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Kyoto University
Component-wise accumulation sets for Axiom A polynomial skew products

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1 Introduction

In this note, we consider Axiom A regular polynomial skew products on $\mathbb{C}^{2}$. It is of the form $f(z, w) = (p(z), q(z, w))$, where $p(z) = z^{d} + \cdots$ and $q_{z}(w) = q(z, w) = w^{d} + \cdots$ are polynomials of degree $d \geq 2$. Then its k-th iterate is expressed by:

$$f^{k}(z, w) = (p^{k}(z), q_{p^{k-1}(z)} \circ \cdots \circ q_{z}(w)) =: (p^{k}(z), Q_{z}^{k}(w)).$$

Hence it preserves the family of fibers $\{z\} \times \mathbb{C}$ and this makes it possible to study its dynamics more precisely. Let $K$ be the set of points with bounded orbits and put $K_{z} := \{w \in \mathbb{C}; (z, w) \in K\}$ and $K_{p} := K \cap (J_{p} \times \mathbb{C})$. The fiber Julia set $J_{z}$ is the boundary of $K_{z}$.

Let $\Omega$ be the set of non-wandering points for $f$. Then $f$ is said to be Axiom A if $\Omega$ is compact, hyperbolic and periodic points are dense in $\Omega$. For polynomial skew products, Jonsson [J2] has shown that $f$ is Axiom A if and only if the following three conditions are satisfied:

(a) $p$ is hyperbolic,
(b) $f$ is vertically expanding over $J_{p}$,
(c) $f$ is vertically expanding over $A_{p} := \{\text{attracting periodic points of } p\}$.

Here $f$ is vertically expanding over $Z \subset \mathbb{C}$ with $p(Z) \subset Z$ if there exist $\lambda > 1$ and $C > 0$ such that $|(Q_{z}^{k})'(w)| \geq C\lambda^{k}$ holds for any $z \in Z, w \in J_{z}$ and $k \geq 0$.

We are interested in the dynamics of $f$ on $J_{p} \times \mathbb{C}$ because the dynamics outside $J_{p} \times \mathbb{C}$ is fairly simple. Consider the critical set

$$C_{J_{p}} = \{(z, w) \in J_{p} \times \mathbb{C}; q_{z}'(w) = 0\}$$

over the base Julia set $J_{p}$. Let $\mu$ be the ergodic measure of maximal entropy for $f$ (see Fornaess and Sibony [FS1]). Its support $J_{2}$ is called the second Julia set of $f$. Let $PC_{J_{p}} := \bigcup_{n \geq 1}f^{n}(C_{J_{p}})$ be the postcritical set of $C_{J_{p}}$. Jonsson [J2] has shown that
(d) $J_2 = \bigcup_{z \in J_\nu} \{z\} \times J_z$ (Corollary 4.4),
(e) the condition (b) \iff $PC_{J_\nu} \cap J_2 = \emptyset$ (Theorem 3.1),
(f) $J_2$ is the closure of the set of repelling periodic points of $f$ (Corollary 4.7).

By the Birkhoff ergodic theorem, $\mu$-a.e. $x$ has a dense orbit in $J_2$. Especially, $J_2 = \text{supp } \mu$ is transitive. Hence $J_2$ coincides with the basic set of unstable dimension two. See also [FS2].

For any subset $X$ in $\mathbb{C}^2$, its accumulation set is defined by
$$A(X) = \cap_{N \geq 0} \overline{\bigcup_{n \geq N} f^n(X)}.$$ DeMarco & Hruska [DH1] defined the pointwise and component-wise accumulation sets of $C_{J_\rho}$ respectively by
$$A_{pt}(C_{J_\rho}) = \overline{\bigcup_{x \in C_{J_\rho}} A(x)}$$ and
$$A_{cc}(C_{J_\rho}) = \overline{\bigcup_{C \in C(C_{J_\rho})} A(C)},$$ where $C(C_{J_\rho})$ denotes the collection of connected components of $C_{J_\rho}$. It follows from the definition that
$$A_{pt}(C_{J_\rho}) \subset A_{cc}(C_{J_\rho}) \subset A(C_{J_\rho}).$$

It also follows that $A_{pt}(C_{J_\rho}) = A_{cc}(C_{J_\rho})$ if $J_\rho$ is a Cantor set and $A_{cc}(C_{J_\rho}) = A(C_{J_\rho})$ if $J_\rho$ is connected.

Let $\Lambda$ be the closure of the set of saddle periodic points in $J_\rho \times \mathbb{C}$. It decomposes into a disjoint union of saddle basic sets: $\Lambda = \bigcup_{i=1}^m \Lambda_i$. Put $\Lambda_z = \{w \in \mathbb{C}; (z, w) \in \Lambda\}$. The stable and unstable manifolds of $\Lambda$ are respectively defined by
$$W^s(\Lambda) = \{y \in \mathbb{C}^2; f^k(y) \to \Lambda\},$$
$$W^u(\Lambda) = \{y \in \mathbb{C}^2; \exists \text{ backward orbit } \hat{y} = (y_{-k}) \text{ tending to } \Lambda\}.$$
Note that
\[ W^u(\Lambda) \cap W^s(\Lambda) = \Lambda \iff W^u(\Lambda_i) \cap W^s(\Lambda_j) = \emptyset \text{ for any } 1 \leq i \neq j \leq m. \tag{4} \]

Sumi [S] gives an example of Axiom A polynomial skew product which does not satisfy the condition in (4). It is also (incorrectly) described as Example 5.10 in [DH1].

We define a relation $\succ$ among saddle basic sets by $\Lambda_i \succ \Lambda_j$ if $(W^u(\Lambda_i) \setminus \Lambda_i) \cap (W^s(\Lambda_j) \setminus \Lambda_j) \neq \emptyset$. A cycle is a chain of basic sets: $\Lambda_i \succ \Lambda_{i+1} \succ \cdots \succ \Lambda_{i_k} = \Lambda_i$. For Axiom A open endomorphisms, there is no trivial cycle because $W^u(\Lambda_i) \cap W^s(\Lambda_i) = \Lambda_i$ holds for any $i$. See [J2], Proposition A.4. Jonsson has also shown that, for Axiom A polynomial skew products on $\mathbb{C}^2$, the non-wandering set $\Omega$ coincides with the chain recurrent set $\mathcal{R}$. This leads to the following lemma, which we use later.

**Lemma 1.** ([J2], Corollary 8.14) Axiom A polynomial skew products on $\mathbb{C}^2$ have no cycles.

Put
\[ C_0 := C_{J_p} \setminus K, \quad C_i := C_{J_p} \cap W^s(\Lambda_i) \quad (1 \leq i \leq m). \]

We will try to give characterizations of the equalities $A_{cc}(C_{J_p}) = A_{pt}(C_{J_p})$ and $A_{pt}(C_{J_p}) = A(C_{J_p})$ in terms of $C_i$.

**Lemma 2.** $C_{J_p} = \bigcup_{i=0}^{m} C_i$.

**proof.** By Proposition 3.1 in Jonsson [J1], $\hat{\Omega}$ has local product structure for open Axiom A endomorphisms. Theorem A implies $A(x) \subset \Lambda$ for any $x \in C_{J_p}$. If $A(x) = \emptyset$, then $x \in C_0$. Otherwise there exist an $n$ and $y \in \Lambda$ such that $f^n(x) \in W^s_{loc}(y)$. Hence $A(x) \subset \Lambda_i$ if $y \in \Lambda_i$. Thus we conclude $C_{J_p} = \bigcup_{i=0}^{m} C_i$. □

If we put $\Lambda_0 = \emptyset$, we have $A(C_i) \supset A_{pt}(C_i) = \Lambda_i$ for any $i \geq 0$.

**Theorem 1.**
\[ A_{cc}(C_{J_p}) = A_{pt}(C_{J_p}) \iff \forall C \in C(C_{J_p}), \ 0 \leq \exists i \leq m \text{ such that } C \subset C_i. \tag{5} \]

In terms of $C_i$, the condition in (1) is expressed by
\[ \forall C \in C(C_{J_p}), \quad C \subset C_0 \text{ or } C \subset \bigcup_{i=1}^{m} C_i. \]

Hence, if $m = 1$, that is, $\Lambda$ itself is a basic set, then the condition in (5) coincides with that in (1). In general, the condition in (5) is stronger than that in (1).

We have another characterization of $A_{pt}(C_{J_p}) = A(C_{J_p})$ in terms of $C_i$. 
Theorem 2. For any $i \geq 0$, we have

$$A(C_i) = \Lambda_i \iff C_i \text{ is closed}.$$  \hfill (6)

Consequently we have

$$A_{pt}(C_{J_{p}}) = A(C_{J_{p}}) \iff C_i \text{ is closed for any } i \geq 0.$$ 

As for the condition in (3), we have

Theorem 3. The following three conditions are equivalent to each other.
(a) $C_0$ is closed,
(b) $A(C_{J_{p}}) = W^u(\Lambda) \cap W^s(\Lambda)$,
(c) the map $z \mapsto K_z$ is continuous in $J_p$.

Note that Theorem 3 reproves the equivalence (3) in Theorem B. We also note that $A_{pt}(C_{J_{p}}) = A(C_{J_{p}})$ is equivalent to

$$W^u(\Lambda) \cap (J_p \times \mathbb{C}) = W^u(\Lambda) \cap W^s(\Lambda) = \Lambda.$$

Corollary 1. Suppose $C_0$ is closed. Then,

$$W^u(\Lambda) \cap W^s(\Lambda) = \Lambda \iff C_i \text{ is closed for any } i \geq 1.$$ 

We do not know whether the assumption that $C_0$ is closed can be removed or not. The $\Rightarrow$ part holds without this assumption.

The author would like to thank Hiroki Sumi for helpful discussion on his example.

2 Proofs of Theorems

First we prove Theorem 1. Note that $A_{pt}(C_{J_{p}}) = A_{cc}(C_{J_{p}})$ if and only if $A(C) \subset \Lambda$ for any $C \subset C(C_{J_{p}})$.

(\Rightarrow) Suppose $C \subset C(C_{J_{p}})$ intersects at least two of $C_i$. By Theorem B, (1), we may assume $C \subset \bigcup_{i=1}^{m} C_i$. Then, by Lemma 2, we have $C = \bigcup_{i=1}^{m} (C \cap C_i)$. If all $C \cap C_i$ are closed, it contradicts the connectivity of $C$. Thus at least one of them is not closed. We may assume that there exists a sequence $x_n \in C \cap C_i$ tending to $x_0 \in C \cap C_j$ for some $i \neq j$. Take a small open neighborhood $U_k$ of $\Lambda_k$ for $1 \leq k \leq m$ so that $f(U_k) \cap U_\ell = \emptyset$ for $k \neq \ell$. Since $x_0 \in C_j$, there exists a $K > 1$ such that $f^K(x_0) \in U_j$ for $k \geq K$. Then $f^K(x_n) \in U_j$ for large $n$. Since $x_n \in C_i$, the orbit of $x_n$ eventually leaves $U_j$. Hence put
Let $k_n := \min\{k \geq K; f^k(x_n) \notin U_j\}$. We will show $k_n \to \infty$. Otherwise, taking a subsequence, we may assume $\{k_n\}$ is bounded. Then there is a subsequence $n_\ell$ such that $k_{n_\ell} = L$ for all $\ell$. That is, $f^L(x_{n_\ell}) \notin U_j$. Taking $\ell \to \infty$, we have $f^L(x_0) \notin U_j$, which contradicts $L \geq K$. Now let $y$ be an accumulation point of the sequence $\{f^{k_n}(x_n)\}$. From the definition of $U_k$, we have $y \notin \cup U_k$, hence $y \notin \Lambda$. Since $y \in A(C)$, this implies $A_{cc}(C_{J_\rho})$ contains a point $y$ outside $\Lambda = A_{pt}(C_{J_\rho})$. Thus we conclude $A_{pt}(C_{J_\rho}) \neq A_{cc}(C_{J_\rho})$.

Moreover we can prove $y \in W^u(\Lambda_j)$. In fact, by taking subsequences if necessary, put $y_\ell = \lim_{n \to \infty} f^{k_n-\ell}(x_n)$. Then $\{y_\ell; \ell \geq 0\}$ forms a backward orbit of $y$ in $\overline{U}_j$. By the local product structure of $\Omega$, we conclude $y_\ell \to \Lambda_j$, hence $y \in W^u(\Lambda_j)$.

(\(
\Rightarrow\n\)) We have only to show that $A(C) \subset \Lambda_i$ if $C \subset C_i$. If $C \subset C_0$, then $A(C) = \emptyset$ since $C$ is compact. Suppose $C \subset C_i$ and there exists $x \in A(C) \setminus \Lambda_i$ for $i \geq 1$. By taking $U_i$ small, there exists a neighborhood $V$ of $x$ such that $V \cap U_i = \emptyset$. Since $x \in \cup_{k \geq N} f^k(C)$ for any $N \geq 0$, there exist $m_n \nearrow \infty$ and $x_n \in C$ such that $f^m(x_n) \in V$, i.e. $f^m(x_n) \notin U_i$ for any $n$. Since $C$ is closed, we may assume $x_n$ tends to some $x_0 \in C \subset C_i$. As above, if we put $k_n := \min\{k \geq K; f^k(x_n) \notin U_i\}$, we have an accumulation point $y$ of $\{f^{k_n}(x_n)\}$ outside $\Lambda$. By the above remark, $y \in W^u(\Lambda_i) \setminus \Lambda_i$. We have $y \notin W^s(\Lambda_i)$ because $W^u(\Lambda_i) \cap W^s(\Lambda_i) = \Lambda_i$. Since $y \in A(C)$, $y \in K_{J_\rho} \setminus J_2 = W^s(\Lambda)$. Thus $y$ must belong to $W^s(\Lambda_i)$ for some $i_1 \neq i$. That is, we have a sequence $\{f^{k_n}(x_n)\}$ in $W^s(\Lambda_i)$ tending to $y \in W^u(\Lambda_i) \cap W^s(\Lambda_i)$, hence we have an order $\Lambda_i \succ \Lambda_{i_1}$.

By successively applying this argument, we have a chain of saddle basic sets:

$$
\Lambda_i \succ \Lambda_{i_1} \succ \Lambda_{i_2} \succ \cdots, \quad i \neq i_1 \neq i_2 \neq \cdots.
$$

Since there exist only finitely many basic sets, we must have a cycle of them, which contradicts Lemma 1. This completes the proof of Theorem 1. \(\Box\)

We will prove Theorem 2. By the same argument as above, we have

**Lemma 3.** Let $i, j \geq 1$. If $\overline{C_i} \cap C_j \neq \emptyset$, then $A(C_i) \cap (W^u(\Lambda_j) \setminus \Lambda) \neq \emptyset$. If $C_i$ is closed, then $A(C_i) = \Lambda_i$.

Note that $A_{pt}(C_{J_\rho}) = A(C_{J_\rho})$ if and only if $A(C_i) \subset \Lambda$ for any $i$. We have only to show (6).

(\(\Rightarrow\)) If $C_i$ for some $i$ is not closed, then there exists a $j \neq i$ such that $\overline{C_i} \cap C_j \neq \emptyset$. If $i \geq 1$, then $j \geq 1$ and by Lemma 3, $A(C_i)$ contains a point outside $\Lambda$. Suppose $C_0$ is not closed. Then there exists a sequence $x_n \in C_0$ tending to a point $x_0 \in C_i$ for some $i \geq 1$. For a fixed large $R > 0$, put $k_n := \min\{k \in \mathbb{N}; ||f^k(x_n)|| > R\}$. It is easy to see $k_n \to \infty$. (Otherwise,
||f^L(x_0)|| \geq R \text{ for some } L \geq K, \text{ which contradicts } x_0 \in C_i.) \text{ Note that } \{f^{k_n}(x_n)\} \text{ is bounded. Thus, if we take any one of its accumulation points } y, \text{ then } y \in A(C_0) \setminus K_{J_p}, \text{ hence } A(C_0) \text{ intersects } W^u(\Lambda) \setminus K_{J_p}.

(\Leftarrow) \text{ By Lemma 3, it follows that, for } i \geq 1, A(C_i) = \Lambda_i \text{ if } C_i \text{ is closed. If } C_0 \text{ is closed, it is compact, hence } A(C_0) = \emptyset. \text{ This completes the proof of Theorem 2. } □

Now we prove Theorem 3.

(a) \Rightarrow (b) \text{ By Theorem 2, } A(C_0) = \emptyset \text{ if } C_0 \text{ is closed. Then }

A(C_{J_p}) = \cup_{i=1}^{m} A(C_i) \subset K_{J_p} \cap (W^u(\Lambda) \cap (J_p \times \mathbb{C})) = W^u(\Lambda) \cap W^s(\Lambda).

(b) \Rightarrow (a) \text{ As is shown in the proof of Theorem 2, if } C_0 \text{ is not closed, then } A(C_0) \text{ intersects } W^u(\Lambda) \setminus K_{J_p}. \text{ Thus } A(C_{J_p}) \neq W^u(\Lambda) \cap W^s(\Lambda).

(c) \Rightarrow (a) \text{ Suppose } C_0 \text{ is not closed. Then there exists a sequence } x_n = (z_n, c_n) \in C_0 \text{ tending to a point } x_0 = (z_0, c_0) \in C_i \text{ for some } i \geq 1. \text{ Then there exists } \delta > 0 \text{ such that } \mathbb{D}(c_0, \delta) \subset int K_{z_0} \text{ since } c_0 \in int K_{z_0}. \text{ Note that the map } z \mapsto J_z \text{ is continuous in } J_p. \text{ Hence, if } z \text{ is close to } z_0, \text{ we have either } \mathbb{D}(c_0, \delta) \subset int K_z \text{ or } \mathbb{D}(c_0, \delta) \cap K_z = \emptyset. \text{ Since for large } n, \text{ } c_n \in \mathbb{D}(c_0, \delta) \text{ is outside } K_{z_n}, \text{ we conclude that } \mathbb{D}(c_0, \delta) \cap K_{z_n} = \emptyset \text{ for large } n. \text{ This implies the discontinuity of the map } z \mapsto K_z \text{ at } z = z_0.

(a) \Rightarrow (c) \text{ We use the argument in Lemma 3.7 of [J2]. Note that } z \mapsto K_z \text{ is upper semi-continuous in } J_p. \text{ Hence if } z \mapsto K_z \text{ is discontinuous at } z = z_0, \text{ then it is not lower semi-continuous there. Thus there exist } w_0 \in int K_{z_0} \text{ and } \delta > 0 \text{ such that } D(w_0, \delta) \cap K_z = \emptyset \text{ for } z \neq z_0 \text{ close to } z_0. \text{ Put } (z_k, w_k) = f^k(z_0, w_0). \text{ By Corollary 3.6 in [J2] (see also Theorem 3.3 and Lemma 3.2 in Comerford [C]), there exist } k \text{ and a critical point } c_k \text{ of } q_{z_k} \text{ in the connected component } U_{w_k} \text{ of } int K_{z_k} \text{ containing } w_k \text{ such that, for any } \epsilon > 0, \text{ there exists an } n \text{ so that } |w_n - Q_{z_k}^{n-k}(c_k)| < \epsilon. \text{ Since } C_0 \text{ is closed, the set } \cup_{i=1}^{m} C_i \ni (z_k, c_k) \text{ is away from } C_0. \text{ Thus the critical point } c_k' \text{ of } q_{p^k(z)} \text{ close to } c_k \text{ for } z \text{ sufficiently close to } z_0 \text{ also belongs to } int K_{p^k(z)}. \text{ For this } n, \text{ take } z \text{ sufficiently close to } z_0 \text{ so that } |Q_{p^k(z)}^{n-k}(c_k') - Q_{z_k}^{n-k}(c_k)| < \epsilon. \text{ Thus we have }

|Q_{z_k}^{n}(w_0) - Q_{p^k(z)}^{n-k}(c_k')| \leq |Q_{z_k}^{n}(w_0) - w_n| + |w_n - Q_{z_k}^{n-k}(c_k)| + |Q_{z_k}^{n-k}(c_k) - Q_{p^k(z)}^{n-k}(c_k')| < 3\epsilon.
Since \( Q_\gamma^n(w_0) \notin K_{p^n(z)} \) and \( Q_{p^k(z)}^{n-k}(c'_k) \in \text{int} K_{p^t(z)} \), this implies the distance of the postcritical set from \( J_2 \) is less than \( 3\epsilon \). Since we can take \( \epsilon \) arbitrarily small, this contradicts the fact that \( f \) is Axiom A. This completes the proof of Theorem 3. \( \square \)

**Remark 1.** [DH1, DH2] has proved \( (c) \Rightarrow (b) \).

**References**


