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RESEARCH ANNOUNCEMENT: STATISTICAL PROPERTIES OF ONE-DIMENSIONAL MAPS UNDER WEAK HYPERBOLICITY ASSUMPTIONS

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1. INTRODUCTION

A general problem in the theory of dynamical systems is to describe the asymptotic behavior of most trajectories of a given system. From a probabilistic point of view, a dynamical system is well understood if the asymptotic distribution of almost all trajectories is described by finitely many invariant probability measures, with good geometric and statistical properties. Such an approach has been popularized by the Russian school since the 1960s and it has been successfully applied to uniformly hyperbolic dynamical systems by the pioneering work of Sinai, Ruelle, and Bowen [Sin72, Rue76, Bow75]. To generalize these results to non-uniformly hyperbolic dynamical systems has been, and continues to be, one of the main themes of research on dynamical systems.

In this paper, we announce results on existence of physical measures and their geometric and statistical properties for a large class of real and complex one-dimensional maps.

Given a continuous map $f : X \to X$ acting on a compact metric space X, an invariant probability Borel measure ν is called *mixing*, if for all $\varphi, \psi \in L^2(\nu)$, we have

$$\mathscr{C}_n(\varphi, \psi) := \int_X \varphi \circ f^n \psi d\nu - \int_X \varphi d\nu \int_X \psi d\nu \to 0 \text{ as } n \to \infty.$$

The sequence

$$\mathscr{C}_n = \sup\{\mathscr{C}_n(\varphi, \psi) : \|\varphi\|_\infty \leq 1, \mathrm{Lip}(\psi) \leq 1\},$$

measures the *speed of mixing (decay of correlations)* of the system (X, ν), where $\mathrm{Lip}(\cdot)$ denotes the best Lipschitz constant. Our main result is that for a real or complex one-dimensional map satisfying a weak hyperbolicity assumption, the speed of mixing is faster than any polynomial.

In the following, we use $\mathscr{A}_\mathbb{R}$ to denote the collection of all C^3 interval maps with non-flat critical points and use $\mathscr{A}_\mathbb{C}$ to denote the collection of all complex rational maps of degree at least 2. For $f \in \mathscr{A} := \mathscr{A}_\mathbb{R} \cup \mathscr{A}_\mathbb{C}$, we

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use $J(f)$ to denote the Julia set, $\mathrm{Crit}(f)$ the set of critical points, and denote $\mathrm{Crit}'(f) = \mathrm{Crit}(f) \cap J(f)$. Moreover, for $c \in \mathrm{Crit}(f)$ we use ℓ_c to denote the order of f at the critical point c and let

$$\ell_{\max}(f) = \max\{\ell_c : c \in \mathrm{Crit}'(f)\}.$$

We use $\mathscr{A}(\ell)$ to denote the collection of maps $f \in \mathscr{A}$ with $\ell_{\max}(f) \leq \ell$.

The main assumption we shall make is the following "backward contraction" condition, introduced in [RL07] in the case of rational maps, and in [BRLSvS08] in the case of interval maps. Let $f \in \mathscr{A}$ be given. When studying $f \in \mathscr{A}_{\mathbb{R}}$, we use the standard metric on the interval $\mathrm{dom}(f) \subset \mathbb{R}$, while when studying $f \in \mathscr{A}_{\mathbb{C}}$, we shall use the spherical metric on $\overline{\mathbb{C}}$. In both cases we use Leb to denote the corresponding Lebesgue measure. For a critical point c and $\delta > 0$ we denote by $\widetilde{B}(c, \delta)$ the connected component of $f^{-1}(B(f(c), \delta))$ containing c.

Definition 1.1. Given a constant $r > 1$ we will say a map $f \in \mathscr{A}$ is *backward contracting with constant* r if there is $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$, every $c \in \mathrm{Crit}'(f)$, every integer $m \geq 1$, and every connected component W of $f^{-m}(\widetilde{B}(c, r\delta))$,

$$W \cap \widetilde{B}(\mathrm{Crit}'(f), \delta) \neq \emptyset \text{ implies } \mathrm{diam}(f(W)) < \delta.$$

Furthermore, we say that f is *backward contracting*, if for every $r > 1$ it is backward contracting with constant r.

Definition 1.2. We say that a map $f \in \mathscr{A}$ is *expanding away from critical points*, if for every neighborhood V' of $\mathrm{Crit}'(f)$ the map f is uniformly expanding on the set

$$\{z \in J(f) \mid \text{ for every } n \geq 0,\ f^n(z) \notin V'\}.$$

It is known that a map f is backward contracting and expanding away from critical points if one of the following holds:

1. $f \in \mathscr{A}_{\mathbb{R}}$, f has no neutral periodic points, and for all $c \in \mathrm{Crit}'(f)$, we have $|Df^n(f(c))| \to \infty$ as $n \to \infty$;
2. $f \in \mathscr{A}_{\mathbb{C}}$ is a polynomial that is at most finitely renormalizable, has only hyperbolic periodic points, and is such that for all $c \in \mathrm{Crit}'(f)$, we have $|Df^n(f(c))| \to \infty$ as $n \to \infty$;

2'. $f \in \mathscr{A}_{\mathbb{C}}$ is a rational map without parabolic periodic points, and such that for all $c \in \mathrm{Crit}'(f)$, we have $\sum_{n=0}^{\infty} |Df^n(f(c))|^{-1} < +\infty$.

Main Theorem (restricted version). *For each $\ell > 1$, $\gamma > 0$ and $p \in (1, \ell/(\ell - 1))$, there exists $r > 1$ such that the following holds. Let $f \in \mathscr{A}(\ell)$ be a map with the following properties:*

- *f is topologically exact on* $\mathrm{dom}(f)$ *(so that $J(f) =$* $\mathrm{dom}(f)$*);*
- *f is expanding away from critical points;*
- *f is backward contracting with constant r.*

Then f has a unique invariant probability measure ν that is absolutely continuous with respect to the Lebesgue measure. Moreover, $d\nu/d\,\mathrm{Leb} \in L^p(\mathrm{Leb})$ and the measure ν is polynomially mixing with exponent γ.

The invariant measure ν is a physical measure. In fact, its basin has full Lebesgue measure in $\mathrm{dom}(f)$, *i.e.*, for Lebesgue a.e. $x \in \mathrm{dom}(f)$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x)) \to \int \varphi d\nu, \text{ as } n \to \infty$$

for every real-valued continuous map φ.

In the real case, the existence of ν and the estimate of the density function was proved in [BRLSvS08] and the result on mixing rate significantly strengthens the previous result in [BLVS03]. In the complex case, the existence of ν strengthens results in [GS09], and the other results are new.

We shall also work with maps whose Julia set has a fractal nature. In this case, under our backward contraction assumption, the Julia set always has Lebesgue measure zero. We prove an analogous result with the Lebesgue measure replaced by a conformal measure of exponent $\mathrm{HD}(J(f))$ supported on the Julia set.

For a map f in $\mathscr{A}$ and $\alpha > 0$, a *conformal measure of exponent* α for f is a Borel probability measure on $\mathrm{dom}(f)$ such that for each Borel set U on which f is injective we have

$$\mu(f(U)) = \int_U |Df|^\alpha d\mu.$$

On the other hand, the *conical Julia set* $J_{\mathrm{con}}(f)$ *of* f is the set of all those points $x \in J(f)$ for which there is $\delta > 0$ and infinitely many integers $m \geq 1$ satisfying the following property: f^m induces a diffeomorphism between the connected component of $f^{-m}(B(f^m(x), \delta))$ containing x and $B(f^m(x), \delta)$.

Main Theorem (general version). *For every $\ell > 1, h > 0, \varepsilon > 0$, $\gamma > 1$ and $p \in \left(0, \frac{\ell}{\ell-1}\right)$ there is $r > 1$ such that the following properties hold. Let $f \in \mathscr{A}(\ell)$ be backward contracting with constant r, expanding away from critical points, and such that $\mathrm{HD}(J(f)) \geq h$. Suppose furthermore in the case $f \in \mathscr{A}_{\mathbb{R}}$ that f is topologically exact on the Julia set. Then the following hold:*

1. *There is a conformal measure μ of exponent $\mathrm{HD}(f)$ which is ergodic, supported on $J_{\mathrm{con}}(f)$ and satisfies $\mathrm{HD}(\mu) = \mathrm{HD}(J(f)) = \mathrm{HD}_{\mathrm{hyp}}(f)$. Any other conformal measure supported on $J(f)$ is of exponent strictly larger than $\mathrm{HD}(J(f))$. Furthermore, for every sufficiently small $\delta > 0$ we have for every $x \in J(f)$,*

$$\mu(B(x, \delta)) \leq \delta^{\mathrm{HD}(J(f)) - \varepsilon}. \tag{1.1}$$

2. *There is a unique invariant probability measure ν that is absolutely continuous with respect to μ, and this invariant measure is polynomially mixing of exponent γ. Furthermore, the density of ν with respect to μ belongs to $L^p(\mu)$.*

Recall that for $f \in \mathscr{A}_{\mathbb{C}}$, the map $f : J(f) \to J(f)$ is always topologically exact. In the above theorem, if $J(f)$ has positive Lebesgue measure, then $J(f) = \mathrm{dom}(f)$, and the measure μ is proportional to the Lebesgue measure. In fact, this is already the case if $J(f)$ has same Hausdorff dimension as the domain of f. See part 1 of Corollary D in §3.1.

The proof of the Main Theorem is divided into two independent parts. In the first part we show two properties of backward contracting maps. The first – and rather surprising – property is that the components of the preimages of a small set shrink at least at a super-polynomial rate, provided the map is "expanding away from critical points" (Theorem A in §2.1). The second property of backward contracting maps that we show is an upper bound on a parameter we call "badness exponent". We show the badness exponent of a backward contracting map is zero (Theorem B in §2.2). In the second part of the proof of the Main Theorem, we study geometric and statistical properties of maps which are polynomially shrinking with a sufficiently large exponent, and that have a sufficiently small badness exponent. We exploit several ideas from [PRL07] and [PRL08] to prove tail estimates for the canonical induced Markov mappings associated with a "nice couple", from which we deduce the existence of a geometric conformal measure and a polynomially mixing absolutely continuous invariant measure. We shall use the badness exponent to give an upper bound on the Poincaré series through which we deduce the L^p estimate on the density of the absolutely continuous invariant measure, see §3.2. As by-products, we prove equality of various fractal dimensions of the Julia set and a removability result of Julia set in the complex case.

2. Two properties of backward contracting maps

In this section we state two results about backward contracting maps (Theorem A in §2.1 and Theorem B in §2.2). We first introduce some notation and terminology.

We say that a map $f \in \mathscr{A}_{\mathbb{R}}$ is *boundary-anchored*, if for each $x \in \partial\,\mathrm{dom}(f)$, we have $f(x) \in \partial\,\mathrm{dom}(f)$ and $Df(x) \neq 0$. Denote by $\mathscr{A}_{\mathbb{R}}^o$ the collection of all boundary-anchored maps in $\mathscr{A}_{\mathbb{R}}$, and denote $\mathscr{A}^o = \mathscr{A}_{\mathbb{R}}^o \cup \mathscr{A}_{\mathbb{C}}$.

We say that $f \in \mathscr{A}_{\mathbb{R}}$ is *essentially topologically exact on* $J(f)$ if there exists a forward invariant compact interval X_0 containing all critical points of f such that $f : J(f|X_0) \to J(f|X_0)$ is topologically exact and such that the interior of the compact interval $\mathrm{dom}(f)$ is contained in $\bigcup_{n=0}^{\infty} f^{-n}(X_0)$. When considering $f_0 : X_0 \to X_0$ in $\mathscr{A}_{\mathbb{R}}$, it is often more convenient to extend it to a boundary-anchored map $f : X \to X$. By choosing the extension carefully, we can assume that

- $\mathrm{Crit}(f) = \mathrm{Crit}(f_0)$, and $|Df(x)| > 1$ for $x \in \partial X$;
- $X \setminus \partial X = \bigcup_{n=0}^{\infty} f^{-n}(X_0)$;
- if $c \in \partial X_0$ is a critical point of f_0, then it is a turning point of f, and $f(c+\varepsilon) = f(c-\varepsilon)$ for $\varepsilon > 0$ small enough.

In particular, if f_0 is backward contracting with constant r, or if f_0 is expanding away from critical points, then so is f. Furthermore, if f_0 is topologically exact on $J(f_0)$, then f is essentially topologically exact on $J(f)$.

2.1. **Polynomial shrinking of components.** To state the first property of the maps appearing in the Main Theorem, we first give the following definition.

Definition 2.1. Given a sequence $\Theta = \{\theta_n\}_{n=1}^{\infty}$ of positive numbers, we say that a map $f \in \mathscr{A}$ satisfies the *Θ-Shrinking Condition*, if there exists constants $\rho > 0$ and $C > 0$ such that for every $x \in J(f)$, and every integer $m \geq 1$, the connected component W of $f^{-m}(B(f^m(x), \rho))$ containing x satisfies

$$\mathrm{diam}(W) \leq C\theta_m.$$

Given $\beta > 0$ we say that f satisfies the *Polynomial Shrinking Condition with exponent β*, if f satisfies the Θ-shrinking condition with $\Theta := \{n^{-\beta}\}_{n=1}^{\infty}$.

Theorem A. *For every $\ell > 1$ and $\beta > 0$ there is $r > 1$ such that each map in $\mathscr{A}(\ell)$ that is expanding away from critical points and that is backward contracting with constant r satisfies the Polynomial Shrinking Condition with exponent β.*

To prove this theorem, we use the backward contracting property to control the size of a pull back of a small ball when it comes close to critical points and use the "expanding away from critical points" property otherwise. In the real case, we introduce a technique called "*preferred quasi-chain*" to deal with the situation that a critical value lies close to the boundary of an interval.

The following result is a direct consequence of Theorem A, of [Mih08, Theorem 2], and of [RL07, Corollary 8.3].

Corollary 2.2 (Local connectivity). *For every integer $\ell \geq 2$ there is $r > 1$ such that for any $f \in \mathscr{A}_{\mathbb{C}}$ with $\ell_{\max}(f) \leq \ell$ that is backward contracting with constant r and has no parabolic periodic points, the Julia set of f is locally connected when it is connected.*

2.2. **Bounding the badness exponent.** The second property of the maps appearing in the Main Theorem that we show is a bound on a parameter we call "badness exponent" (Definition 2.5). In order to define it, let us start introducing "nice sets". For $f \in \mathscr{A}$, a set V, and an integer $m \geq 1$, each connected component W of $f^{-m}(V)$ is called a *pull-back of V by f^m*.

Definition 2.3. For a map $f \in \mathscr{A}_{\mathbb{R}}^{o}$ (resp. $f \in \mathscr{A}_{\mathbb{C}}$), we will say that $V \subset \mathrm{dom}(f)$ is a *nice set* if the following hold:

- $\overline{V}$ is disjoint from the forward orbits of critical points not in $J(f)$ and periodic orbits not in $J(f)$;
- each connected component of V is an open interval (resp. topological disk) and contains precisely one critical point of f in $J(f)$;
- for every integer $n \geq 1$ we have $f^n(\partial V) \cap V = \emptyset$.

For $c \in \mathrm{Crit}'(f)$ we denote by V^c the connected component of V containing c. A nice set V is called *symmetric* if for each $c \in \mathrm{Crit}'(f)$ we have $f(\partial V^c) \subset \partial f(V^c)$. Moreover, a *nice couple for* f is a pair of nice sets $(\widehat{V}, V)$ such that $\overline{V} \subset \widehat{V}$, and such that each pull-back of $\widehat{V}$ intersecting V is contained in V.

The following fact is proved for maps in $\mathscr{A}_{\mathbb{C}}$ in [RL07, Proposition 6.6].

Fact 2.4. *For each $\ell > 1$ there is a constant $r > 1$ such that each $f \in \mathscr{A}^o(\ell)$ that is backward contracting with constant r possesses arbitrarily small (symmetric) nice couples.*

Fix $f \in \mathscr{A}^o$ and a set V. For a component W of V, we define $d_V(W) = 1$. If W is a pull-back of V by f^m, we define an integer $d_V(W) \geq 1$ in the following way:

- If f is a rational map, then $d_V(W)$ is the degree of $f^m : W \to f^m(W)$, *i.e.*, the maximal cardinality of $f^{-m}(x) \cap W$ for $x \in V$.
- If f is an interval map, then $d_V(W) := 2^N$, where N is the number of those $j \in \{0, \ldots, m-1\}$ such that the connected component of $f^{-(m-j)}(V)$ containing $f^j(W)$ intersects $\mathrm{Crit}(f)$.

Let V be an open set and let W be a pull-back of V by f^m. If f^m is a diffeomorphism between W and a component of V, then we say that W is a *diffeomorphic pull-back of* V.

Definition 2.5. Given $f \in \mathscr{A}$ and an open set V, we will say that a pull-back W of V by f^m, $m \geq 1$, is *bad*, if for every integer $m' \in \{1, \ldots, m\}$ such that $f^{m'}(W) \subset V$ the pull-back of V by $f^{m'}$ containing W is not diffeomorphic. Furthermore we denote by $\mathfrak{B}_m(V)$ the collection of all bad pull-backs of V by f^m and put

$$\delta_{\mathrm{bad}}(V) := \inf \left\{ t > 0 : \sum_{m=1}^{\infty} \sum_{W \in \mathfrak{B}_m(V)} d_V(W) \operatorname{diam}(W)^t < \infty \right\}.$$

The *badness exponent of* f is defined as

$$\delta_{\mathrm{bad}}(f) := \inf\{\delta_{\mathrm{bad}}(V) : V \text{ is a nice set of } f\}. \tag{2.1}$$

We prove that $\delta_{\mathrm{bad}}(V) \leq \delta_{\mathrm{bad}}(V')$ for any nice sets $V \subset V'$. Thus if we have a sequence of nice sets $V_1 \supset V_2 \supset \cdots \searrow \mathrm{Crit}'(f)$, then $\delta_{\mathrm{bad}}(f) = \lim_{n\to\infty} \delta_{\mathrm{bad}}(V_n)$.

Theorem B. *For every $\ell > 1$ and $t > 0$ there is a constant $r \geq 2$ such that for each map $f \in \mathscr{A}(\ell)$ that is backward contracting with constant r, we have $\delta_{bad}(f) < t$.*

To prove this theorem, we estimate the total size of "relatively bad pull-backs" of a sequence of nice sets with respect to a fixed one. The following estimate is important in transfering estimates from a larger nice set to a smaller one.

Definition 2.6. For a map $f \in \mathscr{A}$ and an integer $m \geq 1$ we will say that a pull-back W of an open set V by f^m is a *child of* V if it contains precisely one critical point of f, and if f^{m-1} maps a neighborhood of $f(W)$ diffeomorphically onto a component of of V.

In the case of interval maps the following lemma is a variant of [BRLSvS08, Lemma 4].

Lemma 2.7. *For each $s > 0$ and $\ell > 1$ there is a constant $r > 4$ such that for every $f \in \mathscr{A}(\ell)$ that is backward contracting with constant r, the following property holds. For each $\delta > 0$ small there is a nice set $V = \bigcup_{c \in \mathrm{Crit}'(f)} V^c$ such that for each $c \in \mathrm{Crit}'(f)$ we have,*

$$\widetilde{B}(c, \delta) \subset V^c \subset \widetilde{B}(c, 2\delta),$$

and such that,

$$\sum_{Y:\ \textit{child of}\ V} \mathrm{diam}(f(Y))^s \leq \delta^s.$$

3. Geometric and statistic properties

Through an inducing scheme, we convert Theorems A and B into statistical properties of maps $f \in \mathscr{A}$ which are backward contracting. After briefly describing the induced mappings we state in §3.1 a result giving a tail estimate (Theorem C), as well as a corollary of this result on conformal and invariant measures (Corollary D). In §3.2 we state the main ingredient in the proof of the part of the Main Theorem related to the regularity of the invariant density (Proposition 3.3). We first introduce some notation and terminology.

Let $\mathscr{A}^*$ be the set of $f \in \mathscr{A}$ which satisfies the following:

(A1) f is expanding away from critical points;
(A2) $\mathrm{Crit}'(f) \neq \emptyset$ and f has arbitrarily small symmetric nice couples;
(A3) if $f \in \mathscr{A}_{\mathbb{R}}$, then f is boundary-anchored and essentially topologically exact on the Julia set.

Let $\beta_{\max}(f)$ denote the best polynomial shrinking exponent of f, *i.e.*, the supremum of

$\{\beta > 0 : f$ satisfies the polynomial shrinking condition with exponent $\beta\} \cup \{0\}$,

and define

$$\gamma(f) := \beta_{\max}(f)\left(\mathrm{HD}(J(f)) - \delta_{\mathrm{bad}}(f)\right). \tag{3.1}$$

We use the following convention: the product of $+\infty$ with a real number a is $+\infty$ (resp. 0, $-\infty$) if $a > 0$ (resp. $a = 0$, $a < 0$). So $\gamma(f) > 0$ is equivalent to $\delta_{\mathrm{bad}}(f) < \mathrm{HD}(J(f))$ and $\beta_{\max}(f) > 0$.

Since for $f \in \mathscr{A}^*$ we have $\mathrm{HD}(J(f)) \geq \mathrm{HD}_{\mathrm{hyp}}(f) > 0$, Theorems A and B imply that when f is backward contracting we have $\gamma(f) = \infty$.

3.1. **Canonical inducing Markov mapping.** The following definitions appeared first in [PRL07]. Given a nice couple $(\widehat{V}, V)$ of f, we say that an integer $m \geq 1$ is *a good time* for a point x if $f^m(x) \in V$ and if the pull-back of $\widehat{V}$ containing x is diffeomorphic. We denote by D the set of all those points in V having a good time, and for each $x \in D$ we denote by $m(x)$ the least good time of x. Note that $m(x)$ is constant in any component W of D, so $m(W)$ makes sense. The *canonical induced map associated to the nice couple* $(\widehat{V}, V)$ is by definition the map $F : D \to V$ defined by $F(x) = f^{m(x)}(x)$. We denote by $J(F)$ the maximal invariant set of F, which is equal to the set of all those points in V having infinitely many good times.

We say that a sequence $\{\theta_n\}_{n=1}^{\infty}$ of positive numbers is *slowly varying* if $\theta_n/\theta_{n+1} \to 1$ as $n \to \infty$. For instance, $\{n^{-\beta}\}_{n=1}^{\infty}$ and $\{\exp(-\sigma n^{\alpha})\}_{n=1}^{\infty}$ are slowly varying for any $\beta, \sigma, \alpha > 0$, but for each $\theta \in (0,1)$ the sequence $\{\theta^n\}_{n=1}^{\infty}$ is not slowly varying.

Theorem C. *Fix $f \in \mathscr{A}^*$. If $\delta_{bad}(f) < \mathrm{HD}(J(f))$, then $\mathrm{HD}(J(f)) = \mathrm{HD}_{hyp}(f)$, and for each sufficiently small nice couple $(\widehat{V}, V)$, the canonical inducing mapping $F : D \to V$ associated to it satisfies:*

$$\mathrm{HD}(J(F) \cap V^c) = \mathrm{HD}(J(f)), \text{ for all } c \in \mathrm{Crit}'(f).$$

Furthermore, fix $t \in (\delta_{bad}(f), \mathrm{HD}(J(f)))$ and assume that f satisfies the Θ-shrinking condition for some slowly varying and monotone decreasing sequence of positive numbers $\Theta = \{\theta_n\}_{n=1}^{\infty}$. Then for each sufficiently small symmetric nice couple $(\widehat{V}, V)$, there exists a constant $\alpha_0 = \alpha_0(\widehat{V}, V) \in (t, \mathrm{HD}(J(f)))$ such that, for all $\alpha \geq \alpha_0$ and $\sigma \in [0, \alpha - t)$ there is a constant $C > 0$, such that for each $Y \subset V$ and each integer $m \geq 1$, we have

$$\sum_{\substack{W \in \mathfrak{D}, m(W) \geq m \\ W \subset Y}} \mathrm{diam}(W)^{\alpha} \leq C \,\mathrm{diam}(Y)^{\sigma} \sum_{n=m}^{\infty} \theta_n^{\alpha - t - \sigma},$$

where $\mathfrak{D}$ is the collection of all components of D.

The first part of the theorem is needed for the construction of the conformal measure. It follows rather easily from the definition of badness exponent. In fact, provided that $\widehat{V}$ is small enough, we have $\delta_{\mathrm{bad}}(\widehat{V}) < \mathrm{HD}(J(f))$ and the statement follows from the observation that the set $(J(f) \cap V) \setminus \bigcup_{n=0}^{\infty} f^{-n}(J(F) \cup K(V))$ is covered by the elements of $\bigcup_{j=m}^{\infty} \mathfrak{B}_j(\widehat{V})$ for each $m = 1, 2, \ldots$. The same argument shows that for $f \in \mathscr{A}^*$,

$$\mathrm{HD}(J(f) \setminus J_{\mathrm{con}}(f)) \leq \delta_{\mathrm{bad}}(f). \tag{3.2}$$

The proof of the second part is more involved. Let $\mathfrak{L}_V$ be the collection of components of $\mathrm{dom}(f) \setminus K(V)$. Given a component $\widetilde{Y}$ of $f^{-\widetilde{m}}(\widetilde{V})$ for some $\widetilde{m} \geq 0$, we use $\mathfrak{D}_{\widetilde{Y}}$ to denote the collection of all simply connected sets W

for which the following holds: there exist $\widetilde{Y} \supset \widehat{W} \supset W$ and $U \in \mathfrak{L}_V$ such that $U \subset f(V)$ and such that $f^{\widetilde{m}+1}$ maps W diffeomorphically onto U and maps $\widehat{W}$ diffeomorphically onto $\widehat{U}$.

We use the following lemma which is [PRL08, Lemma 3.4]. It is worth noticing that this is the only place where we use a nice couple, as opposed to a nested pair of nice sets.

Lemma 3.1. *Let $F : D \to V$ be the canonical induced map associated to $(\widehat{V}, V)$, let $\mathfrak{D}$ be the collection of all the connected components of D. Then we have,*

$$\mathfrak{D} = \bigcup_{\widetilde{m}=0}^{\infty} \bigcup_{\widetilde{Y} \in \mathfrak{B}_{\widetilde{m}}(\widehat{V})} \mathfrak{D}_{\widetilde{Y}},$$

where $\mathfrak{B}_0(\widehat{V})$ is the collection of connected components of $\widehat{V}$, and for $\widetilde{m} \geq 1$, $\mathfrak{B}_{\widetilde{m}}(\widehat{V})$ is the collection of all bad pull-backs of $\widehat{V}$ by $f^{\widetilde{m}}$.

Applying a technique based on a *Whitney decomposition* of the complement of the critical values of $f^{\widetilde{m}+1} : \widetilde{Y} \to f(\widehat{V})$, we prove the following proposition, which is crucial in the proof of Theorem C.

Proposition 3.2. *Assume that $f \in \mathscr{A}^*$ satisfies the Θ-Shrinking Condition for some slowly varying and monotone decreasing sequence of positive numbers $\Theta = \{\theta_n\}_{n=1}^{\infty}$. Then for each sufficiently small symmetric nice couple $(\widehat{V}, V)$ for f, with $\delta_{bad}(\widehat{V}) < \mathrm{HD}_{\mathrm{hyp}}(f)$, there exists $\alpha_0 \in (\delta_{bad}(\widehat{V}), \mathrm{HD}_{\mathrm{hyp}}(f))$, such that for real numbers α, t, with*

$$\alpha \geq \alpha_0, \ t \in (\delta_{bad}(\widehat{V}), \alpha),$$

the following holds: There is a constant $C_1 > 0$ such that if $\widetilde{Y}$ is a component of $f^{-\widetilde{m}}(\widehat{V})$ for some $\widetilde{m} \geq 0$, $Y \subset \widetilde{Y}$, $m \geq 1$, then we have,

$$\sum_{\substack{W \in \mathfrak{D}_{\widetilde{Y}}, W \subset Y \\ m(W) \geq m}} \mathrm{diam}(W)^{\alpha} \leq C_1 D(\widetilde{Y}) \, \mathrm{diam}(Y)^t \left(\sum_{i=m}^{\infty} \theta_i^{\alpha - t} \right), \tag{3.3}$$

where $D(\widetilde{Y}) := d_{\widehat{V}}(\widetilde{Y}) \left(\log d_{\widehat{V}}(\widetilde{Y}) + 1 \right)$ and $m(W)$ is the canonical inducing time on W with respect to $(\widehat{V}, V)$.

Theorem C has the following consequence.

Corollary D. *For $f \in \mathscr{A}^*$ the following properties hold.*

1. *If $\gamma(f) > 1$, then either $\mathrm{HD}(J(f)) < \mathrm{HD}(\mathrm{dom}(f))$ or $J(f)$ has a non-empty interior. Moreover, there exists a conformal measure μ of exponent $\mathrm{HD}(J(f))$ which is ergodic, supported on the conical Julia set, satisfies $\mathrm{HD}(\mu) = \mathrm{HD}(J(f))$, and it is such that for each $\varepsilon > \delta_{bad}(f) + \beta_{\max}(f)^{-1}$ the following holds: for each sufficiently small $\delta > 0$ we have for every $x \in J(f)$,*

$$\mu(B(x, \delta)) \leq \delta^{\mathrm{HD}(J(f)) - \varepsilon}. \tag{3.4}$$

2. *If $\gamma(f) > 2$, then there is an invariant probability measure ν that is absolutely continuous with respect to μ and this invariant measure ν is polynomially mixing of each exponent $\gamma \in (0, \gamma(f) - 2)$.*

By modifying the proof of [PRL07, Theorem 2], we deduce from Theorem C the existence of a conformal measure supported on $J_{\text{con}}(f)$. The uniform estimate on its local dimension follows from the theorem and the assumption that f is expanding away from critical points. The rest of the corollary is a rather simple application of Young's result [You99].

3.2. **Poincaré series.** The estimate of the density of the absolutely continuous invariant measure is obtained from an analysis of the Poincaré series. Recall that for $s > 0$ and for a point $x_0 \in \text{dom}(f)$, the Poincaré series of f at x_0 with exponent s, is defined as

$$\mathcal{P}(x_0; s) = \sum_{m=0}^{\infty} \mathcal{P}_m(x_0; s),$$

where

$$\mathcal{P}_m(x_0; s) = \sum_{x \in f^{-m}(x_0)} |Df^m(x)|^{-s}.$$

Clearly, if μ is a conformal measure of exponent s and without an atom, then $d((f^m)_*\mu)/d\mu = \mathcal{P}_m(\cdot, s)$ on a set of full measure with respect to μ.

For a subset Q of $\text{dom}(f)$ and an integer $m \geq 0$, let $\mathscr{M}_m(Q)$ be the collection of all connected components of $f^{-m}(Q)$, and let

$$\theta_m(Q) = \sup\{\text{diam}(P) : P \in \mathscr{M}_m(Q)\}, \text{ and } \theta(Q) = \sup_{m=0}^{\infty} \theta_m(Q).$$

Moreover, for $s \geq 0$ we let,

$$\mathscr{L}_m(Q; s) = \sum_{P \in \mathscr{M}_m(Q)} d_Q(P)\, \text{diam}(P)^s, \text{ and } \mathscr{L}(Q; s) = \sum_{m=0}^{\infty} \mathscr{L}_m(Q; s),$$

where d_Q is defined as in §2.2. Note that if $x \in J(f)$ is disjoint from the critical orbits, then

$$\mathcal{P}_m(x; s) = \lim_{\delta \to 0} \frac{\mathscr{L}_m(B(x, \delta); s)}{\text{diam}(B(x, \delta))^s}.$$

For $z \in J(f)$ and $m \geq 0$, let

$$\Delta_m(z) = \text{dist}\left(z, \bigcup_{j=0}^{m} f^j(\text{Crit}'(f))\right),$$

and for $\varepsilon \in (0, 1/2)$, let

$$\xi_m(z; \varepsilon) = \theta_m\left(B\left(z, \varepsilon \Delta_m(z)\right)\right).$$

Given a nice set $\widehat{V}$, let $\mathfrak{B}_0 = \mathscr{M}_0(\widehat{V})$, and for $m \geq 1$, let $\mathfrak{B}_m$ denote the collection of all elements $\widetilde{Y} \in \mathscr{M}_m(\widehat{V})$ which are bad pull-backs of $\widehat{V}$. Moreover, for $s \geq 0$, let

$$\mathscr{L}_m^{\text{bad}}(\widehat{V}; s) = \sum_{\widetilde{Y} \in \mathfrak{B}_m(\widehat{V})} d_{\widehat{V}}(\widetilde{Y}) \operatorname{diam}(\widetilde{Y})^s, \ \mathscr{L}^{\text{bad}}(\widehat{V}; s) = \sum_{m=0}^{\infty} \mathscr{L}_m^{\text{bad}}(\widehat{V}; s).$$

Using the fact that each pull back of a nice set $\widehat{V}$ can be written as composition of a bad pull back with a diffeomorphic pull back, and estimating the diameter of a diffeomorphic pull back of a fixed nice set with bounded distortion by a conformal measure, we obtain the following estimates.

Proposition 3.3. *Assume that $f \in \mathscr{A}^*$ has a conformal measure of exponent $h_0 > \delta_{bad}(f)$. Then for each sufficiently small nice couple $(\widehat{V}, V)$, the following hold:*

1. *For any $s > h_0$, $t \in (0, s)$ and $\varepsilon \in (0, 1]$, there exists a constant $C > 0$ such that for each $z \in V \cap J(f)$, we have*

$$\mathcal{P}(z; s) \leq C \sum_{m=0}^{\infty} \mathscr{L}_m^{bad}(\widehat{V}, t) \xi_m(z; \varepsilon)^{s-t} \Delta_m(z)^{-s}. \tag{3.5}$$

2. *For and each $t \in (0, h_0)$ and $\varepsilon \in (0, 1]$ there exists $C > 0$ such that for each $z \in J(f) \cap V$ and each integer $n \geq 1$,*

$$\mathcal{P}_n(z; h_0) \leq C \sum_{m=0}^{n} \mathscr{L}_m^{bad}(\widehat{V}, t) \xi_m(z; \varepsilon)^{h_0-t} \Delta_m(z)^{-h_0}. \tag{3.6}$$

The L^p estimate of the density of the absolutely continuous invariant measure follows from part 2 of Proposition 3.3, together with the "expanding away from critical points" property. See [RLS10, Theorem G] for a more precise statement.

4. Further results

In this section we state a result related to fractal dimensions (Theorem E in §4.1), and another related to holomorphic removability of Julia sets in the complex setting (Theorem F in §4.2).

4.1. **Fractal dimensions.** Given $f \in \mathscr{A}$, we say that a point $x \in \operatorname{dom}(f)$ is *exceptional* if the set $\bigcup_{n=0}^{\infty} f^{-n}(x)$ is finite, and we say that x is *asymptotically exceptional* if its α-limit set is finite. The *Poincaré exponent of f* is by definition,

$$\delta_{\text{Poin}}(f) := \inf \{s > 0 : \mathcal{P}(x_0; s) < \infty \text{ for some } x_0 \text{ that is not asymptotically exceptional}\} \cup \{0\}.$$

Note that every point in the α-limit set of an asymptotically exceptional point is exceptional. It is well-known that for a rational map of degree at least 2 each asymptotically exceptional point is exceptional, that there are at most 2 exceptional points, and that they are not in the Julia set. Note however that an interval map in $\mathscr{A}_{\mathbb{R}}$ can have infinitely many asymptotically exceptional points.

Theorem E (Equality of fractal dimensions). *If $f \in \mathscr{A}^*$ satisfies $\gamma(f) > 1$, then*

$$\delta_{\mathrm{Poin}}(f) = \overline{BD}(J(f)) = \mathrm{HD}(J(f)) = \mathrm{HD}_{\mathrm{hyp}}(f) > 0.$$

See §3.1 for the definition of $\gamma(f)$.

Equalities of dimensions were shown in [LS08] for backward contracting rational maps, in [Prz98] for rational maps whose derivatives at critical values grow at least as a stretch exponential function, and in [GS09, Theorem 7] for complex rational maps satisfying a summability condition with a small exponent. However, in the latter result a slightly different definition of Poincaré exponent is used: in the case that $J(f) = \overline{\mathbb{C}}$, the exponent $\delta_{\mathrm{Poin}}(f)$ was defined to be 2. Thus our theorem says more in this case. These equalities were shown for a class of infinitely renormalizable quadratic polynomials in [AL08]. See also [Dob06] for the case of interval maps without recurrent critical points.

Equality $\mathrm{HD}(J(f)) = \mathrm{HD}_{\mathrm{hyp}}(f)$ is an easy consequence of Theorem B and (3.2). The proof of the equalities $\delta_{\mathrm{Poin}}(f) = \overline{\mathrm{BD}}(J(f)) = \mathrm{HD}(J(f))$ is more difficult. It relies on the following proposition, whose proof is based on Proposition 3.3, and on Proposition 4.2 stated in §4.2.

Proposition 4.1 (Poincaré series). *Assume that $f \in \mathscr{A}^*$ satisfies $\gamma(f) > 1$. Then $\delta_{\mathrm{Poin}}(f) = \mathrm{HD}(J(f))$. More precisely, we have*

1. *For every $x_0 \in \mathrm{dom}(f)$ that is not asymptotically exceptional, we have $\mathcal{P}(x_0, \mathrm{HD}(J(f))) = \infty$.*
2. *There is a subset E of $J(f)$ with $\mathrm{HD}(E) < \mathrm{HD}(J(f))$ and a neighborhood U of $J(f)$ such that for every $x_0 \in U \setminus E$, and every $s > \mathrm{HD}(J(f))$, the Poincaré series $\mathcal{P}(x_0, s)$ converges.*

4.2. **Holomorphic removability of Julia sets.** We will say that a compact subset J of the Riemann sphere is *holomorphically removable*, if every homeomorphism $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ that is holomorphic outside J is a Möebius transformation.

Theorem F (Holomorphic removability). *If $f \in \mathscr{A}_{\mathbb{C}}^*$ is such that*

$$\beta_{\max}(f)\,(2 - \delta_{bad}(f)) > 1,$$

then the Julia set of f is holomorphically removable. In particular, for every integer $\ell \geq 2$, there is a constant $r > 1$ such that the Julia set of a complex polynomial $f \in \mathcal{A}(\ell)$ that is backward contracting with constant r, and without parabolic periodic points, is holomorphically removable.

See also [Jon95, Kah98] and [GS09, Theorem 8] for other removability results of Julia sets.

In view of the holomorphic removability result [JS00, Theorem 5] (see also [GS09, Fact 9.1]), the theorem above is an easy consequence of the following proposition.

Proposition 4.2. *Assume that $f \in \mathscr{A}^*$ has a conformal measure μ of exponent $h_0 > \delta_{bad}(f)$ such that $\beta_{\max}(f)(h_0 - \delta_{bad}(f)) > 1$ and such that for each open set U intersecting* $\mathrm{Crit}'(f)$ *we have $\mu(U) > 0$. Then there exists $\delta_0 > 0$ such that for each $z \in J(f)$ and each $s > h_0$, $\mathscr{L}(B(z, \delta_0); s) < \infty$. Moreover, if $\mu(J_{\mathrm{con}}(f)) = 0$, then we also have $\mathscr{L}(B(z, \delta_0); h_0) < \infty$ for each $z \in J(f)$.*

Notice that in the proposition above the conformal measure μ might not charge $J(f)$.

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