ENDOMORPHISMS OF PROJECTIVE VARIETIES AND THEIR INVARIANT HYPERSURFACES

DE-QI ZHANG

ABSTRACT. We consider surjective endomorphisms $f$ of degree $> 1$ on projective manifolds $X$ of Picard number one and their $f^{-1}$-stable hypersurfaces $V$. When $X = \mathbb{P}^n$ with $n = 3$, we show that $V$ is a hyperplane (i.e., $\deg(V) = 1$) but with four possible exceptions; it is conjectured that $\deg(V) = 1$ for any $n \geq 2$; cf. [8], [3]. For general $X$, we show that $V$ is rationally chain connected. Also given is an optimal upper bound for the number of $f^{-1}$-stable prime divisors on (not necessarily smooth) projective varieties.

1. ENDOMORPHISMS OF $\mathbb{P}^3$

We work over the field $\mathbb{C}$ of complex numbers. We start with the consideration of endomorphisms of the projective three space. The main result of this section is Theorem 1.1 below.

Theorem 1.1. Let $f : \mathbb{P}^3 \to \mathbb{P}^3$ be an endomorphism of degree $> 1$ and $V$ an irreducible hypersurface such that $f^{-1}(V) = V$. Then either $\deg(V) = 1$, i.e., $V$ is a hyperplane, or $V = V_i := \{S_i = 0\}$ is a cubic hypersurface given by one of the following four defining equations $S_i$ in suitable projective coordinates:

1. $S_1 = X_3^3 + X_0X_1X_2$;
2. $S_2 = X_0^2X_3 + X_0X_1^2 + X_2^3$;
3. $S_3 = X_0^2X_2 + X_1^2X_3$;
4. $S_4 = X_0X_1X_2 + X_0^2X_3 + X_1^3$.

We are unable to rule out the four cases in Theorem 1.1 but see Examples 2.8 (for $V_1$).

Remark 1.2. (1) The non-normal locus of $V_i$ ($i = 3, 4$) is a single line $C$ and stabilized by $f^{-1}$. Let $\sigma : V_i' \to V_i$ ($i = 3, 4$) be the normalization. Then $V_i'$ is the (smooth) Hirzebruch surface $F_1$ (i.e., the one-point blowup of $\mathbb{P}^2$); see [1, Theorem 1.5], [17].

(2) $V_1$ (resp. $V_2$) is unique as a normal cubic (or degree three del Pezzo) surface of Picard number one and with the singular locus $\text{Sing}V_1 = 3A_2$ (resp. $\text{Sing}V_2 = 2000$ Mathematics Subject Classification. 37F10, 32H50, 14E20, 14J45.

Key words and phrases. endomorphism, iteration, projective space.

The author is supported by an ARF of NUS.
$E_6$; see [20, Theorem 1.2] and [10, Theorem 4.4] (for the anti-canonical embedding of $V_i$ in $\mathbb{P}^3$). $V_1$ contains exactly three lines (of triangle-shaped) whose three vertices form the singular locus of $V_1$. And $V_2$ contains a single line on which lies its unique singular point. $f^{-3}$ (replaced by its cube) fixes the singular point(s) of $V_i$ ($i = 1, 2$).

(3) $f^{-1}$ (or its power) does not stabilize the only line $L$ on $V_2$ by using [15, Theorem 4.3.1] since the pair $(V_2, L)$ is not log canonical at the singular point of $V_2$. For $V_1$, we do not know whether $f^{-1}$ (or its power) stabilizes its three lines.

We now sketch the proof of Theorem 1.1.

By [16, Theorem 1.5], we may assume that $V \subset \mathbb{P}^3$ is an irreducible rational singular cubic hypersurface.

We first consider the case where $V$ is non-normal. Such $V$ is classified in [6, Theorem 9.2.1] to the effect that either $V = V_i$ ($i = 3, 4$) or $V$ is a cone over a nodal or cuspidal rational planar cubic curve $B$. The description in Remark 1.2 on $V_3, V_4$ and their normalizations, is given in [17, Theorem 1.1], [1, Theorem 1.5, Case (C), (E1)].

We can rule out the case where $V$ is a cone over $B$.

Next we consider the case where $V \subset \mathbb{P}^3$ is a normal rational singular cubic hypersurface. By the adjunction formula, $-K_V = -(K_{\mathbb{P}^3} + V)|V \sim H|V$ which is ample, where $H \subset \mathbb{P}^3$ is a hyperplane. Since $K_V$ is a Cartier divisor, $V$ has only Du Val (or rational double, or $ADE$) singularities. Let $\sigma : V' \rightarrow V$ be the minimal resolution. Then $K_{V'} = \sigma^*K_V \sim \sigma^*(-H|V)$. For $f : \mathbb{P}^3 \rightarrow \mathbb{P}^3$, we can apply $f|V$ to the result below.

**Lemma 1.3.** Let $V \subset \mathbb{P}^3$ be a normal cubic surface, and $f_V : V \rightarrow V$ an endomorphism such that $f_V^*(H|V) \sim q(H|V)$ for some $q > 1$ and the hyperplane $H \subset \mathbb{P}^3$. Let $S(V) = \{(\text{irreducible})G \subset V \mid G^2 < 0\}$ be the set of negative curves on $V$, and set $E_V := \sum_{E \in S(V)} E$. Replacing $f_V$ by its power, we have:

1. If $f_V^*G \equiv aG$ for some Weil divisor $G \neq 0$, then $a = q$. $f_V^*(L|V) \sim q(L|V)$ for every divisor $L$ on $\mathbb{P}^3$. Especially, $\deg(f_V) = q^2$; $K_V \sim -H|V$ satisfies $f_V^*K_V \sim qK_V$.
2. $S(V)$ is a finite set. $f_V^*E = qE$ for every $E \in S(V)$. So $f_V^*E_V = qE_V$.
3. A curve $E \subset V$ is a line in $\mathbb{P}^3$ if and only if $E$ is equal to $\sigma(E')$ for some $(-1)$-curve $E' \subset V'$.
4. Every curve $E \in S(V)$ is a line in $\mathbb{P}^3$.
5. We have $K_V + E_V = f_V^*(K_V + E_V) + \Delta$ for some effective divisor $\Delta$ containing no line in $S(V)$, so that the ramification divisor $R_{f_V} = (q - 1)E_V + \Delta$. In particular, the cardinality $\#S(V) \leq 3$, and the equality holds exactly when $K_V + E_V \sim_\mathbb{Q} 0$; in this case, $f_V$ is étale outside the three lines of $S(V)$ and $f_V^{-1}(\text{Sing } V)$. 

DE-QI ZHANG
INVARIANT HYPERSURFACES

**Remark 1.4.** In the proof of Theorem 1.1, we can actually show: if $f_V : V \to V$ is an endomorphism (not necessarily the restriction of some $f : \mathbb{P}^3 \to \mathbb{P}^3$) of $\deg(f_V) > 1$ of a Gorenstein normal del Pezzo surface with $K_V^2 = 3$ (i.e., a normal cubic surface), then $V$ is equal to $V_1$ or $V_2$ in Theorem 1.1 in suitable projective coordinates.

**2. SUMMARY OF MAIN RESULTS**

Below is the summary of our recent paper [23]. Theorem 2.1 $\sim$ Theorem 2.4 are our main results.

**Theorem 2.1.** Let $X$ be a locally factorial normal projective variety of dimension $n \geq 2$ and Picard number one, and with only log canonical singularities, and let $f : X \to X$ be a surjective endomorphism with $\deg(f) = q^n > 1$. Then we have:

1. There are at most $n + 1$ prime divisors $V_i \subset X$ with $f^{-1}(V_i) = V_i$.
2. There are $n + 1$ of such $V_i$ if and only if: $X = \mathbb{P}^n$, $V_i = \{X_i = 0\}$ ($1 \leq i \leq n + 1$) (in suitable projective coordinates), and $f$ is given by

$$f : [X_0, \ldots, X_n] \longrightarrow [X_0^q, \ldots, X_n^q].$$

We refer to S. -W. Zhang [21, Conjecture 1.3.1] for the Dynamic Manin-Mumford conjecture etc. solved for the $(X, f)$ in Theorem 2.1 (2).

A projective variety $X$ is **rationally chain connected** if every two points $x_i \in X$ are contained in a connected chain of rational curves on $X$. When $X$ is smooth, $X$ is rationally **chain** connected if and only if $X$ is **rationally connected**, in the sense of Campana, and Kollár-Miyaoka-Mori.

**Theorem 2.2.** Let $X$ be a projective manifold of dimension $n \geq 2$ and Picard number one, $f : X \to X$ an endomorphism of degree $> 1$, and $V \subset X$ a prime divisor with $f^{-1}(V) = V$. Then $X$, $V$ and the normalization $V'$ of $V$ are all rationally chain connected.

In Theorem 2.2, the smoothness and Picard number one assumption on $X$ are necessary (cf. Remark 2.6 and Example 2.9). Theorem 2.2 is known for $X = \mathbb{P}^n$ with $n \leq 3$ (cf. [8], [16]). In Theorem 2.2, $X$ is indeed a Fano manifold. See Remark 2.6 for the case when $X$ is singular.

**Corollary 2.3.** With the notation and assumptions in Theorem 2.2, both $X$ and $V$ are simply connected, while $V'$ has a finite (topological) fundamental group.

A morphism $f : X \to X$ is **polarized** (by $H$) if $f^*H \sim qH$ for some ample line bundle $H$ and some $q > 0$; then $\deg(f) = q^{\dim X}$. For instance, every non-constant endomorphism of a projective variety $X$ of Picard number one, is polarized; an $f$-stable subvariety
$X \subset \mathbb{P}^{n}$ for a non-constant endomorphism $f : \mathbb{P}^{n} \to \mathbb{P}^{n}$, has the restriction $f|X : X \to X$ polarized by the hyperplane; the multiplication map $m_{A} : A \to A, x \mapsto mx$ (with $m \neq 0$) of an abelian variety $A$ is polarized by any $H = L + (-1)^{*}L$ with $L$ an ample divisor, so that $m_{A}^{*}H \sim m^{2}H$.

In Theorems 2.1 and 2.4, we give upper bounds for the number of $f^{-1}$-stable prime divisors on a (not necessarily smooth) projective variety; the bounds are optimal, and the second possibility in Theorem 2.4(2) does occur (cf. Examples 2.8 and 2.9). One may remove the condition (*) in Theorem 2.4, when $\rho(X) = 1$, or $X$ is a weak $\mathbb{Q}$-Fano variety, or the closed cone $\overline{NE}(X)$ of effective curves has only finitely many extremal rays (cf. Remark 2.6); here $N^{1}(X) := \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ is the Néron-Severi group (over $\mathbb{R}$) and $\rho(X) := \text{rank}_{\mathbb{R}}N^{1}(X)$ is the Picard number of $X$. We refer to [11, Definition 2.34] for the definitions of Kawamata log terminal (klt) and log canonical singularities.

**Theorem 2.4.** Let $X$ be a projective variety of dimension $n$ with only $\mathbb{Q}$-factorial Kawamata log terminal singularities, and $f : X \to X$ a polarized endomorphism with $\text{deg}(f) = q^{n} > 1$. Suppose ($*$) : either $f^{*}|N^{1}(X) = q$ id, or $n \leq 3$. Then we have (with $\rho := \rho(X)$):

1. Let $V_{i} \subset X$ ($1 \leq i \leq c$) be prime divisors with $f^{-1}(V_{i}) = V_{i}$. Then $c \leq n + \rho$. Further, if $c \geq 1$, then the pair $(X, \sum V_{i})$ is log canonical and $X$ is uniruled.
2. Suppose that $c \geq n + \rho - 2$. Then either $X$ is rationally connected, or there is a fibration $X \to E$ onto an elliptic curve $E$ so that every fibre is normal rationally connected and some positive power $f^{k}$ descends to an $f_{E} : E \to E$ of degree $q$.
3. Suppose that $c \geq n + \rho - 1$. Then $X$ is rationally connected.
4. Suppose that $c \geq n + \rho$. Then $c = n + \rho$, (for some $t > 0$)

$$K_{X} + \sum_{i=1}^{n+\rho}V_{i} \sim_{\mathbb{Q}} 0, \quad (f^{t})^{*}|\text{Pic}(X) = q^{t} \text{id},$$

$f$ is étale outside $(\cup V_{i}) \cup f^{-1}(\text{Sing} X)$ (and $X$ is a toric surface with $\sum V_{i}$ its boundary divisor, when $\dim X = 2$).

Theorems 2.4 and 2.1 motivate the question below (without assuming the condition (*) in Theorem 2.4), where the last part is also Shokurov's conjecture (cf. [18, Theorem 6.4]).

**Question 2.5.** Suppose that a projective $n$-fold ($n \geq 3$) $X$ has only $\mathbb{Q}$-factorial Kawamata log terminal singularities, $f : X \to X$ a polarized endomorphism of degree $> 1$, and $V_{i} \subset X$ ($1 \leq i \leq s$) prime divisors with $f^{-1}(V_{i}) = V_{i}$. Then, is it true that $s \leq n + \rho(X)$, and equality holds only when $X$ is a toric variety with $\sum V_{i}$ its boundary divisor?

**Remark 2.6.** (1) In Theorem 2.2, it is necessary to assume that $\rho(X) = 1$ (cf. Example 2.9), and $X$ is smooth or at least Kawamata log terminal (klt). Indeed, for every projective
cone $Y$ over an elliptic curve and every section $V \subset Y$ (away from the vertex), there is an endomorphism $f : Y \to Y$ of $\deg(f) > 1$ and with $f^{-1}(V) = V$ (cf. [15, Theorem 7.1.1, or Proposition 5.2.2]). The cone $Y$ has Picard number one and a log canonical singularity at its vertex. Of course, $V$ is an elliptic curve, and is not rationally chain connected. By the way, $Y$ is rationally chain connected, but is not rationally connected.

(2) Let $X$ be a projective variety with only klt singularities. If the closed cone $\overline{\text{NE}}(X)$ of effective curves has only finitely many extremal rays, then every polarized endomorphism $f : X \to X$ satisfies $f^*|N^1(X) = q$ id with $\deg(f) = q^{\dim X}$, after replacing $f$ by its power, so that we can apply Theorem 2.4 (cf. [16, Lemma 2.1]). For instance, if $X$ or $(X, \Delta)$ is $\mathbb{Q}$-Fano, i.e., $X$ (resp. $(X, \Delta)$) has only klt singularities and $-K_X$ (resp. $-(K_X + \Delta)$) is nef and big, then $\overline{\text{NE}}(X)$ has only finitely many extremal rays.

(3) By Example 2.8, it is necessary to assume the local factoriality of $X$ or the Cartier-ness of $V_i$ in Theorem 2.1 (2) even when $X$ has only klt singularities. We remark that a $\mathbb{Q}$-factorial Gorenstein terminal threefold is locally factorial. For Theorem 2.1(2), one may also use Fujita’s theory to prove $X \simeq \mathbb{P}^n$, but our method is useful even when $V_i$’s are only $\mathbb{Q}$-Cartier (cf. Theorem 2.4).

2.7. A motivating conjecture. Here are some motivations for our paper. It is conjectured that every hypersurface $V \subset \mathbb{P}^n$ stabilized by the inverse $f^{-1}$ of an endomorphism $f : \mathbb{P}^n \to \mathbb{P}^n$ of $\deg(f) > 1$, is linear. This conjecture is still open when $n \geq 3$ and $V$ is singular, since the proof of [3] is incomplete as we were informed by an author. The smooth hypersurface case was settled in the affirmative (in any dimension) by Cerveau - Lins Neto [4] and independently by Beauville [2]. See also [16, Theorem 1.5 in its arXiv version: arXiv:0908.1688v1].

From the dynamics point of view, as seen in Dinh-Sibony [5, Theorem 1.3, Corollary 1.4], $f : \mathbb{P}^n \to \mathbb{P}^n$ behaves nicely exactly outside those $f^{-1}$-stabilized subvarieties. We refer to Fornaess-Sibony [8], and [5] for further references.

A smooth hypersurface $X$ in $\mathbb{P}^{n+1}$ with $\deg(X) \geq 3$ and $n \geq 2$, has no endomorphism $f_X : X \to X$ of degree $> 1$ (cf. [2, Theorem]). However, singular $X$ may have plenty of endomorphisms $f_X$ of arbitrary degrees as shown in Example 2.8 below. Conjecture 2.7 asserts that such $f_X$ can not be extended to an endomorphism of $\mathbb{P}^{n+1}$.

**Example 2.8.** We now construct many polarized endomorphisms for some degree $n + 1$ hypersurface $X \subset \mathbb{P}^{n+1}$, with $X$ isomorphic to the $V_i$ in Theorem 1.1 when $n = 2$. Let $f = (F_0, \ldots, F_n) : \mathbb{P}^n \to \mathbb{P}^n$ ($n \geq 2$), with $F_i = F_i(X_0, \ldots, X_n)$ homogeneous, be any endomorphism of degree $q^n > 1$, such that $f^{-1}(S) = S$ for a reduced degree
$n + 1$ hypersurface $S = \{S(X_0, \ldots, X_n) = 0\}$. So $S$ must be normal crossing and linear: $S = \sum_{i=0}^{n} S_i$ (cf. [16, Thm. 1.5 in arXiv version]). Thus we may assume that $f = (X^0_0, \ldots, X^d_0)$ and $S_i = \{X_i = 0\}$. The relation $S \sim (n + 1)H$ with $H \subset \mathbb{P}^n$ a hyperplane, defines

$$
\pi : X = \text{Spec } \oplus_{i=0}^{n} \mathcal{O}(-iH) \rightarrow \mathbb{P}^n
$$

which is a Galois $\mathbb{Z}/(n + 1)$-cover branched over $S$ so that $\pi^*S_i = (n + 1)T_i$ with the restriction $\pi|T_i : T_i \rightarrow S_i$ an isomorphism.

This $X$ is identifiable with the degree $n + 1$ hypersurface $\{Z^{n+1} = S(X_0, \ldots, X_n)\} \subset \mathbb{P}^{n+1}$ and has singularity of type $z^{n+1} = xy$ over the intersection points of $S$ locally defined as $xy = 0$. Thus, when $n = 2$, we have $\text{Sing } X = 3A_2$ and $X$ is isomorphic to the $V_1$ in Theorem 1.1 (cf. Remark 1.2). We may assume that $f^*S(X_0, \ldots, X_n) = S(X_0, \ldots, X_n)^q$ after replacing $S(X_0, \ldots, X_n)$ by a scalar multiple, so $f$ lifts to an endomorphism $g = (Z^q, F_0, \ldots, F_n)$ of $\mathbb{P}^{n+1}$ (with homogeneous coordinates $[Z, X_0, \ldots, X_n]$), stabilizing $X$, so that $g_X := g|X : X \rightarrow X$ is a polarized endomorphism of $\deg(g_X) = q^n$ (cf. [16, Lemma 2.1]). Note that $g^{-1}(X)$ is the union of $q$ distinct hypersurfaces $\{Z = \zeta^iS(X_0, \ldots, X_n)\} \subset \mathbb{P}^{n+1}$ (all isomorphic to $X$), where $\zeta := \exp(2\pi\sqrt{-1}/q)$.

This $X$ has only Kawamata log terminal singularities and $\text{Pic } X = (\text{Pic } \mathbb{P}^{n+1})|X (n \geq 2)$ is of rank one (using Lefschetz type theorem [12, Example 3.1.25] when $n \geq 3$). We have $f^{-1}(S_i) = S_i$ and $g_X^{-1}(T_i) = T_i$, where $0 \leq i \leq n$. Note that $(n + 1)T_i = \pi^*S_i$ is Cartier, but $T_i$ is not Cartier (cf. Theorems 2.1).

When $n = 2$, the relation $(n + 1)(T_1 - T_0) \sim 0$ gives rise to an étale-in-codimension-one $\mathbb{Z}/(n + 1)$-cover $\tau : \mathbb{P}^n \simeq \overline{X} \rightarrow X$ so that $\sum_{i=0}^{n} \tau^*T_i$ is a union of $n + 1$ normal crossing hyperplanes; indeed, $\tau$ restricted over $X \setminus \bigcup T_i$, is its universal cover (cf. [13, Lemma 6]), so that $g_X$ lifts up to $\overline{X}$. A similar result seems to be true for $n \geq 3$, by considering the 'composite' of the $\mathbb{Z}/(n + 1)$-covers given by $(n + 1)(T_i - T_0) \sim 0$ ($1 \leq i < n$); see Question 2.5.

The simple Example 2.9 below shows that the conditions in Theorem 2.4 (2) (3), or the condition $\rho(X) = 1$ in Theorem 2.2, is necessary.

**Example 2.9.** Let $m_A : A \rightarrow A (x \mapsto mx)$ with $m \geq 2$, be the multiplication map of an abelian variety $A$ of dimension $u \geq 1$ and Picard number one, and let $g : \mathbb{P}^u \rightarrow \mathbb{P}^u ([X_0, \ldots, X_u] \mapsto [X^0_0, \ldots, X^u])$ with $v \geq 1$ and $q := m^2$. Then $f = (m_A \times g) : X = A \times \mathbb{P}^u \rightarrow X$ is a polarized endomorphism with $f^*|N^1(X) = \text{diag}[q, q]$, and $f^{-1}$ stabilizes $v + 1$ prime divisors $V_i = A \times \{X_i = 0\} \subset X$ and no others; indeed, $f$ is étale outside $\bigcup V_i$. Note that $X$ and $V_i \simeq A \times \mathbb{P}^{u-1}$ are not rationally chain connected, and $v + 1 = \dim X + \rho(X) - (1 + \dim A)$. 

2.10. The results of Favre [7], Nakayama [15] and Wahl [19] are very inspiring about the restriction of the singularity type of a normal surface imposed by the existence of an endomorphism of degree $> 1$ on the surface. For the proof of our results, the basic ingredients are: a log canonical singularity criterion, a rational connectedness criterion of Qi Zhang [24] and its generalization in Hacon-McKernan [9], the equivariant MMP in our early paper [22], and the characterization in Mori [14] on hypersurfaces in weighted projective spaces.

Acknowledgement. I thank N. Nakayama for the comments and informing me about Shokurov’s conjecture (cf. 2.5) and Wahl’s result [19, Corollary, page 626], and the following colleagues for the opportunity of conference / colloquium talks in December 2009: the organizer H. Sumi of Integrated Research of Complex Dynamics and Related Fields, I. Shimada of Hiroshima University and K. Oguiso of Osaka University.

**REFERENCES**


DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE, 10 LOWER KENT RIDGE ROAD, SINGAPORE 119076
E-mail address: matzdq@nus.edu.sg