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On the Markoff-Painlevé Transcendent

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Abstract

All positive integral solutions to Markoff's equation are in one-to-one correspondence with all analytic continuations of a transcendental solution germ to a special sixth Painlevé equation via the Riemann-Hilbert correspondence. We explicitly determine the parameter value and the initial condition for the Markoff-Painlevé transcendent.

1 Markoff's Diophantine Equation

In 1879 and 1880 A.A. Markoff [9, 10] discussed a Diophantine equation of the form

$$m_1^2 + m_2^2 + m_3^2 = 3m_1m_2m_3 \qquad (m_1, m_2, m_3) \in \mathbb{N}^3, \tag{1}$$

in the study of badly approximable irrational numbers and indefinite binary quadratic forms.

We present some known facts about Markoff's equation (1) (see e.g. [1]). It has the *trivial* solution $(1,1,1)$. It also has another simple solution $(1,1,2)$. These two solutions are referred to as the *exceptional* solutions. Any other solution is called a *regular* solution. Any regular solution has mutually distinct components. There are infinitely many solutions and there is a simple algorithm which produces all of them. It is based on a large symmetry $G = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ leaving equation (1) invariant, where σ_1 is the involution

$$\sigma_1 : (m_1, m_2, m_3) \mapsto (3m_2m_3 - m_1, m_2, m_3), \tag{2}$$

with σ_2 and σ_3 being defined in similar manners. Two solutions are said to be *neighbors* if they share two components. Any regular solution (m_1, m_2, m_3) has exactly three neighbors $\sigma_i(m_1, m_2, m_3)$, $i = 1, 2, 3$, one smaller and two larger, where the ordering is defined by

$$(m_1, m_2, m_3) \prec (m_1', m_2', m_3') \qquad \text{if} \qquad \max\{m_1, m_2, m_3\} < \max\{m_1', m_2', m_3'\}.$$

Starting with the trivial solution $(1,1,1)$, apply σ_1, σ_2, σ_3 recursively in all possible ways to produce infinitely many solutions. This process can be incorporated into a tree in Figure 1, which is known as *Markoff's tree*. Any solution occurs exactly once in the tree and the G-orbit through the trivial solution $(1,1,1)$ constitutes all the solutions to equation (1).

The aim of this note is to throw a bridge between the Markoff orbit and a very special solution to the sixth Painlevé equation via the Riemann-Hilbert correspondence.

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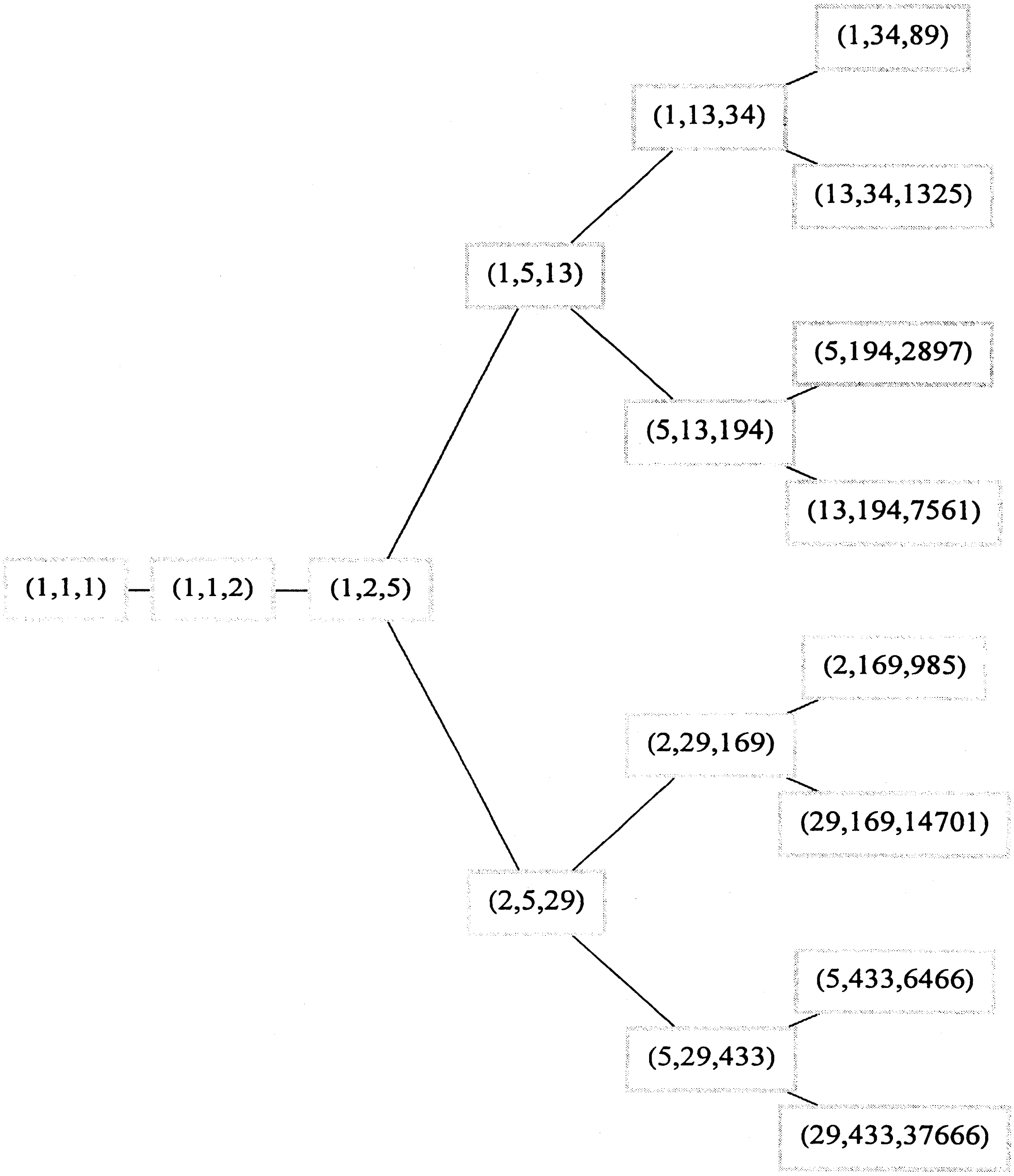

Figure 1: Markoff's tree

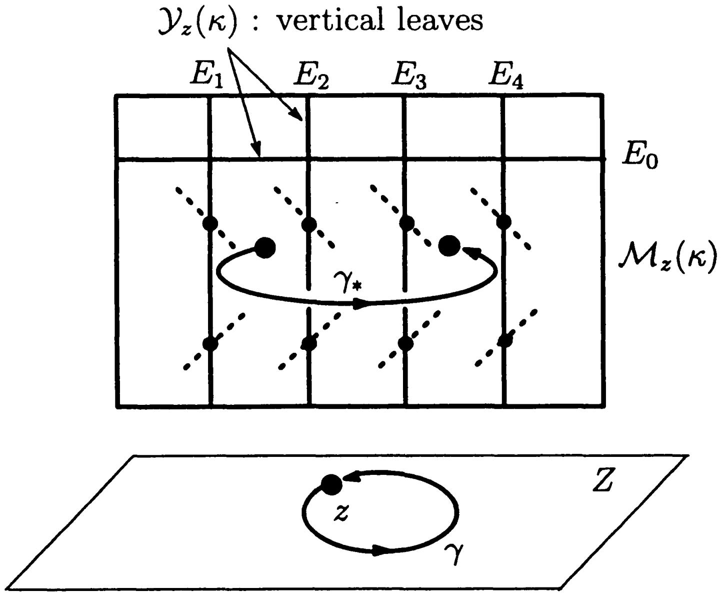

Figure 2: Monodromy map $\gamma_* : \mathcal{M}_z(\kappa) \circlearrowleft$ along a loop $\gamma \in \pi_1(Z, z)$.

2 The Sixth Painlevé Equation

The sixth Painlevé equation $\mathrm{P_{VI}}(\kappa)$ is a Hamiltonian system

$$\frac{dq}{dz} = \frac{\partial H(\kappa)}{\partial p}, \qquad \frac{dp}{dz} = -\frac{\partial H(\kappa)}{\partial q}, \tag{3}$$

with a complex time variable $z \in Z := \mathbb{P}^1 - \{0, 1, \infty\}$ and unknown functions $q = q(z)$ and $p = p(z)$, depending on complex parameters κ in the four-dimensional affine space

$$\mathcal{K} := \{ \kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathbb{C}^5_\kappa \, : \, 2\kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1 \},$$

where the Hamiltonian $H(\kappa) = H(q, p, z; \kappa)$ is given by

$$z(z-1)H(\kappa) = (q_0 q_z q_1)p^2 - \{\kappa_1 q_1 q_z + (\kappa_2 - 1) q_0 q_1 + \kappa_3 q_0 q_z\} p + \kappa_0(\kappa_0 + \kappa_4) q_z,$$

with $q_\nu := q - \nu$ for $\nu \in \{0, z, 1\}$. Any meromorphic solution germ at any point $z \in Z$ admits a global meromorphic continuation along any path in Z emanating from z. This property is known as the *Painlevé property* for the sixth Painlevé equation [2].

Let $\mathcal{M}_z(\kappa)$ be the set of all meromorphic solution germs to equation (3) at a base point $z \in Z$. It is realized as the moduli space of (certain) stable parabolic connections, thereby provided with the structure of a smooth quasi-projective rational complex surface, where a stable parabolic connection is a rank-two vector bundle over $\mathbb{P}^1$ together with a Fuchsian connection having four regular singular points and a parabolic structure that satisfies a sort of stability condition in geometric invariant theory [2, 3, 4].

By the Painlevé property, any solution germ $Q \in \mathcal{M}_z(\kappa)$ continues analytically along any loop $\gamma \in \pi_1(Z, z)$. Let $\gamma_* Q$ be the result of the analytic continuation. Then the map

$$\gamma_* : \mathcal{M}_z(\kappa) \to \mathcal{M}_z(\kappa), \qquad Q \mapsto \gamma_* Q,$$

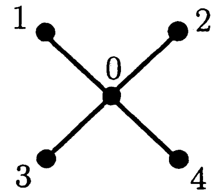

Figure 3: Dynkin diagram of type $D_4^{(1)}$

is a holomorphic automorphism of $\mathcal{M}_z(\kappa)$ (see Figure 2), which is called the monodromy map along the loop γ. It represents the multi-valuedness along γ of the solution germs.

3 Affine Weyl Groups and Stratification

The parameter space $\mathcal{K}$ of Painlevé VI admits some affine Weyl group actions, in terms of which $\mathcal{K}$ carries a natural stratification. We shall now describe these structures [6, 7, 8].

The standard complex Euclidean inner product on $\mathbb{C}_\kappa^4$ induces an inner product on $\mathcal{K}$ through the forgetful isomorphism $\mathcal{K} \to \mathbb{C}_\kappa^4$, $(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \mapsto (\kappa_1, \kappa_2, \kappa_3, \kappa_4)$. For each $i \in \{0, 1, 2, 3, 4\}$ let $w_i : \mathcal{K} \circlearrowleft$ be the orthogonal reflection in the affine hyperplane $H_i := \{\kappa \in \mathcal{K} \, : \, \kappa_i = 0\}$. These five reflections generate an affine Weyl group of type $D_4^{(1)}$,

$$W(D_4^{(1)}) = \langle w_0, w_1, w_2, w_3, w_4 \rangle \curvearrowright \mathcal{K}.$$

Denote the nodes of the Dynkin diagram $D_4^{(1)}$ by $\{0, 1, 2, 3, 4\}$ as in Figure 3. The automorphism group of the Dynkin diagram $D_4^{(1)}$ is the symmetric group S_4 of degree 4 permuting $\{1, 2, 3, 4\}$ while fixing the central node 0. The semi-direct product

$$W(F_4^{(1)}) := W(D_4^{(1)}) \rtimes S_4 \curvearrowright \mathcal{K}$$

is an affine Weyl group of type $F_4^{(1)}$, which is the full symmetry group of Painlevé VI.

Given a proper subdiagram $*$ of the Dynkin diagram $D_4^{(1)}$, let I be a proper subset of $\{0, 1, 2, 3, 4\}\}$ representing $*$. The closed stratum associated with $*$ is then defined by

$$\overline{\mathcal{K}}(*) = \text{the } W(F_4^{(1)})\text{-translates of the affine subspace } H_I := \bigcap_{i \in I} H_i,$$

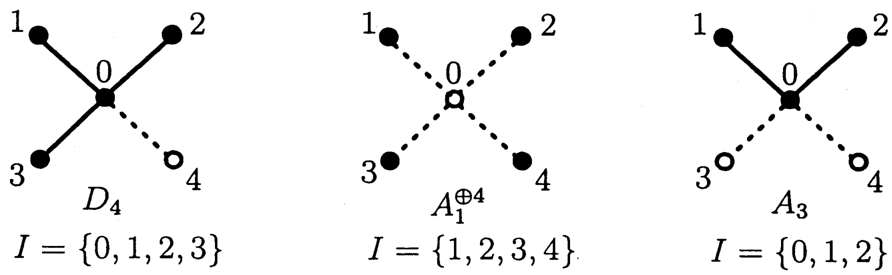

Figure 4: Some strata and their Dynkin subdiagrams

which does not depend on the choice of the representative I. For $I = \emptyset$ one has the big open stratum $\mathcal{K}(\emptyset)$ and some other strata are given in Figure 4. The adjacency relations among the strata are depicted in Figure 5, where $* \longrightarrow \star$ indicates that $\overline{\mathcal{K}}(\star)$ is a subset of $\overline{\mathcal{K}}(*)$. Let $\mathcal{K}(*)$ be the relatively open stratum associated with the closed stratum $\overline{\mathcal{K}}(*)$.

$$\begin{array}{ccccccccc} \emptyset & \longrightarrow & A_1 & \longrightarrow & A_1^{\oplus 2} & \longrightarrow & A_1^{\oplus 3} & \longrightarrow & A_1^{\oplus 4} \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & A_2 & \longrightarrow & A_3 & \longrightarrow & D_4 & & \end{array}$$

Figure 5: Adjacency relations among the strata

4 Riemann-Hilbert Correspondence

The study of Painlevé equation is developed not directly on the moduli space $\mathcal{M}_z(\kappa)$, but by passing to a character variety $\mathcal{S}(\theta)$ via the Riemann-Hilbert correspondence [2, 3, 4, 8],

$$\mathrm{RH}_{z,\kappa} : \mathcal{M}_z(\kappa) \to \mathcal{S}(\theta), \quad Q \mapsto \rho, \qquad \text{with} \quad \theta = \mathrm{rh}(\kappa). \tag{4}$$

Here the character varieties for Painlevé VI can be realized as a four-parameter family of complex affine cubic surfaces $\mathcal{S}(\theta) = \{ x = (x_1, x_2, x_3) \in \mathbb{C}^3 \, : \, f(x, \theta) = 0 \}$ with

$$f(x, \theta) := x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1 x_1 - \theta_2 x_2 - \theta_3 x_3 + \theta_4, \tag{5}$$

parametrized by $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) \in \Theta := \mathbb{C}^4$ and $\mathrm{rh} : \mathcal{K} \to \Theta$ is a holomorphic map which is a branched $W(D_4^{(1)})$-covering ramifying along **Wall** (the union of all reflection hyperplanes) and mapping it onto the discriminant locus $V := \{\theta \in \Theta \, : \, \Delta(\theta) = 0\}$ of the cubic surfaces (see Figure 6). A fundamental fact for the map (4) is the following.

Theorem 1 **([2, 3, 4])** *If $\kappa \in \mathcal{K}(*)$ then the character variety $\mathcal{S}(\theta)$ with $\theta = \mathrm{rh}(\kappa)$ has simple singularities of Dynkin type $*$ and the Riemann-Hilbert correspondence* (4) *is a proper surjective holomorphic map that gives an analytic minimal resolution of $\mathcal{S}(\theta)$.*

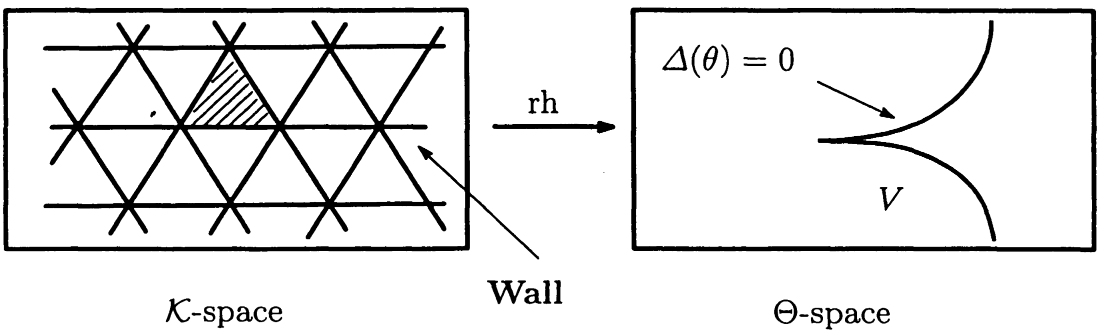

Figure 6: The Riemann-Hilbert correspondence in the parameter level

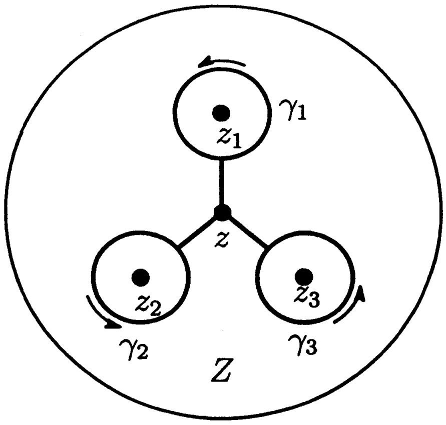

Figure 7: Three basic loops in $\pi_1(Z, z)$, where $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$.

Take an *algebraic* minimal desingularization $\varphi : \widetilde{\mathcal{S}}(\theta) \to \mathcal{S}(\theta)$. Then the Riemann-Hilbert correspondence (4) uniquely lifts to a biholomorphism $\widetilde{\mathrm{RH}}_{z,\kappa} : \mathcal{M}_z(\kappa) \to \widetilde{\mathcal{S}}(\theta)$ such that

$$\begin{array}{ccc} \mathcal{M}_z(\kappa) & \xrightarrow{\widetilde{\mathrm{RH}}_{z,\kappa}} & \widetilde{\mathcal{S}}(\theta) \\ \| & & \downarrow \varphi \\ \mathcal{M}_z(\kappa) & \xrightarrow{\mathrm{RH}_{z,\kappa}} & \mathcal{S}(\theta) \end{array}$$

is commutative. Via the lifted Riemann-Hilbert correspondence $\widetilde{\mathrm{RH}}_{z,\kappa}$, the monodromy map $\gamma_* : \mathcal{M}_z(\kappa) \circlearrowleft$ is strictly conjugate to an automorphism $\sigma : \widetilde{\mathcal{S}}(\theta) \circlearrowleft$ in a way shown below.

The cubic surface $\mathcal{S}(\theta)$ admits three involutive automorphisms σ_i, $i = 1, 2, 3$, where

$$\sigma_1 : (x_1, x_2, x_3) \mapsto (\theta_1 - x_1 - x_2 x_3, x_2, x_3), \qquad (6)$$

with σ_2 and σ_3 being defined in similar manners. They lift in a unique way to automorphisms of the desingularized surface $\widetilde{\mathcal{S}}(\theta)$, which will be denoted by the same symbols σ_i. On the other hand the fundamental group $\pi_1(Z, z)$ is represented as

$$\pi_1(Z, z) = \langle\, \gamma_1, \gamma_2, \gamma_3 \,|\, \gamma_1\gamma_2\gamma_3 = 1 \,\rangle,$$

where γ_i, $i = 1, 2, 3$, are the basic loops as in Figure 7. For each $i = 1, 2, 3$, the monodromy map along the loop γ_i is conjugate to the automorphism $\sigma_{i+1}\sigma_i$ of $\widetilde{\mathcal{S}}(\theta)$, where the index i should be considered modulo 3, via the lifted Riemann-Hilbert correspondence.

Let G be the group generated by the three involutions σ_1, σ_2, σ_3. It is a universal Coxeter group of rank three, having the only relations $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$. Let $G(2)$ be the index-two subgroup of all even words in G. The last paragraph says that the monodromy action $\pi_1(Z, z) \curvearrowright \mathcal{M}_z(\kappa)$ is faithfully represented by the group action $G(2) \curvearrowright \widetilde{\mathcal{S}}(\theta)$. Thus the full group action $G \curvearrowright \widetilde{\mathcal{S}}(\theta)$ may be thought of as faithfully representing the "half-monodromy" action. The corresponding "half-loops" are depicted in Figure 8, where the half-loop corresponding to σ_i is denoted by the same symbol σ_i and $\omega := \exp(2\pi i/3)$. The choice of the two base points $-\omega$ and $-\omega^2$ is just for a later convenience (see Theorem 2).

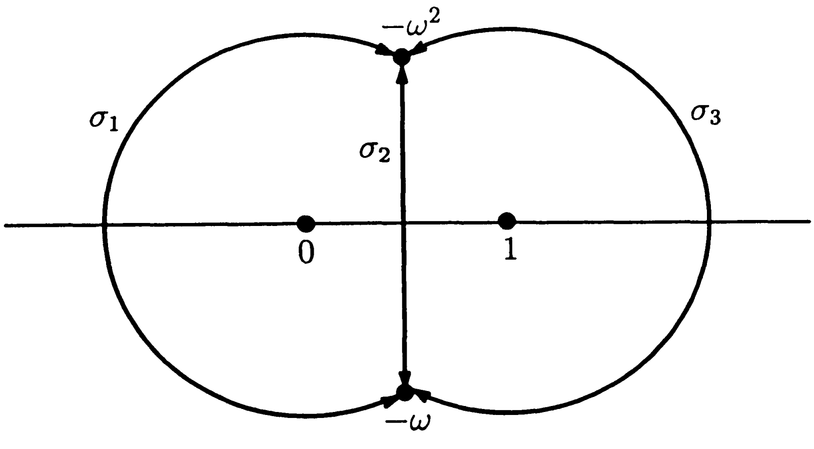

Figure 8: Three half-loops: the point at infinity is invisible

5 The Markoff-Painlevé Transcendent

If we put $(x_1, x_2, x_3) = (-3m_1, -3m_2, -3m_3)$, then formula (5) implies that the Markoff cubic (1) is just the cubic surface $\mathcal{S}(\theta)$ with parameters $(\theta_1, \theta_2, \theta_3, \theta_4) = (0, 0, 0, 0)$ and the involution (2) agrees with the involution (6). Moreover we observe that

$$(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) = \left(-\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\right) \in \mathcal{K}(A_1) \tag{7}$$

lies over $\theta = (0, 0, 0, 0)$ relative to the small Riemann-Hilbert correspondence $\mathrm{rh} : \mathcal{K} \to \Theta$.

The main theorem of this note is now stated as follows.

Theorem 2 *Via the Riemann-Hilbert correspondence* (4)*, the Markoff orbit in Section* 1 *corresponds to all the analytic continuations of the solution germ to equation* (3) *with parameters* (7) *that satisfies the initial condition*

$$(q, p) = \left(\frac{i\omega^2}{\sqrt{3}}, 0\right) \qquad at \quad z = -\omega.$$

The proof of this theorem will be given elsewhere.

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