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“The two-constants theory” and tensors of the original Navier-Stokes equations

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Abstract

The two-constants theory, which was at first, deduced by Laplace in 1806-07, is the one now accepted for isotropic, linear elasticity. The original Navier-Stokes equations [NS equations] or Navier equations were introduced in deducing the two-constants theory. From the view of NS equations, we would like to report the deduction of tensor or equations by Navier, Cauchy, Poisson, Saint-Venant and Stokes[1], and the concurrence between each other. Especially, we would like to take up a subject for discussion on Saint-Venant, however his idea on tensor is, we think, an epock-making for taking the concurrence among three pioneers of NS equations and contributing to Stokes' tensor and equations, which strengthens the frame of NS equations.

1 Preliminary

We use in this report the definition of stress tensor, which is due to I. Imai[7, p.178] as follows : we call the stress tensor such as the matrix P of 3×3 that

$$\begin{bmatrix} P_{nx} \\ P_{ny} \\ P_{nz} \end{bmatrix} = \begin{bmatrix} p_{xx} & p_{yx} & p_{zx} \\ p_{xy} & p_{yy} & p_{zy} \\ p_{xz} & p_{yz} & p_{zz} \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} \quad \Rightarrow \quad \mathbf{P}_n = \mathrm{P} \cdot \mathbf{n}$$

where, by multiplying P by $\mathbf{n}$, it turns into new vector $\mathbf{P}_n$. Moreover, if $p_{xy} = p_{yx}$ & $p_{yz} = p_{zy}$ & $p_{xz} = p_{zx}$, then this tensor is called symmetric, and if not, i.e. if $t_{ij} = -t_{ji}$ then antisymmetric or skew symmetric, where we suppose t_{ij} is the (i, j) element of a matrix.

Below in this paper we show a tensor, in brief, by using two following items :

- δ_{ij} : Kronecker δ,
- $v_{k,k} = \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \cdots$ Einstein's convention

By the way, although Navier does not express the tensor, we can put the tensor of Navier, which is the same method or way as we use today's vectorial expression for brevity. In case of Navier's elastic of (1-1) in Table 4 is as follows :

$$-\varepsilon \begin{bmatrix} \left(3\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) & \left(\frac{du}{dy} + \frac{dv}{dx}\right) & \left(\frac{dw}{dx} + \frac{du}{dz}\right) \\ \left(\frac{du}{dy} + \frac{dv}{dx}\right) & \left(\frac{du}{dx} + 3\frac{dv}{dy} + \frac{dw}{dz}\right) & \left(\frac{dv}{dz} + \frac{dw}{dy}\right) \\ \left(\frac{dw}{dx} + \frac{du}{dz}\right) & \left(\frac{dv}{dz} + \frac{dw}{dy}\right) & \left(\frac{du}{dx} + \frac{dv}{dy} + 3\frac{dw}{dz}\right) \end{bmatrix} = -\varepsilon \begin{bmatrix} \epsilon + 2\frac{du}{dx} & \frac{du}{dy} + \frac{dv}{dx} & \frac{dw}{dx} + \frac{du}{dz} \\ \frac{du}{dy} + \frac{dv}{dx} & \epsilon + 2\frac{dv}{dy} & \frac{dv}{dz} + \frac{dw}{dy} \\ \frac{dw}{dx} + \frac{du}{dz} & \frac{dv}{dz} + \frac{dw}{dy} & \epsilon + 2\frac{dw}{dz} \end{bmatrix},$$

where $\epsilon = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$, which is the same style as Poisson's elastic case (3-1) in Table 4.

Moreover, we can compute easy Navier's t_{ij} in elastic : $t_{ij} = -\varepsilon(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$. In the another case, Stokes' fluid (27) or (5) in Table 4 : $t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i})$, or $\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}$ are equivalent in expression.[2] Below, we mean the tensor as the stress tensor defined by I. Imai. [3]

2 Introduction

We study the original NS equations, in which the formulators[4], Navier, Cauchy, Poisson, Saint-Venant and Stokes, contain their purposes, their ideas to formulate new equations and so on, to deduce the “*indoles*”[5] of the fluid mechanics, as Gauss [5] says in his papers, so we would like to put our two main results as the “*indoles*” of fluid mechanics below.

“The two-constants theory” was introduced at first by Laplace with H and K in the case of capillary action in 1806 (cf. Table 4). After this, the various two-constants were proposed by the formulators through the process of the formulation of NS equations or the equilibrium or the capillary equations. It is called that this theory is the one now accepted for isotropic, linear elasticity.[6]

[1] Navier(1785-1836), Cauchy(1789-1857), Poisson(1781-1840), Saint-Venant(1797-1886), Stokes(1819-1903).

[2] c.f. Schlichting [19], in our footnote(21).

[3] Left-side numbers of equations are by authors in the original papers and right-side numbers are ours. N^e/N^f : elastic/fluid by Navier, etc. If there are the same numbers over the sections, we put it as the double number of “section no.-no. by author”.

[4] This ordering is by the order of the proposition or publication of the paper.

[5] “*indoles*” is the proper Latin, according to a Latin dictionary : Lewis, “*indoles*” means : an inborn quality, natural quality, nature (sic. in Eng.)

[6] Darrigol [4, p.121].

3 Main result 1 (Two-constants theory)

Now, we would like to propose the uniformal methods to describe the kinetic equations for isotropic, linear elasticity such as :

- The partial differential equations of the elastic solid or elastic fluid are expressed by using one or the pair of C_1 and C_2 such that :
  in the elastic solid : $\frac{\partial^2 \mathbf{u}}{\partial t^2} - (C_1T_1 + C_2T_2) = \mathbf{f}$,
  In the elastic fluid : $\frac{\partial \mathbf{u}}{\partial t} - (C_1T_1 + C_2T_2) + \cdots = \mathbf{f}$,
  where T_1, T_2, $\cdots$ are the tensors or terms consisting our equations. For example, in modern notation of the incompressible NS equations, the kinetic equation with the equation of continuity are conventionally described as follows :

$$\frac{\partial \mathbf{u}}{\partial t} - \mu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{div } \mathbf{u} = 0. \tag{1}$$

- Moreover, C_1 and C_2 are described as follows :

$$\begin{cases} C_1 \equiv \mathcal{L}r_1g_1S_1, \\ C_2 \equiv \mathcal{L}r_2g_2S_2, \end{cases} \quad \begin{cases} S_1 = \iint g_3 \to C_3, \\ S_2 = \iint g_4 \to C_4, \end{cases} \Rightarrow \begin{cases} C_1 = C_3Lr_1g_1 = \frac{2\pi}{15}\mathcal{L}r_1g_1, \\ C_2 = C_4Lr_2g_2 = \frac{2\pi}{3}Lr_2g_2. \end{cases}$$

- C_1 and C_2 are two coefficients, for example, ε and E by Navier, or R and G by Cauchy, k and K by Poisson, ε and $\frac{\varepsilon}{3}$ by Saint-Venant, or μ and $\frac{\mu}{3}$ by Stokes, which are expressed by the infinite operator $\mathcal{L}$ ($\sum_0^\infty$ or $\int_0^\infty$) by personal principles or methods, where r_1 and r_2 are the functions related to the radius of the active sphere of the molecules, rised to the power of n, for Poisson's and Navier's case, the relation of function in expressing by logarithm to the base of r exists such that : $\log_r \frac{r_1}{r_2} = 2$.
- g_1 and g_2 are the certain functions which are dependent on r and are described with attraction &/or repulsion.
- S_1 and S_2 are the two expressions which describe the surface of active unit-sphere at the center of a molecule by the double integral (or single sum in case of Poisson's fluid).
- g_3 and g_4 are certain compound trianglar-functions to compute the moment in the unit sphere.
- C_3 and C_4 are indirectly determined as the common coefficients from the invariant tensor. Except for Poisson's fluid case, C_3 of C_1 is $\frac{2\pi}{3}$, and C_4 of C_2 is $\frac{2\pi}{15}$, which are computed from the total moment of the active sphere of the molecules in computing only by integral, and which are independent on personal manner. In Poisson's case, after multiplying by $\frac{1}{4\pi}$, we get the same as above.
- The ratio of the two coefficients including Poisson's case is always same as : $\frac{C_3}{C_4} = \frac{1}{5}$.

4 Main result 2 (A genealogy and convergence of stress tensor)

We show in the figure 1, a genealogy about the tensor of the Navie-Stokes equations including t_{ij}, and Table 4, in which, we distinguish from the material of elastic or fluid. Especially Cauchy[1, 2] may be the inventor or the first user of tensor, and Poisson[16] says he gets the idea of symmetric tensor from Cauchy. Moreover, the idea of Saint-Venant transfer to Stokes. Here, we call the two routes, NCP and PSS pattern, which are in our figure, and by which, we may explain the genealogy of tensor on the NS equations. cf. Table 4.

(fig. 1) A genealogy of stress tensors in the prototypical Navier-Stokes equations

Navier[11, 12] : $t^e_{ij} = -\varepsilon(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$, $t^f_{ij} = (p - \varepsilon u_{k,k})\delta_{ij} - \varepsilon(u_{i,j} + u_{j,i})$
$\nearrow$ ‖ $\Updownarrow$ $\searrow$
(Euler) $\Longrightarrow$ $\cdots$ ‖ Poisson[14, 16]• $\Longrightarrow$ Saint-Venant[18]† $\Longrightarrow$ Stokes[20]‡
$\searrow$ $\Downarrow$ $\Uparrow$ ⊙ $\nearrow$
Cauchy[1, 2] : $t^{e,f}_{ij} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$
← molecular deduction→ ‖ ← non-molecular deduction→
← Navier-Cauchy-Poisson pattern→ ‖ ←Poisson-Saint-Venant-Stokes

• Poisson : $t^e_{ij} = -\frac{a^2}{3}(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$, $t^f_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$
† Saint-Venant : $t^f_{ij} = (\frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i})$, $\frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) = -p$
‡ Stokes : $t^f_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i})$,
⊙ Poisson says his reducing of tensor elements to 6 from 9 is due to Cauchy. (cf.§5.2).

Table 1: C_1, C_2, C_3, C_4 : the constant of definitions and computing of total moment of molecular actions by Poisson, Navier, Cauchy, Saint-Venant & Stokes

no	name/problem	elastic solid	elastic fluid	remark
1	Navier elastic:[11] fluid:[12]	(Navier[11] only :) $C_1 = \varepsilon \equiv \frac{2\pi}{15} \int_0^\infty d\rho \cdot \rho^4 f\rho$ $C_3 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^{2\pi} \cos\varphi d\varphi \ g_3 \Rightarrow \{\frac{16}{15}, \frac{4}{15}, \frac{2}{5}\}$ $\Rightarrow \frac{1}{2}\frac{\pi}{4}\frac{16}{15} = \frac{2\pi}{15}$	(Navier[12] only :) $C_1 = \varepsilon \equiv \frac{2\pi}{15} \int_0^\infty d\rho \cdot \rho^4 f(\rho)$ $C_2 = E \equiv \frac{2\pi}{3} \int_0^\infty d\rho \cdot \rho^2 F(\rho)$ $C_3 = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\psi d\psi \ g_3$ $\Rightarrow \{\frac{\pi}{10}, \frac{\pi}{30}\} \Rightarrow \frac{2\pi}{15}$ $C_4 = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\psi d\psi \ g_4 \Rightarrow \frac{2\pi}{3}$	$\alpha = \rho\cos\psi\cos\varphi$, $\beta = \rho\cos\psi\sin\varphi$, $\gamma = \rho\sin\psi$
2	Cauchy elastic and fluid[2]	(Cauchy[2] :) $C_1 = R = \frac{2\pi\Delta}{15} \int_0^\infty r^3 f(r) dr$ $= \pm\frac{2\pi\Delta}{15} \int_0^\infty [r^4 \mathrm{f}'(r) - r^3 \mathrm{f}(r)] dr$ $C_2 = G = \pm\frac{2\pi\Delta}{3} \int_0^\infty r^3 \mathrm{f}(r) dr$ $C_3 = \frac{1}{2}\int_0^{2\pi} \cos^2 q dq \int_0^\pi \cos^2\alpha\cos^2\beta dp$, $= \frac{1}{2}\int_0^{2\pi} \cos^2 q dq \int_0^\pi \cos^2 p \sin^2 p \sin p dp = \frac{2\pi}{15}$, $C_4 = \frac{1}{2}\int_0^{2\pi} \int_0^\pi \cos^2\alpha \sin p dq dp$ $= \pi \int_0^\pi \cos^2 p \sin \ p dp = \frac{2\pi}{3}$,	(Cauchy[2] :) samely as in elastic solid	$\cos\alpha = \cos p$, $\cos\beta = \sin p \cos q$, $\cos\gamma = \sin p \sin q$ $\Delta = \frac{\mathcal{M}}{\mathcal{V}}$: mass of molecules per volume.
3	Poisson elastic:[14, 16] fluid:[16]	(Poisson[14] only :) $C_1 = k \equiv \frac{2\pi}{15} \sum \frac{r^5}{\alpha^5} \frac{d.\frac{1}{r} fr}{dr}$ $C_2 = K \equiv \frac{2\pi}{3} \sum \frac{r^3}{\alpha^5} fr$ $C_3 = \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} \cos\beta\sin\beta d\beta \ g_3 \Rightarrow \{\frac{2\pi}{5}, \frac{2\pi}{15}\} \Rightarrow \frac{2\pi}{15}$ $C_4 = \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} \cos\beta\sin\beta d\beta \ g_4 \Rightarrow \frac{2\pi}{3}$ Remark: C_3 is choiced as the common factor of $\{\cdot,\cdot\}$	(Poisson[16] : both elastic and fluid) $C_1 = -k \equiv -\frac{1}{30\varepsilon^3} \sum r^3 \frac{d.\frac{1}{r} fr}{dr}$ $= -\frac{2\pi}{15} \sum \frac{r^3}{4\pi\varepsilon^3} \frac{d.\frac{1}{r} fr}{dr}$ $C_2 = -K \equiv -\frac{1}{6\varepsilon^3} \sum r f r$ $= -\frac{2\pi}{3} \sum \frac{r}{4\pi\varepsilon^3} fr$ $C_3 : \begin{cases} G = \frac{1}{10} \sum r^3 \frac{d.\frac{1}{r} fr}{dr}, \\ E = F = \frac{1}{30} \sum r^3 \frac{d.\frac{1}{r} fr}{dr} \end{cases}$ $\Rightarrow \{\frac{1}{10}, \frac{1}{30}\} \Rightarrow \frac{1}{30}$ $C_4 : (3\text{-}2)_{Pf} \quad N = \frac{1}{6\varepsilon^3} \sum r f r \Rightarrow \frac{1}{6}$	In Poisson[16], he treates as the same as both elastic and fluid. $x_1 = r\cos\beta\cos\gamma$, $y_1 = r\sin\beta\sin\gamma$, $\zeta = -r\cos\beta$
4	Saint-Venant[18]		$C_1 = \varepsilon, \quad C_2 = \frac{\varepsilon}{3}$	
5	Stokes[20]	$C_1 = A, \quad C_2 = B$	$C_1 = \mu, \quad C_2 = \frac{\mu}{3}$	

Table 2: The two constants in the kinetic equations

no	name	problem	C_1	C_2	C_3	C_4	$\mathcal{L}$	r_1	r_2	g_1	g_2	remark
1	Navier [11]	elastic solid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f\rho$		ρ : radius
2	Navier [12]	fluid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f(\rho)$		ρ : radius
				E		$\frac{2\pi}{3}$	$\int_0^\infty d\rho$		ρ^2		$F(\rho)$	
3	Cauchy [2]	system of particles in elastic and fluid	R		$\frac{2\pi}{15}\Delta$		$\int_0^\infty dr$	r^3		$f(r)$		$f(r) \equiv \pm[r\mathrm{f}'(r) - \mathrm{f}(r)]$
				G		$\frac{2\pi}{3}\Delta$	$\int_0^\infty dr$		r^3		$\pm\mathrm{f}(r)$	$f(r) \neq \mathrm{f}(r)$, $\Delta = \frac{\mathcal{M}}{\mathcal{V}}$: mass of molecules per volume.
4	Poisson [14]	elastic solid	k		$\frac{2\pi}{15}$		$\sum \frac{1}{\alpha^5}$	r^5		$\frac{d.\frac{1}{r} fr}{dr}$		
				K		$\frac{2\pi}{3}$	$\sum \frac{1}{\alpha^5}$		r^3		fr	
5	Poisson [16]	elastic solid and fluid	k		$\frac{1}{30}$		$\sum \frac{1}{\varepsilon^3}$	r^3		$\frac{d.\frac{1}{r} fr}{dr}$		$C_3 = \frac{1}{4\pi}\frac{2\pi}{15} = \frac{1}{30}$
				K		$\frac{1}{6}$	$\sum \frac{1}{\varepsilon^3}$		r		fr	$C_4 = \frac{1}{4\pi}\frac{2\pi}{3} = \frac{1}{6}$
6	Saint-Venant [18]	fluid	ε	$\frac{\varepsilon}{3}$								
7	Stokes [20]	fluid	μ	$\frac{\mu}{3}$								
8	Stokes [20]	elastic solid	A	B								$A = 5B$

Table 3: The two constants in equilibrium equation

no	name	problem	C_1	C_2	C_3	C_4	$\mathcal{L}$	r_1	r_2	g_1	g_2	remark
1	Laplace [9, p.700]	capillary action	H		2π		$\int_0^\infty dz$	z		$\Psi(z)$		z : distance
				K		2π	$\int_0^\infty dz$				$\Psi(z)$	cf.Gauss[5]
2	Poisson [17]	capillary action	H		$\frac{\pi}{4}\rho^2$		$\int_0^\infty dr$	r^4		φr		[17, p.14]
				K		$\frac{2\pi}{3}\rho^2$	$\int_0^\infty dr$		r^3		φr	[17, p.12]
3	Navier fluid [12]	equiliblium of fluid	p		$\frac{4\pi}{3}$		$\int_0^\infty d\rho$	ρ^3		$f(\rho)$		ρ : radius
4	Poisson [16]	equiliblium of elastic body	q		$\frac{1}{4}$		$\sum \frac{1}{\varepsilon^3}$	$\frac{1}{r}$		$r_i^2 z' R$		$C_3 = \frac{1}{4\pi}\pi = \frac{1}{4}$
				p		$\frac{1}{6}$	$\sum \frac{1}{\varepsilon^3}$		r		R	$C_4 = \frac{1}{4\pi}\frac{2\pi}{3} = \frac{1}{6}$

5 Deductions of two constants and tensor

Not only today's Darrigol [4, p.121] says : "it is called that the two-constants theory is the one now accepted for isotropic, linear elasticity," but also Poisson [17, p.4] said already in 1831 : "elles renferment les deux constantes spéciales donc j'ai parlé tout à l'heure. (we says already about two equations contain two special constants.) Moreover, we think, first of all, the proposer of two constants is Laplace [9] in Table 3.

5.1 Navier's two constants and tensor

The corresponding Navier-Stokes equations by Navier himself on the incompressible fluid (1) are as follow :

$$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = X + \varepsilon\left(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dxdz}\right) - \frac{du}{dt} - \frac{du}{dx}\cdot u - \frac{du}{dy}\cdot v - \frac{du}{dz}\cdot w\ ; \\ \frac{1}{\rho}\frac{dp}{dy} = Y + \varepsilon\left(\frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dydz}\right) - \frac{dv}{dt} - \frac{dv}{dx}\cdot u - \frac{dv}{dy}\cdot v - \frac{dv}{dz}\cdot w\ ; \\ \frac{1}{\rho}\frac{dp}{dz} = Z + \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\right) - \frac{dw}{dt} - \frac{dw}{dx}\cdot u - \frac{dw}{dy}\cdot v - \frac{dw}{dz}\cdot w\ ; \end{cases} \quad (2)$$

and the equation of continuity :

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \quad (3)$$

Navier supposes two constants as follows :

$$(3\text{-}10)_{Nf} \quad \varepsilon \equiv \frac{8\pi}{30}\int_0^\infty d\rho\rho^4 f(\rho) = \frac{4\pi}{15}\int_0^\infty d\rho\rho^4 f(\rho), \quad E \equiv \frac{4\pi}{6}\int_0^\infty d\rho\rho^2 F(\rho) = \frac{2\pi}{3}\int_0^\infty d\rho\rho^2 F(\rho). \quad (4)$$

Navier is always necessary (3), because he must get $\varepsilon\Delta$ from (2), by derivating (3) with $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$.

By the way, Navier always uses his well-worn devices through the four steps to solve the three equations such as the equilibrium equation for the fluid [12], the kinetic equation for the elastic [11], and the kinetic equation for the fluid [12] with the common methods as follows :

1. at first, to deduce one or two constants including uncomputable function,
2. then, to make the indeterminate equation,
3. then, to take Taylor expansion and partial integral, exchanging d and δ, and pairing with the same integral operator,
4. and at last, to solve the indeterminate equation from the two points of view of the inner point and the boundary condition.

Here we show only the case of fluid by Navier [12].

5.1.1 Indeterminate equation

The indeterminate equation, which Navier says it is called then, is as follows :

$$
\begin{aligned}
(3\text{-}24)_{Nf}\quad 0 \quad = & \iiint dxdydz \begin{cases} [P - \frac{dp}{dx} - \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right)]\delta u \\ [Q - \frac{dp}{dy} - \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right)]\delta v \\ [R - \frac{dp}{dz} - \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right)]\delta w \end{cases} \\
- & \varepsilon \iiint dxdydz \begin{cases} \left(3\frac{du}{dx}\frac{\delta du}{dx} + \frac{du}{dy}\frac{\delta du}{dy} + \frac{du}{dz}\frac{\delta du}{dz}\right) + \left(\frac{dv}{dy}\frac{\delta du}{dx} + \frac{dv}{dx}\frac{\delta du}{dy}\right) + \left(\frac{dw}{dz}\frac{\delta du}{dx} + \frac{dw}{dx}\frac{\delta du}{dz}\right) \\ \left(\frac{du}{dx}\frac{\delta dv}{dy} + \frac{du}{dy}\frac{\delta dv}{dx}\right) + \left(\frac{dv}{dx}\frac{\delta dv}{dx} + 3\frac{dv}{dy}\frac{\delta dv}{dy} + \frac{dv}{dz}\frac{\delta dv}{dz}\right) + \left(\frac{dw}{dy}\frac{\delta dv}{dz} + \frac{dw}{dz}\frac{\delta dv}{dy}\right) \\ \left(\frac{du}{dx}\frac{\delta dw}{dz} + \frac{du}{dz}\frac{\delta dw}{dx}\right) + \left(\frac{dv}{dy}\frac{\delta dw}{dz} + \frac{dv}{dz}\frac{\delta dw}{dy}\right) + \left(\frac{dw}{dx}\frac{\delta dw}{dx} + \frac{dw}{dy}\frac{\delta dw}{dy} + 3\frac{dw}{dz}\frac{\delta dw}{dz}\right) \end{cases} \\
+ & \mathbf{S}ds^2 E(u\delta u + v\delta v + w\delta w). \qquad (5)
\end{aligned}
$$

5.1.2 Taylor expansion and partial integral

Considering $\mathbf{S}ds^2 E(u\delta u + v\delta v + w\delta w) = 0$ of indeterminate equation (5) and neglecting the total of the rest terms of below the 2 orders of Taylor's expansion, we get as follows :

$$
(3\text{-}29)_{Nf}\quad 0 = \iiint dxdydz \begin{cases} [P - \frac{dp}{dx} - \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right) + \varepsilon\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right)]\delta u \\ [Q - \frac{dp}{dy} - \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right) + \varepsilon\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right)]\delta v \\ [R - \frac{dp}{dz} - \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right) + \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right)]\delta w \end{cases} \qquad (6)
$$

From (6) we get (2) i.e. the kinetic equation which is the first expression of (1).

5.1.3 Boundary condition

As the same way of above notice, Navier explains as follows : regarding the conditions which react at the points of the surface of the fluid, if we substitute

- $dydz \quad \rightarrow \quad ds^2 \cos l,\quad l$: the angles by which the tangent plane makes on the surface frame with the plane yz,
- $dxdz \quad \rightarrow \quad ds^2 \cos m,\quad m$: samely, the angles with the plane xz,
- $dxdy \quad \rightarrow \quad ds^2 \cos n,\quad n$: samely, the angles with the plane xy,
- $\iint dydz, \iint dxdz, \iint dxdy \quad \rightarrow \quad \mathbf{S}ds^2$,

then because the affected terms by the quantities $\delta u, \delta v$ and δw respectively reduce to zero, the following determinated equations should hold for any points of the surface of the fluid :

$$
(3\text{-}32)_{Nf}\quad \begin{cases} Eu + \varepsilon[\cos l 2\frac{du}{dx} + \cos m\left(\frac{du}{dy} + \frac{dv}{dx}\right) + \cos n\left(\frac{du}{dz} + \frac{dw}{dx}\right)] = 0, \\ Ev + \varepsilon[\cos l\left(\frac{du}{dy} + \frac{dv}{dx}\right) + \cos m 2\frac{dv}{dy} + \cos n\left(\frac{dv}{dz} + \frac{dw}{dy}\right)] = 0, \\ Ew + \varepsilon[\cos l\left(\frac{du}{dz} + \frac{dw}{dx}\right) + \cos m\left(\frac{dv}{dz} + \frac{dw}{dy}\right) + \cos n 2\frac{dw}{dz}] = 0, \end{cases} \qquad (7)
$$

here the value of the constant E must vary in accordance with the nature of solid with which the fluid contacts. (7) express the boundary condition. The first terms of the left-hand side of (7) are defined by (4) for the expression which we seek for the sum of the moments of the total actions which caused between the molecules of the boundary and the fluid, and the second terms are the normal derivatives. Here, (7) is put by :

$$
E\begin{bmatrix} u \\ v \\ w \end{bmatrix} + \varepsilon \begin{bmatrix} 2\frac{du}{dx} & \frac{du}{dy} + \frac{dv}{dx} & \frac{du}{dz} + \frac{dw}{dx} \\ \frac{du}{dy} + \frac{dv}{dx} & 2\frac{dv}{dy} & \frac{dv}{dz} + \frac{dw}{dy} \\ \frac{du}{dz} + \frac{dw}{dx} & \frac{dv}{dz} + \frac{dw}{dy} & 2\frac{dw}{dz} \end{bmatrix} \begin{bmatrix} \cos l \\ \cos m \\ \cos n \end{bmatrix} = 0 \qquad (8)
$$

If putting the basis of the tensor as $\begin{bmatrix} \cos l & \cos m & \cos n \end{bmatrix}^T$, then the tensor part of (8) is expressed as follows :

$$
t_{ij} = \varepsilon[\{2v_{i,j} - (v_{i,j} + v_{j,i})\}\delta_{ij} + (v_{i,j} + v_{j,i})] = \varepsilon\{0\delta_{ij} + (v_{i,j} + v_{j,i})\} = \varepsilon(v_{i,j} + v_{j,i}).
$$

5.2 Cauchy's two constants and tensor

(Invariant value of tensor)

Cauchy says about the elements of tensor i.e. the invariable values : $G, H, I, L, M, N, P, Q, R$:

If we suppose that the molecules $m, m', m'', \cdots$ are originally allocated by the same way in relation to the three planes made by the molecule m in parallel with the plane coordinates, then the values of these quantities come to remain invariable, even though a series of changes are made among the three angles : α, β, γ.

Cauchy considers symmetric tensor such that :

$$(41)_C \quad G = H = I, \quad L = M = N, \quad P = Q = R, \quad (45)_C \quad L = 3R.$$

Cauchy may be the inventor of the nomenclature[7] of "tensor", and Poisson backs up this fact such that his idea reducing from 9 to 6 elements is due to Cauchy, as follows :

> D'un autre côté, il faut, pour l'equilibre d'un parallélépipède rectangle d'une étendue insensible, que les neuf composantes des pressions appliquées à ses trois faces non-parallélles, se réduisent à six forces qui peuvent être inégales. Cette proposition est due à M.Cauchy, et se déduit de la considération des momens. [16, §38, p.83]

We assume the mass of molecules per volume such as : $(48)_C \quad \Delta = \frac{\mathcal{M}}{\mathcal{V}}$, where, $\mathcal{M}$: the sum of the mass of molecules comtained in the sphere, $\mathcal{V}$: the volume of the sphere.

$$(50)_C \quad \begin{cases} G = \pm\frac{\Delta}{2}\int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} r^3 \mathrm{f}(r)\cos^2\alpha\sin p drdqdp, \\ R = \frac{\Delta}{2}\int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} r^3 f(r)\cos^2\alpha\cos^2\beta\sin p drdqdp \end{cases} \tag{9}$$

We compute in general case such that :

$$(51)_C \quad \cos\alpha = \cos p, \quad \cos\beta = \sin p\cos q, \quad \cos\gamma = \sin p\sin q$$

then (9) turns out as follows : [8]

$$(52)_C \quad \begin{cases} G = \pm\frac{2\pi\Delta}{3}\int_0^{\infty} r^3 \mathrm{f}(r)dr, \\ R = \frac{2\pi\Delta}{15}\int_0^{\infty} r^3 f(r)dr = \pm\frac{2\pi\Delta}{15}\int_0^{\infty}\left[r^4\mathrm{f}'(r) - r^3\mathrm{f}(r)\right]dr \end{cases} \tag{10}$$

> D'ailleurs, si, pour des valeurs croissantes de la distance r, la fonction $\mathrm{f}(r)$ décroit plus rapidment que la fonction que $\frac{1}{r^4}$, si de plus le produit $r^4\mathrm{f}(r)$ s'évanouit pour $r = 0$, on trouvera, en supposant la fonction $\mathrm{f}'(r)$ continue, et en intégrant par parties,

$$(53)_C \quad \int_0^{\infty} r^4\mathrm{f}'(r)dr = -4\int_0^{\infty} r^3\mathrm{f}(r)dr$$

It turns out from $(52)_C$

$$(54)_C \quad R = -\frac{2\pi\Delta}{15}\left(5\int_0^{\infty} r^3\mathrm{f}(r)dr\right) = -G,$$

and after all, we get the followings from the formulae $(46)_C$:

$$(46)_C \quad \begin{cases} X = (R+G)\left(\frac{\partial^2\xi}{\partial a^2} + \frac{\partial^2\xi}{\partial b^2} + \frac{\partial^2\xi}{\partial c^2}\right) + 2R\frac{\partial v}{\partial a}, \\ Y = (R+G)\left(\frac{\partial^2\eta}{\partial a^2} + \frac{\partial^2\eta}{\partial b^2} + \frac{\partial^2\eta}{\partial c^2}\right) + 2R\frac{\partial v}{\partial b}, \\ Z = (R+G)\left(\frac{\partial^2\zeta}{\partial a^2} + \frac{\partial^2\zeta}{\partial b^2} + \frac{\partial^2\zeta}{\partial c^2}\right) + 2R\frac{\partial v}{\partial c} \end{cases}$$

$$(55)_C \quad X = 2R\frac{\partial v}{\partial a}, \quad Y = 2R\frac{\partial v}{\partial b}, \quad Z = 2R\frac{\partial v}{\partial c}$$

[7] George Prange, the editors of Hamilton's works says in a comment [6].

[8] We get as follows :

$$\begin{cases} \int_0^{2\pi}\int_0^{\pi}\cos^2\alpha\sin p dqdp = 2\pi\int_0^{\pi}\cos^2 p\sin\ pdp = 2\pi\left[-\frac{\cos^3 p}{3}\right]_0^{\pi} = \frac{4\pi}{3}, \\ \int_0^{2\pi}\int_0^{\pi}\cos^2\alpha\cos^2\beta\sin\ pdp = \int_0^{2\pi}\cos^2 qdq\int_0^{\pi}\cos^2 p(1-\cos^2 p)\sin pdp = \left[\frac{q}{2} + \frac{1}{4}\sin 2q\right]_0^{2\pi}\left[-\frac{\cos^5 p}{5}\right]_0^{\pi} = (\frac{2\pi}{2} - 0)(\frac{2}{3} - \frac{2}{5}) = \frac{4\pi}{15} \\ C_3 = \frac{1}{2}\frac{4\pi}{15} = \frac{2\pi}{15}, \quad C_4 = \frac{1}{2}\frac{4\pi}{3} = \frac{2\pi}{3}, \end{cases}$$

Lorsque les quantités, désinées dans les formules $(40)_C$ et $(48)_C$ par les lettres $G, H, I, L, M, N, P, Q, R$ et Δ, deviennent constantes, c'est-à-dire, indépendantes des coordonées a, b, c, ou, ce qui revient au même, de la place qu'occupe la molécule m, alors, en faisant, pour plus de commodité,

$$(56)_C \quad \begin{cases} A = \left[(L+G)\frac{\partial\xi}{\partial a} + (R-G)\frac{\partial\eta}{\partial b} + (Q-G)\frac{\partial\zeta}{\partial c}\right]\Delta, \\ B = \left[(R-H)\frac{\partial\xi}{\partial a} + (M+H)\frac{\partial\eta}{\partial b} + (P-H)\frac{\partial\zeta}{\partial c}\right]\Delta, \\ C = \left[(Q-I)\frac{\partial\xi}{\partial a} + (P-I)\frac{\partial\eta}{\partial b} + (N+I)\frac{\partial\zeta}{\partial c}\right]\Delta, \end{cases} \Rightarrow \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \Delta \begin{bmatrix} L+G & R-G & Q-G \\ R-H & M+H & P-H \\ Q-I & P-I & N+I \end{bmatrix} \begin{bmatrix} \frac{\partial\xi}{\partial a} \\ \frac{\partial\eta}{\partial b} \\ \frac{\partial\zeta}{\partial c} \end{bmatrix}$$

$$(57)_C \quad \begin{cases} D = \left[(P+I)\frac{\partial\eta}{\partial c} + (P+H)\frac{\partial\zeta}{\partial b}\right]\Delta, \\ E = \left[(Q+G)\frac{\partial\zeta}{\partial a} + (Q+I)\frac{\partial\xi}{\partial c}\right]\Delta, \\ F = \left[(R+H)\frac{\partial\xi}{\partial b} + (R+G)\frac{\partial\eta}{\partial a}\right]\Delta, \end{cases} \Rightarrow \begin{bmatrix} D \\ E \\ F \end{bmatrix} = \Delta \begin{bmatrix} 0 & P+I & P+H \\ Q+I & 0 & Q+G \\ R+H & R+G & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{\partial\xi}{\partial c} & \frac{\partial\xi}{\partial b} \\ \frac{\partial\eta}{\partial c} & 0 & \frac{\partial\eta}{\partial a} \\ \frac{\partial\zeta}{\partial b} & \frac{\partial\zeta}{\partial a} & 0 \end{bmatrix}$$

We can reduce $(40)_C$ as follows :

$$(58)_C \quad \begin{cases} X = \frac{1}{\Delta}\left(\frac{\partial A}{\partial a} + \frac{\partial F}{\partial b} + \frac{\partial E}{\partial c}\right), \\ Y = \frac{1}{\Delta}\left(\frac{\partial F}{\partial a} + \frac{\partial B}{\partial b} + \frac{\partial D}{\partial c}\right), \\ X = \frac{1}{\Delta}\left(\frac{\partial E}{\partial a} + \frac{\partial D}{\partial b} + \frac{\partial C}{\partial c}\right), \end{cases} \Rightarrow \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \Delta \begin{bmatrix} A & F & E \\ F & B & D \\ E & D & C \end{bmatrix} \begin{bmatrix} \frac{1}{\partial a} \\ \frac{1}{\partial b} \\ \frac{1}{\partial c} \end{bmatrix}$$

By $(41)_C$ and $(45)_C$, we get the followings :
$\frac{A}{\Delta} = 2(R+G)\frac{\partial\xi}{\partial a} + (R-G)v, \quad \frac{B}{\Delta} = 2(R+G)\frac{\partial\eta}{\partial b} + (R-G)v, \quad \frac{C}{\Delta} = 2(R+G)\frac{\partial\zeta}{\partial c} + (R-G)v,$
$\frac{D}{\Delta} = (R+G)\left(\frac{\partial\eta}{\partial b} + \frac{\partial\zeta}{\partial c}\right), \quad \frac{E}{\Delta} = (R+G)\left(\frac{\partial\zeta}{\partial a} + \frac{\partial\xi}{\partial c}\right), \quad \frac{F}{\Delta} = (R+G)\left(\frac{\partial\xi}{\partial b} + \frac{\partial\eta}{\partial a}\right)$
For convenience's sake, in the paticular case, for $(41)_C$ and $(45)_C$ to hold, it is sufficient to be as follows :

$$(59)_C \quad (R+G)\Delta \equiv \frac{1}{2}k, \quad (R-G)\Delta \equiv K$$

For the equations $(56)_C$ and $(57)_C$,

$$(60)_C \quad \Rightarrow \quad \begin{bmatrix} A & F & E \\ F & B & D \\ E & D & C \end{bmatrix} = \begin{bmatrix} k\frac{\partial\xi}{\partial a} + Kv & \frac{1}{2}k\left(\frac{\partial\xi}{\partial b} + \frac{\partial\eta}{\partial a}\right) & \frac{1}{2}k\left(\frac{\partial\zeta}{\partial a} + \frac{\partial\xi}{\partial c}\right) \\ \frac{1}{2}k\left(\frac{\partial\xi}{\partial b} + \frac{\partial\eta}{\partial a}\right) & k\frac{\partial\eta}{\partial b} + Kv & \frac{1}{2}k\left(\frac{\partial\eta}{\partial c} + \frac{\partial\zeta}{\partial b}\right) \\ \frac{1}{2}k\left(\frac{\partial\zeta}{\partial a} + \frac{\partial\xi}{\partial c}\right) & \frac{1}{2}k\left(\frac{\partial\eta}{\partial c} + \frac{\partial\zeta}{\partial b}\right) & k\frac{\partial\zeta}{\partial c} + Kv \end{bmatrix} \quad (11)$$

Here, we must remark that the layout of symmetric tensor of $(58)_C$ or $(60)_C$ is the Cauchy's invented nomenclature. If, moreover, the condition $(54)_C$: $R = -G$ holds, then $k = 0$ holds , and the followings hold :
$(61)_C \quad A = B = C = Kv, \quad D = E = F = 0.$

5.2.1 Equilibrium and kinetic equation of elastic fluid by Cauchy

We show the equation number of fluid by Cauchy in below, with $(\cdot)_{C^*}$ instead by $(\cdot)_C$ for discrimination with the equations of elastic in above.
(Assumption of elastic fluid.)
As the equations in equiliblium and as the equations in motion :

$$(67)_{C^*} \quad \begin{cases} (L+G)\frac{\partial^2\xi}{\partial x^2} + (R+H)\frac{\partial^2\xi}{\partial y^2} + (Q+I)\frac{\partial^2\xi}{\partial z^2} + 2R\frac{\partial^2\eta}{\partial x\partial y} + 2Q\frac{\partial^2\zeta}{\partial z\partial x} + X = 0, \\ (R+G)\frac{\partial^2\eta}{\partial x^2} + (M+H)\frac{\partial^2\eta}{\partial y^2} + (P+I)\frac{\partial^2\eta}{\partial z^2} + 2P\frac{\partial^2\zeta}{\partial y\partial z} + 2R\frac{\partial^2\xi}{\partial x\partial y} + Y = 0, \\ (Q+G)\frac{\partial^2\zeta}{\partial x^2} + (P+H)\frac{\partial^2\zeta}{\partial y^2} + (N+I)\frac{\partial^2\zeta}{\partial z^2} + 2Q\frac{\partial^2\xi}{\partial z\partial x} + 2P\frac{\partial^2\eta}{\partial y\partial z} + Z = 0, \end{cases} \quad (68)_{C^*} \quad \begin{cases} = \frac{\partial^2\xi}{\partial t^2}, \\ = \frac{\partial^2\eta}{\partial t^2}, \\ = \frac{\partial^2\zeta}{\partial t^2} \end{cases}$$

Si de plus les valeurs de G, H, I, L, M, N, P, Q, R deviennent indépecdantes en chaque point des directions assignées aux des x, y et z, les conditions $(41)_C$ et $(45)_C$ seront vérifiées, et, en supposant la quantité v déterminée par l'equation $(47)_C$, ou, ce qui revient au même, par la suivante

$$(69)_{C^*} \quad v = \frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z} = \nabla\cdot\mathbf{u} = div\ \mathbf{u}, \quad \mathbf{u} = (\xi,\ \eta,\ \zeta).$$

The equilibrium equation of fluid and the kinetic equation :

$$(70)_{C^*}\quad \begin{cases} (R+G)\left(\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}+\frac{\partial^2\xi}{\partial z^2}\right)+2R\frac{\partial v}{\partial x}+X=0,\\ (R+G)\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}+\frac{\partial^2\eta}{\partial z^2}\right)+2R\frac{\partial v}{\partial y}+Y=0,\\ (R+G)\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}+\frac{\partial^2\zeta}{\partial z^2}\right)+2R\frac{\partial v}{\partial z}+Z=0,\end{cases} \qquad (71)_{C^*}\quad \begin{cases} =\frac{\partial^2\xi}{\partial t^2},\\ =\frac{\partial^2\eta}{\partial t^2},\\ =\frac{\partial^2\zeta}{\partial t^2},\end{cases}$$

By $(54)_C$

$$(72)_{C^*}\quad 2R\frac{\partial v}{\partial x}+X=0,\quad 2R\frac{\partial v}{\partial y}+Y=0,\quad 2R\frac{\partial v}{\partial z}+Z=0$$

$$(73)_{C^*}\quad 2R\frac{\partial v}{\partial x}+X=\frac{\partial^2\xi}{\partial t^2},\quad 2R\frac{\partial v}{\partial y}+Y=\frac{\partial^2\eta}{\partial t^2},\quad 2R\frac{\partial v}{\partial z}+Z=\frac{\partial^2\zeta}{\partial t^2}$$

On doit observer

- que la quantité v, déterminée par formule $(69)_{C^*}$, représente la dilatation qu'éprouve un volume très petet, mais choisi de manière à renfermer avec la molécule **m** un grand nombre de molécules voisines, tandis que ces molécules changent de position dans l'espace. [2, p.248]

(Verification of equations in elastic fluid.)
By replacing $(a,\ b,\ c)$ of $(56)_C$ and $(57)_C$ with $(x,\ y,\ z)$, we get $(74)_{C^*}$ and $(75)_{C^*}$ of the equivalence of $(56)_C$ and $(57)_C$.

$$(67)_C \;\Rightarrow\; (76)_{C^*}\quad \begin{cases} \frac{\partial A}{\partial x}+\frac{\partial F}{\partial y}+\frac{\partial E}{\partial z}+X\Delta=0,\\ \frac{\partial F}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial D}{\partial z}+Y\Delta=0,\\ \frac{\partial E}{\partial x}+\frac{\partial D}{\partial y}+\frac{\partial C}{\partial z}+Z\Delta=0,\end{cases} \;\Rightarrow\; \begin{bmatrix} A & F & E\\ F & B & D\\ E & D & C\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{bmatrix}+\Delta\begin{bmatrix}X\\ Y\\ Z\end{bmatrix}=\mathbf{0}$$

The layout of Cauchy's symmetric tensor appears in $(76)_{C^*}$.
We reduce $(74)_C$ and $(75)_C$ into as follows :

$$(60)_C \;\Rightarrow\; (78)_{C^*}\quad \begin{bmatrix} A & F & E\\ F & B & D\\ E & D & C\end{bmatrix} = \begin{bmatrix} k\frac{\partial\xi}{\partial x}+Kv & \frac{1}{2}k\left(\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial x}\right) & \frac{1}{2}k\left(\frac{\partial\zeta}{\partial x}+\frac{\partial\xi}{\partial z}\right)\\ \frac{1}{2}k\left(\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial x}\right) & k\frac{\partial\eta}{\partial y}+Kv & \frac{1}{2}k\left(\frac{\partial\eta}{\partial z}+\frac{\partial\zeta}{\partial y}\right)\\ \frac{1}{2}k\left(\frac{\partial\zeta}{\partial x}+\frac{\partial\xi}{\partial z}\right) & \frac{1}{2}k\left(\frac{\partial\eta}{\partial z}+\frac{\partial\zeta}{\partial y}\right) & k\frac{\partial\zeta}{\partial z}+Kv\end{bmatrix}$$

By replacing $R+G$ and $2R$ of $(70)_C$ and $(71)_C$ with followings :[9]

$$C_1\equiv R+G=\frac{k}{2\Delta},\quad C_2\equiv 2R=\frac{k+2K}{2\Delta}$$

As the equations in equilibrium and in motion of fluid :

$$(79)_{C^*}\quad \begin{cases} C_1\left(\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}+\frac{\partial^2\xi}{\partial z^2}\right)+C_2\frac{\partial v}{\partial x}+X=0,\\ C_1\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}+\frac{\partial^2\eta}{\partial z^2}\right)+C_2\frac{\partial v}{\partial y}+Y=0,\\ C_1\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}+\frac{\partial^2\zeta}{\partial z^2}\right)+C_2\frac{\partial v}{\partial z}+Z=0,\end{cases} \qquad (80)_{C^*}\quad \begin{cases} =\frac{\partial^2\xi}{\partial t^2},\\ =\frac{\partial^2\eta}{\partial t^2},\\ =\frac{\partial^2\zeta}{\partial t^2}.\end{cases}$$

(Comparison with Navier's equation in elasticity.)
Cauchy says : for the reduction of the equations $(79)_{C^*}$ and $(80)_{C^*}$ to Navier's equations([11]) to determine the law of equilibrium and elasticity, it is necessary to assume such as the condition which we have mentioned above : $k=2K$.
(Comments to Navier's equations in elasticity.)
If $G=0$ then we get as the equations of equiliblium and the kinetic equations in equal elasticity:

$$(83)_{C^*}\quad \begin{cases} L\frac{\partial^2\xi}{\partial x^2}+R\frac{\partial^2\xi}{\partial y^2}+Q\frac{\partial^2\xi}{\partial z^2}+2R\frac{\partial^2\eta}{\partial x\partial y}+2Q\frac{\partial^2\zeta}{\partial z\partial x}+X=0,\\ R\frac{\partial^2\eta}{\partial x^2}+M\frac{\partial^2\eta}{\partial y^2}+P\frac{\partial^2\eta}{\partial z^2}+2P\frac{\partial^2\zeta}{\partial y\partial z}+2R\frac{\partial^2\xi}{\partial x\partial y}+Y=0,\\ Q\frac{\partial^2\zeta}{\partial x^2}+P\frac{\partial^2\zeta}{\partial y^2}+N\frac{\partial^2\zeta}{\partial z^2}+2Q\frac{\partial^2\xi}{\partial z\partial x}+2P\frac{\partial^2\eta}{\partial y\partial z}+Z=0,\end{cases} \qquad (84)_{C^*}\quad \begin{cases} =\frac{\partial^2\xi}{\partial t^2},\\ =\frac{\partial^2\eta}{\partial t^2},\\ =\frac{\partial^2\zeta}{\partial t^2}\end{cases}$$

The tensor in $(84)_{C^*}$ is equivalent with the tensor not only of the elastic but also of ε in Navier's fluid equation (2) (c.f. Table 4).

[9] Here, C_1 and C_2 are not the two-constants by ours but named temporarily by Cauchy.

5.3 Poisson's two constants and tensor

5.3.1 Principle and equations in elastic solid

We deduce K and k in accordance with Poisson[14, p.368-405, §1-§16] in followings. For brevity, we put as follows :

$$\begin{cases} ax_1 + by_1 + c(z_1 - \zeta_1) \equiv \phi, \\ a'x_1 + b'y_1 + c'(z_1 - \zeta_1) \equiv \psi, \\ a''x_1 + b''y_1 + c''(z_1 - \zeta_1) \equiv \theta, \end{cases} \qquad \begin{cases} \phi\frac{du}{dx} + \psi\frac{du}{dy} + \theta\frac{du}{dz} \equiv \phi', \\ \phi\frac{dv}{dx} + \psi\frac{dv}{dy} + \theta\frac{dv}{dz} \equiv \psi', \\ \phi\frac{dw}{dx} + \psi\frac{dw}{dy} + \theta\frac{dw}{dz} \equiv \theta' \end{cases} \tag{12}$$

Namely,

$$\begin{bmatrix} \phi \\ \psi \\ \theta \end{bmatrix} \equiv \begin{bmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 - \zeta \end{bmatrix}, \quad \begin{bmatrix} \phi' \\ \psi' \\ \theta' \end{bmatrix} \equiv \begin{bmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz} \\ \frac{dw}{dx} & \frac{dw}{dy} & \frac{dw}{dz} \end{bmatrix} \begin{bmatrix} \phi \\ \psi \\ \theta \end{bmatrix} = \nabla\mathbf{u} \cdot \begin{bmatrix} \phi \\ \psi \\ \theta \end{bmatrix}$$

We assume that α : the average molecular interval, ω : surface, $\frac{\omega}{\alpha^2}$: the number of molecules in ω.

$$P = \sum \frac{(\phi + \phi')\zeta}{\alpha^3 r'} fr', \quad Q = \sum \frac{(\psi + \psi')\zeta}{\alpha^3 r'} fr' \quad R = \sum \frac{(\theta + \theta')\zeta}{\alpha^3 r'} fr'. \tag{13}$$

It turns out from (12) and (13) as follows :[10]

$$(1)_{Pe} \quad \begin{cases} P = \sum \frac{(\phi+\phi')\zeta}{\alpha^3 r} fr + \sum(\phi\phi' + \psi\psi' + \theta\theta')\frac{\phi\zeta}{\alpha^3 r}\frac{d.\frac{1}{r}fr}{dr}, \\ Q = \sum \frac{(\psi+\psi')\zeta}{\alpha^3 r} fr + \sum(\phi\phi' + \psi\psi' + \theta\theta')\frac{\psi\zeta}{\alpha^3 r}\frac{d.\frac{1}{r}fr}{dr}, \\ R = \sum \frac{(\theta+\theta')\zeta}{\alpha^3 r} fr + \sum(\phi\phi' + \psi\psi' + \theta\theta')\frac{\theta\zeta}{\alpha^3 r}\frac{d.\frac{1}{r}fr}{dr}, \end{cases} \tag{14}$$

We donate :
β : the angle between the vectoriel rayon of one of molecules : r and the axis of ζ, and
γ : the angle which the projection of the rayon on the $x-y$ plane makes with the axis of x. We have :

$$x_1 = r\cos\beta\cos\gamma, \quad y_1 = r\sin\beta\sin\gamma, \quad \zeta = r\cos\beta,$$

The quantities which majored under the $\sum$ take the form : pFr, which is expressed by
p : an entire function with sines and cosines of β and γ,
Fr : a same function as fr, of which value are insensible for total sensible value of the variable, and moreover, which is equal to 0 for the paticular value of $r = 0$.
We consider that the summation which is the question is composed by the parties of the form : $\sum\left[\left(\sum\sum p\right)Fr\right]$, here, the outer $\sum$ corresponds to r and can extend to $r = \infty$, and the inner double $\sum$s correspond to β and γ.

$$\phi = gr, \quad \psi = hr, \quad \theta = lr, \quad \phi' = g'r, \quad \psi' = h'r, \quad \theta' = l'r,$$

$$\begin{cases} g = a\sin\beta\cos\gamma + b\sin\beta\sin\gamma - c\cos\beta, & g' = g\frac{du}{dx} + h\frac{du}{dy} + l\frac{du}{dz}, \\ h = a'\sin\beta\cos\gamma + b'\sin\beta\sin\gamma - c'\cos\beta, & h' = g\frac{dv}{dx} + h\frac{dv}{dy} + l\frac{dv}{dz}, \\ l = a''\sin\beta\cos\gamma + b''\sin\beta\sin\gamma - c''\cos\beta, & l' = g\frac{dw}{dx} + h\frac{dw}{dy} + l\frac{dw}{dz} \end{cases}$$

In brief :

$$\begin{bmatrix} g \\ h \\ l \end{bmatrix} = \begin{bmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{bmatrix} \begin{bmatrix} \sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ -\cos\beta \end{bmatrix}, \quad \begin{bmatrix} g' \\ h' \\ l' \end{bmatrix} = \begin{bmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz} \\ \frac{dw}{dx} & \frac{dw}{dy} & \frac{dw}{dz} \end{bmatrix} \begin{bmatrix} g \\ h \\ l \end{bmatrix} = \nabla\mathbf{u} \cdot \begin{bmatrix} g \\ h \\ l \end{bmatrix}$$

By using the *effective transformation* by Poisson we get from (13) as follows :

$$\begin{cases} P = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \left[(g + g')\sum\frac{r^3}{\alpha^5}fr + (gg' + hh' + ll')g\sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \\ Q = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \left[(h + h')\sum\frac{r^3}{\alpha^5}fr + (gg' + hh' + ll')h\sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \\ R = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \left[(l + l')\sum\frac{r^3}{\alpha^5}fr + (gg' + hh' + ll')l\sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \end{cases} \tag{15}$$

[10] We use $_{Pe}$ in the left-side equation number as Poisson's equation number in [14]. And $_{Pf}$ means Poisson[16]

(15) implies as follows :

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \Delta \begin{bmatrix} g+g' & (gg'+hh'+ll')g \\ h+h' & (gg'+hh'+ll')h \\ l+l' & (gg'+hh'+ll')l \end{bmatrix} \begin{bmatrix} \sum \frac{r^3 fr}{\alpha^5} \\ \sum \frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr} \end{bmatrix} \equiv \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \Delta \begin{bmatrix} g+g' & P' \\ h+h' & Q' \\ l+l' & R' \end{bmatrix} \begin{bmatrix} K' \\ k' \end{bmatrix},$$

where

$$\begin{bmatrix} P' \\ Q' \\ R' \end{bmatrix} = \begin{bmatrix} (g^3\nabla_x u + g^2h\nabla_y u + g^2lg\nabla_z u) + (g^2h\nabla_x v + gh^2\nabla_y v + ghl\nabla_z v) + (g^2l\nabla_x w + ghl\nabla_y w + gl^2\nabla_z w) \\ (g^2h\nabla_x u + gh^2\nabla_y u + ghl\nabla_z u) + (gh^2\nabla_x v + h^3\nabla_y v + h^2l\nabla_z v) + (ghl\nabla_x w + h^2l\nabla_y w + hl^2\nabla_z w) \\ (g^2l\nabla_x u + ghl\nabla_y u + gl^2g\nabla_z u) + (ghl\nabla_x v + h^2l\nabla_y v + hl^2\nabla_z v) + (gl^2\nabla_x w + hl^2\nabla_y w + l^3\nabla_z w) \end{bmatrix},$$

$\Delta := \cos\beta \cdot \sin\beta \; d\beta \; d\gamma, \quad \nabla_x u := \frac{du}{dx}$, etc , $\quad K' := \sum \frac{r^3 fr}{\alpha^5}, \quad k' := \sum \frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}.$

$$K \equiv \frac{2\pi}{3}\sum \frac{r^3}{\alpha^5} fr, \quad k \equiv \frac{2\pi}{15}\sum \frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}. \qquad (16)$$

By using (16), we get the following from (15) :

$$\begin{cases} P = -K\left(c + \frac{du}{dx}c + \frac{du}{dy}c' + \frac{du}{dz}c''\right) - k\left(3\frac{du}{dx}c + \frac{du}{dy}c' + \frac{du}{dz}c'' + \frac{dv}{dx}c' + \frac{dv}{dy}c + \frac{dw}{dx}c'' + \frac{dw}{dz}c\right), \\ Q = -K\left(c' + \frac{dv}{dx}c + \frac{dv}{dy}c' + \frac{dv}{dz}c''\right) - k\left(\frac{dv}{dx}c + 3\frac{dv}{dy}c' + \frac{dv}{dz}c'' + \frac{du}{dx}c' + \frac{du}{dy}c + \frac{dw}{dy}c'' + \frac{dw}{dz}c'\right), \\ R = -K\left(c'' + \frac{dw}{dx}c + \frac{dw}{dy}c' + \frac{dw}{dz}c''\right) - k\left(\frac{dw}{dx}c + \frac{dw}{dy}c' + 3\frac{dw}{dz}c'' + \frac{du}{dx}c'' + \frac{du}{dz}c + \frac{dv}{dy}c'' + \frac{dv}{dz}c'\right). \end{cases} \qquad (17)$$

Afterward, Poisson re-calculates this problem in [16][11] as follows :

$$\begin{cases} P = \left[K(1+\frac{du}{dx}) + k\left(3\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right]c + \left[K\frac{du}{dy} + k\left(\frac{du}{dy} + \frac{dv}{dx}\right)\right]c' + \left[K\frac{du}{dz} + k\left(\frac{du}{dz} + \frac{dw}{dx}\right)\right]c'', \\ Q = \left[K(1+\frac{dv}{dy}) + k\left(\frac{du}{dx} + 3\frac{dv}{dy} + \frac{dw}{dz}\right)\right]c' + \left[K\frac{dv}{dx} + k\left(\frac{dv}{dx} + \frac{du}{dy}\right)\right]c + \left[K\frac{dv}{dz} + k\left(\frac{dv}{dz} + \frac{dw}{dy}\right)\right]c'', \\ R = \left[K(1+\frac{dw}{dz}) + k\left(\frac{du}{dx} + \frac{dv}{dy} + 3\frac{dw}{dz}\right)\right]c'' + \left[K\frac{dw}{dy} + k\left(\frac{dw}{dy} + \frac{dv}{dz}\right)\right]c' + \left[K\frac{dw}{dx} + k\left(\frac{dw}{dx} + \frac{du}{dz}\right)\right]c, \end{cases} \qquad (18)$$

where, for abbreviation, he uses samely K and k, instead of α in (16), newly by ε as the average interval of molecules around the points, and from the following considerations :

- on voit que la pression N restera la même en tous sens autour de ce point : elle sera normale à ce plan et dirigée de dehors en dedans de A, ou de dedans en dehors, selon que sa value sera positive ou negative, [$\Rightarrow$ we see that the pressure orients samely in all of the direction around an arbitrary point : A, and from outward into inward or from inward to outward, according to the positive or negative value, (then we ought to consider as $\frac{1}{2}$) ;]
- from the supposition of homogeneity, $r^2 = x^2 + y^2 + z^2, \quad z^2 \equiv \frac{r^2}{3} \quad \Rightarrow \quad \Sigma\frac{z^2}{r}fr = \Sigma\frac{1}{3}rfr$, (cf. Poisson [16], pp. 32-34) :

$$(3\text{-}8)_{Pf} \qquad K \equiv \frac{1}{6\varepsilon^3}\sum rfr = \frac{2\pi}{3}\sum\frac{rfr}{4\pi\varepsilon^3}, \quad k \equiv \frac{1}{30\varepsilon^3}\sum r^3\frac{d.\frac{1}{r}fr}{dr} = \frac{2\pi}{15}\sum\frac{1}{4\pi\varepsilon^3}r^3\frac{d.\frac{1}{r}fr}{dr}, \qquad (19)$$

et étendant les sommes Σ à tous les points matériels du corps qui sont compris dans la sphère d'activité de M. [$\Rightarrow$ and extending the summation Σ to all the material points contained in the active sphere by M.] (cf. Poisson [16], p. 46) :

5.3.2 Fluid pressure in motion

[12] Poisson's tensor of the pressures in fluid reads as follows :

$$(7\text{-}7)_{Pf} \begin{bmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{bmatrix} = \begin{bmatrix} \beta\left(\frac{du}{dz} + \frac{dw}{dx}\right) & \beta\left(\frac{du}{dy} + \frac{dv}{dx}\right) & p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{du}{dx} \\ \beta\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{dv}{dy} & \beta\left(\frac{du}{dy} + \frac{dv}{dx}\right) \\ p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{dw}{dz} & \beta\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & \beta\left(\frac{du}{dz} + \frac{dw}{dx}\right) \end{bmatrix},$$

[11] In Poisson [16], the title of the chaper 3 reads "*Calcul des Pressions dans les Corps élastiques ; équations defferentielles de l'équilibre et du mouvement de ces Corps.*"

[12] In Poisson [16], the title of the chaper 7 reads "*Calcul des Pressions dans les Fluides en mouvement ; équations defferentielles de ce mouvement.*"

$$(k+K)\alpha=\beta. \quad (k-K)\alpha=\beta', \quad p=\psi t=K, \quad \text{then} \quad \beta+\beta'=2k\alpha, \tag{20}$$

where χt is the density of the fluid around the point M, and ψt is the pressure. Here K and k are the same one as in $(3\text{-}8)_{Pf}(=(19))$ of the elastic body. The elements of velocity $\mathbf{u}=(u,v,w)$ are :

$$\frac{dx}{dt}=u, \quad \frac{dy}{dt}=v, \quad \frac{dz}{dt}=w, \quad \begin{cases} \frac{d^2x}{dt^2}=\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}, \\ \frac{d^2y}{dt^2}=\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz}, \\ \frac{d^2z}{dt^2}=\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz} \end{cases}, \quad \varpi\equiv-\alpha\frac{d\psi t}{dt}-\frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt},$$

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$$(7\text{-}9)_{Pf} \quad \begin{cases} \rho(X-\frac{d^2x}{dt^2})=\frac{d\varpi}{dx}+\beta(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}), \\ \rho(Y-\frac{d^2y}{dt^2})=\frac{d\varpi}{dy}+\beta(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}), \\ \rho(Z-\frac{d^2z}{dt^2})=\frac{d\varpi}{dz}+\beta(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}). \end{cases} \tag{21}$$

If we put $\mathbf{f}=(X,Y,Z)$ then (21) becomes as follows : $\frac{\partial \mathbf{u}}{\partial t}+\frac{\beta}{\rho}\Delta\mathbf{u}+\frac{1}{\rho}\nabla\varpi=\mathbf{f}$.

5.4 Saint-Venant's tensor

Saint-Venant[14] explains the object of his paper [18] to simplfy the description and calculation of molecular relation without setting the molecular function. His method is an epoc-making method of tensor :

> Cette Note a pour objet de faciliter l'examen du Mémoire de 1834 et de ce qui y a été ajouté en 1837, en simplifiant, comme on va le dire, l'exposition du point principal, qui est la recherche des formules des pressions dans l'intérieur des fluides en mouvement, sans faire de supposition sur la grandeur des atractions et répulsions des molécules en founction, soit de leurs distances, soit de leurs vitesses relatives. [18, p.1240]

We show Saint-Venant's tensor, which seems to hint Stokes, from the extract [18]. ξ,η,ζ : velocities on the arbitrary point m of a fluid in motion of paralleled direction of the coordinate x,y,z respectively. P_{xx},P_{yy},P_{zz} : normal pressure and P_{yz},P_{zx},P_{xy} : tangential pressure with double sub-indices showing perpendicular plane and direction of decomposition.

$$(1)_{SV} \quad \frac{P_{xx}-P_{yy}}{2(\frac{d\xi}{dx}-\frac{d\eta}{dy})}=\frac{P_{zz}-P_{xx}}{2(\frac{d\zeta}{dx}-\frac{d\eta}{dz})}=\frac{P_{yy}-P_{zz}}{2(\frac{d\eta}{dy}-\frac{d\zeta}{dz})}=\frac{P_{yz}}{\frac{d\eta}{dz}+\frac{d\zeta}{dy}}=\frac{P_{zx}}{\frac{d\zeta}{dx}+\frac{d\xi}{dz}}=\frac{P_{xy}}{\frac{d\xi}{dy}+\frac{d\eta}{dx}}=\varepsilon,$$

where, $\frac{1}{3}\Big(P_{xx}+P_{yy}+P_{zz}\Big)-\frac{2\varepsilon}{3}\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big)=\pi$. We put normal pressure respectively as follows :

$$(2)_{SV} \quad P_{xx}=\pi+2\varepsilon\frac{d\xi}{dx}, \quad P_{yy}=\pi+2\varepsilon\frac{d\eta}{dy}, \quad P_{zz}=\pi+2\varepsilon\frac{d\zeta}{dz},$$

From $(1)_{SV}$, we get tangential pressure respectively as follows :

$$(3)_{SV} \quad P_{yz}=\varepsilon\Big(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\Big), \quad P_{zx}=\varepsilon\Big(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\Big), \quad P_{xy}=\varepsilon\Big(\frac{d\xi}{dy}+\frac{d\eta}{dx}\Big).$$

From $(2)_{SV}$, we get π as follows : $\pi=\frac{1}{3}\Big(P_{xx}+P_{yy}+P_{zz}\Big)-\frac{2\varepsilon}{3}\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big)$.

$$\begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} = \begin{bmatrix} \pi+2\varepsilon\frac{d\xi}{dx}, & \varepsilon\Big(\frac{d\xi}{dy}+\frac{d\eta}{dx}\Big) & \varepsilon\Big(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\Big) \\ \varepsilon\Big(\frac{d\xi}{dy}+\frac{d\eta}{dx}\Big) & \pi+2\varepsilon\frac{d\eta}{dy} & \varepsilon\Big(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\Big) \\ \varepsilon\Big(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\Big) & \varepsilon\Big(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\Big) & \pi+2\varepsilon\frac{d\zeta}{dz} \end{bmatrix}, \tag{22}$$

Saint-Venant says by using his theory, we can deduce the concurrence with Navier, Cauchy and Poisson :

> Si l'on remplace π par $\varpi-\varepsilon(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz})$, et si l'on substitue les équations $(2)_{SV}$ et $(3)_{SV}$ dans les relations connues entre les pressions et les forces accélératrices, on obtient, en supposant ε le même en tous les points du fluide, les équations différentielles données le 18 mars 1822 par M.Navier (*Mémoires de l'Institut*, t.VI), en 1828 par M.Cauchy (*Exercices de Mathématiques*, p.187)[15], et le 12 octobre 1829 par M.Poisson (même *Mémoire*, p.152)[16].
>
> La quantité variable ϖ ou π n'est autre chose, dans les liquides, que la *pression normale moyenne* en chaque point. [18, p.1243]

[13] $(7\text{-}9)_{Pf}$ means the equation number : (9) with chapter number : 7 of Poisson [16]

[14] Adhémar Jean Claude Barré de Saint-Venant (1797-1886).

[15] Cauchy [1, p.226]

[16] Poisson [16, p.152] $(7\text{-}9)_{Pf}$.

This paper[18] seems to give Stokes a hint of tensor (27), because we can see by comparing[17] t_{ij} with Stokes' t_{ij} (28) :

$$\begin{aligned} t_{ij} &= (\pi + 2\varepsilon v_{i,j} - \gamma)\delta_{ij} + \gamma, \quad (\text{where} \quad \gamma = \varepsilon(v_{i,j} + v_{j,i})), \\ &= \Big(\frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) - \frac{2\varepsilon}{3}\Big(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\Big) + 2\varepsilon v_{i,j} - \gamma\Big)\delta_{ij} + \gamma \\ &= (\frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) - \frac{2\varepsilon}{3} v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i}) \quad \Leftarrow \quad 2\varepsilon v_{i,j}\delta_{ij} = \varepsilon(v_{i,j} + v_{j,i})\delta_{ij} = \gamma\delta_{ij} \end{aligned} \tag{23}$$

Here, using (23), if we put[18] $P_{xx} = P_{yy} = P_{zz} = -p$ by Stokes principle in § 5.5, then (23) is equivalent to Stokes' t_{ij} as follows : if we pick up P_1 from (22) of Saint-Venant's tensor

$$\begin{aligned} \pi + 2\varepsilon\frac{d\xi}{dx} &= -p + \Big(2 - \frac{2}{3}\varepsilon\frac{d\xi}{dx}\Big) - \frac{2\varepsilon}{3}\Big(\frac{d\eta}{dy} + \frac{d\zeta}{dz}\Big) = -p + 2\varepsilon\Big\{\frac{2}{3}\frac{d\xi}{dx} - \frac{1}{3}\Big(\frac{d\eta}{dy} + \frac{d\zeta}{dz}\Big)\Big\} \\ &= -p + 2\varepsilon\Big\{\frac{d\xi}{dx} - \frac{1}{3}\Big(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\Big)\Big\} = -p + 2\varepsilon\Big(\frac{d\xi}{dx} - \delta\Big) \quad \Rightarrow \quad P_1 \text{ of Stokes' (27).} \end{aligned}$$

In the cases of else members are the same and we would like to omit. Moreover, Saint-Venant assume that : if we put $\pi = \varpi - \varepsilon\Big(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\Big) = \varpi - \varepsilon v_{k,k}$ then $t_{ij} = (\varpi - \varepsilon v_{k,k} + 2\varepsilon v_{i,j} - \gamma)\delta_{ij} + \gamma = (\varpi - \varepsilon v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i})$.

5.5 Stokes' principle, equations and tensor

Stokes says in [20, p.80] :[19]

> If the molecules of E were in a state of relative equilibrium, the pressure would be equal in all directions about P, as in the case of fluids at rest. Hence I shall assume the following principle :
>
> - That the difference between the pressure on a plane in a given direction passing through any point P of a fluid in motion and the pressure which would exist in all directions about P if the fluid in its neighbourhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about P ; and
> - that the relative motion due to any motion of rotation may be eliminated without affecting the differences of the pressures above mentioned.

Stokes comments on Navier's equation :

> The same equations have also been obtained by Navier in the case of an incompressible fluid (Mém. de l'Académie, t. VI. p.389)[20], but his principles differ from mine still more than do Poisson's. [20, p.77, footnote]

$$(12)_S \quad \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu\Big(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\Big) - \frac{\mu}{3}\frac{d}{dx}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu\Big(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\Big) - \frac{\mu}{3}\frac{d}{dy}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu\Big(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\Big) - \frac{\mu}{3}\frac{d}{dz}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0. \end{cases} \tag{24}$$

where Stokes says the coincidence with Poisson :

$$\varpi = p + \frac{\alpha}{3}(K + k)\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) \Rightarrow \nabla\varpi = \nabla p + \frac{\beta}{3}\nabla\cdot(\nabla\cdot\mathbf{u}). \tag{25}$$

> Observing that $\alpha(K + k) \equiv \beta$, this value of ϖ reduces Poisson's equation $(7\text{-}9)_{Pf}$ (=(21) in our renumbering) to the equation $(12)_S$ of this paper.

[17] In our paper, we cite the description of t_{ij} of the tensor : of Poisson and Cauchy, from C.Truesdell[22], of Navier, from G.Darrigol [4]. in else case by ourself or Schlichting[19].

[18] cf.I.Imai [7, p.185].

[19] Stokes [20, pp.78-105] Section 1. Explanation of the Theory of Fluid Motion proposed. Formulation of the Differential Equations. Application of these Equations to a few simple cases.

[20] Navier[12].

By the way, the linear part : $C_1T_1 + C_2T_2$ in $(12)_S$ (=(24)) is equivalent with the followings :

$$\begin{cases} -\mu\Big(\frac{4}{3}\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + \frac{1}{3}\frac{d^2v}{dxdy} + \frac{1}{3}\frac{d^2w}{dxdz}\Big) = 0, \\ -\mu\Big(\frac{d^2v}{dx^2} + \frac{4}{3}\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + \frac{1}{3}\frac{d^2u}{dxdy} + \frac{1}{3}\frac{d^2w}{dydz}\Big) = 0, \\ -\mu\Big(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{4}{3}\frac{d^2w}{dz^2} + \frac{1}{3}\frac{d^2u}{dxdz} + \frac{1}{3}\frac{d^2w}{dydz}\Big) = 0. \end{cases} \quad \text{or} \quad \begin{cases} -\frac{\mu}{3}\Big(4\frac{d^2u}{dx^2} + 3\frac{d^2u}{dy^2} + 3\frac{d^2u}{dz^2} + \frac{d^2v}{dxdy} + \frac{d^2w}{dxdz}\Big) = 0, \\ -\frac{\mu}{3}\Big(3\frac{d^2v}{dx^2} + 4\frac{d^2v}{dy^2} + 3\frac{d^2v}{dz^2} + \frac{d^2u}{dxdy} + \frac{d^2w}{dydz}\Big) = 0, \\ -\frac{\mu}{3}\Big(3\frac{d^2w}{dx^2} + 3\frac{d^2w}{dy^2} + 4\frac{d^2w}{dz^2} + \frac{d^2u}{dxdz} + \frac{d^2w}{dydz}\Big) = 0, \end{cases}$$

where when we use vectoriel notation after replacing with $\mathbf{f} \equiv (X, Y, Z)$, we get :

$$\rho(\frac{D\mathbf{u}}{Dt} - \mathbf{f}) + \nabla p - \mu\Big(\Delta\mathbf{u} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{u})\Big) = 0 \quad \text{or} \quad \frac{D\mathbf{u}}{Dt} - \frac{\mu}{\rho}\Delta\mathbf{u} - \frac{1}{3\rho}\nabla(\nabla\cdot\mathbf{u}) + \frac{1}{\rho}\nabla p = \mathbf{f}$$

Stokes proposes the Stokes' approximate equations in [20, p.93] , which is the same as $(7\text{-}9)_{Pf}$(=(21) :

$$(13)_S \quad \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}) = 0, \end{cases} \quad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \tag{26}$$

Stokes proposes that :

> These equations are applicable to the determination of the motion of water in pipes and canala, to the calculation of the effect of friction on the motions of tides and waves, and such questions.

Here we shall trace his deduction with Stokes' tensor :

$$\begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} = \begin{bmatrix} p - 2\mu\Big(\frac{du}{dx} - \delta\Big) & -\mu\Big(\frac{du}{dy} + \frac{dv}{dx}\Big) & -\mu\Big(\frac{dw}{dx} + \frac{du}{dz}\Big) \\ -\mu\Big(\frac{du}{dy} + \frac{dv}{dx}\Big) & p - 2\mu\Big(\frac{dv}{dy} - \delta\Big) & -\mu\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) \\ -\mu\Big(\frac{dw}{dx} + \frac{du}{dz}\Big) & -\mu\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) & p - 2\mu\Big(\frac{dw}{dz} - \delta\Big) \end{bmatrix}, \quad \text{where} \quad 3\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \tag{27}$$

Here, he reads, "it may also be very easily provided directly that the value of 3δ, the rate of cubical dilation". By the way, Stokes' tensor is described compactly as follows :

$$\begin{aligned} -t_{ij} &= \{p - 2\mu(v_{i,j} - \delta) + \gamma\}\delta_{ij} - \gamma \\ &= \{p - 2\mu v_{i,j}\}\delta_{ij} + \gamma(-\delta_{ij} + \delta_{ij} - 1) \quad \Leftarrow \quad 2\mu\delta\delta_{ij} = \mu(v_{i,j} + v_{j,i})\delta_{ij} = \gamma\delta_{ij} \\ &= (p + 2\mu\gamma)\delta_{ij} - \gamma = (p + \frac{2}{3}\mu v_{k,k})\delta_{ij} - \mu(v_{i,j} + v_{j,i}), \end{aligned} \tag{28}$$

Here, the sign of $-t_{ij}$ depends on the location of the tensor in the equation, and we consider the coincident with (24). [21] We see Stokes' tensor comes from Saint-Venant's tensor. From here, the article by J.J.O'Connor and E.F.Robertson point out this resemblance.[22] Moreover Stokes reports on the then academic activities of hydromechanics [21], in which he cites Saint-Venant[18].
By D'Alembert's principle

$$\begin{cases} \rho\Big(\frac{Du}{Dt} - X\Big) + \frac{dP_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz} = \rho\Big(\frac{Du}{Dt} - X\Big) + P = 0, \\ \rho\Big(\frac{Dv}{Dt} - Y\Big) + \frac{dP_2}{dy} + \frac{dT_3}{dx} + \frac{dT_1}{dz} = \rho\Big(\frac{Dv}{Dt} - Y\Big) + Q = 0, \\ \rho\Big(\frac{Dw}{Dt} - Z\Big) + \frac{dP_3}{dz} + \frac{dT_2}{dx} + \frac{dT_1}{dy} = \rho\Big(\frac{Dw}{Dt} - Z\Big) + R = 0 \end{cases} \tag{29}$$

By (27) and (29), we get (26).
By the modern vectoriel expression, if we take $\mathbf{f} = (X, Y, Z)$, $\nu \equiv \frac{\mu}{\rho}$, and as Stokes says that we may put $Du/Dt = \partial\mathbf{u}/\partial t$, then (26) turns out as follows : $\frac{\partial\mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \frac{1}{\rho}\nabla p = \mathbf{f}$, $\quad \text{div}\,\mathbf{u} = 0$.

6 Summary

It is called that "the two-constants theory" is the one now accepted for isotropic, linear elasticity. (Darrigol[4, p.121]). We can, however, insist that Poisson already uses, "elles renferment les deux constantes spéciales donc j'ai parlé tout à l'heure", in 1831 [17, p.4]. Poisson is one of persons who have an awareness of this issue. We showed in our report : the original evidences in mathematics ; the geneaology of tensors or equations are transferred by Navier, Cauchy, Poisson, Saint-Venant and Stokes (sic orderly) ; especially, we would like to intensify the work of Saint-Venant, whose tensor is an epock-making, for taking the concurrence among these pioneers of NS equations, and for contributing to Stokes.

[21] Schlichting writes Stokes' tensor with the minus sign as follows : $\sigma_{ij} = -p\delta_{ij} + \mu(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}$ [19, p.58, in footnote].

[22] cf. J.J.O'Connor, E.F.Robertson,→ http://www-groups.dcs.st-and.ac.uk/ history/Printonly/Saint-Venant.html.[13]

Table 4: Concurrences and variations of tensors

1	name	tensor (3×3)	coefficient matrix (3 × 5) in equations
1-1	Navier elastic	$t_{ij} = -\varepsilon(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$ $(5\text{-}4)_{N^e}$ $-\varepsilon\begin{bmatrix} \left(3\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right) & \left(\frac{du}{dy}+\frac{dv}{dx}\right) & \left(\frac{dw}{dx}+\frac{du}{dz}\right) \\ \left(\frac{du}{dy}+\frac{dv}{dx}\right) & \left(\frac{du}{dx}+3\frac{dv}{dy}+\frac{dw}{dz}\right) & \left(\frac{dv}{dz}+\frac{dw}{dy}\right) \\ \left(\frac{dw}{dx}+\frac{du}{dz}\right) & \left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \left(\frac{du}{dx}+\frac{dv}{dy}+3\frac{dw}{dz}\right) \end{bmatrix}$ $= -\varepsilon\begin{bmatrix} \epsilon+2\frac{du}{dx} & \frac{du}{dy}+\frac{dv}{dx} & \frac{dw}{dx}+\frac{du}{dz} \\ \frac{du}{dy}+\frac{dv}{dx} & \epsilon+2\frac{dv}{dy} & \frac{dv}{dz}+\frac{dw}{dy} \\ \frac{dw}{dx}+\frac{du}{dz} & \frac{dv}{dz}+\frac{dw}{dy} & \epsilon+2\frac{dw}{dz} \end{bmatrix}$, where $\epsilon = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$	We define the coefficient matrix in elastics : C^e_T as follows : C^e_T : the coefficient of $\begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^2 u}{\partial z^2} & \frac{\partial^2 v}{\partial x\partial y} & \frac{\partial^2 w}{\partial z\partial x} \\ \frac{\partial^2 v}{\partial x^2} & \frac{\partial^2 v}{\partial y^2} & \frac{\partial^2 v}{\partial z^2} & \frac{\partial^2 w}{\partial y\partial z} & \frac{\partial^2 u}{\partial x\partial y} \\ \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial y^2} & \frac{\partial^2 w}{\partial z^2} & \frac{\partial^2 u}{\partial z\partial x} & \frac{\partial^2 v}{\partial y\partial z} \end{bmatrix}$, then $(6\text{-}1)_{N^e} \Rightarrow C^e_T = -\varepsilon\begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix} \Rightarrow (2)$
1-2	Navier fluid (only linear part of (2))	$t_{ij} = (p - \varepsilon u_{k,k})\delta_{ij} - \varepsilon(u_{i,j} + u_{j,i})$ (2) $\begin{bmatrix} \epsilon' - 2\varepsilon\frac{du}{dx} & -\varepsilon(\frac{du}{dy}+\frac{dv}{dx}) & -\varepsilon(\frac{dw}{dx}+\frac{du}{dz}) \\ -\varepsilon(\frac{du}{dy}+\frac{dv}{dx}) & \epsilon' - 2\varepsilon\frac{dv}{dy} & -\varepsilon(\frac{dv}{dz}+\frac{dw}{dy}) \\ -\varepsilon(\frac{dw}{dx}+\frac{du}{dz}) & -\varepsilon(\frac{dv}{dz}+\frac{dw}{dy}) & \epsilon' - 2\varepsilon\frac{dw}{dz} \end{bmatrix}$, where $\epsilon' = p - \varepsilon(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz})$	Samely, we define the coefficient matrix in fluid : C^f_T , which contains p in (1,1)-, (2,2)- and (3,3)-element. $(2) \Rightarrow C^f_T = \begin{bmatrix} p-3\varepsilon & -\varepsilon & -\varepsilon & -2\varepsilon & -2\varepsilon \\ -\varepsilon & p-3\varepsilon & -\varepsilon & -2\varepsilon & -2\varepsilon \\ -\varepsilon & -\varepsilon & p-3\varepsilon & -2\varepsilon & -2\varepsilon \end{bmatrix}$
2	Cauchy system	$t_{ij} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$ $(60)_C$ $\begin{bmatrix} k\frac{\partial\xi}{\partial a}+K\nu & \frac{k}{2}\left(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\right) & \frac{k}{2}\left(\frac{\partial\zeta}{\partial a}+\frac{\partial\xi}{\partial c}\right) \\ \frac{k}{2}\left(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\right) & k\frac{\partial\eta}{\partial b}+K\nu & \frac{k}{2}\left(\frac{\partial\eta}{\partial c}+\frac{\partial\zeta}{\partial b}\right) \\ \frac{k}{2}\left(\frac{\partial\zeta}{\partial a}+\frac{\partial\xi}{\partial c}\right) & \frac{k}{2}\left(\frac{\partial\eta}{\partial c}+\frac{\partial\zeta}{\partial b}\right) & k\frac{\partial\zeta}{\partial c}+K\nu \end{bmatrix}$, where $\nu = \frac{\partial\xi}{\partial a}+\frac{\partial\eta}{\partial b}+\frac{\partial\zeta}{\partial c}$	$(46)_C \Rightarrow C^e_T = \begin{bmatrix} L & R & Q & 2R & 2Q \\ R & M & P & 2P & 2R \\ Q & P & N & 2Q & 2P \end{bmatrix}$ $\Rightarrow R\begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix}$, where $P = Q = R,\quad L = M = N,\quad L = 3R.$
3-1	Poisson elastic	$t_{ij} = -\frac{a^2}{3}(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$ $(6)_{P^e}$ $-\frac{a^2}{3}\begin{bmatrix} \epsilon+2\frac{du}{dx} & \frac{du}{dy}+\frac{dv}{dx} & \frac{dw}{dx}+\frac{du}{dz} \\ \frac{du}{dy}+\frac{dv}{dx} & \epsilon+2\frac{dv}{dy} & \frac{dv}{dz}+\frac{dw}{dy} \\ \frac{dw}{dx}+\frac{du}{dz} & \frac{dv}{dz}+\frac{dw}{dy} & \epsilon+2\frac{dw}{dz} \end{bmatrix}$, where $\epsilon = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$	$(6)_{P^e}$ $\begin{cases} X - \frac{d^2u}{dt^2} + a^2\left(\frac{d^2u}{dx^2} + \frac{2}{3}\frac{d^2v}{dydx} + \frac{2}{3}\frac{d^2w}{dzdx} + \frac{1}{3}\frac{d^2u}{dy^2} + \frac{1}{3}\frac{d^2u}{dz^2}\right) = 0, \\ Y - \frac{d^2v}{dt^2} + a^2\left(\frac{d^2v}{dy^2} + \frac{2}{3}\frac{d^2u}{dxdy} + \frac{2}{3}\frac{d^2w}{dzdy} + \frac{1}{3}\frac{d^2v}{dx^2} + \frac{1}{3}\frac{d^2v}{dz^2}\right) = 0, \\ Z - \frac{d^2w}{dt^2} + a^2\left(\frac{d^2w}{dz^2} + \frac{2}{3}\frac{d^2u}{dxdz} + \frac{2}{3}\frac{d^2v}{dydz} + \frac{1}{3}\frac{d^2w}{dx^2} + \frac{1}{3}\frac{d^2w}{dy^2}\right) = 0, \end{cases}$ $\Rightarrow C^e_T = -\frac{a^2}{3}\begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix}$
3-2	Poisson fluid	$t_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$ $(7\text{-}7)_{Pf}$ $\begin{bmatrix} \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right) & \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right) & \pi+2\beta\frac{du}{dx} \\ \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \pi+2\beta\frac{dv}{dy} & \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right) \\ \pi+2\beta\frac{dw}{dz} & \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right) \end{bmatrix}$, where $\pi = p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt}$	$(7\text{-}9)_{Pf} \Rightarrow C^f_T = \begin{bmatrix} \varpi+\beta & \beta & \beta & 0 & 0 \\ \beta & \varpi+\beta & \beta & 0 & 0 \\ \beta & \beta & \varpi+\beta & 0 & 0 \end{bmatrix}$. According to Stokes : if we put $\varpi = p + \frac{\alpha}{3}(K+k)\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)$ $\Rightarrow C^f_T = \begin{bmatrix} p+\frac{4\beta}{3} & \beta & \beta & \frac{\beta}{3} & \frac{\beta}{3} \\ \beta & p+\frac{4\beta}{3} & \beta & \frac{\beta}{3} & \frac{\beta}{3} \\ \beta & \beta & p+\frac{4\beta}{3} & \frac{\beta}{3} & \frac{\beta}{3} \end{bmatrix} \Rightarrow (12)_S\ (= (24)).$ Remark : $\alpha(K+k) = \beta$.
4	Saint-Venant fluid	$t_{ij} = (\frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i})$ $= (-p - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i})$ $\begin{bmatrix} \pi+2\varepsilon\frac{d\xi}{dx}, & \varepsilon\left(\frac{d\xi}{dy}+\frac{d\eta}{dx}\right) & \varepsilon\left(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\right) \\ \varepsilon\left(\frac{d\xi}{dy}+\frac{d\eta}{dx}\right) & \pi+2\varepsilon\frac{d\eta}{dy} & \varepsilon\left(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\right) \\ \varepsilon\left(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\right) & \varepsilon\left(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\right) & \pi+2\varepsilon\frac{d\zeta}{dz} \end{bmatrix}$, where $\pi = \frac{1}{3}\left(P_{xx} + P_{yy} + P_{zz}\right) - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)$ $\equiv -p - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)$ (22)	non description in [18].
5	Stokes fluid	$t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i})$, tensor $= -1\ \times$ $\begin{bmatrix} p-2\mu\left(\frac{du}{dx}-\delta\right) & -\mu\left(\frac{du}{dy}+\frac{dv}{dx}\right) & -\mu\left(\frac{dw}{dx}+\frac{du}{dz}\right) \\ -\mu\left(\frac{du}{dy}+\frac{dv}{dx}\right) & p-2\mu\left(\frac{dv}{dy}-\delta\right) & -\mu\left(\frac{dv}{dz}+\frac{dw}{dy}\right) \\ -\mu\left(\frac{dw}{dx}+\frac{du}{dz}\right) & -\mu\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & p-2\mu\left(\frac{dw}{dz}-\delta\right) \end{bmatrix}$ where $3\delta = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$ (27)	$(12)_S \Rightarrow C^f_T = \begin{bmatrix} -p+\frac{4\mu}{3} & \mu & \mu & \frac{\mu}{3} & \frac{\mu}{3} \\ \mu & -p+\frac{4\mu}{3} & \mu & \frac{\mu}{3} & \frac{\mu}{3} \\ \mu & \mu & -p+\frac{4\mu}{3} & \frac{\mu}{3} & \frac{\mu}{3} \end{bmatrix} \Rightarrow (27).$ Remark : $\frac{4}{3}\mu = 2\mu(1 - \frac{1}{3})$

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