On the Markoff-Painlevé Transcendent

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Abstract

All positive integral solutions to Markoff’s equation are in one-to-one correspondence with all analytic continuations of a transcendental solution germ to a special sixth Painlevé equation via the Riemann-Hilbert correspondence. We explicitly determine the parameter value and the initial condition for the Markoff-Painlevé transcendent.

1 Markoff’s Diophantine Equation

In 1879 and 1880 A.A. Markoff [9, 10] discussed a Diophantine equation of the form

\[ m_1^2 + m_2^2 + m_3^2 = 3m_1m_2m_3 \quad (m_1, m_2, m_3) \in \mathbb{N}^3, \]

in the study of badly approximable irrational numbers and indefinite binary quadratic forms.

We present some known facts about Markoff’s equation (1) (see e.g. [1]). It has the trivial solution (1, 1, 1). It also has another simple solution (1, 1, 2). These two solutions are referred to as the exceptional solutions. Any other solution is called a regular solution. Any regular solution has mutually distinct components. There are infinitely many solutions and there is a simple algorithm which produces all of them. It is based on a large symmetry

\[ G = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \]

leaving equation (1) invariant, where \( \sigma_1 \) is the involution

\[ \sigma_1 : (m_1, m_2, m_3) \mapsto (3m_2m_3 - m_1, m_2, m_3), \]

with \( \sigma_2 \) and \( \sigma_3 \) being defined in similar manners. Two solutions are said to be neighbors if they share two components. Any regular solution \((m_1, m_2, m_3)\) has exactly three neighbors \(\sigma_i(m_1, m_2, m_3)\), \(i = 1, 2, 3\), one smaller and two larger, where the ordering is defined by

\[ (m_1, m_2, m_3) \prec (m'_1, m'_2, m'_3) \quad \text{if} \quad \max\{m_1, m_2, m_3\} < \max\{m'_1, m'_2, m'_3\}. \]

Starting with the trivial solution (1, 1, 1), apply \( \sigma_1, \sigma_2, \sigma_3 \) recursively in all possible ways to produce infinitely many solutions. This process can be incorporated into a tree in Figure 1, which is known as Markoff’s tree. Any solution occurs exactly once in the tree and the \(G\)-orbit through the trivial solution (1, 1, 1) constitutes all the solutions to equation (1).

The aim of this note is to throw a bridge between the Markoff orbit and a very special solution to the sixth Painlevé equation via the Riemann-Hilbert correspondence.

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Figure 1: Markoff's tree
The Sixth Painlevé Equation

The sixth Painlevé equation $P_{VI}(\kappa)$ is a Hamiltonian system

$$\begin{align*}
\frac{dq}{dz} &= \frac{\partial H(\kappa)}{\partial p}, \\
\frac{dp}{dz} &= -\frac{\partial H(\kappa)}{\partial q},
\end{align*}$$

with a complex time variable $z \in Z := \mathbb{P}^1 - \{0, 1, \infty\}$ and unknown functions $q = q(z)$ and $p = p(z)$, depending on complex parameters $\kappa$ in the four-dimensional affine space $\mathcal{K} := \{\kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathbb{C}^5_\kappa : 2\kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1\}$, where the Hamiltonian $H(\kappa) = H(q, p, z; \kappa)$ is given by

$$z(z - 1)H(\kappa) = (q_0q_zq_1)p^2 - \{\kappa_1q_1q_z + (\kappa_2 - 1)q_0q_1 + \kappa_3q_0q_z\}p + \kappa_0(\kappa_0 + \kappa_4)q_z,$$

with $q_\nu := q - \nu$ for $\nu \in \{0, z, 1\}$. Any meromorphic solution germ at any point $z \in Z$ admits a global meromorphic continuation along any path in $Z$ emanating from $z$. This property is known as the Painlevé property for the sixth Painlevé equation [2].

Let $\mathcal{M}_z(\kappa)$ be the set of all meromorphic solution germs to equation (3) at a base point $z \in Z$. It is realized as the moduli space of (certain) stable parabolic connections, thereby provided with the structure of a smooth quasi-projective rational complex surface, where a stable parabolic connection is a rank-two vector bundle over $\mathbb{P}^1$ together with a Fuchsian connection having four regular singular points and a parabolic structure that satisfies a sort of stability condition in geometric invariant theory [2, 3, 4].

By the Painlevé property, any solution germ $Q \in \mathcal{M}_z(\kappa)$ continues analytically along any loop $\gamma \in \pi_1(Z, z)$. Let $\gamma_\ast Q$ be the result of the analytic continuation. Then the map

$$\gamma_\ast : \mathcal{M}_z(\kappa) \to \mathcal{M}_z(\kappa), \quad Q \mapsto \gamma_\ast Q,$$
is a holomorphic automorphism of $\mathcal{M}_z(\kappa)$ (see Figure 2), which is called the monodromy map along the loop $\gamma$. It represents the multi-valuedness along $\gamma$ of the solution germs.

3 Affine Weyl Groups and Stratification

The parameter space $\mathcal{K}$ of Painlevé VI admits some affine Weyl group actions, in terms of which $\mathcal{K}$ carries a natural stratification. We shall now describe these structures [6, 7, 8].

The standard complex Euclidean inner product on $\mathbb{C}_\kappa^4$ induces an inner product on $\mathcal{K}$ through the forgetful isomorphism $\mathcal{K}\to \mathbb{C}_\kappa^4$, $(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \mapsto (\kappa_1, \kappa_2, \kappa_3, \kappa_4)$. For each $i \in \{0,1,2,3,4\}$ let $w_i : \mathcal{K} \to \mathcal{K}$ be the orthogonal reflection in the affine hyperplane $H_i := \{ \kappa \in \mathcal{K} : \kappa_i = 0 \}$. These five reflections generate an affine Weyl group of type $D_{4}^{(1)}$, $W(D_{4}^{(1)}) = \langle w_0, w_1, w_2, w_3, w_4 \rangle \cap \mathcal{K}$.

Denote the nodes of the Dynkin diagram $D_{4}^{(1)}$ by $\{0,1,2,3,4\}$ as in Figure 3. The automorphism group of the Dynkin diagram $D_{4}^{(1)}$ is the symmetric group $S_4$ of degree 4 permuting $\{1,2,3,4\}$ while fixing the central node 0. The semi-direct product $W(F_{4}^{(1)}) := W(D_{4}^{(1)}) \rtimes S_4 \subset \mathcal{K}$ is an affine Weyl group of type $F_{4}^{(1)}$, which is the full symmetry group of Painlevé VI.

Given a proper subdiagram $\bullet$ of the Dynkin diagram $D_{4}^{(1)}$, let $I$ be a proper subset of $\{0,1,2,3,4\}$ representing $\bullet$. The closed stratum associated with $\bullet$ is then defined by

$\overline{\mathcal{K}}(\bullet) = \text{the } W(F_{4}^{(1)})\text{-translates of the affine subspace } H_I := \bigcap_{i \in I} H_i,$

Figure 3: Dynkin diagram of type $D_{4}^{(1)}$

Figure 4: Some strata and their Dynkin subdiagrams
which does not depend on the choice of the representative $I$. For $I = \emptyset$ one has the big open stratum $\mathcal{K}(\emptyset)$ and some other strata are given in Figure 4. The adjacency relations among the strata are depicted in Figure 5, where $\star \to \star$ indicates that $\overline{\mathcal{K}}(\star)$ is a subset of $\overline{\mathcal{K}}(\star)$. Let $\mathcal{K}(\star)$ be the relatively open stratum associated with the closed stratum $\overline{\mathcal{K}}(\star)$.

\[
\emptyset \longrightarrow A_1 \longrightarrow A_1^{\oplus 2} \longrightarrow A_1^{\oplus 3} \longrightarrow A_1^{\oplus 4} \\
\downarrow \quad \downarrow \quad \downarrow \\
A_2 \longrightarrow A_3 \longrightarrow D_4
\]

Figure 5: Adjacency relations among the strata

4 Riemann-Hilbert Correspondence

The study of Painlevé equation is developed not directly on the moduli space $\mathcal{M}_z(\kappa)$, but by passing to a character variety $S(\theta)$ via the Riemann-Hilbert correspondence [2, 3, 4, 8],

\[
\text{RH}_{z,\kappa} : \mathcal{M}_z(\kappa) \to S(\theta), \quad Q \mapsto \rho, \quad \text{with } \theta = \text{rh}(\kappa).
\]

(4)

Here the character varieties for Painlevé VI can be realized as a four-parameter family of complex affine cubic surfaces $S(\theta) = \{ x = (x_1, x_2, x_3) \in \mathbb{C}^3 : f(x, \theta) = 0 \}$ with

\[
f(x, \theta) := x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1 x_1 - \theta_2 x_2 - \theta_3 x_3 + \theta_4,
\]

(5)

parametrized by $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) \in \Theta := \mathbb{C}^4$ and $\text{rh} : \mathcal{K} \to \Theta$ is a holomorphic map which is a branched $W(D_4^{(1)})$-covering ramifying along Wall (the union of all reflection hyperplanes) and mapping it onto the discriminant locus $V := \{ \theta \in \Theta : \Delta(\theta) = 0 \}$ of the cubic surfaces (see Figure 6). A fundamental fact for the map (4) is the following.

Theorem 1 ([2, 3, 4]) If $\kappa \in \mathcal{K}(\star)$ then the character variety $S(\theta)$ with $\theta = \text{rh}(\kappa)$ has simple singularities of Dynkin type $\star$ and the Riemann-Hilbert correspondence (4) is a proper surjective holomorphic map that gives an analytic minimal resolution of $S(\theta)$.

![Diagram](image_url)

Figure 6: The Riemann-Hilbert correspondence in the parameter level
Take an algebraic minimal desingularization $\varphi : \tilde{\mathcal{S}}(\theta) \to S(\theta)$. Then the Riemann-Hilbert correspondence (4) uniquely lifts to a biholomorphism $\overline{\text{RH}}_{z,\kappa} : \mathcal{M}_{z}(\kappa) \to \tilde{\mathcal{S}}(\theta)$ such that

$$
\mathcal{M}_{z}(\kappa) \xrightarrow{\overline{\text{RH}}_{z,\kappa}} \tilde{\mathcal{S}}(\theta) \xrightarrow{\varphi} S(\theta)
$$

is commutative. Via the lifted Riemann-Hilbert correspondence $\overline{\text{RH}}_{z,\kappa}$, the monodromy map $\gamma : \mathcal{M}_{z}(\kappa) \to \tilde{S}(\theta)$ is strictly conjugate to an automorphism $\sigma : \tilde{S}(\theta) \to S(\theta)$ in a way shown below.

The cubic surface $S(\theta)$ admits three involutive automorphisms $\sigma_{i}, i = 1, 2, 3$, where

$$
\sigma_{1} : (x_{1}, x_{2}, x_{3}) \mapsto (\theta_{1} - x_{1} - x_{2}x_{3}, x_{2}, x_{3}), \quad (6)
$$

with $\sigma_{2}$ and $\sigma_{3}$ being defined in similar manners. They lift in a unique way to automorphisms of the desingularized surface $\tilde{S}(\theta)$, which will be denoted by the same symbols $\sigma_{i}$. On the other hand the fundamental group $\pi_{1}(Z, z)$ is represented as

$$
\pi_{1}(Z, z) = \langle \gamma_{1}, \gamma_{2}, \gamma_{3} | \gamma_{1}\gamma_{2}\gamma_{3} = 1 \rangle,
$$

where $\gamma_{i}, i = 1, 2, 3$, are the basic loops as in Figure 7. For each $i = 1, 2, 3$, the monodromy map along the loop $\gamma_{i}$ is conjugate to the automorphism $\sigma_{i+1}\sigma_{i}$ of $\tilde{S}(\theta)$, where the index $i$ should be considered modulo 3, via the lifted Riemann-Hilbert correspondence.

Let $G$ be the group generated by the three involutions $\sigma_{1}, \sigma_{2}, \sigma_{3}$. It is a universal Coxeter group of rank three, having the only relations $\sigma_{1}^{2} = \sigma_{2}^{2} = \sigma_{3}^{2} = 1$. Let $G(2)$ be the index-two subgroup of all even words in $G$. The last paragraph says that the monodromy action $\pi_{1}(Z, z) \curvearrowright \mathcal{M}_{z}(\kappa)$ is faithfully represented by the group action $G(2) \curvearrowright \tilde{S}(\theta)$. Thus the full group action $G \curvearrowright \tilde{S}(\theta)$ may be thought of as faithfully representing the “half-monodromy” action. The corresponding “half-loops” are depicted in Figure 8, where the half-loop corresponding to $\sigma_{i}$ is denoted by the same symbol $\sigma_{i}$ and $\omega := \exp(2\pi i/3)$. The choice of the two base points $-\omega$ and $-\omega^{2}$ is just for a later convenience (see Theorem 2).
5 The Markoff-Painlevé Transcendent

If we put \((x_1, x_2, x_3) = (-3m_1, -3m_2, -3m_3)\), then formula (5) implies that the Markoff cubic (1) is just the cubic surface \(S(\theta)\) with parameters \((\theta_1, \theta_2, \theta_3, \theta_4) = (0, 0, 0, 0)\) and the involution (2) agrees with the involution (6). Moreover we observe that

\[
(k_0, k_1, k_2, k_3, k_4) = \left(-\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\right) \in \mathcal{K}(A_1)
\]

lies over \(\theta = (0, 0, 0, 0)\) relative to the small Riemann-Hilbert correspondence \(\text{rh} : \mathcal{K} \to \Theta\).

The main theorem of this note is now stated as follows.

**Theorem 2** Via the Riemann-Hilbert correspondence (4), the Markoff orbit in Section 1 corresponds to all the analytic continuations of the solution germ to equation (3) with parameters (7) that satisfies the initial condition

\[
(q, p) = \left(\frac{i\omega^2}{\sqrt{3}}, 0\right) \quad \text{at} \quad z = -\omega.
\]

The proof of this theorem will be given elsewhere.

**References**


