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Kyoto University
Component-wise accumulation sets for Axiom A polynomial skew products

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1 Introduction

In this note, we consider Axiom A regular polynomial skew products on $\mathbb{C}^2$. It is of the form: $f(z, w) = (p(z), q(z, w))$, where $p(z) = z^d + \cdots$ and $q_z(w) = q(z, w) = w^d + \cdots$ are polynomials of degree $d \geq 2$. Then its k-th iterate is expressed by:

$$f^k(z, w) = (p^k(z), q^k_z(w)) =: (p^k(z), Q^k_z(w)).$$

Hence it preserves the family of fibers $\{z\} \times \mathbb{C}$ and this makes it possible to study its dynamics more precisely. Let $K$ be the set of points with bounded orbits and put $K_z := \{w \in \mathbb{C}; (z, w) \in K\}$ and $K_{p} := K \cap (J_p \times \mathbb{C})$. The fiber Julia set $J_z$ is the boundary of $K_z$.

Let $\Omega$ be the set of non-wandering points for $f$. Then $f$ is said to be Axiom A if $\Omega$ is compact, hyperbolic and periodic points are dense in $\Omega$. For polynomial skew products, Jonsson [J2] has shown that $f$ is Axiom A if and only if the following three conditions are satisfied:

(a) $p$ is hyperbolic,
(b) $f$ is vertically expanding over $J_p$,
(c) $f$ is vertically expanding over $A_p := \{\text{attracting periodic points of } p\}$.

Here $f$ is vertically expanding over $Z \subset \mathbb{C}$ with $p(Z) \subset Z$ if there exist $\lambda > 1$ and $C > 0$ such that $|(Q^k_z)'(w)| \geq C\lambda^k$ holds for any $z \in Z, w \in J_z$ and $k \geq 0$.

We are interested in the dynamics of $f$ on $J_p \times \mathbb{C}$ because the dynamics outside $J_p \times \mathbb{C}$ is fairly simple. Consider the critical set

$$C_{J_p} = \{(z, w) \in J_p \times \mathbb{C}; q^1_z(w) = 0\}$$

over the base Julia set $J_p$. Let $\mu$ be the ergodic measure of maximal entropy for $f$ (see Fornaess and Sibony [FS1]). Its support $J_2$ is called the second Julia set of $f$. Let $PC_{J_p} := \bigcup_{n \geq 1} f^n(C_{J_p})$ be the postcritical set of $C_{J_p}$. Jonsson [J2] has shown that
(d) \( J_2 = \bigcup_{z \in J_p} \{z\} \times J_z \) (Corollary 4.4),
(e) the condition (b) \( \iff \) \( PC_{J_p} \cap J_2 = \emptyset \) (Theorem 3.1),
(f) \( J_2 \) is the closure of the set of repelling periodic points of \( f \) (Corollary 4.7).

By the Birkhoff ergodic theorem, \( \mu \)-a.e. \( x \) has a dense orbit in \( J_2 \). Especially, \( J_2 = \text{supp} \mu \) is transitive. Hence \( J_2 \) coincides with the basic set of unstable dimension two. See also [FS2].

For any subset \( X \) in \( \mathbb{C}^2 \), its accumulation set is defined by
\[
A(X) = \cap_{n \geq 0} \overline{\bigcup_{n \geq 0} f^n(X)}.
\]

DeMarco & Hruska [DH1] defined the pointwise and component-wise accumulation sets of \( C_{J_p} \) respectively by
\[
A_{pt}(C_{J_p}) = \bigcup_{x \in C_{J_p}} A(x) \quad \text{and} \quad A_{cc}(C_{J_p}) = \bigcup_{C \in C(C_{J_p})} A(C),
\]
where \( C(C_{J_p}) \) denotes the collection of connected components of \( C_{J_p} \). It follows from the definition that
\[
A_{pt}(C_{J_p}) \subset A_{cc}(C_{J_p}) \subset A(C_{J_p}).
\]

It also follows that \( A_{pt}(C_{J_p}) = A_{cc}(C_{J_p}) \) if \( J_p \) is a Cantor set and \( A_{cc}(C_{J_p}) = A(C_{J_p}) \) if \( J_p \) is connected.

Let \( \Lambda \) be the closure of the set of saddle periodic points in \( J_p \times \mathbb{C} \). It decomposes into a disjoint union of saddle basic sets: \( \Lambda = \bigcup_{i=1}^m \Lambda_i \). Put \( \Lambda_z = \{w \in \mathbb{C}; (z, w) \in \Lambda\} \). The stable and unstable manifolds of \( \Lambda \) are respectively defined by
\[
W^s(\Lambda) = \{y \in \mathbb{C}^2; f^k(y) \to \Lambda\},
\]
\[
W^u(\Lambda) = \{y \in \mathbb{C}^2; \exists \text{ backward orbit } \hat{y} = (y_{-k}) \text{ tending to } \Lambda\}.
\]

**Theorem A.** ([DH1])
\[
A_{pt}(C_{J_p}) = \Lambda, \quad A(C_{J_p}) = W^u(\Lambda) \cap (J_p \times \mathbb{C}).
\]

**Theorem B.** ([DH1, DH2])
\[
A_{cc}(C_{J_p}) = A_{pt}(C_{J_p}) \quad \iff \quad \forall C \in C(C_{J_p}), C \cap K = \emptyset \text{ or } C \subset K. \quad (1)
\]
\[
A_{pt}(C_{J_p}) = A(C_{J_p}) \quad \iff \quad \text{the map } z \mapsto \Lambda_z \text{ is continuous in } J_p. \quad (2)
\]

Under the assumption \( W^u(\Lambda) \cap W^s(\Lambda) = \Lambda \),
\[
A_{pt}(C_{J_p}) = A(C_{J_p}) \iff \text{the map } z \mapsto \Lambda_z \text{ is continuous in } J_p. \quad (3)
\]
Note that
\[ W^u(\Lambda) \cap W^s(\Lambda) = \Lambda \iff W^u(\Lambda_i) \cap W^s(\Lambda_j) = \emptyset \text{ for any } 1 \leq i \neq j \leq m. \] (4)

Sumi [S] gives an example of Axiom A polynomial skew product which does not satisfy the condition in (4). It is also (incorrectly) described as Example 5.10 in [DH1].

We define a relation \( \succ \) among saddle basic sets by \( \Lambda_i \succ \Lambda_j \) if \( (W^u(\Lambda_i) \setminus \Lambda_i) \cap (W^s(\Lambda_j) \setminus \Lambda_j) \neq \emptyset \). A cycle is a chain of basic sets: \( \Lambda_i \succ \Lambda_j \succ \cdots \succ \Lambda_{i_\iota} = \Lambda_i \). For Axiom A open endomorphisms, there is no trivial cycle because \( W^u(\Lambda_i) \cap W^s(\Lambda_i) = \Lambda_i \) holds for any \( i \). See [J2], Proposition A.4. Jonsson has also shown that, for Axiom A polynomial skew products on \( \mathbb{C}^2 \), the non-wandering set \( \Omega \) coincides with the chain recurrent set \( \mathcal{R} \). This leads to the following lemma, which we use later.

**Lemma 1.** ([J2], Corollary 8.14) Axiom A polynomial skew products on \( \mathbb{C}^2 \) have no cycles.

Put
\[ C_0 := C_{J_p} \setminus K, \quad C_i := C_{J_p} \cap W^s(\Lambda_i) \ (1 \leq i \leq m). \]

We will try to give characterizations of the equalities \( A_{cc}(C_{J_p}) = A_{pt}(C_{J_p}) \) and \( A_{pt}(C_{J_p}) = A(C_{J_p}) \) in terms of \( C_i \).

**Lemma 2.** \( C_{J_p} = \sqcup_{i=0}^{m} C_i \).

**proof.** By Proposition 3.1 in Jonsson [J1], \( \hat{\Omega} \) has local product structure for open Axiom A endomorphisms. Theorem A implies \( A(x) \subset \Lambda \) for any \( x \in C_{J_p} \). If \( A(x) = \emptyset \), then \( x \in C_0 \). Otherwise there exist an \( n \) and \( y \in \Lambda \) such that \( f^n(x) \in W^s_{loc}(y) \). Hence \( A(x) \subset \Lambda_i \) if \( y \in \Lambda_i \). Thus we conclude \( C_{J_p} = \sqcup_{i=0}^{m} C_i \).

If we put \( \Lambda_0 = \emptyset \), we have \( A(C_i) \supset A_{pt}(C_i) = \Lambda_i \) for any \( i \geq 0 \).

**Theorem 1.**

\[ A_{cc}(C_{J_p}) = A_{pt}(C_{J_p}) \iff \forall C \in C(C_{J_p}), \ 0 \leq i \leq m \text{ such that } C \subset C_i. \] (5)

In terms of \( C_i \), the condition in (1) is expressed by
\[ \forall C \in C(C_{J_p}), \ C \subset C_0 \text{ or } C \subset \sqcup_{i=1}^{m} C_i. \]

Hence, if \( m = 1 \), that is, \( \Lambda \) itself is a basic set, then the condition in (5) coincides with that in (1). In general, the condition in (5) is stronger than that in (1).

We have another characterization of \( A_{pt}(C_{J_p}) = A(C_{J_p}) \) in terms of \( C_i \).
Theorem 2. For any $i \geq 0$, we have

$$A(C_i) = \Lambda_i \iff C_i \text{ is closed}.$$  \hfill (6)

Consequently we have

$$A_{pt}(C J_{\rho}) = A(C_{J_{\rho}}) \iff C_i \text{ is closed for any } i \geq 0.$$

As for the condition in (3), we have

Theorem 3. The following three conditions are equivalent to each other.
(a) $C_0$ is closed,
(b) $A(C_{J_{\rho}}) = W^u(\Lambda) \cap W^s(\Lambda)$,
(c) the map $z \mapsto K_z$ is continuous in $J_{\rho}$.

Note that Theorem 3 reproves the equivalence (3) in Theorem B. We also note that $A_{pt}(C J_{\rho}) = A(C_{J_{\rho}})$ is equivalent to

$$W^u(\Lambda) \cap (J_{\rho} \times \mathbb{C}) = W^u(\Lambda) \cap W^s(\Lambda) = \Lambda.$$

Corollary 1. Suppose $C_0$ is closed. Then,

$$W^u(\Lambda) \cap W^s(\Lambda) = \Lambda \iff C_i \text{ is closed for any } i \geq 1.$$

We do not know whether the assumption that $C_0$ is closed can be removed or not. The (⇒) part holds without this assumption.

The author would like to thank Hiroki Sumi for helpful discussion on his example.

2 Proofs of Theorems

First we prove Theorem 1. Note that $A_{pt}(C_{J_{\rho}}) = A_{cc}(C_{J_{\rho}})$ if and only if $A(C) \subset \Lambda$ for any $C \in C(C_{J_{\rho}})$.

(⇒) Suppose $C \in C(C_{J_{\rho}})$ intersects at least two of $C_i$. By Theorem B, (1), we may assume $C \subset \bigcup_{i=1}^{m} C_i$. Then, by Lemma 2, we have $C = \bigcup_{i=1}^{m} (C \cap C_i)$. If all $C \cap C_i$ are closed, it contradicts the connectivity of $C$. Thus at least one of them is not closed. We may assume that there exists a sequence $x_n \in C \cap C_i$ tending to $x_0 \in C \cap C_j$ for some $i \neq j$. Take a small open neighborhood $U_k$ of $\Lambda_k$ for $1 \leq k \leq m$ so that $f(U_k) \cap U_\ell = \emptyset$ for $k \neq \ell$. Since $x_0 \in C_j$, there exists a $K > 1$ such that $f^k(x_0) \in U_j$ for $k \geq K$. Then $f^K(x_n) \in U_j$ for large $n$. Since $x_n \in C_i$, the orbit of $x_n$ eventually leaves $U_j$. Hence put
$k_n := \min\{k \geq K; f^k(x_n) \notin U_j\}$. We will show $k_n \to \infty$. Otherwise, taking a subsequence, we may assume \{$k_n$\} is bounded. Then there is a subsequence $n_\ell$ such that $k_{n_\ell} = L$ for all $\ell$. That is, $f^L(x_{n_\ell}) \notin U_j$. Taking $\ell \to \infty$, we have $f^L(x_0) \notin U_j$, which contradicts $L \geq K$. Now let $y$ be an accumulation point of the sequence \{$f^{k_n}(x_n)$\}. From the definition of $U_k$, we have $y \notin U_k$, hence $y \notin \Lambda$. Since $y \in A(C)$, this implies $A_{cc}(C_{J_p})$ contains a point $y$ outside $\Lambda = A_{pt}(C_{J_p})$. Thus we conclude $A_{pt}(C_{J_p}) \neq A_{cc}(C_{J_p})$.

Moreover we can prove $y \in W^u(\Lambda_j)$. In fact, by taking subsequences if necessary, put $y_{-\ell} = \lim_{n \to \infty} f^{k_n-\ell}(x_n)$. Then \{$y_{-\ell}; \ell \geq 0$\} forms a backward orbit of $y$ in $\overline{U_j}$. By the local product structure of $\Omega$, we conclude $y_{-\ell} \to \Lambda_j$, hence $y \in W^u(\Lambda_j)$.

$(\Leftarrow)$ We have only to show that $A(C) \subset \Lambda_i$ if $C \subset C_i$. If $C \subset C_0$, then $A(C) = \emptyset$ since $C$ is compact. Suppose $C \subset C_i$ and there exists $x \in A(C) \setminus \Lambda_i$ for $i \geq 1$. By taking $U_i$ small, there exists a neighborhood $V$ of $x$ such that $V \cap U_i = \emptyset$. Since $x \in \bigcup_{k \geq N} f^k(C)$ for any $N \geq 0$, there exist $m_n \not\to \infty$ and $x_n \in C$ such that $f^{m_n}(x_n) \in V$, i.e. $f^{m_n}(x_n) \notin U_i$ for any $n$. Since $C$ is closed, we may assume $x_n$ tends to some $x_0 \in C \subset C_i$. As above, if we put $k_n := \min\{k \geq K; f^k(x_n) \notin U_i\}$, we have an accumulation point $y$ of \{$f^{k_n}(x_n)$\} outside $\Lambda$. By the above remark, $y \in W^u(\Lambda_i) \setminus \Lambda_i$. We have $y \notin W^s(\Lambda_i)$ because $W^u(\Lambda_i) \cap W^s(\Lambda_i) = \emptyset$. Since $y \in A(C)$, $y \in K_{J_p} \setminus J_2 = W^s(\Lambda)$. Thus $y$ must belong to $W^s(\Lambda_i)$ for some $i_1 \neq i$. That is, we have a sequence \{$f^{k_n}(x_n)$\} in $W^s(\Lambda_i)$ tending to $y \in W^u(\Lambda_i) \cap W^s(\Lambda_i)$, hence we have an order $\Lambda_i \succ \Lambda_{i_1}$.

By successively applying this argument, we have a chain of saddle basic sets:

$$\Lambda_i \succ \Lambda_{i_1} \succ \Lambda_{i_2} \succ \cdots, \quad i \neq i_1 \neq i_2 \neq \cdots.$$  

Since there exist only finitely many basic sets, we must have a cycle of them, which contradicts Lemma 1. This completes the proof of Theorem 1. $\Box$

We will prove Theorem 2. By the same argument as above, we have

**Lemma 3.** Let $i, j \geq 1$. If $\overline{C_i} \cap C_j \neq \emptyset$, then $A(C_i) \cap (W^u(\Lambda_j) \setminus \Lambda) \neq \emptyset$. If $C_i$ is closed, then $A(C_i) = \Lambda_i$.

Note that $A_{pt}(C_{J_p}) = A(C_{J_p})$ if and only if $A(C_i) \subset \Lambda$ for any $i$. We have only to show (6).

$(\Rightarrow)$ If $C_i$ for some $i$ is not closed, then there exists a $j \neq i$ such that $\overline{C_i} \cap C_j \neq \emptyset$. If $i \geq 1$, then $j \geq 1$ and by Lemma 3, $A(C_i)$ contains a point outside $\Lambda$. Suppose $C_0$ is not closed. Then there exists a sequence $x_n \in C_0$ tending to a point $x_0 \in C_i$ for some $i \geq 1$. For a fixed large $R > 0$, put $k_n := \min\{k \in \mathbb{N}; ||f^k(x_n)|| > R\}$. It is easy to see $k_n \to \infty$. (Otherwise,
\[ ||f^k(x_0)|| \geq R \text{ for some } L \geq K, \text{ which contradicts } x_0 \in C_i.\] Note that \( \{f^k(x_n)\} \) is bounded. Thus, if we take any one of its accumulation points \( y \), then \( y \in A(C_0) \setminus K_{J_p} \), hence \( A(C_0) \) intersects \( W^u(\Lambda) \setminus K_{J_p} \).

(\( \Leftarrow \)) By Lemma 3, it follows that, for \( i \geq 1 \), \( A(C_i) = \Lambda_i \) if \( C_i \) is closed. If \( C_0 \) is closed, it is compact, hence \( A(C_0) = \emptyset \). This completes the proof of Theorem 2. \( \Box \)

Now we prove Theorem 3.

(a) \( \Rightarrow \) (b) By Theorem 2, \( A(C_0) = \emptyset \) if \( C_0 \) is closed. Then

\[ A(C_{J_p}) = \bigcup_{i=1}^{n} A(C_i) \subset K_{J_p} \cap (W^u(\Lambda) \cap (J_p \times \mathbb{C})) = W^u(\Lambda) \cap W^s(\Lambda). \]

(b) \( \Rightarrow \) (a) As is shown in the proof of Theorem 2, if \( C_0 \) is not closed, then \( A(C_0) \) intersects \( W^u(\Lambda) \setminus K_{J_p} \). Thus \( A(C_{J_p}) \neq W^u(\Lambda) \cap W^s(\Lambda) \).

(c) \( \Rightarrow \) (a) Suppose \( C_0 \) is not closed. Then there exists a sequence \( x_n = (z_n, c_n) \in C_0 \) tending to a point \( x_0 = (z_0, c_0) \in C_i \) for some \( i \geq 1 \). Then there exists \( \delta > 0 \) such that \( \mathbb{D}(c_0, \delta) \subset \text{int } K_{z_0} \) since \( c_0 \in \text{int } K_{z_0} \). Note that the map \( z \mapsto J_z \) is continuous in \( J_p \). Hence, if \( z \) is close to \( z_0 \), we have either \( \mathbb{D}(c_0, \delta) \subset \text{int } K_z \) or \( \mathbb{D}(c_0, \delta) \cap K_z = \emptyset \). Since for large \( n \), \( c_n \in \mathbb{D}(c_0, \delta) \) is outside \( K_{z_n} \), we conclude that \( \mathbb{D}(c_0, \delta) \cap K_{z_n} = \emptyset \) for large \( n \). This implies the discontinuity of the map \( z \mapsto K_z \) at \( z = z_0 \).

(a) \( \Rightarrow \) (c) We use the argument in Lemma 3.7 of [J2]. Note that \( z \mapsto K_z \) is upper semi-continuous in \( J_p \). Hence if \( z \mapsto K_z \) is discontinuous at \( z = z_0 \), then it is not lower semi-continuous there. Thus there exist \( w_0 \in \text{int } K_{z_0} \) and \( \delta > 0 \) such that \( D(w_0, \delta) \cap K_z = \emptyset \) for \( z \neq z_0 \) close to \( z_0 \). Put \( (z_k, w_k) = f^k(z_0, w_0) \).

By Corollary 3.6 in [J2] (see also Theorem 3.3 and Lemma 3.2 in Comerford [C]), there exist \( k \) and a critical point \( c_k \) of \( q_{z_k} \) in the connected component \( U_{w_k} \) of \( \text{int } K_{z_k} \) containing \( w_k \) such that, for any \( \epsilon > 0 \), there exists an \( n \) so that \( |w_n - Q_{z_k}^{n-k}(c_k)| < \epsilon \). Since \( C_0 \) is closed, the set \( \bigcup_{i=1}^{n} C_i \ni (z_k, c_k) \) is away from \( C_0 \). Thus the critical point \( c'_k \) of \( q_{p^k(z)} \) close to \( c_k \) for \( z \) sufficiently close to \( z_0 \) also belongs to \( \text{int } K_{p^k(z)} \). For this \( n \), take \( z \) sufficiently close to \( z_0 \) so that \( |Q_{z}^{n}(w_0) - w_n| < \epsilon \) and that \( |Q_{p^k(z)}^{n-k}(c'_k) - Q_{z_k}^{n-k}(c_k)| < \epsilon \). Thus we have

\[
|Q_{z}^{n}(w_0) - Q_{p^k(z)}^{n-k}(c'_k)| \leq |Q_{z}^{n}(w_0) - w_n| + |w_n - Q_{z_k}^{n-k}(c_k)| + |Q_{z_k}^{n-k}(c_k) - Q_{p^k(z)}^{n-k}(c'_k)| < 3\epsilon.
\]
Since $Q_{\gamma}^{n}(w_0) \notin K_{p^n(z)}$ and $Q_{p^n(z)}^{n-k}(c'_k) \in \text{int} K_{p^n(z)}$, this implies the distance of the postcritical set from $J_2$ is less than $3\epsilon$. Since we can take $\epsilon$ arbitrarily small, this contradicts the fact that $f$ is Axiom A. This completes the proof of Theorem 3. \( \square \)

**Remark 1.** [DH1, DH2] has proved $(c) \Rightarrow (b)$.

**References**


