, $\theta \in \mathbb{R}/2\pi \mathbb{Z}$. ¹ We denote the torus and the circle

$$\{ \tau < \epsilon \} = \mathcal{T}_\epsilon, \quad \{ \tau = 0 \} = C \quad (1.3)$$

for $0 < \epsilon \leq a$. If we fix two of the coordinates $(\tau, \phi, \theta)$ and vary the rest one, we have a linear orbit or a circular one. We denote the unit tangent vector of these orbits as

$$e_\tau = \begin{pmatrix} \cos \phi \cos \theta \\ \cos \phi \sin \theta \\ \sin \phi \end{pmatrix}, \quad e_\phi = \begin{pmatrix} -\sin \phi \cos \theta \\ -\sin \phi \sin \theta \\ \cos \phi \end{pmatrix}, \quad e_\theta = \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix}.$$

We shall need the following conditions:

¹$\mathbb{R}/2\pi \mathbb{Z}$ is the quotient Lie group equipped with local coordinates $\mathbb{R}/2\pi \mathbb{Z} \ni \theta = r + 2\pi \mathbb{Z} \mapsto r \in (r_0, r_0 + 2\pi)$, $r_0 \in \mathbb{R}$. In these coordinates, the trigonometric functions and the derivatives are well-defined as $\sin \theta = \sin r$, $\cos \theta = \cos r$ and $\partial_r f = \partial_r f$. 
(A1) Let $E := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$ and $b \in C_0^\infty(E; \mathbb{R})$ satisfying
\[ \int_E b(x_1, x_2) dx_1 dx_2 = 2\pi \alpha, \quad \alpha \in \mathbb{R} \setminus \mathbb{Z}. \]  

(1.4)

(A2) For $0 < \epsilon < \epsilon_0$, we assume $B_\epsilon \in C^\infty(\mathbb{R}^3; \mathbb{R})^3$, supp $B_\epsilon$ is contained in the open torus $\mathcal{T}_\epsilon$ and in this torus
\[ B_\epsilon = -\frac{1}{\epsilon^2} b \left( \frac{\tau \cos \phi}{\epsilon}, \frac{\tau \sin \phi}{\epsilon} \right) e_\theta. \]  

(1.5)

(A3) For $\epsilon = 0$, we assume $B_\epsilon \in \mathcal{D}(\mathbb{R}^3; \mathbb{R})^3$ (the vector-valued distributions on $\mathbb{R}^3$) and
\[ B_0 = -2\pi \alpha \delta_C e_\theta, \]  

(1.6)

where $\delta_C$ is the delta measure on the circle $C$, that is,
\[ \langle B_0, \varphi \rangle = -2\pi \alpha \int_0^{2\pi} \varphi(a \cos \theta, a \sin \theta, 0) e_\theta a \, d\theta \]  

for any test function $\varphi \in C_0^\infty(\mathbb{R}^3)$, where $\langle \cdot, \cdot \rangle$ denotes the coupling of a distribution and a test function.

(A4) For $0 < \epsilon < \epsilon_0$, $A_\epsilon \in C_0^\infty(\mathbb{R}^3; \mathbb{R})^3$. For $\epsilon = 0$, $A_0 \in C^\infty(\mathbb{R}^3 \setminus C; \mathbb{R})^3 \cap L^1(\mathbb{R}^3; \mathbb{R})^3$, and supp $A_0$ is compact in $\mathbb{R}^3$. $A_\epsilon$ satisfies (1.1) for $0 \leq \epsilon < \epsilon_0$.

Remark 1.1 Let $\Pi = \{x_2 = 0, x_1 > 0\}$ (the half $x_3x_1$ plane). We have the flux $\Phi$ through the plane $\Pi$ of $B_\epsilon$ equals $2\pi \alpha$ independently of $\epsilon > 0$:
\[ \Phi = \int_{\Pi \cap \{	au \leq \epsilon\}} B_{\epsilon,2} \, dx_3 \wedge dx_1 = \oint_{\Pi \cap \{	au = \epsilon\}} (A_{\epsilon,1} \, dx_1 + A_{\epsilon,3} \, dx_3) = 2\pi \alpha \]  

by (A1), (A2) and by the Stokes theorem. The minus sign before $1/\epsilon^2$ in (1.5) and (1.6) is added since $(e_\tau, e_\phi, -e_\theta)$ makes a right-hand system.

For given magnetic fields $\{B_\epsilon\}_{0 \leq \epsilon < \epsilon_0}$ satisfying (A2) and (A3), we show in section 3 that there exist vector potentials $\{A_\epsilon\}_{0 \leq \epsilon < \epsilon_0}$ satisfying (A4).

Then we can define self-adjoint realizations of $\{\mathcal{L}_\epsilon\}_{0 \leq \epsilon < \epsilon_0}$ as follows. When $0 < \epsilon < \epsilon_0$, the vector potential $A_\epsilon$ has no singularity. Then it is well-known that $L_\epsilon = \mathcal{L}_\epsilon|_{C_0^\infty(\mathbb{R}^3)}$ is essentially self-adjoint (see e.g. [IK] or [LS]), so we define $H_\epsilon = \overline{L_\epsilon}$. When $\epsilon = 0$, our vector potential $A_0$ has strong singularities on the circle $C$ so that $A_0$ does not belong to $L^2(\mathbb{R}^3)^3$ (see (3.3); see also Proposition 5.1). Then $L_0 = \mathcal{L}_0|_{C_0^\infty(\mathbb{R}^3 \setminus C)}$ is positive, symmetric, but not essentially self-adjoint. As a self-adjoint realization, we choose the Friedrichs extension of $L_0$, and denote it by $H_0$.

\[ C_0^\infty(\Omega) \]  

\[ \text{denotes the space of } C^\infty \text{ functions on } \mathbb{R}^d \text{ with compact support contained in an open set } \Omega \subset \mathbb{R}^d \]

\[ \text{In fact, we can prove the deficiency indices of } L_0 \text{ are } (\infty, \infty) \]
We further define the operators $H_{e}^{D}$ for $0 < \epsilon \leq \epsilon_{0}$ in the exterior region $\Omega(\epsilon) := \mathbb{R}^{3} \setminus \mathcal{T}_{\epsilon}$ with Dirichlet boundary conditions on its boundary as the Friedrichs extension of the operator $L_{\delta}^{D} = L_{0}|_{C_{c}^{\infty}(\Omega(\epsilon))}$. Note that $A_{0}$ is smooth outside of $C$ so that $A_{0}$ is smooth in a neighborhood of $\Omega(\epsilon)$.

The main results of this paper are the following:

**Theorem 1.1** Suppose that $b$ satisfies (A1) and $\{B_{\epsilon}\}_{0 \leq \epsilon < \epsilon_{0}}$ are given by (A2) and (A3). Then there exist vector potentials $\{A_{\epsilon}\}_{0 \leq \epsilon < \epsilon_{0}}$ satisfying (A4) such that $H_{\epsilon}$ converges to $H_{0}$ in the norm resolvent sense, as $\epsilon$ tends to 0.

**Theorem 1.2** Suppose $B_{0}$ is given by (A3). Then there exists a vector potential $A_{0}$ satisfying (A4) such that $\chi^{*}(H_{\epsilon}^{D} + E)^{-1}\chi$ converges to $(H_{0} + E)^{-1}$ in the operator norm of $L^{2}(\mathbb{R}^{3})$ as $\epsilon$ tends to 0, where $\chi$ is the restriction operator $L^{2}(\mathbb{R}^{3}) \rightarrow L^{2}(\Omega)$ and $\chi^{*}$ is its adjoint which is the extension operator $L^{2}(\Omega) \rightarrow L^{2}(\mathbb{R}^{3})$.

These result are analogies of the result by Tamura [Ta]. There remains a natural question:

If we add some scalar potential $V_{\epsilon}$ to $H_{\epsilon}$, then the norm resolvent limit exists?

If it exists, what are the boundary conditions on $C$ of the limit operator?

Tamura's result suggests the conclusion is true if we choose suitable $V_{\epsilon}$ and the boundary conditions depend on the existence of the zero-energy resonance. We will discuss this problem elsewhere in the future.

## 2 Torus Coordinate

Let us give several formulas for the coordinate $(\tau, \phi, \theta)$ defined in (1.2). By direct computation, we have

$$
\frac{\partial x}{\partial \tau} = e_{\tau}, \quad \frac{\partial x}{\partial \phi} = \tau e_{\phi}, \quad \frac{\partial x}{\partial \theta} = (a + \tau \cos \phi)e_{\theta}.
$$

(2.1)

Since $(e_{\tau}, e_{\phi}, e_{\theta})$ is an orthogonal matrix, we have

$$
\det \left( \begin{array}{c}
\frac{\partial(x_{1}, x_{2}, x_{3})}{\partial(\tau, \phi, \theta)}
\end{array} \right) = \tau(a + \tau \cos \phi).
$$

Thus we have

$$
\int_{\mathcal{T}_{\epsilon}} u \, dx_{1} dx_{2} dx_{3} = \int_{\mathcal{T}_{\epsilon}} u \tau(a + \tau \cos \phi) \, d\tau d\phi d\theta
$$

(2.2)

for any function $u \in L^{1}(\mathcal{T}_{\epsilon})$. For the derivatives, we have by (2.1)

$$
\partial_{\tau} u = \nabla u \cdot e_{\tau}, \quad \partial_{\phi} u = \tau \nabla u \cdot e_{\phi}, \quad \partial_{\theta} u = (a + \tau \cos \phi) \nabla u \cdot e_{\theta},
$$

where $\partial_{\tau} = \frac{\partial}{\partial \tau}$, etc. Thus we have in $\mathcal{T}_{\epsilon}$

$$
\nabla u = (\nabla u \cdot e_{\tau}) e_{\tau} + (\nabla u \cdot e_{\phi}) e_{\phi} + (\nabla u \cdot e_{\theta}) e_{\theta}
$$

$$
= (\partial_{\tau} u) e_{\tau} + \left( \frac{1}{\tau} \partial_{\phi} u \right) e_{\phi} + \left( \frac{1}{a + \tau \cos \phi} \partial_{\theta} u \right) e_{\theta}.
$$

(2.3)
3 Vector potentials

There are many ways of constructing vector potentials giving the toroidal magnetic fields: see [A] and [BW]. Especially, Afanasiev gives compactly supported vector potentials by using the Riemann toroidal coordinate [A, section 2.2.6]. In this section, we shall give vector potentials giving the toroidal magnetic fields, by using the coordinate defined in section 1. The resulting vector potentials can be compactly supported for a suitable choice of the functions given in the following construction. And in fact we shall assume that $A_0$ is compactly supported throughout the paper as is stated in the condition (A4).

In section 2, the coordinate functions $\tau, \phi, \theta$ are defined only in the torus $T_{c_0}$. However, (1.2) is still valid in $R^3 \setminus (C \cap X_3)$ and $\tau, \phi, \theta$ are smooth there, where $X_3 = \{x_1 = x_2 = 0\}$ is the $x_3$-axis. Mollifying the functions $\tau$ and $\phi$ near $X_3$, we can construct new functions $\tau_1$ and $\phi_1$ satisfying the following conditions:

(i) $\tau_1 \in C^\infty(R^3 \setminus C; R_+), \phi_1 \in C^\infty(R^3 \setminus C; R/2\pi Z)$.

(ii) $\tau_1 = \tau$ and $\phi_1 = \phi$ in the torus $T_{c_0} = \{\tau < \epsilon_0\}$.

(iii) $\tau_1 > \epsilon_0$ on $R^3 \setminus \overline{T_{c_0}}$.

(iv) Let $\Lambda_\kappa = \{|x_3| < \kappa(a^2 - x_1^2 - x_2^2)\}$ for $\kappa > 0$. $|\phi_1 - \pi| > \eta_0$ on $R^3 \setminus \Lambda_\kappa$ for some $\kappa > 0$ and $\eta_0 > 0$, where we choose $0 \leq \phi_1 < 2\pi$ as the branch of $R/2\pi Z$.

In the sequel, we use only new functions $\tau_1$ and $\phi_1$, so we omit the subscript 1 and write $\tau = \tau_1$ and $\phi = \phi_1$.

Let $\psi \in C^\infty(R/2\pi Z; R)$ satisfying

$$\int_0^{2\pi} \psi(s)ds = 2\pi \alpha.$$  \hspace{1cm} (3.1)

Define the vector potential $A_0$ by

$$A_0 = \psi(\phi)\nabla \phi.$$  \hspace{1cm} (3.2)

Then, with the use of (2.3), we have

$$A_0 = \psi(\phi)\nabla \phi = \frac{1}{\tau} \psi(\phi)e_\phi \text{ in } T_{c_0}$$  \hspace{1cm} (3.3)

and, by (2.2), $A_0 \in C^\infty(R^3 \setminus C; R)^3 \cap L^1_{loc}(R^3; R)^3$. We assume also

(v) $\text{supp } \psi \subset (\pi - \eta_0, \pi + \eta_0)$ for some $\eta_0$ with $0 < \eta_0 << 1$.

Then, by the condition (iv) above, $\text{supp } A_0 = \text{supp } \psi(\phi)\nabla \phi \subset \Lambda_\kappa$ and hence is compact.

We have also

$$B_0 = \nabla \times A_0 = \psi'(\phi)\nabla \phi \times \nabla \phi + \psi(\phi)(\nabla \times \nabla)\phi = 0 \text{ on } R^3 \setminus C,$$
so supp $B_0 \subset C$. Let $\rho \in C^\infty(R; R)$ such that $0 \leq \rho(r) \leq 1$ for every $r$, $\rho(r) = 0$ for $r \leq 1$ and $\rho(r) = 1$ for $r \geq 2$. Put $\rho_n(\tau) = \rho(n\tau)$ for $n = 1, 2, \ldots$. For any $\varphi \in C^\infty_0(R^3)$, we have

\[
\langle B_0, \varphi \rangle = \int_{\mathbb{R}^3} A_0 \times \nabla \varphi \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^3} \rho_n A_0 \times \nabla \varphi \, dx
\]

\[
= \lim_{n \to \infty} \int_{\mathbb{R}^3} A_0 \times \nabla (\rho_n \varphi) \, dx - \lim_{n \to \infty} \int_{\mathbb{R}^3} \varphi A_0 \times \nabla \rho_n \, dx
\]

\[
= \lim_{n \to \infty} \langle B_0, \rho_n \varphi \rangle - \lim_{n \to \infty} \int_{1/n}^{2/n} d\tau \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta \varphi(a \cos \theta, a \sin \theta, 0) e_\theta \, ad\theta
\]

\[
= - (2\pi \alpha \delta_C e_\theta, \varphi),
\]

where we used (2.2), (2.3) and (3.3) in the third equality, supp $B_0 \subset C$, $e_\phi \times e_\tau = e_\theta$, and the Lebesgue dominated convergence theorem in the fourth, and (3.1) in the last. Thus we see $B_0 = \nabla \times A_0$ satisfies the condition (A3).

As for the vector potentials $A_\epsilon$ for $\epsilon > 0$, we only give a remark that we can construct them by modifying $A_0$ in $\mathcal{T}_\epsilon$ so that $A_\epsilon$ satisfies (A4).

Let us discuss the gauge invariance for the potential $A_0$.

**Proposition 3.1** Let $\alpha_1$, $\alpha_2 \in \mathbb{R}$, $\psi_1$, $\psi_2 \in C^\infty(R/2\pi Z; \mathbb{R})$ satisfying

\[
\int_0^{2\pi} \psi_j(s) \, ds = \alpha_j
\]

for $j = 1, 2$. Let $A_j = \psi_j(\phi) \nabla \phi$. Assume

\[
\alpha_1 - \alpha_2 \in \mathbb{Z}. \tag{3.4}
\]

Then, there exists $\Phi \in C^\infty(\mathbb{R}^3 \setminus C; \mathbb{C})$ such that $|\Phi(x)| = 1$ and

\[
(D - A_1)\Phi u = \Phi(D - A_2) u \tag{3.5}
\]

for $u \in C^\infty_0(\mathbb{R}^3 \setminus C)$.

**Proof.** Put

\[
\Phi(x) = \exp \left( i \int_{0}^{\phi(x)} (\psi_1(s) - \psi_2(s)) \, ds \right).
\]

The right hand side is independent of the choice of the representative of $\phi(x) \in \mathbb{R}/2\pi \mathbb{Z}$ by the assumption (3.4), and is smooth in $\mathbb{R}^3 \setminus C$. The equation (3.5) can be checked by direct computation. \(\square\)
By this proposition, there is some arbitrariness in the choice of the function \( \psi \) satisfying (3.1). The simplest choice is the constant function \( \psi(\phi) = \alpha \), then
\[
A_0 = \alpha \nabla \phi.
\]
However, we have chosen \( \psi \) so that \( \text{supp} \psi \subset [\pi - \eta_0, \pi + \eta_0] \) for some small positive \( \eta_0 \), to obtain a compactly supported vector potential \( A_0 = \psi(\phi) \nabla \phi \).

Especially in the torus \( \mathcal{T}_{\epsilon_0} \), we have \( \nabla \phi = (1/\tau)e_\phi \) by (2.3). So
\[
A_\epsilon = A_0 = \frac{1}{\tau} \psi(\phi) e_\phi
\]
for \( \epsilon < \tau < \epsilon_0 \). Then, for \( 0 < \epsilon < \epsilon_0 \) and \( u \in C^\infty_0(\mathbb{R}^3) \), we have by (2.2) and (2.3)
\[
\int_{\epsilon<\tau<\epsilon_0} |(D-A_\epsilon)u|^2 dx = \int_{\epsilon<\tau<\epsilon_0} (|D_\tau u|^2 + |\tau^{-1}(D_\phi - \psi(\phi))u|^2
+ |(a + \tau \cos \phi)^{-1}D_\theta u|^2)\tau(a + \tau \cos \phi) d\tau d\phi d\theta,
\]
where \( D_\tau = \frac{1}{i} \frac{\partial}{\partial \tau} \), etc. When \( \epsilon = 0 \), the equality (3.7) holds for \( u \in C^\infty_0(\mathbb{R}^3 \setminus \{0\}) \). By an integration by parts, we have the explicit form of the operator \( \mathcal{L}_0 \) in the torus \( \mathcal{T}_{\epsilon_0} \) in terms of the coordinate \((\tau, \phi, \theta)\):
\[
\mathcal{L}_0 = \frac{1}{\tau(a + \tau \cos \phi)}(D_\tau \tau(a + \tau \cos \phi)D_\tau
+ (D_\phi - \psi(\phi))\tau^{-1}(a + \tau \cos \phi)(D_\phi - \psi(\phi))
+ (a + \tau \cos \phi)^{-2}D_\theta^2).
\]

4 Hardy type inequality

The Hardy type inequality is first proved by Laptev and Weidl [LW], for the two-dimensional Aharonov-Bohm type magnetic field. An analogy of their result holds for our operators, as stated below.

**Proposition 4.1** Let \( \alpha \in \mathbb{R} \). Put
\[
C_\alpha = (a + \epsilon_0)^{-1}(a - \epsilon_0) \min_{m \in \mathbb{Z}} |m - \alpha|^2.
\]
Then, we have
\[
\int_{\epsilon<\tau<\epsilon_0} |(D-A_\epsilon)u|^2 dx \geq C_\alpha \int_{\epsilon<\tau<\epsilon_0} \frac{1}{\tau^2} |u|^2 dx
\]
for any \( 0 < \epsilon < \epsilon_0 \) and any \( u \in C^\infty_0(\mathbb{R}^3) \). When \( \epsilon = 0 \), (4.1) holds for any \( u \in C^\infty_0(\mathbb{R}^3 \setminus C) \).

**Remark.** The constant \( C_\alpha \) is positive if and only if \( \alpha \in \mathbb{R} \setminus \mathbb{Z} \).
Proof. By Proposition 3.1, we may assume $\psi$ is the constant function $\psi = \alpha$. For $0 < \tau < \epsilon_0$, we have

$$0 < a - \epsilon_0 < a + \tau \cos \phi < a + \epsilon_0.$$ 

Thus we have by (2.2) and (3.7)

$$\int_{\epsilon < \tau < \epsilon_0} \frac{1}{\tau^2} |u|^2 d \tau \leq (a + \epsilon_0) \int_{\epsilon < \tau < \epsilon_0} \frac{1}{\tau} |u|^2 d \tau d \phi d \theta, \quad (4.2)$$

$$\int_{\epsilon < \tau < \epsilon_0} |(D - A_\epsilon)u|^2 d \tau \geq (a - \epsilon_0) \int_{\epsilon < \tau < \epsilon_0} |(D_\phi - \alpha)u|^2 \frac{1}{\tau} d \tau d \phi d \theta. \quad (4.3)$$

Using the Fourier expansion $u = \sum_{m \in \mathbb{Z}} u_m(\tau, \theta)e^{im\phi}$, we have

$$\int_0^{2\pi} |(D_\phi - \alpha)u|^2 d\phi = 2\pi \sum_{m \in \mathbb{Z}} |(m - \alpha)^2 u_m(\tau, \theta)|^2 \geq 2\pi \min_{m \in \mathbb{Z}} |m - \alpha|^2 \sum_{m \in \mathbb{Z}} |u_m(\tau, \theta)|^2 \geq \min_{m \in \mathbb{Z}} |m - \alpha|^2 \int_0^{2\pi} |u|^2 d\phi. \quad (4.4)$$

Integrating (4.4) with respect to the measure $\frac{1}{\tau} d\tau d\theta$ on $(\tau, \theta) \in (\epsilon, \epsilon_0) \times (0, 2\pi)$ and combining it with (4.2) and (4.3), we have (4.1). \quad \square

5 Diamagnetic inequality

In this section we state known facts about the diamagnetic inequality (see [LS], [DIM]) which holds for very wide class of potentials and for general open set:

Proposition 5.1 Suppose $A \in (L_{10c}^2(\mathbb{R}^d))^d$, $V \in L_{10c}^1(\mathbb{R}^d)$, $V \geq 0$, $\Omega$ is an open set in $\mathbb{R}^d$. Define sesqui-linear form $h_\Omega = h_{A,V,\Omega}$ and $h_\Omega^D$ as $h_\Omega(u,v) = ((D - A)u,(D - A)v) + (Vu,v)$ with form domain $Q(h_\Omega) = \{u \in L^2(\Omega)|(D - A)u \in (L^2(\Omega))^3, V^{1/2}u \in L^2(\Omega)\}$, $h_\Omega^D = \text{the form closure of } h_\Omega|_{C_{0}^\infty(\Omega)}$. Denote $H_\Omega^D = H_{A,V,\Omega}^D$ the selfadjoint operator associated with $h_\Omega^D$. Then we have the following:

(1) For $\Omega = \mathbb{R}^d$, $C_{0}^\infty(\mathbb{R}^d)$ is a form core for $h_{R^d}$, i.e. $h_{R^d} = h_{R^d}^D$.

(2) Let $E > 0$ and $f \in L^2(\Omega)$. Then

$$|(H_\Omega^D + E)^{-1}f(x)| \leq \chi_\Omega(K_0 + E)^{-1}\chi_\Omega^*|f|(x) \quad \text{a.e. } x \in \Omega$$

where $\chi_\Omega$ is the restriction operator $L^2(\mathbb{R}^d) \to L^2(\Omega)$ and $K_0 = -\Delta$ with domain $H^2(\mathbb{R}^d)$. 

6 Cauchy sequence

The following lemma says the resolvent of our operators forms a Cauchy sequence in the operator norm.

**Lemma 6.1** Let \( \{A_\varepsilon\}_{0<\varepsilon<\varepsilon_0} \) be the vector potentials defined in section 3 and \( \{H_\varepsilon\}_{0<\varepsilon<\varepsilon_0} \) the corresponding self-adjoint operators defined in section 1. Then, we have

\[
\lim_{\varepsilon,\varepsilon'\to 0} \|(H_\varepsilon + 1)^{-1} - (H_{\varepsilon'} + 1)^{-1}\| = 0
\]

We omit the detail of the proof of Lemma 6.1 since it is very similar to that in [IMS]. We only give several propositions needed and would like only note that the use of the resolvent equation \((H_\varepsilon + 1)^{-1} - (H_{\varepsilon'} + 1)^{-1} = (H_\varepsilon + 1)^{-1}(H_{\varepsilon'} - H_{\varepsilon})(H_{\varepsilon'} + 1)^{-1}\) and the function \(L_\varepsilon(\tau)\) given by (6.3) below is key to our proof.

Let \( \chi \in C^\infty(\mathbb{R};\mathbb{R}) \) such that \(0 \leq \chi(t) \leq 1\) and

\[
\chi(t) = \begin{cases} 
1 & (t \geq 2), \\
0 & (t \leq 1).
\end{cases}
\]

Put \( \chi_\varepsilon(\tau) = \chi(\tau/\varepsilon) \) for \( \varepsilon > 0 \).

**Proposition 6.2** Assume \( \alpha \in \mathbb{R} \setminus \mathbb{Z} \). Then, there exists \( C_1 > 0 \) dependent only on \( \alpha \) and \( \varepsilon_0 \) (independent of \( \varepsilon \)), such that

\[
\left\| \frac{\chi_\varepsilon(\tau)}{\tau}(H_\varepsilon + 1)^{-\frac{1}{2}} \right\| \leq C_1
\]

for any \( \varepsilon \) with \( 0 < 2\varepsilon \leq \varepsilon_0 \).

This proposition is shown by using the Hardy type inequality.

**Proposition 6.3** Let \( M \in L^2(\mathbb{R}^3) \). Then, we have

\[
\|M(H_\varepsilon + 1)^{-1}\| \leq C_2\|M\|_{L^2(\mathbb{R}^3)}, \tag{6.1}
\]

where \( C_2 = (\int_{\mathbb{R}^3}(\xi^2 + 1)^{-2}d\xi)^{1/2}/(2\pi)^{3/2} \).

**Proof.** It is sufficient to show that

\[
\|M(H_\varepsilon + 1)^{-1}\|_{HS} \leq C_2\|M\|_{L^2(\mathbb{R}^3)}, \tag{6.2}
\]

where \( \|\cdot\|_{HS} \) denotes the Hilbert-Schmidt norm. By the diamagnetic inequality, we have

\[
|M(H_\varepsilon + 1)^{-1}f| \leq |M|(-\Delta + 1)^{-1}|f| \quad \text{a.e.}
\]

The operator \(|M|(-\Delta + 1)^{-1}\) has the integral kernel \(|M(x)|g(x - y)/(2\pi)^{3/2} \), where \( g \) is the inverse Fourier transform of the function \((\xi^2 + 1)^{-1}\). Thus (6.2) follows from the Plancherel theorem. \(\square\)
Take \( \eta \in C^\infty(\mathbb{R}_+) \) such that \( 0 \leq \eta \leq 1 \) and
\[
\eta(s) = \begin{cases} 
0 & (s \geq \epsilon_0), \\
1 & (s \leq \epsilon_0/2).
\end{cases}
\]

For \( 0 < 4\epsilon < \epsilon_0 \), put
\[
L_{\epsilon}(\tau) = \eta(\tau) \int_{\tau}^{\epsilon_0/2} \frac{\chi_{\epsilon}(s)}{s} ds.
\]
(6.3)

Then we have
\[
|L_{\epsilon}(\tau)| \leq \left| \log\frac{\epsilon_0}{2\tau} \right|
\]
(6.4)
for \( 0 < \tau \leq \epsilon_0 \), and
\[
L_{\epsilon}(\tau) \geq \begin{cases} 
\log(\epsilon_0/4\epsilon) & (0 < \tau < 2\epsilon), \\
\log(\epsilon_0/2\tau) & (2\epsilon \leq \tau < \epsilon_0/2).
\end{cases}
\]
(6.5)

**Proposition 6.4** There exists a constant \( C_3 > 0 \) independent of \( \epsilon \) and \( \gamma \) such that
\[
\|L_{\epsilon}^{2\gamma}(H_{\epsilon} + 1)^{-\gamma}\| \leq C_3
\]
(6.6)
for \( 0 < 4\epsilon < \epsilon_0 \) and \( 0 \leq \gamma \leq 1 \).

We can show this proposition by the interpolation theorem from Proposition 6.3.

### 7 Form domain of \( H_0 \)

We can specify explicitly the form domain of the operator \( H_0 \). Define a sesqui-linear form \( h_0 \) by
\[
h_0(u, v) = (\mathcal{L}_0 u, v) = ((D - A_0)u, (D - A_0)v),
\]
\[
Q(h_0) = C_0^\infty(\mathbb{R}^3 \setminus \mathcal{C}).
\]

Let \( \overline{h_0} \) be the closure of the form \( h_0 \). The operator \( H_0 \) is the self-adjoint operator associated with the form \( \overline{h_0} \).

**Proposition 7.1** Suppose \( \alpha \in \mathbb{R} \setminus \mathbb{Z} \). Then, we have
\[
Q(\overline{h_0}) = \left\{ u \in L^2(\mathbb{R}^3) \left| (D - A_0)u \in L^2(\mathbb{R}^3)^3, \frac{1}{\tau} u \in L^2(\mathbb{R}^3) \right\}
\]
where the distribution \( Du = -i\nabla u \) is defined as an element of \( \mathcal{D}'(\mathbb{R}^3 \setminus \mathcal{C})^3 \).

For the proof of Proposition 7.1, we shall use a lemma.

**Lemma 7.2** Assume \( u \in L^2(\mathbb{R}^3), (D - A_0)u \in L^2(\mathbb{R}^3)^3 \) and \( \text{supp} u \cap \mathcal{C} = \emptyset \). Then \( u \in Q(\overline{h_0}) \).

Proposition 7.1 and Lemma 7.2 can be shown by using usual cutoff argument and making the approximating sequence of functions (see [IMS]).
8 Sketch of the proof of the main theorems

By Lemma 6.1, there exists a bounded, self-adjoint operator $R$ on $L^2(\mathbb{R}^3)$ such that

$$R = \lim_{\epsilon \to 0} (H_\epsilon + 1)^{-1}.$$  

Thus the proof of Theorem 1.1 is completed if we prove

$$R = (H_0 + 1)^{-1}.$$  

(8.1)

For the proof we use a series of lemmas which are shown with the use of the Hardy type inequality.

**Lemma 8.1** The operator $R$ is injective.

By Lemma 8.1, we can define a self-adjoint operator $T$ by

$$T = R^{-1} - 1, \quad D(T) = \text{Ran } R.$$  

Then $T$ is self-adjoint and $R = (T + 1)^{-1}$. Thus it suffices to prove $T = H_0$.

**Lemma 8.2** For $u \in D(T)$, we have

$$Tu = L_0 u = (D - A_0)^2 u,$$  

(8.2)

where $L_0 u$ is defined as an element of $\mathcal{D}'(\mathbb{R}^3 \setminus \mathcal{C})$.

**Lemma 8.3** We have $D(T) \supset C_0^\infty(\mathbb{R}^3 \setminus \mathcal{C})$.

**Lemma 8.4** Suppose $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. Then, we have $D(T) \subset Q(\overline{h_0})$.

Lemma 8.3 and Lemma 8.3 implies $T$ is a self-adjoint extension of $L_0|_{C_0^\infty(\mathbb{R}^3 \setminus \mathcal{C})}$. Since the Friedrichs extension $H_0$ is the unique self-adjoint extension of $L_0$ with the property $D(H_0) \subset Q(\overline{h_0})$, we have $H = T_0$ by Lemma 8.4. Thus Theorem 1.1 is proved. The proof of Theorem 1.2 is quite similar.

9 Spectral and Scattering theory

In this section, we study the spectral properties of $H_0$ and develop the scattering theory for the pair $(H_0, K_0)$, where $K_0$ denotes the free hamltomian $K_0 = -\Delta$ with $D(K_0) = H^2(\mathbb{R}^3)$.

**Theorem 9.1** The operator $H_0$ has no eigenvalue.
Proof. It suffices to show that $H_0$ has no nonnegative eigenvalue, since $H_0$ is nonnegative. First assume that there exist $u \in D(H_0)$ and $\lambda > 0$ such that $H_0u = \lambda u$. Then, since $A_0$ is smooth except the circle $C = \{\tau = 0\}$, it follows from Lemma 8.3 and the elliptic regularity that $u$ is smooth except $C$. Moreover, since $A_0$ has a compact support, $u$ satisfies the Helmholtz equation $(\Delta + \lambda)u = 0$ on $\{|x| > R\}$ (some large $R > 0$), which implies $u$ must vanish on that exterior region by [M, Lemma 8.4]. Thus, noting the unique continuation property for the elliptic equations (e.g. [H]), we have $u = 0$ in $L^2(\mathbb{R}^3)$. Next assume that there exists $u \in D(H_0)$ such that $H_0u = 0$. Then, we have

$$0 = (H_0u, u) = \overline{h_0}(u, u) = ||(D - A_0)u||^2.$$ 

Hence $Du(x) = 0$ on $\{|x| > R\}$ (some large $R > 0$), which implies $u$ must vanish on that exterior region, since $u \in L^2(\mathbb{R}^3)$. Thus, the unique continuation property again shows $u = 0$ in $L^2(\mathbb{R}^3)$. □

Let us proceed to the scattering problems for the pair $(H_0, K_0)$. The wave operators $W_\pm(H_0, K_0)$ are defined by

$$W_\pm(H_0, K_0) = s - \lim_{t \to \pm \infty} e^{itH_0}e^{-itK_0},$$

if they exist. We use the Enss method and know the following ([P, p.106, Theorem 8.1; p.108, Proposition 8.1]).

**Theorem 9.2** Let $H$ be a self-adjoint operator on $L^2(\mathbb{R}^d)$ such that

- (s1) $(H - z)^{-1} - (K_0 - z)^{-1}$ is compact.
- (s2) The function $h(R) = ||j_R(K_0 + i)^{-1} - (H + i)^{-1}(K_0 + i)j_R(K_0 + i)^{-1}||$ is integrable on $(0, \infty)$, where $j_R(x) = \varphi(\frac{|x|}{R})$ and $\varphi \in C^\infty(\mathbb{R}; \mathbb{R})$ is taken such that

$$0 \leq \varphi(s) \leq 1 (\forall s \in \mathbb{R}), \quad \varphi(s) = 0 (|s| \leq 1), \quad = 1 (|s| \geq 2).$$

Then:

(i) $\sigma_{ess}(H) = [0, \infty)$, where $\sigma_{ess}(H)$ denotes the essential spectrum of $H$.

(ii) $H$ has empty singular continuous spectrum.

(iii) The wave operators $W_\pm(H, K_0)$ exist and are complete.

We apply the above theorem to obtain the following.

**Theorem 9.3** We have:

(i) $\sigma(H_0) = \sigma_{abs}(H_0) = \sigma_{ess}(H_0) = [0, \infty)$, where $\sigma(H_0)$ and $\sigma_{abs}(H_0)$ denote the spectrum of $H_0$ and the absolutely continuous spectrum of $H_0$, respectively.
(ii) The wave operators $W_\pm(H_0, K_0)$ exist and are complete.

Proof. We first show (s1) for $(H_0, K_0)$ with $z = -1$. We write

$$(H_0 + 1)^{-1} - (K_0 + 1)^{-1} = \{(H_0 + 1)^{-1} - (H_\epsilon + 1)^{-1}\} + \{(H_\epsilon + 1)^{-1} - (K_0 + 1)^{-1}\} = I_\epsilon + J_\epsilon.$$ 

In view of Theorem 1.1, $I_\epsilon \to 0$ in the operator norm as $\epsilon \downarrow 0$. On the other hand, the resolvent equation reads as

$$J_\epsilon = (H_\epsilon + 1)^{-1}\{(D \cdot A_\epsilon + 2A_\epsilon \cdot D - |A_\epsilon|^2)(K_0 + 1)^{-1}\}$$

which implies $J_\epsilon$ is compact, since $V(x, \partial)(K_0 + 1)^{-1}$ is compact by (A4). So, (s1) holds, since the set of compact operators is closed in $B(L^2(R^3))$.

Next, we show that (s2) holds for $(H_0, K_0)$. It suffices to show that $B(L^2(R^3))$-valued function

$$B(R) = j_R(K_0 + i)^{-1} - (H_0 + i)^{-1}(K_0 + i)j_R(K_0 + i)^{-1}$$

vanishes for large $R$. Take $R_0 > 0$ such that $A_0(x) = 0$ for $|x| > R_0$ and put $u = (K_0 + i)^{-1}f, v = (H_0 - i)^{-1}g$ for $f, g \in L^2(R^3)$. Then we have

$$(B(R)f, g) = \{(j_R(K_0 + i)^{-1} - (H_0 + i)^{-1}(K_0 + i)j_R(K_0 + i)^{-1}\}f, g)$$

$$= (j_Ru, (H_0 - i)v) - ((K_0 + i)(j_Ru), v)$$

$$= (j_Ru, H_0v) - (K_0(j_Ru), v).$$

Now, let us show that for $R > R_0$

$$(j_Ru, H_0v) = (K_0(j_Ru), v).$$

In fact, since $u \in D(K_0) = H^2(R^3)$, we can find a sequence $\{u_n\} \subset H^2(R^3)$ such that $u_n \to u$ in $H^2(R^3)$ as $n \to \infty$. Then, noting that $j_Ru_n \in C_0^\infty(|x| > R_0 \} \subset C_0^\infty(R^3 \setminus C)$, we have by Lemma 7.2

$$(j_Ru, H_0v) = \lim_{n \to \infty} (j_Ru_n, H_0v)$$

$$= \lim_{n \to \infty} ((D - A_0)^2(j_Ru_n), v)$$

$$= \lim_{n \to \infty} (K_0(j_Ru_n), v)$$

$$= (K_0(j_Ru), v).$$

The above argument shows $B(R) = 0$ for $R > R_0$, and hence (s2) holds. Therefore the second part of the theorem has been proven. The first part follows from Theorems 9.1, 9.2 (i), (ii). $\square$


P. A. Perry; *Scattering theory by the Enss method*, Harwood academic publishers gmbh, 1983.
