GLOBAL EXISTENCE AND FULL REGULARITY OF THE BOLTZMANN EQUATION WITHOUT ANGULAR CUTOFF

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Abstract. We prove the global existence and uniqueness of classical solutions around an equilibrium to the Boltzmann equation without angular cutoff in some Sobolev spaces. In addition, the solutions thus obtained are shown to be non-negative and C^∞ in all variables for any positive time. In this paper, we study the Maxwellian molecule type collision operator with mild singularity. One of the key observations is the introduction of a new important norm related to the singular behavior of the cross section in the collision operator. This norm captures the essential properties of the singularity and yields precisely the dissipation of the linearized collision operator through the celebrated H-theorem.

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1. Introduction

We consider the Cauchy problem for the inhomogeneous Boltzmann equation

$$f_t + v \cdot \nabla_x f = Q(f, f), \quad f|_{t=0} = f_0, \tag{1.1}$$

where $f = f(t, x, v)$ is the density distribution function of particles, having position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$ at time t. Here, the right hand side of (1.1) is given by the Boltzmann bilinear collision operator, which is given in the classical $\sigma-$representation by

$$Q(g, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) \{g'_* f' - g_* f\} d\sigma dv_*,$$

where $f'_* = f(t, x, v'_*)$, $f' = f(t, x, v')$, $f_* = f(t, x, v_*)$, $f = f(t, x, v)$, and for $\sigma \in \mathbb{S}^2$,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \;\; v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma,$$

which gives the relation between the post and pre collisional velocities. Recall that we have conservation of momentum and kinetic energy, that is, $v + v_* = v' + v'_*$ and $|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$. The kernel B is the cross-section which can be computed in different physical settings.

In particular, the non-negative cross section $B(z, \sigma)$ depends only on $|z|$ and the scalar product $\langle \frac{z}{|z|}, \sigma\rangle$. In most cases, the kernel B cannot be expressed explicitly, but to capture its main properties, one may assume that it takes the form

$$B(|v - v_*|, \cos\theta) = \Phi(|v - v_*|) b(\cos\theta), \quad \cos\theta = \Big\langle \frac{v - v_*}{|v - v_*|}, \sigma \Big\rangle, \;\; 0 \le \theta \le \frac{\pi}{2}.$$

An important example is the inverse power law potential ρ^{-r} with $r > 1$, ρ being the distance between two particles, in which the cross section has a kinetic factor given by

$$\Phi(|v - v_*|) \approx |v - v_*|^\gamma, \qquad \gamma = 1 - \frac{4}{r},$$

and a factor related to the collision angle containing a singularity,

$$b(\cos\theta) \approx K\theta^{-2-2s} \;\text{ when }\; \theta \to 0+,$$

for some constants $K > 0$ and $0 < s = \frac{1}{r} < 1$.

The cases with $1 < r < 4$, $r = 4$ and $r > 4$ correspond to so-called soft, Maxwellian molecule and hard potentials respectively. In the following discussion, this type of cross sections, with the parameters γ and s given above, will be kept in mind.

As a fundamental equation in kinetic theory and a key stone in statistical physics, the Boltzmann equation has attracted, and is still attracting, a lot of research investigations since its derivation in 1872.

A large number of mathematical works have been performed under the Grad's cutoff assumption, avoiding the non-integrable angular singularity of the cross-sections, see for example [22, 23, 34, 39, 40, 46, 55, 48, 49, 70] to cite only a few, further references being given in the review [73].

However, except for the hard sphere model, for most of the other molecule interaction potentials such as the inverse power laws recalled above, the cross section $B(v - v_*, \sigma)$ is a non-integral function in angular variable and the collision operator $Q(f, f)$ is a nonlinear singular integral operator in velocity variable.

By no means to be complete, let us now review some previous works related to the Boltzmann equation in the context of such singular (or non-cutoff) cross-sections. For other references and comments, readers are referred to [5, 73].

The mathematical study for the Boltzmann equation, without assuming Grad's cutoff assumption, can be traced back at least to the work by Pao in 1970s [64] which is about the spectrum of the linearized operator. In 1980s, the existence of weak solutions to the spatially homogeneous case was proved by Arkeryd in [17] for the mild singular case, that is, when $0 < s < \frac{1}{2}$, and by using an abstract Cauchy-Kovalevskaya theorem, Ukai in [69] proved the local existence of solutions to the inhomogeneous equation, in the space of functions which are analytic in x and Gevrey in v.

For a long time, the mathematical study of singular cross-sections was limited to these results and a few others, most of them related to the spatially homogeneous case concerning only the existence. An important step was initiated by the works of Desvillettes and his collaborators in 1990s, showing partial regularization results for some simplified kinetic models, cf. [26, 27, 28, 29, 33, 31, 72].

After the well known result of DiPerna and Lions [34] for the cutoff case, Lions was able to show the gain of regularity of solutions in the Landau case [50], which is a model arising as a grazing limit of the Boltzmann equation. It was then expected that this kind of singular cross sections should lead to smoothing effect on solutions, that is, the solutions have higher regularity than the initial data. For example, it should be similar to the case when one replaces the collision operator in the Boltzmann equation by a fractional Laplacian in velocity variable, that is, a fractional Kolmogorov-type equation [61].

Certainly, the results of Lions [51] and Desvillettes have influenced the research in this direction. It is therefore not surprising that a systematic approach, using the entropy dissipation and/or the smoothing property of the gain part of the collision operator, was initiated and has been developed to an almost optimal stage through the efforts of many researchers, such as Alexandre, Bouchut, Desvillettes, Golse, Lions, Villani and Wennberg. The underlying tools have proven to be very useful for the study on the mathematical theory regarding the regularizing effect for the spatially homogeneous problems for which the theory can now be considered as quite satisfactory, cf. [6, 7, 16, 24, 29, 30, 32, 47, 59, 60, 71], and the references therein, see also for a much more detailed discussion [5].

Compared to the spatially homogeneous problems, the original spatially inhomogeneous Boltzmann equation is of course physically more interesting and mathematically more challenging. For existence of weak solutions, we mention two results regarding the Cauchy problem. One is about the local existence between two moving Maxwellians proved in [3] by constructing the upper and lower solutions, another is the global existence of renormalized solutions with defect measures shown in [16] where the solutions become weak solutions if the defect measures vanish. On the other hand, the local existence of classical solutions was proved in [12] in some weighted Sobolev spaces.

However, in view of the above available results, the mathematical theory for non-cutoff cross-sections is so far not satisfactory. This is in sharp contrast to the cutoff case, for which the theories have been well developed, see [19, 20, 21, 34, 36, 46, 52, 53, 67, 68, 70] and the references therein.

For the study of the regularizing effect, one of the main difficulties comes from the coupling of the transport operator with the collision operator, which is similar to the Landau equation studied in [25]. To overcome this difficulty, a generalized uncertainty principle à la Fefferman [38] (see also [56, 57, 58]) was introduced in [8, 9] for the study of smoothing effects of the linearized and spatially inhomogeneous Boltzmann equation with non-cutoff cross sections.

In order to complete the full regularization process, recently, in [12], by using suitable pseudo-differential operators and harmonic analysis, we have developed sharp coercivity and upper bounds of the collision operators in Sobolev space, together with the estimation on the commutators with these pseudo-differential operators. More precisely, in [10, 11, 12], for classical solutions, we established the hypo-ellipticity of the Boltzmann operator, using the generalized version of the uncertainty principle.

The present work is a continuation of our collaborative program since 2006 [9, 10, 11, 12]. Comparing to the cutoff case, we aim to settle a mathematical framework similar to the studies first proved by Ukai, see [67, 68], and fitted into an energy method by Liu and collaborators [52, 53] and Guo [46] which has led to a clean theory for the Cauchy problem in the cutoff case, for solutions close to a global equilibrium.

In this paper, we will establish the global existence of non-negative solutions in some Sobolev space for the Boltzmann equation near a global equilibrium and prove the full regularity in all variables for any positive time.

As mentioned in the abstract, one of the main ingredients in the proof is the introduction of a new non-isotropic norm which captures the main feature of the singularity in the cross-section. This new norm is in fact the counterpart of the coercive norm which was introduced by Guo [45] as an essential step for Landau equation.

It is not known if there is any equivalence of this norm to some Sobolev norm, in contrast to the case of the Landau equation. However, since it is designed to be equivalent to, and to have much simpler expression than, the Dirichlet form of the linearized collision operator, this norm not only works extremely well for the description of the dissipative effect of the linearized collision operator through the H-theorem, but also well fits for the upper bound estimation on the nonlinear collision operator. Here, we would like to mention the work by Mouhot and Strain [62, 63] about the gain of moment in a linearized context due to the singularity in the cross-section. Such a gain of moment which is well described by the new non-isotropic norm is in fact crucial for the proof of global existence.

We now come back to the problem considered in this paper. To make the presentation as simple as possible, and to concentrate on the singularity of the grazing effect, we shall study the Maxwellian molecule type cross-sections with mild singularity, that is, the case when

$$B(|v-v_*|,\cos\theta)=b(\cos\theta),\quad \cos\theta=\Big\langle\frac{v-v_*}{|v-v_*|}\,,\,\sigma\Big\rangle,\ \ 0\le\theta\le\frac{\pi}{2},$$

and

$$b(\cos\theta)\approx K\theta^{-2-2s},\quad \theta\,\to\,0^+, \tag{1.2}$$

with $0<s<\frac{1}{2}$. The general case will be left to our future work.

In order to prove the global existence, we need to use the complete dissipative effect of the collision operator. Similar to angular cutoff case, such dissipative effect can be fully represented by the dissipation of the linearized collision operator on the microscopic component of the solution through the H-theorem.

Thus, as usual, we consider the Boltzmann equation around a normalized Maxwellian distribution

$$\mu(v)=(2\pi)^{-\frac{3}{2}}e^{-\frac{|v|^2}{2}}.$$

Since μ is the global equilibrium state satisfying $Q(\mu,\mu)=0$, by setting $f=\mu+\sqrt{\mu}g$, we have

$$Q(\mu+\sqrt{\mu}g,\,\mu+\sqrt{\mu}\,g)=Q(\mu,\,\sqrt{\mu}\,g)+Q(\sqrt{\mu}\,g,\,\mu)+Q(\sqrt{\mu}\,g,\,\sqrt{\mu}\,g).$$

Denote

$$\Gamma(g,\,h)=\mu^{-1/2}Q(\sqrt{\mu}\,g,\,\sqrt{\mu}\,h).$$

Then the linearized Boltzmann operator takes the form

$$\mathcal{L}g = \mathcal{L}_1\, g + \mathcal{L}_2\, g = -\Gamma(\sqrt{\mu}\,,\, g) - \Gamma(g,\ \sqrt{\mu}\,).$$

And the original problem (1.1) is now reduced to the Cauchy problem for the perturbation g

$$\left\{\begin{array}{l} g_t + v\cdot\nabla_x g + \mathcal{L}g = \Gamma(g,\, g), \quad t>0\,;\\ g|_{t=0} = g_0. \end{array}\right. \tag{1.3}$$

This problem will be considered in the following weighted Sobolev spaces. For $k,\ \ell \in \mathbb{R}$, set

$$H^k_\ell(\mathbb{R}^6_{x,v}) = \left\{ f\in \mathcal{S}'(\mathbb{R}^6_{x,v})\,;\ W^\ell f \in H^k(\mathbb{R}^6_{x,v})\right\},$$

where $\mathbb{R}^6_{x,v} = \mathbb{R}^3_x\times\mathbb{R}^3_v$ and $W^\ell(v) = \langle v\rangle^\ell = (1+|v|^2)^{\ell/2}$ is the weight with respect to the velocity variable $v\in\mathbb{R}^3_v$.

The main theorem can be stated as follows.

Theorem 1.1. *Assume that the cross-section satisfies (1.2) with $0<s<1/2$. Let $g_0\in H^k_\ell(\mathbb{R}^6)$ for some $k\geq 3, \ell\geq 3$ and*

$$f_0(x,v) = \mu + \sqrt{\mu}\, g_0(x,v)\geq 0.$$

Then there exists $\varepsilon_0>0$, such that if $\|g_0\|_{H^k_\ell(\mathbb{R}^6)}\leq\varepsilon_0$, the Cauchy problem (1.3) admits a unique global solution

$$g\in L^\infty([0,+\infty[\,;\ H^k_\ell(\mathbb{R}^6)).$$

Moreover, $f(t,x,v) = \mu + \sqrt{\mu}\ g(t,x,v)\geq 0$ and

$$g\in C^\infty(]0,+\infty[\,\times\mathbb{R}^6).$$

Actually, for the uniqueness, we can prove the following stronger result, which might be of independent interest. Note that here we do not need to assume that f is a small perturbation of μ.

Theorem 1.2. *Under the same condition on the cross-section, for $0<T\leq+\infty$ and $l>2s+7/2$, let $f_0\geq 0,\ f_0\in L^\infty(\mathbb{R}^3_x; H^{2s}_{l+2}(\mathbb{R}^3_v))$. Suppose that $f_1,\ f_2\in L^\infty(]0,T[\times\mathbb{R}^3_x;\ H^{2s}_{l+2}(\mathbb{R}^3_v))$ are two solutions to the Cauchy problem (1.1). If one solution is non-negative, then $f_1\equiv f_2$.*

Throughout this paper, we assume that the cross-section satisfies the condition (1.6) with $0<s<1/2$ except otherwise stated.

The rest of the paper will be organized as follows. In the next section, we will introduce a new non-isotropic norm and prove some essential coercivity and upper bound estimates on the collision operators with respect to this new norm. In order to study the gain of regularity of the solution, we need to apply some pseudo-differential operators on the Boltzmann equation. For this purpose, in Section 3, we study the commutators of the collision operators with the pseudo-differential operators. In Section 4, we will apply the energy method for the Boltzmann equation and obtain the local existence theorem. In Section 5, we will study the uniqueness and the non-negativity of the solutions. This new method for proving non-negativity can be applied to the case with angular cutoff. For more detail discussion on the non-negativity problem, refer to [15]. In Section 6, the full regularity is proved along the approach of [12]. Finally, the global existence of the solution will be given in the last section. For this, the macro-micro decomposition introduced by Guo [45] will be used for the estimation on the macroscopic component.

Note: After finishing this paper, we were informed by R. Strain of his recent paper in collaboration with P. Gressmann [41], showing also the existence of global solutions to the Cauchy problem by using different approach. Notice that their solution is in different function space which does not lead to full regularity because of the weak regularity in the velocity variable.

Note added in September, 2010: Several new results have been announced along the same line of development since the submission of the current paper. For the reader's references we mention [42, 43, 44, 13, 14, 15]. The main difference of the results is the range of admissible values of γ: $\gamma > -1-2s$ in the first 3 papers and $\gamma > \max(-3, -3/2-2s)$ in the latter 4 paper.

2. Non-isotropic norms

In this section, we study the bilinear collision operator given by

$$Q(g, f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\{g'_* f' - g_* f\}\, d\sigma dv_*\,,$$

through harmonic analysis. Since the collision operator acts only with respect to the velocity variable $v \in \mathbb{R}^3$, (t, x) is regarded as a parameter in this section.

2.1. **Coercivity and upper bound estimates.** Let $g \ge 0$, $g \not\equiv 0$, $g \in L^1_2 \bigcap L\log L(\mathbb{R}^3_v)$. It was shown in [6] that there exists a constant $c_g > 0$ depending only on the values of $\|g\|_{L^1_2}$ and $\|g\|_{L\log L}$ such that for any smooth function $f \in H^s(\mathbb{R}^3_v)$, we have

$$c_g\|f\|^2_{H^s(\mathbb{R}^3_v)} \le (-Q(g, f), f)_{L^2(\mathbb{R}^3_v)} + C\|g\|_{L^1(\mathbb{R}^3_v)}\|f\|^2_{L^2(\mathbb{R}^3_v)}. \tag{2.1.1}$$

Besides this, we still need some functional estimates on the Boltzmann collision operators. The first one, given below, is about the boundedness of the collision operator in weighted Sobolev spaces, see [1, 2, 4, 5, 12, 47] .

Theorem 2.1. *Assume that the cross-section satisfies (1.2) with* $0 < s < 1$ *. Then for any* $m \in \mathbb{R}$ *and any* $\alpha \in \mathbb{R}$*, there exists* $C > 0$ *such that*

$$\|Q(f, g)\|_{H^m_\alpha(\mathbb{R}^3_v)} \le C\|f\|_{L^1_{\alpha^+ + 2s}(\mathbb{R}^3_v)}\|g\|_{H^{m+2s}_{(\alpha+2s)^+}(\mathbb{R}^3_v)} \tag{2.1.2}$$

for all $f \in L^1_{\alpha^+ + 2s}(\mathbb{R}^3_v)$ *and* $g \in H^{m+2s}_{(\alpha+2s)^+}(\mathbb{R}^3_v)$.

We now turn to the linearized operator. First of all, by using the conservation of energy

$$|v'_*|^2 + |v'|^2 = |v_*|^2 + |v|^2,$$

we have $\mu(v_*) = \mu^{-1}(v)\,\mu(v'_*)\,\mu(v')$. Thus,

$$\begin{aligned}\Gamma(f, g)(v) &= \mu^{-1/2}\iint b(\cos\theta)\Big(\sqrt{\mu'_*}\, f'_*\,\sqrt{\mu'}g' - \sqrt{\mu_*}\, f_*\,\sqrt{\mu}\ g\Big)dv_* d\sigma\\ &= \iint b(\cos\theta)\,\sqrt{\mu_*}\,\Big(f'_* g' - f_* g\Big)dv_* d\sigma.\end{aligned} \tag{2.1.3}$$

It is well-known that $\mathcal{L}$ (acting with respect to the velocity variable) is an unbounded symmetric operator on $L^2(\mathbb{R}^3_v)$. Moreover, its Dirichlet form satisfies

$$
\begin{aligned}
\left(\mathcal{L}g,\, g\right)_{L^2(\mathbb{R}^3_v)} &= -\left(\Gamma(\sqrt{\mu}\,, g) + \Gamma(g,\ \sqrt{\mu}\,),\, g\right)_{L^2(\mathbb{R}^3_v)}\\
&= \iiint b(\cos\theta)\Big((\mu_*)^{1/2}g - (\mu'_*)^{1/2}g' + g_*(\mu)^{1/2} - g'_*(\mu')^{1/2}\Big)(\mu_*)^{1/2}\, g dv_* d\sigma dv\\
&= \iiint b(\cos\theta)\Big((\mu'_*)^{1/2}g' - (\mu_*)^{1/2}g + g'_*(\mu')^{1/2} - g_*(\mu)^{1/2}\Big)(\mu'_*)^{1/2}\, g' dv_* d\sigma dv\\
&= \iiint b(\cos\theta)\Big((\mu)^{1/2}g_* - (\mu')^{1/2}g'_* + g(\mu_*)^{1/2} - g'(\mu'_*)^{1/2}\Big)(\mu)^{1/2}\, g_* dv_* d\sigma dv \qquad (2.1.4)\\
&= \iiint b(\cos\theta)\Big((\mu')^{1/2}g'_* - (\mu)^{1/2}g_* + g'(\mu'_*)^{1/2} - g(\mu_*)^{1/2}\Big)(\mu')^{1/2}\, g'_* dv_* d\sigma dv\\
&= \frac{1}{4}\iiint b(\cos\theta)\left(\left((\mu_*)^{1/2}g - (\mu'_*)^{1/2}g'\right) + \left((\mu)^{1/2}g_* - (\mu')^{1/2}g'_*\right)\right)^2 dv_* d\sigma dv\\
&\geq 0.
\end{aligned}
$$

The third line in the above equation is obtained by using the change of variables $(v, v_*) \to (v', v'_*)$. The fourth line follows from the change of variables $(v, v_*) \to (v_*, v)$ and then the fifth line follows from the fourth one by using the change of variables $(v, v_*) \to (v', v'_*)$. And the second last line is just the summation of the previous four lines. Note that the Jacobians of the above coordinate transformations are equal to 1.

Moreover, it follows from the above formula that $\left(\mathcal{L}g,\, g\right)_{L^2(\mathbb{R}^3_v)} = 0$ if and only if $\mathbf{P}g = g$ where

$$\mathbf{P}g = \left(a + b\,\cdot\, v + c|v|^2\right)\sqrt{\mu},$$

with $a, c \in \mathbb{R}, b \in \mathbb{R}^3$. Here, $\mathbf{P}$ is the L^2-orthogonal projection onto the null space

$$\mathcal{N} = \text{Span}\left\{\sqrt{\mu}\,,\ v_1\sqrt{\mu}\,,\ v_2\sqrt{\mu}\,,\ v_3\sqrt{\mu}\,,\ |v|^2\sqrt{\mu}\ \right\}.$$

The following result on the gain of moment of order s in the linearized framework is essential in the sequent analysis.

Theorem 2.2. *(Theorem 1.1 of* [63]*)*

Assume that the cross-section satisfies (1.2) with $0 < s < 1$. *Then there exists a constant* $C > 0$ *such that*

$$\left(\mathcal{L}g,\, g\right)_{L^2(\mathbb{R}^3_v)} \geq C\,\|(\mathbf{I} - \mathbf{P})g\|^2_{L^2_s(\mathbb{R}^3_v)}\ .$$

For the bilinear operator $\Gamma(\,\cdot,\,\cdot)$, we need the following two formulas. For suitable functions f, g, the first formula coming from (2.1.3) is

$$\Gamma(f,\, g)(v) = Q(\sqrt{\mu}\, f,\, g) + \iint b(\cos\theta)\Big(\sqrt{\mu_*}\, - \,\sqrt{\mu'_*}\,\Big) f'_* g' dv_* d\sigma\,. \qquad (2.1.5)$$

On the other hand, applying the change of variables $(v, v_*) \to (v', v'_*)$ in (2.1.3) gives

$$
\begin{aligned}
\left(\Gamma(f,\, g),\, h\right)_{L^2(\mathbb{R}^3_v)} &= \iiint b(\cos\theta)\sqrt{\mu_*}\,(f'_*g' - f_*g)h\\
&= \iiint b(\cos\theta)\sqrt{\mu'_*}\,(f_*g - f'_*g')h'\,.
\end{aligned}
$$

By adding these two lines, the second formula is

$$\left(\Gamma(f,\, g),\, h\right)_{L^2(\mathbb{R}^3_v)} = \frac{1}{2}\iiint b(\cos\theta)\Big(f'_*g' - f_*g\Big)\Big(\sqrt{\mu_*}\, h - \sqrt{\mu'_*}\, h'\Big)\,. \qquad (2.1.6)$$

The following lemma shows that $\mathcal{L}_1$ controls $\mathcal{L}$.

Lemma 2.3. *Under the condition (1.2) on the cross-section with* $0 < s < 1$*, we have*

$$\Big(\mathcal{L}_1 g,\, g\Big)_{L^2(\mathbb{R}^3_v)} \geq \frac{1}{2}\Big(\mathcal{L} g,\, g\Big)_{L^2(\mathbb{R}^3_v)}\,. \tag{2.1.7}$$

Proof. From (2.1.3) and similar changes of variables, we have

$$\begin{aligned}
&\Big(\mathcal{L}_1 g,\, g\Big)_{L^2(\mathbb{R}^3_v)} = -\Big(\Gamma(\sqrt{\mu}\,, g),\, g\Big)_{L^2(\mathbb{R}^3_v)}\\
&= \frac{1}{2}\iiint b(\cos\theta)\Big((\mu'_*)^{1/2} g' - (\mu_*)^{1/2} g\Big)^2 dv_* d\sigma dv\\
&= \frac{1}{2}\iiint b(\cos\theta)\Big((\mu')^{1/2} g'_* - (\mu)^{1/2} g_*\Big)^2 dv_* d\sigma dv\\
&= \frac{1}{4}\iiint b(\cos\theta)\Big\{\Big((\mu'_*)^{1/2} g' - (\mu_*)^{1/2} g\Big)^2 + \Big((\mu')^{1/2} g'_* - (\mu)^{1/2} g_*\Big)^2\Big\} dv_* d\sigma dv\,.
\end{aligned}$$

Therefore, (2.1.7) follows from $(A+B)^2 \leq 2(A^2+B^2)$ and (2.1.4). □

2.2. **Definition and properties of the non-isotropic norm.**

The non-isotropic norm associated with the cross-section $b(\cos\theta)$ is defined by

$$|||g|||^2 = \iiint b(\cos\theta)\mu_*\,(g' - g\,)^2 + \iiint b(\cos\theta) g_*^2 (\,\sqrt{\mu'}\; - \;\sqrt{\mu}\;)^2\,, \tag{2.2.1}$$

where the integration is over $\mathbb{R}^3_v \times \mathbb{R}^3_{v_*} \times \mathbb{S}^2_\sigma$. Thus, it is a norm with respect to the velocity variable $v \in \mathbb{R}^3$ only. As we will see later, the reason that this norm is called non-isotropic is because it combines both derivative and weight of order s due to the singularity of cross-section $b(\cos\theta)$.

The following lemma gives an upper bound of this non-isotropic norm by some weighted Sobolev norm.

Lemma 2.4. *Assume that the cross-section satisfies (1.2) with* $0 < s < 1$*. Then there exists* $C > 0$ *such that*

$$|||g|||^2 \leq C\|g\|^2_{H^s_s} \tag{2.2.2}$$

for any $g \in H^s_s(\mathbb{R}^3_v)$.

Proof. Applying (2.1.2) with $\alpha = -s$ and $m = -s$ gives

$$\left|\Big(Q(f^2,\, g),\, g\Big)_{L^2(\mathbb{R}^3_v)}\right| \leq C\|f^2\|_{L^1_{2s}}\|g\|_{H^s_s}\|g\|_{H^s_s} \leq C\|f\|^2_{L^2_s}\|g\|^2_{H^s_s}. \tag{2.2.3}$$

On the other hand,

$$\begin{aligned}
\Big(Q(f^2,\, g),\, g\Big)_{L^2(\mathbb{R}^3_v)} &= \iiint b(\cos\theta)\Big(f^{2'}_* g' - f^2_* g\Big) g\\
&= \iiint b(\cos\theta) f^{2'}_*\Big(g' - g\Big) g + \int g^2 \iint b(\cos\theta)\Big(f^{2'}_* - f^2_*\Big).
\end{aligned}$$

For the first term in the last equation, using $b(a-b) = \frac{1}{2}(a^2 - b^2) - \frac{1}{2}(a-b)^2$ yields

$$\begin{aligned}
\Big(Q(f^2,\, g),\, g\Big)_{L^2(\mathbb{R}^3_v)} &= \frac{1}{2}\iiint b(\cos\theta) f^{2'}_*\Big(g^{2'} - g^2\Big)\\
&- \frac{1}{2}\iiint b(\cos\theta) f^{2'}_*\Big(g' - g\Big)^2 + \int g^2 \iint b(\cos\theta)\Big(f^{2'}_* - f^2_*\Big).
\end{aligned}$$

By the change of variables $(v'_*, v') \to (v_*, v)$, the first term above is also $\frac{1}{2}\iiint bg^2(f_*^2 - f_*^{2'})$. Thus, it follows that

$$\left(Q(f^2, g), g\right)_{L^2(\mathbb{R}^3_v)} = -\frac{1}{2}\iiint b f_*^{2'}(g'-g)^2 + \frac{1}{2}\int g^2 \iint b\left(f_*^{2'} - f_*^2\right),$$

and then

$$\iiint b f_*^2 (g'-g)^2 \le 2\left|\left(Q(f^2, g), g\right)\right| + \left|\iiint bg^2\left(f_*^{2'} - f_*^2\right)\right|.$$

By using (2.2.3) and the cancellation lemma from [6], we get

$$\iiint b f_*^2(g'-g)^2 \le C\|f\|^2_{L^2_s}\|g\|^2_{H^s_s} + C\|g\|^2_{L^2}\|f\|^2_{L^2} \le C\|f\|^2_{L^2_s}\|g\|^2_{H^s_s}. \tag{2.2.4}$$

Thus, choosing $f = \sqrt{\mu}$ gives

$$|||g|||^2 \le C(\|\sqrt{\mu}\|^2_{L^2_s}\|g\|^2_{H^s_s} + \|g\|^2_{L^2_s}\|\sqrt{\mu}\|^2_{H^s_s}) \le C\|g\|^2_{H^s_s}.$$

This completes the proof of the lemma. □

In the context of usual weighted Sobolev spaces, this last result is likely to be optimal. Next we will show that this non-isotropic norm is controlled by the linearized operator. First of all, we shall need the following preliminary computation.

Lemma 2.5. *For any $\phi \in C^1_b$, we have*

$$\int_\sigma b(\cos\theta)|\phi(v_*) - \phi(v'_*)|d\sigma \le C_\phi|v - v_*|^{2s} \le C\langle v\rangle^{2s}\langle v_*\rangle^{2s},$$

where C_ϕ depends on $\|\phi\|_{C^1_b} = \|\phi\|_{L^\infty} + \|\nabla\phi\|_{L^\infty}$.

Proof. It follows from Taylor's formula that

$$|\phi(v_*) - \phi(v'_*)| \le C_\phi|v_* - v'_*| \le C_\phi \sin\left(\frac{\theta}{2}\right)|v - v_*|,$$

and $|\phi(v_*) - \phi(v'_*)| \le C_\phi$. Then for any $\delta \in (0, \pi/2)$,

$$\begin{aligned}\int_\sigma b(\cos\theta)|\phi(v_*) - \phi(v'_*)|d\sigma &\le C_\phi\left\{|v - v_*|\int_0^\delta \frac{\sin(\theta/2)}{\theta^{1+2s}}d\theta + \int_\delta^{\pi/2}\frac{1}{\theta^{1+2s}}d\theta\right\}\\ &\le C_\phi\left\{|v - v_*|\delta^{-2s+1} + \delta^{-2s}\right\}.\end{aligned}$$

If $|v - v_*|^{-1} \le \frac{\pi}{2}$, by choosing $\delta = |v - v_*|^{-1}$, we get

$$\int_\sigma b(\cos\theta)|\phi(v_*) - \phi(v'_*)|d\sigma \le C_\phi|v - v_*|^{2s} \le C\langle v\rangle^{2s}\langle v_*\rangle^{2s}.$$

If $|v - v_*| \le \frac{2}{\pi}$, we have

$$\int_\sigma b(\cos\theta)|\phi(v_*) - \phi(v'_*)|d\sigma \le C_\phi|v - v_*| \le C_\phi\frac{2}{\pi} \le C\langle v\rangle^{2s}\langle v_*\rangle^{2s}.$$

And this completes the proof of the lemma. □

Up to the kernel of $\mathcal{L}$, the following lemma gives the equivalence between the non-isotropic norm and the Dirichlet form of $\mathcal{L}$.

Lemma 2.6. *For $g \in \mathcal{N}^\perp$, we have*

$$\left(\mathcal{L}g, g\right)_{L^2(\mathbb{R}^3_v)} \sim |||g|||^2. \tag{2.2.5}$$

Here $A \sim B$ means that there exists two generic constants $C_1, C_2 > 0$ such that $C_1 A \le B \le C_2 A$.

Proof. We first deal with the lower bound estimate starting with the terms linked to $\mathcal{L}_2$. Since

$$-\Big(\mathcal{L}_2 g,\, g\Big)_{L^2(\mathbb{R}^3_v)} = \Big(\Gamma(g,\ \sqrt{\mu}\,), g\Big)_{L^2(\mathbb{R}^3)},$$

we get from (2.1.5) that

$$-\Big(\mathcal{L}_2 g,\, g\Big)_{L^2(\mathbb{R}^3_v)} = \Big(Q(\sqrt{\mu} g,\ \sqrt{\mu}),\, g\Big)_{L^2(\mathbb{R}^3_v)} + \iiint b(\cos\theta)\Big(\sqrt{\mu_*} - \sqrt{\mu'_*}\Big) g'_* \sqrt{\mu'}\, g. \tag{2.2.6}$$

Using (2.1.2) with $\alpha = 0, m = 0$, the first term on the right hand side of (2.2.6) can be estimated by

$$\begin{aligned}\left|\Big(Q(\sqrt{\mu} g,\ \sqrt{\mu}),\, g\Big)_{L^2(\mathbb{R}^3_v)}\right| &\leq \|Q(\sqrt{\mu} g,\ \sqrt{\mu})\|_{L^2}\|g\|_{L^2}\\ &\leq C\|\sqrt{\mu} g\|_{L^1_{2s}}\|\sqrt{\mu}\|_{H^{2s}_{2s}}\|g\|_{L^2} \leq C\|g\|^2_{L^2}.\end{aligned}$$

For the second term on the right hand side of (2.2.6), we have

$$\begin{aligned}&\iiint b(\cos\theta)\Big(\sqrt{\mu_*} - \sqrt{\mu'_*}\Big) g'_* \sqrt{\mu'}\, g\, dv dv_* d\sigma\\ &= \iiint b(\cos\theta)\Big(\sqrt{\mu_*} - \sqrt{\mu'_*}\Big) g'_* (\mu')^{1/4}\Big((\mu')^{1/4} - (\mu)^{1/4}\Big) g\\ &+ \iiint b(\cos\theta)\Big(\sqrt{\mu_*} - \sqrt{\mu'_*}\Big) g'_* (\mu')^{1/4}\,(\mu)^{1/4} g.\end{aligned}$$

Thus,

$$\begin{aligned}&\left|\iiint b(\cos\theta)\Big(\sqrt{\mu_*} - \sqrt{\mu'_*}\Big) g'_* \sqrt{\mu'}\, g\right|\\ &\leq \left(\iiint b(\cos\theta)\Big(\sqrt{\mu_*} - \sqrt{\mu'_*}\Big)^2 |g|^2 (\mu')^{1/4}\right)^{1/2}\\ &\qquad\times\left(\iiint b(\cos\theta)\Big((\mu')^{1/4} - (\mu)^{1/4}\Big)^2 |g'_*|^2 (\mu')^{1/4}\right)^{1/2}\\ &+\left(\iiint b(\cos\theta)\Big|\sqrt{\mu_*} - \sqrt{\mu'_*}\Big| |g|^2 (\mu')^{1/4}(\mu)^{1/4}\right)^{1/2}\\ &\qquad\times\left(\iiint b(\cos\theta)\Big|\sqrt{\mu_*} - \sqrt{\mu'_*}\Big| |g'_*|^2 (\mu')^{1/4}(\mu)^{1/4}\right)^{1/2}\\ &\leq I_1^{1/2}\times I_2^{1/2} + I_3^{1/2}\times I_4^{1/2}.\end{aligned}$$

Using Lemma 2.5 with $\phi = \mu^{1/4}$ gives

$$\int_\sigma b(\cos\theta)\Big|(\mu'_*)^{1/4} - (\mu_*)^{1/4}\Big| d\sigma \leq C|v - v_*|^{2s} \leq C < v >^{2s} < v_* >^{2s}.$$

Since $(\mu'_*)^{1/4}(\mu')^{1/2} = (\mu'_*)^{1/4}(\mu')^{1/4}(\mu')^{1/4} = (\mu_*)^{1/4}\mu^{1/4}(\mu')^{1/4}$, we get

$$\begin{aligned}
I_1 + I_3 &\le C \iiint b(\cos\theta)|(\mu_*)^{1/2} - (\mu'_*)^{1/2}|\ |g|^2(\mu')^{1/2} dv dv_* d\sigma \\
&\le C \iiint b(\cos\theta)\Big|(\mu_*)^{1/4} - (\mu'_*)^{1/4}\Big|\Big((\mu_*)^{1/4} + (\mu'_*)^{1/4}\Big)|g|^2(\mu')^{1/2} \\
&\le C \iiint b(\cos\theta)\Big|(\mu_*)^{1/4} - (\mu'_*)^{1/4}\Big|(\mu_*)^{1/4}|g|^2 \\
&+ C \iiint b(\cos\theta)\Big|(\mu_*)^{1/4} - (\mu'_*)^{1/4}\Big|(\mu'_*)^{1/4}(\mu')^{1/2}|g|^2 \\
&\le C \iint \Big(\langle v_*\rangle^{2s}(\mu_*)^{1/4}\langle v\rangle^{2s}|g|^2 + \langle v_*\rangle^{2s}(\mu_*)^{1/4}\langle v\rangle^{2s}\mu^{1/4}|g|^2\Big) dv dv_* \\
&\le C(\|g\|^2_{L^2_s(\mathbb{R}^3)} + \|g\|^2_{L^2(\mathbb{R}^3)}).
\end{aligned}$$

For I_2, by using the change of variables $(v, v_*) \to (v_*, v)$ and then $(v', v'_*) \to (v, v_*)$, one has

$$\begin{aligned}
&\iiint b(\cos\theta)\left((\mu')^{1/4} - (\mu)^{1/4}\right)^2\ |g'_*|^2(\mu')^{1/4} \\
&= \iiint b(\cos\theta)\left((\mu'_*)^{1/4} - (\mu_*)^{1/4}\right)^2\ |g|^2(\mu_*)^{1/4} \\
&\le C \iint \Big(\langle v_*\rangle^{2s}(\mu_*)^{1/4}\langle v\rangle^{2s}|g|^2\Big) dv dv_* \le C\|g\|^2_{L^2_s(\mathbb{R}^3)}.
\end{aligned}$$

For I_4, using the change of variables $(v, v_*) \to (v', v'_*)$ implies that

$$\begin{aligned}
&\iiint b(\cos\theta)\Big|\sqrt{\mu_*} - \sqrt{\mu'_*}\ \Big|\ |g'_*|^2(\mu')^{1/4}(\mu)^{1/4} \\
&= \iiint b(\cos\theta)\Big|\sqrt{\mu_*} - \sqrt{\mu'_*}\ \Big|\ |g_*|^2(\mu')^{1/4}(\mu)^{1/4} \\
&\le C \iint \Big(\langle v\rangle^{2s}(\mu)^{1/4}\langle v_*\rangle^{2s}|g_*|^2\Big) dv dv_* \le C\|g\|^2_{L^2_s(\mathbb{R}^3)}.
\end{aligned}$$

In summary, we obtain

$$|(\mathcal{L}_2 g, g)| \le C\|g\|^2_{L^2_s}. \tag{2.2.7}$$

For the term involving $\mathcal{L}_1$, using (2.1.6) yields

$$\begin{aligned}
&\Big(\mathcal{L}_1 g,\ g\Big)_{L^2(\mathbb{R}^3_v)} = -\Big(\Gamma(\sqrt{\mu},\ g),\ g\Big)_{L^2(\mathbb{R}^3_v)} \\
&= \frac{1}{2}\iiint b(\cos\theta)\left((\mu'_*)^{1/2}g' - (\mu_*)^{1/2}\ g\right)^2 \\
&= \frac{1}{2}\iiint b(\cos\theta)\left((\mu'_*)^{1/2}(g' - g) + g((\mu'_*)^{1/2} - (\mu_*)^{1/2})\right)^2 \\
&\ge \frac{1}{4}\iiint b(\cos\theta)\mu'_*(g' - g)^2 - \frac{1}{2}\iiint b(\cos\theta)g^2\Big((\mu'_*)^{1/2} - (\mu_*)^{1/2}\Big)^2,
\end{aligned}$$

where we used the inequality $(a + b)^2 \ge \frac{1}{2}a^2 - b^2$. Then

$$\begin{aligned}
\Big(\mathcal{L}_1 g,\ g\Big)_{L^2(\mathbb{R}^3_v)} \ge \frac{1}{4}&\left(\iiint b(\cos\theta)\mu'_*(g' - g)^2 + \iiint b(\cos\theta)g^2\Big((\mu'_*)^{1/2} - (\mu_*)^{1/2}\Big)^2\right) \\
&- \frac{3}{4}\iiint b(\cos\theta)g^2\Big((\mu'_*)^{1/2} - (\mu_*)^{1/2}\Big)^2.
\end{aligned}$$

We now apply (2.2.4) and the change of variables $(v, v_*) \to (v_*, v)$ to get

$$\iiint b(\cos\theta) g^2\Big((\mu'_*)^{1/2} - (\mu_*)^{1/2}\Big)^2 \le C\|g\|^2_{L^2_s}\|\mu^{1/2}\ \|^2_{H^s_s} \le C\|g\|^2_{L^2_s}\,.$$

Therefore,

$$\Big(\mathcal{L}_1 g,\, g\Big)_{L^2} \ge \frac{1}{4}|||g|||^2 - C\|g\|^2_{L^2_s}.$$

Thus, we have from (2.2.7)

$$\begin{aligned}\Big(\mathcal{L}g,\, g\Big)_{L^2} &= \Big(\mathcal{L}_1 g, g\Big)_{L^2} + \Big(\mathcal{L}_2 g, g\Big)_{L^2}\\ &\ge \frac{1}{4}|||g|||^2 - C\|g\|^2_{L^2_s}.\end{aligned}$$

By Theorem 2.2, we have from the assumption $g \in \mathcal{N}^\perp$ that

$$|||g|||^2 \le 4\Big(\mathcal{L}g,\, g\Big)_{L^2} + C\|g\|^2_{L^2_s} \le \tilde{C}\Big(\mathcal{L}g,\, g\Big)_{L^2}\,,$$

which gives the lower bound estimation.

For the upper bound estimate, we have

$$\begin{aligned}&\Big(\mathcal{L}_1 g,\, g\Big)_{L^2(\mathbb{R}^3_v)} = \frac{1}{2}\iiint b(\cos\theta)\Big((\mu'_*)^{1/2} g' - (\mu_*)^{1/2}\, g\Big)^2\\ &= \frac{1}{2}\iiint b(\cos\theta)\Big((\mu'_*)^{1/2}(g' - g) + g((\mu_*)^{1/2}\ - (\mu_*)^{1/2})\Big)^2\\ &\le \iiint b(\cos\theta)\mu'_*(g'-g)^2 + \iiint b(\cos\theta) g^2\Big((\mu'_*)^{1/2} - (\mu_*)^{1/2}\Big)^2\\ &\le |||g|||^2\,.\end{aligned}$$

By (2.1.7), we have

$$\Big(\mathcal{L}g,\, g\Big)_{L^2(\mathbb{R}^3_v)} \le 2|||g|||^2\,.$$

The proof of Lemma 2.6 is then completed. ◻

The next result shows that the non-isotropic norm controls the Sobolev norm of both derivative and weight of order s.

Lemma 2.7. *There exists $C > 0$ such that*

$$|||g|||^2 \ge C(\|g\|^2_{H^s} + \|g\|^2_{L^2_s}). \tag{2.2.8}$$

Proof. Write

$$\begin{aligned}|||g|||^2 = &\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta)\mu_*\Big(g(v) - g(v')\Big)^2 d\sigma dv_* dv\\ &+ \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta) g_*^2\Big(\mu^{1/2}(v) - \mu^{1/2}(v')\Big)^2 d\sigma dv_* dv \equiv A + B.\end{aligned}$$

According to the calculation of Propositions 1 and 2 in [6], we have

$$
\begin{aligned}
A &= (2\pi)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big)\Big\{\hat{\mu}(0)|\hat{g}(\xi)|^2 + \hat{\mu}(0)|\hat{g}(\xi^+)|^2 \\
&\qquad - 2Re\, \hat{\mu}(\xi^-)\hat{g}(\xi^+)\overline{\hat{g}}(\xi)\Big\} d\sigma d\xi \\
&\geq \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} |\hat{g}(\xi)|^2 \left\{ \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big)(\hat{\mu}(0) - |\hat{\mu}(\xi^-)|) d\sigma \right\} d\xi \\
&\geq C_1 \int_{|\xi|\geq 1} |\xi|^{2s} |\hat{g}(\xi)|^2 d\xi \geq C_1 2^{-2s} \int_{|\xi|\geq 1} (1 + |\xi|^2)^s |\hat{g}(\xi)|^2 d\xi \\
&\geq C_1 2^{-2s} \|g\|^2_{H^s(\mathbb{R}^3_v)} - C_1 \|g\|^2_{L^2(\mathbb{R}^3_v)},
\end{aligned}
$$

where we have used Lemma 3 in [6] that

$$
\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big)(\hat{\mu}(0) - |\hat{\mu}(\xi^-)|) d\sigma \geq C_1 |\xi|^{2s}, \qquad \forall |\xi| \geq 1. \tag{2.2.9}
$$

Similarly,

$$
\begin{aligned}
B &= (2\pi)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big)\Big\{\widehat{g^2}(0)|\widehat{\mu^{1/2}}(\xi)|^2 + \widehat{g^2}(0)|\widehat{\mu^{1/2}}(\xi^+)|^2 \\
&\qquad - 2Re\, \widehat{g^2}(\xi^-)\widehat{\mu^{1/2}}(\xi^+)\,\overline{\widehat{\mu^{1/2}}}(\xi)\Big\} d\sigma d\xi \\
&= \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big) \widehat{g^2}(0) \Big|\widehat{\mu^{1/2}}(\xi^+) - \widehat{\mu^{1/2}}(\xi)\Big|^2 d\sigma d\xi \\
&+ \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big)\Big(\widehat{g^2}(0) - Re\, \widehat{g^2}(\xi^-)\Big)\overline{\widehat{\mu^{1/2}}}(\xi)\,\widehat{\mu^{1/2}}(\xi^+) d\sigma d\xi \\
&= B_1 + B_2\,.
\end{aligned}
$$

For B_1, one has

$$
\begin{aligned}
B_1 &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big) \widehat{g^2}(0) \Big|\widehat{\mu^{1/2}}(\xi^+) - \widehat{\mu^{1/2}}(\xi)\Big|^2 d\sigma d\xi \\
&= C_1 \|g\|^2_{L^2(\mathbb{R}^3_v)} \int_{\mathbb{R}^3_\xi} \widehat{\mu}(2\xi) \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big) \Big|\widehat{\mu^{1/2}}(\xi^-) - 1\Big|^2 d\sigma d\xi \\
&\geq C_2 \|g\|^2_{L^2(\mathbb{R}^3_v)},
\end{aligned}
$$

where

$$
C_2 = C_1 \int_{\mathbb{R}^3_\xi} \widehat{\mu}(2\xi) \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big) \Big|\widehat{\mu^{1/2}}(\xi^-) - 1\Big|^2 d\sigma d\xi > 0.
$$

For the second term on the right hand side, by using

$$
\overline{\widehat{\mu^{1/2}}}(\xi)\,\widehat{\mu^{1/2}}(\xi^+) \geq C\widehat{\mu}(2\xi),
$$

for some positive constant C, we have

$$
\begin{aligned}
B_2 &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big)\Big(\widehat{g^2}(0) - Re\, \widehat{g^2}(\xi^-)\Big)\overline{\widehat{\mu^{1/2}}}(\xi)\,\widehat{\mu^{1/2}}(\xi^+) d\sigma d\xi \\
&\geq C \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big)\Big(\widehat{g^2}(0) - Re\, \widehat{g^2}(\xi^-)\Big)\widehat{\mu}(2\xi) d\sigma d\xi. \\
&= C \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big) \int_{\mathbb{R}^3_v} g^2(v)\Big(1 - \cos(\xi^- \cdot v)\Big) dv \widehat{\mu}(2\xi) d\sigma d\xi.
\end{aligned}
$$

We now use Bobylev's technique [18] to have

$$\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\,\cdot\,\sigma\Big)\psi(\xi^-\cdot v)d\sigma = \int_{\mathbb{S}^2} b\Big(\frac{v}{|v|}\,\cdot\,\sigma\Big)\psi(\xi\cdot v^-)d\sigma,$$

so that

$$\begin{aligned} B_2 &\geq C\int_{\mathbb{R}^3_v} g^2(v)\left(\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\Big(\frac{v}{|v|}\,\cdot\,\sigma\Big)\Big(1-\cos(\xi\,\cdot\,v^-)\Big)\widehat{\mu}(2\xi)d\sigma d\xi\right)dv\\ &= C\int_{\mathbb{R}^3_v} g^2(v)\left(\int_{\mathbb{S}^2} b\Big(\frac{v}{|v|}\,\cdot\,\sigma\Big)\Big(\mu(0)-\mu(\frac{v^-}{2})\Big)d\sigma\right)dv\\ &\geq C\int_{|v|\geq 1} g^2(v)|v|^{2s}dv \geq C2^{-2s}\|g\|^2_{L^2_s(\mathbb{R}^3_v)} - C\|g\|^2_{L^2(\mathbb{R}^3_v)}. \end{aligned}$$

where we have used (2.2.9) and the change of variables in σ by exchanging $\xi/|\xi|$ and $v/|v|$.

Finally, by choosing a suitably small constant $0<\lambda<1$,

$$\begin{aligned} |||g|||^2 = A + B_1 + B_2 &\geq \lambda A + B_1 + \lambda B_2\\ &\geq C(\|g\|^2_{H^s(\mathbb{R}^3_v)} + \|g\|^2_{L^2(\mathbb{R}^3_v)}), \end{aligned}$$

and this concludes the proof of the lemma. □

2.3. **Upper bound estimates.** To apply the energy method, we need some upper bound estimate on the collision operator in terms of the non-isotropic norm which will be given in the following proposition. For this, we first prove

Lemma 2.8. *There exists $C>0$ such that*

$$\iiint b(\cos\theta)\, f_*^2(g'-g)^2 \leq C\,\|f\|^2_{L^2_s}\, |||g|||^2\,. \tag{2.3.1}$$

Proof. Different from Lemma 2.4, we apply Bobylev formula [18] to have

$$\begin{aligned} &\iiint b(\cos\theta)\mu_*(g'-g)^2dv_*d\sigma dv\\ &= \frac{1}{(2\pi)^3}\iint b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\Big(\hat\mu(0)(|\hat g(\xi)|^2+|\hat g(\xi^+)|^2) - 2Re\;\hat\mu(\xi^-)\hat g(\xi^+)\bar{\hat g}(\xi)\Big)d\xi d\sigma\\ &= \frac{1}{(2\pi)^3}\iint b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big(\hat\mu(0)|\hat g(\xi)-\hat g(\xi^+)|^2 + 2Re\left(\hat\mu(0)-\hat\mu(\xi^-)\right)\hat g(\xi^+)\bar{\hat g}(\xi)\Big)d\xi d\sigma, \end{aligned}$$

and

$$\begin{aligned} &\iiint b(\cos\theta)\, f_*^2(g'-g)^2dv_*d\sigma dv\\ &= \frac{1}{(2\pi)^3}\iint b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\left(\widehat{f^2}(0)|\hat g(\xi)-\hat g(\xi^+)|^2 + 2Re\left(\widehat{f^2}(0)-\widehat{f^2}(\xi^-)\right)\hat g(\xi^+)\bar{\hat g}(\xi)\right)d\xi d\sigma\,. \end{aligned}$$

Since $\hat\mu(0) = 1$, $\widehat{f^2}(0) = \|f\|^2_{L^2}$, we obtain

$$
\begin{aligned}
&\iiint b(\cos\theta) f_*^2 (g' - g)^2 dv_* d\sigma dv \\
&= \|f\|^2_{L^2} \iiint b(\cos\theta)\mu_*(g'-g)^2 dv_* d\sigma dv \\
&\quad - \frac{2}{(2\pi)^3}\|f\|^2_{L^2}\iint b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) Re\Big(\hat\mu(0) - \hat\mu(\xi^-)\Big)\hat g(\xi^+)\bar{\hat g}(\xi)\Big) d\xi d\sigma \\
&\qquad + \frac{2}{(2\pi)^3}\iint b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) Re\Big(\widehat{f^2}(0) - \widehat{f^2}(\xi^-)\Big)\hat g(\xi^+)\bar{\hat g}(\xi)\Big) d\xi d\sigma\,.
\end{aligned}
$$

For the last term, we note that

$$\int b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)|\widehat{f^2}(0) - \widehat{f^2}(\xi^-)|d\sigma \le \int b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\left(\int_v f^2(v)\Big|1 - e^{-iv\cdot\xi^-}\Big|dv\right)d\sigma.$$

Now consider

$$\int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\Big|1 - e^{-iv\cdot\xi^-}\Big|d\sigma.$$

If $|v||\xi| \ge \frac{2}{\pi}$, we choose $\delta = \frac{1}{|\xi||v|} \le \pi/2$ to have $|1 - e^{-iv.\xi^-}| \le |v||\xi|\sin\theta$ for any $0 \le \theta \le \delta$. And if $\frac{\pi}{2} \ge \theta \ge \delta$, we have $|1 - e^{-iv.\xi^-}| \le 2$. Hence,

$$
\begin{aligned}
\int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\Big|1 - e^{-iv\cdot\xi^-}\Big|d\sigma &\le C|v||\xi|\int_0^\delta \frac{1}{\theta^{1+2s}}\sin\theta\, d\theta + C'\int_\delta^{\pi/2}\frac{1}{\theta^{1+2s}}d\theta \\
&\le C|v||\xi|\delta^{-2s+1} + C'\delta^{-2s} \le \tilde C|v|^{2s}|\xi|^{2s}.
\end{aligned}
$$

On the other hand, if $|v||\xi| \le \frac{2}{\pi}$, we have directly

$$
\begin{aligned}
\int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\Big|1 - e^{-iv\cdot\xi^-}\Big|d\sigma &\le C\int_0^{\pi/2}\frac{1}{\theta^{1+2s}}|v|\,|\xi|\sin\theta d\theta \\
&\le \tilde C|v||\xi| \le \tilde C|v|^{2s}|\xi|^{2s}.
\end{aligned}
$$

Thus, we have

$$\iint b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\Big|\widehat{f^2}(0) - \widehat{f^2}(\xi^-)\Big||\hat g|^2(\xi)d\xi d\sigma \le C\|f\|^2_{L^2_s}\|g\|^2_{H^s}.$$

By using the regular change of variables $\xi \to \xi^+$, and by noticing that

$$\xi^- = \phi(\xi^+,\sigma) = \xi^+ - \frac{|\xi^+|}{\cos\frac{\theta}{2}}\sigma, \qquad |\xi^-| = |\xi^+|\tan\frac{\theta}{2}, \qquad \cos\frac{\theta}{2} = \frac{\xi^+}{|\xi^+|}\cdot\sigma,$$
$$\left|\frac{\partial(\xi^+)}{\partial(\xi)}\right| = \frac{1}{4}\cos^2\theta/2,$$

we have

$$
\begin{aligned}
&\iint b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\Big|\widehat{f^2}(0) - \widehat{f^2}(\xi^-)\Big||\hat g|^2(\xi^+)d\xi d\sigma \\
&= \iint \frac{1}{\cos^2\theta/2} b\left(2(\frac{\xi^+}{|\xi^+|}\cdot\sigma)^2 - 1\right)\Big|\widehat{f^2}(0) - \widehat{f^2}(\phi(\xi^+,\sigma))\Big||\hat g|^2(\xi^+)d\xi^+ d\sigma \\
&\le C\|f\|^2_{L^2_s}\|g\|^2_{H^s}.
\end{aligned}
$$

Hence,

$$\Big|\iint b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) Re\Big(\widehat{f^2}(0) - \widehat{f^2}(\xi^-)\Big)\hat g(\xi^+)\bar{\hat g}(\xi)d\xi d\sigma\Big| \le C\|f\|^2_{L^2_s}\|g\|^2_{H^s}.$$

Similarly, we have

$$\left| \iint b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) Re\left(\hat{\mu}(0)-\hat{\mu}(\xi^-)\right)\hat{g}(\xi^+)\bar{\hat{g}}(\xi)d\xi d\sigma \right| \le C\|\sqrt{\mu}\|^2_{L^2_s}\|g\|^2_{H^s}.$$

Therefore, we have proved (2.3.1) by using (2.2.8). □

In view of future application of the energy method, the scalar product of the collision operator with a test function is given by

Proposition 2.9. *There exists $C > 0$ such that*

$$\left|\left(\Gamma(f,\, g),\, h\right)_{L^2(\mathbb{R}^3)}\right| \le C\left(\|f\|_{L^2_s}|||g||| + \|g\|_{L^2_s}|||f|||\right)\, |||h|||\,.$$

Proof. Note that

$$\begin{aligned}
\left(\Gamma(f,\, g),\, h\right)_{L^2(\mathbb{R}^3)} &= \left(\mu^{-1/2}Q(\mu^{1/2}f,\, \mu^{1/2}g),\, h\right)_{L^2(\mathbb{R}^3)}\\
&= \iiint b(\cos\theta)\mu_*^{1/2}\left(f'_*g' - f_*g\right)\, h\\
&= \frac{1}{2}\iiint b(\cos\theta)\left(f'_*g' - f_*g\right)\left(\mu_*^{1/2}h - \mu_*^{1/2'}h'\right)\\
&\le \frac{1}{2}\left(\iiint b(\cos\theta)\left(f'_*g' - f_*g\right)^2\right)^{1/2}\\
&\qquad\qquad \times\left(\iiint b(\cos\theta)\left((\mu_*)^{1/2}h - (\mu'_*)^{1/2}h'\right)^2\right)^{1/2}\\
&\le \frac{1}{2}A^{1/2}\times B^{1/2}.
\end{aligned}$$

For B, we have

$$B \le 2\iiint b(\cos\theta)\mu_*(h'-h)^2 + 2\iiint b(\cos\theta)h_*^2\left((\mu')^{1/2}-\mu^{1/2}\right)^2 = 2|||h|||^2,$$

where we have used the change of variables $(v, v_*)\to(v', v'_*)$ for the first term and $(v, v_*)\to(v_*, v)$ for the second term. Similarly,

$$A \le 2\iiint b(\cos\theta)f_*{}^2(g'-g)^2 + 2\iiint b(\cos\theta)g_*{}^2(f'-f)^2.$$

Then (2.3.1) implies that

$$A \le C\left(\|f\|^2_{L^2_s}|||g|||^2 + \|g\|^2_{L^2_s}|||f|||^2\right),$$

which completes the proof of the proposition. □

3. Commutator estimates

3.1. **Non-isotropic norm in $\mathbb{R}^6_{x,v}$.** We now define the norm associated with the collision operator on the space of (x, v). For $m\in\mathbb{N}$, $\ell\in\mathbb{R}$, set

$$\mathcal{B}^m_\ell(\mathbb{R}^6_{x,v}) = \left\{g\in\mathcal{S}'(\mathbb{R}^6_{x,v});\ |||g|||^2_{\mathcal{B}^m_\ell(\mathbb{R}^6)} = \sum_{|\alpha|\le m}\int_{\mathbb{R}^3_x}|||W^\ell\,\partial^\alpha_{x,v}g(x,\,\cdot\,)|||^2dx < +\infty\right\},$$

where $|||\cdot|||$ is the non-isotropic norm defined in (2.2.1).

First of all, one has

Lemma 3.1. *For any* $\ell \geq 0, \gamma, \beta \in \mathbb{N}^3$,

$$|||W^\ell \partial_x^\gamma \partial_v^\beta \mathbf{P} g|||_{\mathcal{B}_0^0(\mathbb{R}^6)} + |||\mathbf{P}(W^\ell \partial_x^\gamma \partial_v^\beta g)|||_{\mathcal{B}_0^0(\mathbb{R}^6)} \leq C_{\ell,\beta} \|\partial_x^\gamma g\|^2_{L^2(\mathbb{R}^6)}, \tag{3.1.1}$$

$$C_0|||g|||^2_{\mathcal{B}_0^0(\mathbb{R}^6)} - C_2\|g\|^2_{L^2(\mathbb{R}^6)} \leq \Big(\mathcal{L}g,\, g\Big)_{L^2(\mathbb{R}^6_{x,v})} \leq C_3|||g|||^2_{\mathcal{B}_0^0(\mathbb{R}^6)}, \tag{3.1.2}$$

and

$$\|g\|^2_{L^2_{l+s}(\mathbb{R}^6)} + \|g\|^2_{L^2(\mathbb{R}^3_x; H^s_l(\mathbb{R}^3_v))} \leq C|||g|||^2_{\mathcal{B}^0_l(\mathbb{R}^6)} \leq C\|g\|^2_{L^2(\mathbb{R}^3_x; H^s_{l+s}(\mathbb{R}^3_v))}. \tag{3.1.3}$$

Proof. By definition of the projection operator $\mathbf{P}$, we have

$$\mathbf{P}g = a_g(t,x)\mu^{1/2} + \sum_{j=1}^3 b_{g,j}(t,x)\, v_j \mu^{1/2} + c_g(t,x)|v|^2\mu^{1/2},$$

with

$$a_g(t,x) = \int_{\mathbb{R}^3_v} g(t,x,v)\mu^{1/2}(v)dv,\ \ c_g(t,x) = \int_v g(t,x,v)|v|^2\mu^{1/2}(v)dv,$$

and

$$b_{g,j}(t,x) = \int_v g(t,x,v)\, v_j \mu^{1/2}(v)dv, \qquad\qquad j = 1,2,3.$$

Thus (3.1.1) can be obtained by integration by parts. To get (3.1.2), we use (2.2.2) and (2.2.5) to obtain

$$\begin{aligned}|||g|||^2_{\mathcal{B}_0^0(\mathbb{R}^6)} &\geq C\Big(\mathcal{L}g,\, g\Big)_{L^2(\mathbb{R}^6_{x,v})} \geq C_0|||(\mathbf{I}-\mathbf{P})g|||^2_{\mathcal{B}_0^0(\mathbb{R}^6)}\\ &\geq \frac{C_0}{2}|||g|||^2_{\mathcal{B}_0^0(\mathbb{R}^6)} - C_0|||\mathbf{P}g|||^2_{\mathcal{B}_0^0(\mathbb{R}^6)}\\ &\geq \frac{C_0}{2}|||g|||^2_{\mathcal{B}_0^0(\mathbb{R}^6)} - C_2\|g\|^2_{L^2(\mathbb{R}^6)}.\end{aligned}$$

Finally, (3.1.3) follows directly from (2.2.2) and (2.2.8). □

3.2. **Weighted estimates on commutators.** We will use the following notation, for $\gamma \in \mathbb{N}^3$,

$$\mathcal{T}(F,\, G,\, \mu_\gamma) = Q(\mu_\gamma F,\, G) + \iint b(\cos\theta)\Big((\mu_\gamma)_* - (\mu_\gamma)'_*\Big)F'_*G'dv_*d\sigma\,, \tag{3.2.1}$$

where $\mu_\gamma = p_\gamma(v)\sqrt{\mu(v)} = \partial^\gamma(\sqrt{\mu})$ is a Maxwellian type function of variable v.

In this notation, (2.1.5) is equivalent to

$$\Gamma(f,g) = \mathcal{T}(f,g,\sqrt{\mu}).$$

And the Leibniz formula gives

$$\partial_x^\alpha \partial_v^\beta \Gamma(f,\, g) = \sum_{\alpha_1+\alpha_2=\alpha,\ \beta_1+\beta_2+\beta_3=\beta} C^{\alpha_1,\alpha_2}_{\beta_1,\beta_2,\beta_3}\mathcal{T}(\partial_x^{\alpha_1}\partial_v^{\beta_1} f,\, \partial_x^{\alpha_2}\partial_v^{\beta_2} g,\, \mu_{\beta_3})\,. \tag{3.2.2}$$

First of all, let us recall the following lemma from [12].

Lemma 3.2. *Let* $\ell \geq 0, 0 < s < 1/2$. *There exists* $C > 0$ *such that*

$$\Big|\Big((W^\ell\ Q(f,\ g) - Q(f,\ W^\ell\ g)),\ \ h\Big)_{L^2(\mathbb{R}^3)}\Big| \leq C\|f\|_{L^1_\ell(\mathbb{R}^3_v))}\|g\|_{L^2_\ell(\mathbb{R}^3)}\|h\|_{L^2(\mathbb{R}^3)}.$$

Using this result, we shall show that

Proposition 3.3. *For any* $\ell \geq 0$,

$$\Big(W^\ell\mathcal{T}(F,\, G,\, \mu_\gamma) - \mathcal{T}(F,\, W^\ell G,\, \mu_\gamma),\ h\Big)_{L^2(\mathbb{R}^3_v)} \leq C\|F\|_{L^2_\ell}\|G\|_{L^2_\ell}\|h\|_{L^2_s}. \tag{3.2.3}$$

Proof. From (3.2.1), it follows that

$$\begin{aligned}&\Big(W^\ell \mathcal{T}(F,\, G,\, \mu_\gamma\,) \,-\mathcal{T}(F,\, W^\ell\, G,\, \mu_\gamma\,),\;\, h\Big)_{L^2(\mathbb{R}^3_v)}\\ &= \Big(W^\ell\, Q(\mu_\gamma\, F,\, G) - Q(\mu_\gamma F, W^\ell G),\, h\Big)_{L^2(\mathbb{R}^3_v)}\\ &+ \iiint b(\cos\theta)(\mu_{\gamma *} - \mu'_{\gamma *})F'_* G'(W^\ell - W'^\ell)\, h\\ &= B_1 + B_2.\end{aligned}$$

Lemma 3.2 implies that

$$B_1 \le C\|\mu_\gamma F\|_{L^1_\ell}\|G\|_{L^2_\ell}\|h\|_{L^2} \le C\|F\|_{L^2}\|G\|_{L^2_\ell}\|h\|_{L^2}.$$

For B_2, since we have assumed that $0 < s < 1/2$, we get

$$B_2 \le \left(\iiint b(\cos\theta)|F'_*|^2|G'|^2\frac{|W^\ell - W'^\ell|^2}{\sin\theta}\right)^{1/2}\left(\iiint b(\cos\theta)\sin\theta\Big(\mu_{\gamma *} - \mu'_{\gamma *}\Big)^2|h|^2\right)^{1/2}.$$

(2.2.4) implies that

$$\iiint b(\cos\theta)\Big(\mu_{\gamma *} - \mu'_{\gamma *}\Big)^2|h|^2 \le C\|\mu_\gamma\|^2_{H^s_s}\|h\|^2_{L^2_s},$$

while, using

$$|W^\ell - W'^\ell|^2 \le \sin^2\theta\Big((W^\ell_*)^2 + (W'^\ell)^2\Big) \le \sin^2\theta\; (W^\ell_*)^2(W'^\ell)^2,$$

we get

$$\begin{aligned}\iiint b(\cos\theta)|F'_*|^2|G'|^2\frac{|W^\ell - W'^\ell|^2}{\sin\theta} &\le \iiint b(\cos\theta)\sin\theta\,(W^\ell\, F)'^2_*(W^\ell\, G)'^2\\ &\le C\|F\|^2_{L^2_\ell}\|G\|^2_{L^2_\ell},\end{aligned}$$

which leads to completion of the proof of the proposition. □

Similarly, we have also

Proposition 3.4. *There exists a constant $C > 0$ such that*

$$\left|\Big(\mathcal{T}(F,\, G,\, \mu_\gamma\,),\;\, h\Big)_{L^2(\mathbb{R}^3_v)}\right| \le C\left(\|F\|_{L^2_s}\, |||G||| + \|G\|_{L^2_s}\, |||F|||\right)|||h|||. \tag{3.2.4}$$

Proof. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\Big(\mathcal{T}(F,\, G,\, \mu_\gamma\,),\;\, h\Big)_{L^2(\mathbb{R}^3_v)} &= \iiint b(\cos\theta)(\mu_{\gamma *})^{1/2}\Big(F'_*G' - F_*G\Big)\; h\\ &= \frac{1}{2}\iiint b(\cos\theta)\Big(F'_*G' - F_*G\Big)\Big((\mu_{\gamma *})^{1/2}h - (\mu'_{\gamma *})^{1/2}h'\Big)\\ &\le \frac{1}{2}\left(\iiint(\cos\theta)\Big(F'_*G' - F_*G\Big)^2\right)^{1/2}\\ &\qquad\times\left(\iiint b(\cos\theta)\Big((\mu_{\gamma *})^{1/2}h - (\mu'_{\gamma *})^{1/2}h'\Big)^2\right)^{1/2}\\ &\le \frac{1}{2}\tilde{A}^{1/2}\times\tilde{B}^{1/2}.\end{aligned}$$

By using the estimation of the term A in the proof of Proposition 2.9, it follows that

$$\tilde{A} \le C\left(\|F\|^2_{L^2_s}\, |||G|||^2 + \|G\|^2_{L^2_s}\, |||F|||^2\right)$$

and

$$\tilde{B} \le C\Big(\|\mu_\gamma\|^2_{L^2_s} |||h|||^2 + \|h\|^2_{L^2_s} |||\mu_\gamma|||^2\Big) \le C|||h|||^2.$$

□

We are now ready to prove the following estimate with differentiation and weight.

Proposition 3.5. *For any $\ell \ge 3$, and $N \ge 3$, we have, for all $\beta \in \mathbb{N}^6$, $|\beta| \le N$,*

$$\Big|\Big(W^\ell\, \partial^\beta_{x,v}\Gamma(f,\, g\,),\;\, h\Big)_{L^2(\mathbb{R}^6_{x,v})}\Big| \le C\|f\|_{H^N_\ell(\mathbb{R}^6)}\; |||g|||_{\mathcal{B}^N_\ell(\mathbb{R}^6)}\; |||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)}.$$

Remark 3.6. *In fact, this proposition holds even when $\ell > \frac{3}{2} + 2s$, and $N > \frac{3}{2} + 2s$. Here, we consider the case when $\ell \ge 3, N \ge 3$ with $0 < s < 1/2$ for the simplicity of the notations.*

Proof. Using the Leibniz formula (3.2.2) gives

$$\begin{aligned}
&\Big(W^\ell\partial^\beta_{x,v}\Gamma(f,\, g),\;\, h\Big)_{L^2(\mathbb{R}^6_{x,v})} = \sum C^{\beta_1}_{\beta_2,\beta_3}\Big(\mathcal{T}(\partial^{\beta_1} f,\, W^\ell\partial^{\beta_2}g,\, \mu_{\beta_3}\,),\;\, h\Big)_{L^2(\mathbb{R}^6_{x,v})}\\
&+\sum C^{\beta_1}_{\beta_2,\beta_3}\Big(W^\ell\mathcal{T}(\partial^{\beta_1} f,\, \partial^{\beta_2}g,\, \mu_{\beta_3}\,) - \mathcal{T}(\partial^{\beta_1} f,\, W^\ell\partial^{\beta_2}g,\, \mu_{\beta_3}\,),\;\, h\Big)_{L^2(\mathbb{R}^6_{x,v})}.
\end{aligned}$$

Then from (3.2.3), we get

$$\begin{aligned}
&\Big|\Big(W^\ell\mathcal{T}(\partial^{\beta_1} f,\, \partial^{\beta_2}g,\, \mu_{\beta_3}\,) - \mathcal{T}(\partial^{\beta_1} f,\, W^\ell\partial^{\beta_2}g,\, \mu_{\beta_3}\,),\;\, h\Big)_{L^2(\mathbb{R}^6_{x,v})}\Big|\\
&\le C\left(\int_{\mathbb{R}^3_x}\|\partial^{\beta_1} f\|^2_{L^2_\ell(\mathbb{R}^3_v)}\|\partial^{\beta_2}g\|^2_{L^2_\ell(\mathbb{R}^3_v)}dx\right)^{1/2}\|h\|_{L^2_s(\mathbb{R}^6_{x,v})}\\
&\le C\begin{cases}\|\partial^{\beta_1} f\|_{L^\infty(\mathbb{R}^3_x;\, L^2_\ell(\mathbb{R}^3_v))}\|\partial^{\beta_2}g\|_{L^2_\ell(\mathbb{R}^6_{x,v})}\,\|h\|_{L^2_s(\mathbb{R}^6_{x,v})}, & \text{if } |\beta_1| \le 1\,;\\
\|\partial^{\beta_1} f\|_{L^2_\ell(\mathbb{R}^6_{x,v})}\|\partial^{\beta_2}g\|_{L^\infty(\mathbb{R}^3_x;L^2_\ell(\mathbb{R}^3_v))}\,\|h\|_{L^2_s(\mathbb{R}^6_{x,v})}, & \text{if } |\beta_1| \ge 2\,.\end{cases}
\end{aligned}$$

Since $|\beta_1| \le 1$ implies $|\beta_1| + 3/2 < 3 \le N$ and $|\beta_1| \ge 2$ implies $|\beta_2| + 3/2 < |\beta|$, it follows that

$$\begin{aligned}
&\Big|\Big(W^\ell\mathcal{T}(\partial^{\beta_1} f,\, \partial^{\beta_2}g,\, \mu_{\beta_3}\,) - \mathcal{T}(\partial^{\beta_1} f,\, W^\ell\partial^{\beta_2}g,\, \mu_{\beta_3}\,),\;\, h\Big)_{L^2(\mathbb{R}^6_{x,v})}\Big|\\
&\le C\|f\|_{H^N_\ell(\mathbb{R}^6)}\|g\|_{H^{|\beta|}_\ell(\mathbb{R}^6)}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6_{x,v})}\,.
\end{aligned}\tag{3.2.5}$$

On the other hand, if $|\beta_1| \le 1$ so that $|\beta_1| + \frac{3}{2} + s < 3 \le N$, we get from (3.2.4)

$$\begin{aligned}
&\Big|\Big(\mathcal{T}(\partial^{\beta_1} f,\, W^\ell\partial^{\beta_2}g,\, \mu_{\beta_3}\,),\;\, h\Big)_{L^2(\mathbb{R}^6_{x,v})}\Big|\\
&\le C\left(\int_{\mathbb{R}^3_x}\|\partial^{\beta_1} f\|^2_{L^2_s(\mathbb{R}^3_v)}\Big(|||W^\ell\partial^{\beta_2}g|||^2 + \|W^\ell\partial^{\beta_2}g\|^2_{H^s(\mathbb{R}^3_v)}\Big)dx\right.\\
&\qquad\left. + \int_{\mathbb{R}^3_x}\|W^\ell\partial^{\beta_2}g\|^2_{L^2_s(\mathbb{R}^3_v)}\Big(|||\partial^{\beta_1} f|||^2 + \|\partial^{\beta_1} f\|^2_{H^s(\mathbb{R}^3_v)}\Big)dx\right)^{1/2}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)}\\
&\le C\Big(\|\partial^{\beta_1} f\|_{L^\infty(\mathbb{R}^3_x;\, L^2_s(\mathbb{R}^3_v))} + \|\partial^{\beta_1} f\|^2_{L^\infty(\mathbb{R}^3_x;\, H^s_s(\mathbb{R}^3_v))}\Big)|||g|||_{\mathcal{B}^{|\beta_2|}_\ell(\mathbb{R}^6)}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)}\\
&\le C\|f\|_{H^{|\beta_1|+3/2+s+\epsilon}_s(\mathbb{R}^6)}|||g|||_{\mathcal{B}^{|\beta_2|}_\ell(\mathbb{R}^6)}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)},
\end{aligned}$$

Hence, for $|\beta_1| \le 1$, we have

$$\Big|\Big(\mathcal{T}(\partial^{\beta_1} f,\, W^\ell\partial^{\beta_2}g,\, \mu_{\beta_3}\,),\;\, h\Big)_{L^2(\mathbb{R}^6_{x,v})}\Big| \le C\|f\|_{H^3_s(\mathbb{R}^6)}|||g|||_{\mathcal{B}^{|\beta_2|}_\ell(\mathbb{R}^6)}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)}\,.$$

We now consider the case when $|\beta_1| \geq 2$. First of all, assume $2 \leq |\beta_1| \leq |\beta| - 1$ so that $|\beta_2| = |\beta| - |\beta_1| - |\beta_3| \leq |\beta| - 2$. Then, we get

$$\begin{aligned}
&\Big|\Big(\mathcal{T}(\partial^{\beta_1} f,\ W^\ell \partial^{\beta_2} g,\ \mu_{\beta_3}),\ h\Big)_{L^2(\mathbb{R}^6_{x,v})}\Big| \\
&\leq C\Big(\|f\|_{H^{|\beta_1|}_s(\mathbb{R}^6)}\|W^\ell \partial^{\beta_2} g\|_{L^\infty(\mathbb{R}^3_x;\ H^s_s(\mathbb{R}^3_v))} \\
&\qquad + \|W^\ell \partial^{\beta_2} g\|_{L^\infty(\mathbb{R}^3_x;\ L^2_s(\mathbb{R}^3_v))}\|\partial^{\beta_1} f\|_{L^2(\mathbb{R}^3_x;\ H^s_s(\mathbb{R}^3_v))}\Big)|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)} \\
&\leq C\|f\|_{H^{|\beta_1|+s}_s(\mathbb{R}^6)}\ \|W^\ell \Lambda_x^{3/2+\epsilon}\Lambda_v^s \partial^{\beta_2} g\|^2_{L^2_s(\mathbb{R}^6)}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)} \\
&\leq C\|f\|_{H^{|\beta|-1+s}_s(\mathbb{R}^6)}|||g|||_{\mathcal{B}^{|\beta|-2+3/2+s+\epsilon}_\ell(\mathbb{R}^6)}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)} \\
&\leq C\|f\|_{H^{|\beta|}_s(\mathbb{R}^6)}|||g|||_{\mathcal{B}^{|\beta|}_\ell(\mathbb{R}^6)}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)}.
\end{aligned}$$

We turn next to the case when $\beta_1 = \beta$, for which we have

$$\Big(\mathcal{T}(\partial^\beta f,\ W^\ell g,\ \sqrt{\mu}),\ h\Big)_{L^2(\mathbb{R}^6_{x,v})} = \Big(\Gamma(\partial^\beta f,\ W^\ell g),\ h\Big)_{L^2(\mathbb{R}^6_{x,v})}.$$

Since we want to avoid using the non-isotropic norm of f on the right hand side, we can not use the estimate (2.2.3) to complete the proof. So we proceed in a different way, use firstly (2.1.5) to get

$$\begin{aligned}
&\Big(\Gamma(\partial^\beta f,\ W^\ell g),\ h\Big)_{L^2(\mathbb{R}^6_{x,v})} = \Big(Q(\sqrt{\mu}\,\partial^\beta f,\ W^\ell g),\ h\Big)_{L^2(\mathbb{R}^6_{x,v})} \\
&+ \int\iiint b(\cos\theta)\Big(\sqrt{\mu_*} - \sqrt{\mu'_*}\Big)(\partial^\beta f)'_*(W^\ell g)' h\, dv_* d\sigma dv dx\,.
\end{aligned}$$

On one hand, (2.1.2) with $m = 0, \alpha = -s$, implies that

$$\begin{aligned}
\Big|\Big(Q(\sqrt{\mu}\,\partial^\beta f,\ W^\ell g),\ h\Big)_{L^2(\mathbb{R}^6_{x,v})}\Big| &\leq C\|h\|_{L^2_s(\mathbb{R}^6)}\|\sqrt{\mu}\,\partial^\beta f\|_{L^2(\mathbb{R}^3_x;\ L^1_{2s}(\mathbb{R}^3_v))}\|W^\ell g\|_{L^\infty(\mathbb{R}^3_x;\ H^{2s}_s(\mathbb{R}^3_v))} \\
&\leq C\|f\|_{H^{|\beta|}(\mathbb{R}^6)}\|W^\ell g\|_{H^{3/2+2s+\epsilon}_s(\mathbb{R}^6)}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)}.
\end{aligned}$$

On the other hand, we can write

$$\begin{aligned}
&\int\iiint b(\cos\theta)\Big(\sqrt{\mu_*} - \sqrt{\mu'_*}\Big)(\partial^\beta f)'_*(W^\ell g)' h\, dv_* d\sigma dv dx \\
&= \int\iiint b(\cos\theta)\Big(\sqrt{\mu_*} - \sqrt{\mu'_*}\Big)(\partial^\beta f)'_*(W^\ell g)'\Big(h - h'\Big) dv_* d\sigma dv dx \\
&+ \int\iiint b(\cos\theta)\Big(\sqrt{\mu_*} - \sqrt{\mu'_*}\Big)(\partial^\beta f)'_*(W^\ell g)' h'\, dv_* d\sigma dv dx \\
&= D_1 + D_2\,.
\end{aligned}$$

By the Cauchy-Schwarz inequality, one has

$$\begin{aligned}
|D_1| \leq &\Big(\int\iiint b(\cos\theta)|(\partial^\beta f)'_*|^2|(W^\ell g)'|^2\Big((\mu_*)^{1/4} - (\mu'_*)^{1/4}\Big)^2 dv_* d\sigma dv dx\Big)^{1/2} \\
&\times\Big(\int\iiint b(\cos\theta)\Big(\mu_*^{1/4} + (\mu'_*)^{1/4}\Big)^2(h - h')^2 dv_* d\sigma dv dx\Big)^{1/2}.
\end{aligned}$$

Lemma 2.5 yields

$$
\begin{aligned}
&\int\iiint b(\cos\theta)|(\partial^\beta f)'_*|^2|(W^\ell g)'|^2\Big((\mu_*)^{1/4}-(\mu'_*)^{1/4}\Big)^2 dv_*d\sigma dvdx\\
&\le C\int_{\mathbb{R}^3_x}\int_{\mathbb{R}^6_{v,v_*}}|(\partial^\beta f)_*|^2|(W^\ell g)|^2\langle v\rangle^{2s}\langle v_*\rangle^{2s}dv_*ddvdx\\
&\le C\int_{\mathbb{R}^3_x}\|\partial^\beta f\|^2_{L^2_s(\mathbb{R}^3_v)}\|W^\ell g\|^2_{L^2_s(\mathbb{R}^3_v)}dx\le C\|\partial^\beta f\|^2_{L^2_s(\mathbb{R}^6)}\|W^\ell g\|^2_{L^\infty(\mathbb{R}^3_x;\,L^2_s(\mathbb{R}^3_v))}\\
&\le C\|f\|^2_{H^N_s(\mathbb{R}^6)}\|W^\ell\Lambda_x^{3/2+\epsilon}g\|^2_{L^2_s(\mathbb{R}^6)}\le C\|f\|^2_{H^{|\beta|}_s(\mathbb{R}^6)}|||g|||^2_{\mathcal{B}^2_\ell(\mathbb{R}^6)}\,,
\end{aligned}
$$

while from Lemma 2.8, we get

$$
\begin{aligned}
&\int\iiint b(\cos\theta)\Big(\mu_*^{1/4}+(\mu'_*)^{1/4}\Big)^2(h-h')^2dv_*d\sigma dvdx\\
&\le 4\int\iiint b(\cos\theta)\mu_*^{1/2}(h-h')^2dv_*d\sigma dvdx\le C|||h|||^2_{\mathcal{B}^0_0(\mathbb{R}^6)}.
\end{aligned}
$$

Therefore, we obtain

$$
|D_1|\le C\|f\|_{H^{|\beta|}_s(\mathbb{R}^6)}|||g|||_{\mathcal{B}^2_\ell(\mathbb{R}^6)}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)}.
$$

For the term D_2, we have

$$
\begin{aligned}
&\Big|\int\iiint b(\cos\theta)\Big(\sqrt{\mu_*}-\sqrt{\mu'_*}\Big)(\partial^\beta f)'_*(W^\ell g)'h'dv_*d\sigma dvdx\Big|\\
&=\Big|\int\iiint b(\cos\theta)\Big(\sqrt{\mu'_*}-\sqrt{\mu_*}\Big)(\partial^\beta f)_*(W^\ell g)hdv_*d\sigma dvdx\Big|\\
&\le C\int_{\mathbb{R}^3_x}\int_{\mathbb{R}^6_{v,v_*}}\Big|(\partial^\beta f)_*\Big|\,\Big|W^\ell g\Big|\,|h|\langle v\rangle^{2s}\langle v_*\rangle^{2s}dv_*ddvdx\\
&\le C\int_{\mathbb{R}^3_x}\|\partial^\beta f\|_{L^1_{2s}(\mathbb{R}^3_v)}\|W^\ell g\|_{L^2_s(\mathbb{R}^3_v)}\|h\|_{L^2_s(\mathbb{R}^3_v)}dx\\
&\le C\|\partial^\beta f\|_{L^2(\mathbb{R}^3_x;\,L^2_{3/2+2s+\epsilon}(\mathbb{R}^3_v))}\|W^\ell g\|_{L^\infty(\mathbb{R}^3_x;\,L^2_s(\mathbb{R}^3_v))}\|h\|_{L^2_s(\mathbb{R}^6)},
\end{aligned}
$$

so that

$$
|D_2|\le C\|f\|_{H^{|\beta|}_{3/2+2s+\epsilon}(\mathbb{R}^6)}|||g|||_{\mathcal{B}^2_\ell(\mathbb{R}^6)}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)}.
$$

Therefore, it follows that

$$
\Big|\Big(\Gamma(\partial^\beta f,\,W^\ell g),\;h\Big)_{L^2(\mathbb{R}^6_{x,v})}\Big|\le C\|f\|_{H^{|\beta|}_{3/2+2s+\epsilon}(\mathbb{R}^6)}|||g|||_{\mathcal{B}^2_\ell(\mathbb{R}^6)}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)}. \tag{3.2.6}
$$

Finally, for the case $|\beta_1|\ge 2$, since $3/2+2s<3\le N$, we have also

$$
\Big|\Big(\mathcal{T}(\partial^{\beta_1}f,\,W^\ell\partial^{\beta_2}g,\,\mu_{\beta_3}),\;h\Big)_{L^2(\mathbb{R}^6_{x,v})}\Big|\le C\|f\|_{H^N_\ell(\mathbb{R}^6)}|||g|||_{\mathcal{B}^N_\ell(\mathbb{R}^6)}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)}\,.
$$

The proof of the proposition is then completed. □

By using the argument in the proof of the above proposition, the following proposition follows from the Sobolev imbedding theorems.

Proposition 3.7. *For any $\ell\ge 3$, we have, for all $\beta\in\mathbb{N}^6$, $|\beta|\le 2$,*

$$
\Big|\Big(W^\ell\,\partial^\beta_{x,v}\Gamma(f,\,g),\;h\Big)_{L^2(\mathbb{R}^6_{x,v})}\Big|\le C\|f\|_{H^3_\ell(\mathbb{R}^6)}\;|||g|||_{\mathcal{B}^3_\ell(\mathbb{R}^6)}\;|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)}. \tag{3.2.7}
$$

Finally, the linear operators can be also estimated as follows.

Proposition 3.8. *For $\ell \geq 3$, we have for any $\beta \in \mathbb{N}^6$,*

$$\left|\Big(W^\ell\, \partial^\beta_{x,v}\mathcal{L}_2(f),\ h\Big)_{L^2(\mathbb{R}^6)}\right| \leq C_{|\beta|,\ell}\|f\|_{H^{|\beta|}_\ell(\mathbb{R}^6)}\ |||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)}. \tag{3.2.8}$$

If $|\beta| \geq 1$, we have

$$\begin{aligned}&\left|\Big(\mathcal{L}_1(W^\ell\, \partial^\beta_{x,v}g) - W^\ell\, \partial^\beta_{x,v}\mathcal{L}_1(g),\ h\Big)_{L^2(\mathbb{R}^6)}\right|\\ &\qquad \leq C_{|\beta|,\ell}\Big(\|g\|_{H^{|\beta|}_\ell(\mathbb{R}^6)} + |||g|||_{\mathcal{B}^{|\beta|-1}_\ell(\mathbb{R}^6)}\Big)\ |||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)},\end{aligned} \tag{3.2.9}$$

and for $|\beta| = 0$,

$$\left|\Big(\mathcal{L}_1(W^\ell\, g) - W^\ell\, \mathcal{L}_1(g),\ h\Big)_{L^2(\mathbb{R}^6)}\right| \leq C\|g\|_{L^2_\ell(\mathbb{R}^6)}\ |||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)}. \tag{3.2.10}$$

Remark 3.9. *On the right hand side of (3.2.7), the term $|||g|||_{\mathcal{B}^3_\ell(\mathbb{R}^6)}$ comes from the Sobolev imbedding*

$$L^\infty(\mathbb{R}^3_x;\ H^{2s}_s(\mathbb{R}^3_v)) \supset H^{3/2+2s+\epsilon}_s(\mathbb{R}^6) \supset \mathcal{B}^3_0(\mathbb{R}^6),$$

where ϵ is any small positive number. Thus the order of differentiation is equal to 3. Note that this is due to the nonlinearity in the operator $\Gamma(\cdot,\cdot)$. For the linear operators, the estimates given in (3.2.9) and (3.2.10) do not involve this term.

Proof. For the proof of (3.2.8), by using the Leibniz formula (3.2.2), we have

$$\begin{aligned}&-\Big(W^\ell\, \partial^\beta_{x,v}\mathcal{L}_2(f),\ h\Big)_{L^2(\mathbb{R}^6_{x,v})} = \Big(W^\ell\, \partial^\beta_{x,v}\Gamma(f,\ \sqrt{\mu}),\ h\Big)_{L^2(\mathbb{R}^6_{x,v})}\\ &= \sum C^{\beta_1}_{\beta_2,\beta_3}\Big(\mathcal{T}(\partial^{\beta_1} f,\ W^\ell\partial^{\beta_2}\sqrt{\mu},\ \mu_{\beta_3}),\ h\Big)_{L^2(\mathbb{R}^6_{x,v})}\\ &\quad + \sum C^{\beta_1}_{\beta_2,\beta_3}\Big(W^\ell\mathcal{T}(\partial^{\beta_1} f,\ \partial^{\beta_2}\sqrt{\mu},\ \mu_{\beta_3}) - \mathcal{T}(\partial^{\beta_1} f,\ W^\ell\partial^{\beta_2}\sqrt{\mu},\ \mu_{\beta_3}),\ h\Big)_{L^2(\mathbb{R}^6_{x,v})}\\ &= E_1 + E_2.\end{aligned}$$

Then (3.2.5) implies

$$\begin{aligned}|E_2| &= \left|\sum C^{\beta_1}_{\beta_2,\beta_3}\Big(W^\ell\mathcal{T}(\partial^{\beta_1} f,\ \partial^{\beta_2}\sqrt{\mu},\ \mu_{\beta_3}) - \mathcal{T}(\partial^{\beta_1} f,\ W^\ell\partial^{\beta_2}\sqrt{\mu},\ \mu_{\beta_3}),\ h\Big)_{L^2(\mathbb{R}^6_{x,v})}\right|\\ &\leq C\|f\|_{H^{|\beta|}_\ell(\mathbb{R}^6)}\|\sqrt{\mu}\|_{H^{|\beta|}_\ell(\mathbb{R}^3)}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)} \leq C\|f\|_{H^{|\beta|}_\ell(\mathbb{R}^6)}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)},\end{aligned}$$

and (3.2.4) implies also,

$$\begin{aligned}|E_1| &= \left|\sum C^{\beta_1}_{\beta_2,\beta_3}\Big(\mathcal{T}(\partial^{\beta_1} f,\ W^\ell\partial^{\beta_2}\sqrt{\mu},\ \mu_{\beta_3}),\ h\Big)_{L^2(\mathbb{R}^6_{x,v})}\right|\\ &\leq C\|f\|_{H^{|\beta|}_\ell(\mathbb{R}^6)}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)},\end{aligned}$$

where for the case when $\beta_1 = \beta$, we have used (3.2.6).

For (3.2.9), since $-\mathcal{L}_1(g) = \Gamma(\sqrt{\mu}, g)$, by using again the Leibniz formula (3.2.2), we have

$$\begin{aligned}&-\Big(W^\ell\, \partial^\beta_{x,v}\,\mathcal{L}_1(g) - \mathcal{L}_1(W^\ell\, \partial^\beta_{x,v}\, g),\ h\Big)_{L^2(\mathbb{R}^6)}\\ &= \Big(W^\ell\, \partial^\beta_{x,v}\Gamma(\sqrt{\mu},\ , g) - \Gamma(\sqrt{\mu},\ W^\ell\, \partial^\beta_{x,v},\ g),\ h\Big)_{L^2(\mathbb{R}^6)}\\ &= \sum_{|\beta_2|\leq|\beta|-1} C^{\beta_1}_{\beta_2,\beta_3}\Big(\mathcal{T}(\partial^{\beta_1}\sqrt{\mu},\ W^\ell\partial^{\beta_2}g,\ \mu_{\beta_3}),\ h\Big)_{L^2(\mathbb{R}^6_{x,v})}\\ &\qquad + \sum C^{\beta_1}_{\beta_2,\beta_3}\Big(W^\ell\mathcal{T}(\partial^{\beta_1}\sqrt{\mu},\ \partial^{\beta_2}g,\ \mu_{\beta_3}) - \mathcal{T}(\partial^{\beta_1}\sqrt{\mu},\ W^\ell\partial^{\beta_2}g,\ \mu_{\beta_3}),\ h\Big)_{L^2(\mathbb{R}^6_{x,v})}\\ &= F_1 + F_2.\end{aligned}$$

Then (3.2.5) implies

$$
\begin{aligned}
|F_2| &= \Big|\sum C^{\beta_1}_{\beta_2,\beta_3}\Big(W^\ell \mathcal{T}(\partial^{\beta_1}\sqrt{\mu}\,,\,\partial^{\beta_2}g,\,\mu_{\beta_3}\,) - \mathcal{T}(\partial^{\beta_1}\sqrt{\mu}\,,\,W^\ell\partial^{\beta_2}g,\,\mu_{\beta_3}\,),\;h\Big)_{L^2(\mathbb{R}^6_{x,v})}\Big|\\
&\le C\|\sqrt{\mu}\,\|_{H^{|\beta|}_\ell(\mathbb{R}^3)}\|g\|_{H^{|\beta|}_\ell(\mathbb{R}^6)}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6_{x,v})} \le C\|g\|_{H^{|\beta|}_\ell(\mathbb{R}^6)}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6_{x,v})}\,,
\end{aligned}
$$

which also gives (3.2.10) .

On the other hand, for F_1, (3.2.4) implies that, when $|\beta_2| \le |\beta| - 1$,

$$
\begin{aligned}
&\Big|\Big(\mathcal{T}(\partial^{\beta_1}\sqrt{\mu}\,,\,W^\ell\partial^{\beta_2}g,\,\mu_{\beta_3}\,),\;h\Big)_{L^2(\mathbb{R}^6_{x,v})}\Big|\\
&\le C\Bigg(\int_{\mathbb{R}^3_x}\|\partial^{\beta_1}\sqrt{\mu}\,\|^2_{L^2_s(\mathbb{R}^3_v)}\Big(|||W^\ell\partial^{\beta_2}g|||^2 + \|W^\ell\partial^{\beta_2}g\|^2_{H^s(\mathbb{R}^3_v)}\Big)dx\\
&+\int_{\mathbb{R}^3_x}\|W^\ell\partial^{\beta_2}g\|^2_{L^2_s(\mathbb{R}^3_v)}\Big(|||\partial^{\beta_1}\sqrt{\mu}\,|||^2 + \|\partial^{\beta_1}\sqrt{\mu}\,\|^2_{H^s(\mathbb{R}^3_v)}\Big)dx\Bigg)^{1/2}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)}\\
&\le C\|\sqrt{\mu}\,\|_{H^{|\beta_1|+s}_s(\mathbb{R}^3_v)}|||g|||_{\mathcal{B}^{|\beta_2|}_\ell(\mathbb{R}^6)}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)}\\
&\le C|||g|||_{\mathcal{B}^{|\beta|-1}_\ell(\mathbb{R}^6)}|||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)}.
\end{aligned}
$$

Then the proof of the proposition is completed. □

4. Local existence

4.1. **Energy estimates for a linear equation.** We now consider the following Cauchy problem for a linear Boltzmann equation with a given function f,

$$
\partial_t g + v\cdot\nabla_x g + \mathcal{L}_1 g = \Gamma(f,\,g) - \mathcal{L}_2 f,\qquad g|_{t=0} = g_0\,, \tag{4.1.1}
$$

which is equivalent to the problem:

$$
\partial_t G + v\cdot\nabla_x G = Q(F,\,G),\qquad G|_{t=0} = G_0,
$$

with $F = \mu + \sqrt{\mu}\,f$ and $G = \mu + \sqrt{\mu}\,g$.

We shall now study the energy estimates on (4.1.1) in the function space H^N_ℓ. For $N \ge 3$, $\ell \ge 3$ and $\beta \in \mathbb{N}^6$, $|\beta| \le N$, taking

$$
\varphi(t,x,v) = (-1)^{|\beta|}\Big(\partial^\beta_{x,v}W^{2\ell}\partial^\beta_{x,v}g\Big)(t,x,v),
$$

as a test function on $\mathbb{R}^3_x\times\mathbb{R}^3_v$, we get

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|\partial^\beta g\|^2_{L^2(\mathbb{R}^6)} + \Big(W^\ell\Big[\partial^\beta_{x,v},\,v\Big]\cdot\nabla_x g,\,W^\ell\partial^\beta_{x,v}g\Big)_{L^2(\mathbb{R}^6)} + \Big(W^\ell\partial^\beta_{x,v}\mathcal{L}_1(g),\,W^\ell\partial^\beta_{x,v}g\Big)_{L^2(\mathbb{R}^6)}\\
&= \Big(W^\ell\partial^\beta_{x,v}\Gamma(f,\,g),\,W^\ell\partial^\beta_{x,v}g\Big)_{L^2(\mathbb{R}^6)} - \Big(W^\ell\partial^\beta_{x,v}\mathcal{L}_2(f),\,W^\ell\partial^\beta_{x,v}g\Big)_{L^2(\mathbb{R}^6)},
\end{aligned}
$$

where we have used the fact that

$$
\Big(v\cdot\nabla_x\Big(W^\ell\partial^\beta_{x,v}\,g\Big),\,W^\ell\partial^\beta_{x,v}g\Big)_{L^2(\mathbb{R}^6)} = 0\,.
$$

Applying now Propositions 3.5 and 3.8, we get for any $3 \le k \le N$ and $|\beta| \le k$,

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|\partial^\beta g\|^2_{L^2_\ell(\mathbb{R}^6)} + \Big(\mathcal{L}_1\Big(W^\ell\partial^\beta_{x,v}g\Big),\,W^\ell\partial^\beta_{x,v}g\Big)_{L^2(\mathbb{R}^6)}\\
&\le C\Big\{\|f\|_{H^k_\ell(\mathbb{R}^6)}\,|||g|||^2_{\mathcal{B}^k_\ell(\mathbb{R}^6)} + \|g\|^2_{H^k_\ell(\mathbb{R}^6)} + \|f\|_{H^k_\ell(\mathbb{R}^6)}\,|||g|||_{\mathcal{B}^k_\ell(\mathbb{R}^6)}\\
&\qquad + \Big(\|g\|_{H^k_\ell(\mathbb{R}^6)} + |||g|||_{\mathcal{B}^{k-1}_\ell(\mathbb{R}^6)}\Big)|||g|||_{\mathcal{B}^k_\ell(\mathbb{R}^6)}\Big\}.
\end{aligned}
$$

By taking summation over $|\beta| \leq k$, Lemma 2.3 together with (3.1.2) and the Cauchy-Schwarz inequality imply that

$$\frac{d}{dt}\|g\|^2_{H^k_\ell(\mathbb{R}^6)} + \frac{C_0}{2}|||g|||^2_{\mathcal{B}^k_\ell(\mathbb{R}^6)} \leq C_{k,\ell}\|f\|_{H^k_\ell(\mathbb{R}^6)}|||g|||^2_{\mathcal{B}^k_\ell(\mathbb{R}^6)} + C_{k,\ell}\left(\|g\|^2_{H^k_\ell(\mathbb{R}^6)} + \|f\|^2_{H^k_\ell(\mathbb{R}^6)} + |||g|||^2_{\mathcal{B}^{k-1}_\ell(\mathbb{R}^6)}\right), \qquad 3 \leq k \leq N. \tag{4.1.2}$$

For $k = 1, 2$, Proposition 3.7 is used to get

$$\frac{d}{dt}\|g\|^2_{H^k_\ell(\mathbb{R}^6)} + \frac{C_0}{2}|||g|||^2_{\mathcal{B}^k_\ell(\mathbb{R}^6)} \leq C_{k,\ell}\|f\|_{H^3_\ell(\mathbb{R}^6)}|||g|||^2_{\mathcal{B}^3_\ell(\mathbb{R}^6)} + C_{k,\ell}\left(\|g\|^2_{H^k_\ell(\mathbb{R}^6)} + \|f\|^2_{H^3_\ell(\mathbb{R}^6)} + |||g|||^2_{\mathcal{B}^{k-1}_\ell(\mathbb{R}^6)}\right), \tag{4.1.3}$$

while for $k = 0$

$$\frac{d}{dt}\|g\|^2_{L^2_\ell(\mathbb{R}^6)} + \frac{C_0}{2}|||g|||^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)} \leq C_{0,\ell}\|f\|_{H^3_\ell(\mathbb{R}^6)}|||g|||^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)} + C_{0,\ell}\left(\|g\|^2_{L^2_\ell(\mathbb{R}^6)} + \|f\|^2_{H^3_\ell(\mathbb{R}^6)}\right), \tag{4.1.4}$$

where C_0 is the constant in (3.1.2), which is independent on k, ℓ and N.

Take $N \geq 3$, when $k \geq 2$, by taking a linear combination of (4.1.2) and (4.1.3), we have

$$\begin{aligned}
&\frac{d}{dt}\left(\|g\|^2_{H^{k-1}_\ell(\mathbb{R}^6)} + \frac{C_0}{2C_{k,\ell}}\|g\|^2_{H^k_\ell(\mathbb{R}^6)}\right) + \frac{C_0^2}{2^2 C_{k,\ell}}|||g|||^2_{\mathcal{B}^k_\ell(\mathbb{R}^6)} \\
&\leq \frac{C_0}{2}\left(\|f\|_{H^N_\ell(\mathbb{R}^6)}|||g|||^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)} + \|g\|^2_{H^N_\ell(\mathbb{R}^6)} + \|f\|^2_{H^N_\ell(\mathbb{R}^6)}\right) \\
&\qquad + \frac{d}{dt}\|g\|^2_{H^{k-1}_\ell(\mathbb{R}^6)} + \frac{C_0}{2}|||g|||^2_{\mathcal{B}^{k-1}_\ell(\mathbb{R}^6)} \\
&\leq \frac{C_0}{2}\left(\|f\|_{H^N_\ell(\mathbb{R}^6)}|||g|||^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)} + \|g\|^2_{H^N_\ell(\mathbb{R}^6)} + \|f\|^2_{H^N_\ell(\mathbb{R}^6)}\right) \\
&\quad + C_{k-1,\ell}\left(\|f\|_{H^N_\ell(\mathbb{R}^6)}|||g|||^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)} + \|g\|^2_{H^N_\ell(\mathbb{R}^6)} + \|f\|^2_{H^N_\ell(\mathbb{R}^6)} + |||g|||^2_{\mathcal{B}^{k-2}_\ell(\mathbb{R}^6)}\right).
\end{aligned}$$

By induction and by using (4.1.4), we have the following estimate

$$\frac{d}{dt}\left(\sum_{0\leq k\leq N} c_{k,l}\ \|g\|^2_{H^k_\ell(\mathbb{R}^6)}\right) + \widetilde{C_0}\ |||g|||^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)} \leq \widetilde{C}_{N,\ell}\left(\|f\|_{H^N_\ell(\mathbb{R}^6)}|||g|||^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)} + \|g\|^2_{H^N_\ell(\mathbb{R}^6)} + \|f\|^2_{H^N_\ell(\mathbb{R}^6)}\right), \tag{4.1.5}$$

for some positive constants $\widetilde{C_0} < C_0$, $c_{k,\ell}$ and $\widetilde{C}_{N,\ell}$. Notice that

$$\|g\|^2_{H^N_\ell(\mathbb{R}^6)} \sim \sum_{0\leq k\leq N} c_{k,l}\|g\|^2_{H^k_\ell(\mathbb{R}^6)}. \tag{4.1.6}$$

We are now ready to prove the following theorem.

Theorem 4.1. *Let $N \geq 3, \ell \geq 3$. Assume that $g_0 \in H^N_\ell(\mathbb{R}^6)$ and $f \in L^\infty([0,T];\ H^N_\ell(\mathbb{R}^6))$. If $g \in L^\infty([0,T];\ H^N_\ell(\mathbb{R}^6)) \bigcap L^2([0,T];\ \mathcal{B}^N_\ell(\mathbb{R}^6))$ is a solution of Cauchy problem (4.1.1), then there exists $\epsilon_0 > 0$ such that if*

$$\|f\|_{L^\infty([0,T];\ H^N_\ell(\mathbb{R}^6))} \leq \epsilon_0,$$

we have

$$\|g\|^2_{L^\infty([0,T];\ H^N_\ell(\mathbb{R}^6))} + \|g\|^2_{L^2([0,T];\ \mathcal{B}^N_\ell(\mathbb{R}^6))} \leq Ce^{CT}(\|g_0\|^2_{H^N_\ell(\mathbb{R}^6)} + \epsilon_0^2 T), \tag{4.1.7}$$

for a constant $C > 0$ depending only on N, ℓ .

Proof. Choosing $\epsilon_0 = \frac{\widetilde{C_0}}{2\widetilde{C}_{N,\ell}}$, we have, from (4.1.5),

$$\frac{d}{dt}\left(\sum_{0\le k\le N} c_{k,l}\ \|g\|^2_{H^k_\ell(\mathbb{R}^6)}\right) + \frac{\widetilde{C_0}}{2}\ |||g|||^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)}$$
$$\le 2\widetilde{C}_{N,\ell}(\|g\|^2_{H^N_\ell(\mathbb{R}^6)} + \epsilon_0^2) \le C(\sum_{0\le k\le N} c_{k,l}\ \|g\|^2_{H^k_\ell(\mathbb{R}^6)} + \epsilon_0^2),$$

and

$$\frac{d}{dt}\left(e^{-Ct}\sum_{0\le k\le N} c_{k,l}\ \|g\|^2_{H^k_\ell(\mathbb{R}^6)}\right) + \frac{\widetilde{C_0}}{2}\ e^{-C\,t}\,|||g|||^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)} \le C\epsilon_0^2 e^{-Ct}\,.$$

Thus we get (4.1.7) for some constant $C > 0$ and this completes the proof of the theorem. □

4.2. **Existence for the linear equation.** With the energy estimate given in the above subsection, we can now prove the following local existence theorem by using the Hahn-Banach theorem.

Theorem 4.2. *Let $\ell \ge 3, N \ge 3$ and $g_0 \in H^N_\ell(\mathbb{R}^6)$. There exists $\epsilon_0 > 0$ such that if*

$$\|f\|_{L^\infty([0,T];\,H^N_\ell(\mathbb{R}^6))} \le \epsilon_0,$$

then the Cauchy problem (4.1.1) admits a unique solution

$$g \in L^\infty([0,T];\ H^N_\ell(\mathbb{R}^6)) \cap L^2([0,T];\ \mathcal{B}^N_\ell(\mathbb{R}^6)).$$

Proof. We consider the following Cauchy problem :

$$\mathcal{P}g \equiv \partial_t g + v\,\cdot\,\nabla_x g + \mathcal{L}_1 g - \Gamma(f,\ g) = H,\ \ g(0) = g_0. \tag{4.2.1}$$

For $h \in C^\infty([0,T];\ \mathcal{S}(\mathbb{R}^6_{x,v}))$ with $h(T) = 0$, we define

$$\left(g,\ \mathcal{P}^*_{N,\,\ell}\, h\right)_{L^2([0,\,T];H^N_\ell(\mathbb{R}^6))} = \left(\mathcal{P}\, g,\ h\right)_{L^2([0,\,T];H^N_\ell(\mathbb{R}^6))}\ ,$$

so that $\mathcal{P}^*_{N,\,\ell}$ is the adjoint of the linear operator $\mathcal{P}$ in the Hilbert space $L^2([0,\ T]; H^N_\ell(\mathbb{R}^6))$.
Set

$$\mathbb{W} = \left\{w = \mathcal{P}^*_{N,\,\ell}\, h;\ \ h \in C^\infty([0,T];\ \mathcal{S}(\mathbb{R}^6_{x,v}))\ \text{with}\ h(T) = 0\right\},$$

which is a dense subspace of $L^2([0,\ T];\ H^N_\ell(\mathbb{R}^6))$. And we also have

$$\mathcal{P}^*_{N,\,\ell}(h) = -\partial_t h + (v\,\cdot\,\nabla_x)^* h + \mathcal{L}^*_1\, h + \Gamma^*(f,\ h)\,.$$

Then

$$\left(h,\ \mathcal{P}^*_{N,\,\ell}\, h\right)_{H^N_\ell(\mathbb{R}^6)} = -\frac{1}{2}\frac{d}{dt}\|h(t)\|^2_{H^N_\ell(\mathbb{R}^6)} + \left(v\,\cdot\,\nabla_x\, h,\ h\right)_{H^N_\ell(\mathbb{R}^6)}$$
$$+ \left(\mathcal{L}_1(h), h\right)_{H^N_\ell(\mathbb{R}^6)} - \left(\Gamma(f,\ h),\ h\right)_{H^N_\ell(\mathbb{R}^6)}.$$

Same as Theorem 4.1, for $\|f\|_{L^\infty([0,T];\,H^N_\ell(\mathbb{R}^6))} \le \epsilon_0$, we have

$$\int_t^T e^{2C(s-t)}\left|\left(h,\ \mathcal{P}^*_{N,\,\ell}\, h\right)_{H^N_\ell(\mathbb{R}^6)}\right| dt \ge \|h(t)\|^2_{H^N_\ell(\mathbb{R}^6)} + \int_t^T Ce^{2C(s-t)}|||h(s)|||^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)} ds\,.$$

Thus, for all $0 < t < T$,

$$\begin{aligned}&\|h(t)\|^2_{H^N_\ell(\mathbb{R}^6)} + C\|h\|^2_{L^2([t,T];\,\mathcal{B}^N_\ell(\mathbb{R}^6))} \leq C\Big(h,\ \mathcal{P}^*_{N,\ell}\, h\Big)_{L^2([t,T];\,H^N_\ell(\mathbb{R}^6))}\\ &\leq C\|\mathcal{P}^*_{N,\ell}(h)\|_{L^2([t,T];H^N_\ell(\mathbb{R}^6))}\|h\|_{L^2([t,T];H^N_\ell(\mathbb{R}^6))}\,.\end{aligned}$$

Hence, we get

$$\|h\|_{L^2([0,T];\,H^N_\ell(\mathbb{R}^6))} + \|h\|_{L^\infty([0,T];\,H^N_\ell(\mathbb{R}^6))} \leq CT\|\mathcal{P}^*_{N,\ell}(h)\|_{L^2([0,T];\,H^N_\ell(\mathbb{R}^6))}\,. \tag{4.2.2}$$

Since

$$\|h\|_{L^2([t,T];H^N_\ell(\mathbb{R}^6))} \leq C\|h\|_{L^2([t,T];\,\mathcal{B}^N_\ell(\mathbb{R}^6))},$$

we also have

$$\|h\|_{L^2([0,T];\,\mathcal{B}^N_\ell(\mathbb{R}^6))} \leq C\|\mathcal{P}^*_{N,\ell}(h)\|_{L^2([0,T];\,H^N_\ell(\mathbb{R}^6))}\,. \tag{4.2.3}$$

Next, we define a functional $\mathcal{G}$ on $\mathbb{W}$ as follows

$$\mathcal{G}(w) = (H,\ h)_{L^2([0,T];\,H^N_\ell(\mathbb{R}^6))} + (g_0, h(0))_{H^N_\ell(\mathbb{R}^6)}.$$

Then, if $H \in L^2([0,\ T];\ H^N_{\ell-s}(\mathbb{R}^6))$, (3.1.3) gives

$$\begin{aligned}|\mathcal{G}(w)| &\leq \|H\|_{L^2([0,T];\,H^N_{\ell-s}(\mathbb{R}^6))}\|h\|_{L^2([0,T];\,H^N_{\ell+s}(\mathbb{R}^6))} + \|g_0\|_{H^N_\ell(\mathbb{R}^6)}\|h(0)\|_{H^N_\ell(\mathbb{R}^6)}\\ &\leq C\|H\|_{L^2([0,T];\,H^N_{\ell-s}(\mathbb{R}^6))}\|h\|_{L^2([0,T];\,\mathcal{B}^N_\ell(\mathbb{R}^6))} + \|g_0\|_{H^N_\ell(\mathbb{R}^6)}\|h(0)\|_{H^N_\ell(\mathbb{R}^6)}\\ &\leq C\|\mathcal{P}^*_{N,\ell}(h)\|_{L^2([0,T];\,H^N_\ell(\mathbb{R}^6))} \leq C\|w\|_{L^2([0,T];\,H^N_\ell(\mathbb{R}^6))}\,,\end{aligned}$$

where we have used (4.2.2) and (4.2.3).

Thus, $\mathcal{G}$ is a continuous linear functional on $\Big(\mathbb{W};\ \|\ \cdot\ \|_{L^2([0,T];\,H^N_\ell(\mathbb{R}^6))}\Big)$. Now, there exists $g \in L^2([0,\ T];\ H^N_\ell(\mathbb{R}^6))$ such that for any $w \in \mathbb{W}$,

$$\mathcal{G}(w) = (g,\ w)_{L^2([0,T];\,H^N_\ell(\mathbb{R}^6))},$$

by Hahn-Banach Theorem. For any $h \in C^\infty([0,T];\mathcal{S}(\mathbb{R}^6_{x,v}))$ with $h(T) = 0$, we have

$$\Big(g,\ \mathcal{P}^*_{N,\ell}\, h\Big)_{L^2([0,T];H^N_\ell(\mathbb{R}^6))} = (H,\ h)_{L^2([0,T];H^N_\ell(\mathbb{R}^6))} + (g_0,\ h(0))_{H^N_\ell(\mathbb{R}^6)},$$

and by the definition of the operator $\mathcal{P}^*_{N,\ell}$, we have also

$$\Big(\mathcal{P}\, g,\ \tilde h\Big)_{L^2([0,T];L^2(\mathbb{R}^6))} = (H,\ \tilde h)_{L^2([0,T];L^2(\mathbb{R}^6))} + (g_0,\ \tilde h(0))_{L^2(\mathbb{R}^6)},$$

where

$$\tilde h = \Lambda^N W^{2\ell}\Lambda^N h \in C^\infty([0,T];\ \mathcal{S}(\mathbb{R}^6))\ \text{ with }\ \tilde h(T) = 0,$$

where $\Lambda = (1-\Delta_{x,v})^{\frac12}$. Since $\Lambda^N W^{2\ell}\Lambda^N$ is an isomorphism on $\{h : h \in C^\infty([0,T];\ \mathcal{S}(\mathbb{R}^6))$ with $h(T) = 0\}$, we have shown that if $H \in L^2([0,\ T];H^N_{\ell-s}(\mathbb{R}^6))$, then $g \in L^2([0,\ T];H^N_\ell(\mathbb{R}^6))$ is a solution of the Cauchy problem (4.2.1).

It remains to take

$$H = -\mathcal{L}_2(f) = \Gamma(f,\ \sqrt{\mu}),$$

to get

$$\Big|(H,\ h)_{L^2([0,T];\,H^N_\ell(\mathbb{R}^6))}\Big| \leq C\|f\|_{L^2([0,T];\,H^N_\ell(\mathbb{R}^6))}\|h\|_{L^2([0,T];\,\mathcal{B}^N_\ell(\mathbb{R}^6))}.$$

Then $\mathcal{G}$ is also continuous on $\mathbb{W}$. And this completes the proof of Theorem 4.2. $\square$

4.3. **Convergence of approximate solutions.** In this subsection, we prove the local existence theorem.

Theorem 4.3. *Let $N \geq 3, \ell \geq 3$. There exist ϵ_1, $T > 0$ such that if $g_0 \in H^N_\ell(\mathbb{R}^6)$ and*

$$\|g_0\|_{H^N_\ell(\mathbb{R}^6)} \leq \epsilon_1 \,,$$

then the Cauchy problem (1.3) *admits a solution*

$$g \in L^\infty([0, T];\ H^N_\ell(\mathbb{R}^6)) \cap L^2([0, T];\ \mathcal{B}^N_\ell(\mathbb{R}^6)).$$

Remark 4.4. *By the equation in* (1.3)*, we have, for $0 < s < 1/2$, By using the equation* (1.3)*, we have, for $0 < s < 1/2$,*

$$\partial_t g,\ v \cdot \nabla_x g \in L^2([0, T]; H^{N-1}_{\ell-1}(\mathbb{R}^6)).$$

Moreover, if we go back to the equation (1.1), we have that

$$f = \mu + \mu^{1/2} g \in H^N_\ell([0, T] \times \Omega \times \mathbb{R}^3)),$$

for any $\ell \in \mathbb{N}$ and any bounded domain $\Omega \subset \mathbb{R}^3_x$, and thus the Sobolev embedding implies that f is a classical solution of equation (1.1) if $N > 7/2 + 1$. We will use this properties for the smoothing effect of Theorem 1.1.

For the proof of Theorem 4.3, we consider the sequence of approximate solutions defined by the following Cauchy problem, $n \in \mathbb{N}$,

$$\partial_t f^{n+1} + v \cdot \nabla_x f^{n+1} = Q(f^n, f^{n+1})\,, \qquad f^{n+1}|_{t=0} = f_0,$$

where $f^n = \mu + \mu^{1/2} g^n$ and $f^0 = f_0$. Note that it is also equivalent to

$$\partial_t g^{n+1} + v \cdot \nabla_x g^{n+1} + \mathcal{L}_1 g^{n+1} - \Gamma(g^n, g^{n+1}) = -\mathcal{L}_2 g^n\,, \qquad g^{n+1}|_{t=0} = g_0. \tag{4.3.1}$$

Proposition 4.5. *Let $N \geq 3, \ell \geq 3$. There exist ϵ_1, $T > 0$ such that if $g_0 \in H^N_\ell(\mathbb{R}^6)$ and*

$$\|g_0\|_{H^N_\ell(\mathbb{R}^6)} \leq \epsilon_1 \,,$$

the Cauchy problem (4.3.1) admits a sequence of solutions

$$\{g^n, n \in \mathbb{N}\} \subset L^\infty([0, T];\ H^N_\ell(\mathbb{R}^6)) \cap L^2([0, T];\ \mathcal{B}^N_\ell(\mathbb{R}^6)).$$

Moreover, for all $n \in \mathbb{N}$,

$$\|g^n\|_{L^\infty([0,T];\, H^N_\ell(\mathbb{R}^6))} + \|g^n\|_{L^2([0,T];\, \mathcal{B}^N_\ell(\mathbb{R}^6))} \leq \epsilon_0, \tag{4.3.2}$$

where ϵ_0 is the constant in Theorem 4.1.

Proof. (4.3.2) will be proven by induction on n. Firstly, consider the equation

$$\partial_t g^1 + v \cdot \nabla_x g^1 + \mathcal{L}_1 g^1 - \Gamma(g_0, g^1) = -\mathcal{L}_2 g_0\,, \qquad g^1|_{t=0} = g_0.$$

When $\epsilon_1 < \epsilon_0$, the existence of g^1 is given by Theorem 4.2 satisfying

$$g^1 \in L^\infty([0, T];\ H^N_\ell(\mathbb{R}^6)) \cap L^2([0, T];\ \mathcal{B}^N_\ell(\mathbb{R}^6)).$$

From Theorem 4.1, we can deduce

$$\|g^1\|_{L^\infty([0,T];\, H^N_\ell(\mathbb{R}^6))} + \|g^1\|_{L^2([0,T];\, \mathcal{B}^N_\ell(\mathbb{R}^6))} \leq C e^{CT} \|g_0\|_{H^N_\ell(\mathbb{R}^6)}.$$

Thus (4.3.2) holds when ϵ_1 is chosen to be small compared to ϵ_0.

For $n \geq 1$, under the assumption that

$$\|g^n\|_{L^\infty([0,T];\, H^N_\ell(\mathbb{R}^6))} + \|g^n\|_{L^2([0,T];\, \mathcal{B}^N_\ell(\mathbb{R}^6))} \leq \epsilon_0,$$

Theorem 4.2 yields the existence of

$$g^{n+1} \in L^\infty([0, T];\ H^N_\ell(\mathbb{R}^6)) \cap L^2([0, T];\ \mathcal{B}^N_\ell(\mathbb{R}^6)).$$

From Theorem 4.1, we can deduce

$$\|g^{n+1}\|^2_{L^\infty([0,T];\,H^N_\ell(\mathbb{R}^6))} + \|g^{n+1}\|^2_{L^2([0,T];\,\mathcal{B}^N_\ell(\mathbb{R}^6))} \le Ce^{CT}(\|g_0\|^2_{H^N_\ell(\mathbb{R}^6)} + \epsilon_0^2 T).$$

and this gives

$$\|g^{n+1}\|_{L^\infty([0,T];\,H^N_\ell(\mathbb{R}^6))} + \|g^{n+1}\|_{L^2([0,T];\,\mathcal{B}^N_\ell(\mathbb{R}^6))} \le \epsilon_0,$$

when $T > 0$ is sufficiently small,

Thus we prove (4.3.2) for all $n \in \mathbb{N}$, and this completes the proof of the proposition. □

It remains to prove the convergence. Set $w^n = g^{n+1} - g^n$ and deduce from (4.3.1) that

$$\partial_t w^n + v\cdot\nabla_x w^n + \mathcal{L}_1 w^n - \Gamma(g^n, w^n) = \Gamma(w^{n-1}, g^n) - \mathcal{L}_2 w^{n-1}\,, \qquad w^n|_{t=0} = 0.$$

Similar to the computation for (4.1.4), we obtain

$$\begin{aligned}\frac{d}{dt}\|w^n\|^2_{L^2_\ell(\mathbb{R}^6)} + \frac{C_0}{2}|||w^n|||^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)} &\le C_{0,\ell}\|g^n\|_{H^3_\ell(\mathbb{R}^6)}\,|||w^n|||^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)}\\ &\quad + C_{0,\ell}\|w^{n-1}\|_{L^2_\ell(\mathbb{R}^6)}(|||g^n|||_{\mathcal{B}^3_\ell(\mathbb{R}^3)} + 1)|||w^n|||_{\mathcal{B}^0_\ell(\mathbb{R}^6)}.\end{aligned}$$

If ϵ_0 is sufficiently small, this yields,

$$\frac{d}{dt}\|w^n\|^2_{L^2_\ell(\mathbb{R}^6)} + C_1|||w^n|||^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)} \le C_2\|w^{n-1}\|_{L^2_\ell(\mathbb{R}^3)},$$

which, in turn, gives, if T is sufficiently small,

$$\|w^n\|_{L^\infty([0,T];L^2_\ell(\mathbb{R}^6))} \le \lambda\|w^{n-1}\|_{L^\infty([0,T];L^2_\ell(\mathbb{R}^6))},$$

for some $\lambda \in (0,1)$. Thus we conclude that the sequence $\{g^n\}$ is a Cauchy sequence in $L^\infty([0,T]; L^2_\ell(\mathbb{R}^6))$. Let g be the limit function.

By interpolation with the uniform estimates (4.3.2), we see that the sequence is strongly convergent in

$$L^\infty([0,T];\, H^{N-\delta}_\ell(\mathbb{R}^6)) \cap L^2([0,T];\, \mathcal{B}^{N-\delta}_\ell(\mathbb{R}^6))$$

for any $\delta > 0$. Furthermore, by using equation (4.3.1) and Proposition 4.5, we see that $\{\partial_t g^n\}$ is uniformly bounded in $L^\infty([0,T]; H^{N-1}_{\ell-1})$, so that it is a compact set in the function space

$$C^{1-\delta}(]0,T_*[\,;\; H^{N-1-2\delta}_{l-1}(\Omega\times\mathbb{R}^3_v))$$

for any bounded domain $\Omega \subset \mathbb{R}^3_x$. Now we can take the limit in equation (4.3.1) and thus g is a solution of Cauchy problem (1.3).

Finally, by a standard weak compactness argument, we can extract a subsequence of approximate solutions such that

$$g^n \to g \in L^\infty([0,T];\, H^N_\ell(\mathbb{R}^6)) \quad \text{weakly*},$$
$$g^n \to g \in L^2([0,T];\, \mathcal{B}^N_\ell(\mathbb{R}^6)) \quad \text{weakly},$$

which shows that

$$g \in L^\infty([0,T];\, H^N_\ell(\mathbb{R}^6)) \cap L^2([0,T];\, \mathcal{B}^N_\ell(\mathbb{R}^6)).$$

Now the proof of Theorem 4.3 is complete.

5. Qualitative study on the solutions

In this section, we will prove two main qualitative properties of the solutions to the problem considered in this paper, that is, the uniqueness and non-negativity.

5.1. **Uniqueness.** The uniqueness of solutions can be proved in a larger function space as stated in Theorem 1.2. To obtain this theorem, we now first prove two preliminary results in the following lemmas.

Set $\varphi(v,x) = \langle v, x\rangle^2 = 1 + |v|^2 + |x|^2$ and

$$W_{\varphi,l} = \frac{\langle v\rangle^l}{\varphi(v,x)} = \frac{(1+|v|^2)^{l/2}}{1+|v|^2+|x|^2}.$$

Lemma 5.1. *For $l \geq 4$, we have*

$$\begin{aligned} |W_{\varphi,l} - W'_{\varphi,l}| &\leq C \sin\left(\frac{\theta}{2}\right)\left(\frac{W'_l + W'_{l-3}W'_{3,*}}{\varphi(v'_*, x)} + \sin^{l-3}\left(\frac{\theta}{2}\right) W'_{\varphi,l,*}\right) \\ &\leq C\left(\theta W'_l W'_{\varphi,3,*} + \theta^{l-2} W'_{\varphi,l,*}\right), \end{aligned} \tag{5.1.1}$$

where $W'_{\varphi,l,*} = \dfrac{W'_{l,*}}{\varphi(v'_*, x)}$, and also for $l \geq 1$,

$$|W_{\varphi,l} - W'_{\varphi,l}| \leq C \sin\left(\frac{\theta}{2}\right)\frac{W'_l + W'_{l,*}}{\varphi(v', x)} \leq C\theta \frac{W'_l W'_{l,*}}{\varphi(v',x)}. \tag{5.1.2}$$

Proof. For $k \geq 0, a \geq 0$, set

$$F_k(\lambda) = \frac{\lambda^k}{\lambda + a}.$$

Then for $\lambda \in [1, \infty[$, we have $\frac{d}{d\lambda}F_k(\lambda) \geq 0$ if $k \geq 1$ and $\frac{d^2}{d\lambda^2}F_k(\lambda) \geq 0$ if $k \geq 2$. Since $\frac{d}{d\lambda}F_k(\lambda)$ is positive and increasing on $[1, \infty[$ if $k \geq 2$, it follows from the mean value theorem that for $\lambda, \lambda' \geq 1$

$$|F_k(\lambda) - F_k(\lambda')| \leq \frac{d}{d\lambda}F_k(\lambda + |\lambda - \lambda'|)|\lambda - \lambda'|.$$

Setting $\lambda = \langle v\rangle^2, \lambda' = \langle v'\rangle^2$, we have

$$\begin{aligned} |F_k(\langle v\rangle^2) - F_k(\langle v'\rangle^2)| &\leq \frac{d}{d\lambda}F_k(2(\langle v\rangle^2 + |v-v'|^2))(2|v| + |v-v'|)|v-v'| \\ &\leq 2kF_{k-1/2}(2(\langle v\rangle^2 + |v-v'|^2))|v-v'|, \end{aligned}$$

because $|\lambda - \lambda'| \leq 2|v-v'||v| + |v-v'|^2 \leq |v|^2 + 2|v-v'|^2$ and $\sqrt{\lambda}\frac{d}{d\lambda}F_k(\lambda) \leq kF_{k-1/2}(\lambda)$. Therefore, we obtain, choosing $a = |x|^2$

$$\begin{aligned} |W_{\varphi,l} - W'_{\varphi,l}| &\leq C_l\left(\frac{\langle v\rangle^{l-1}|v-v'|}{\langle v\rangle^2 + |v-v'|^2 + a} + \frac{|v-v'|^l}{\langle v\rangle^2 + |v-v'|^2 + a}\right) \\ &\leq C_l\ |v-v'|\langle v\rangle^{l-3}F_1(\langle v\rangle^2) + C_l\frac{|v-v'|^l}{\langle v\rangle^2 + |v-v'|^2 + a} \\ &= I + II. \end{aligned} \tag{5.1.3}$$

Note that $\langle v\rangle^2 \leq 2\langle v_*\rangle^2 + 2|v-v_*|^2$. Since F_1 is increasing, we have

$$\begin{aligned} I &\leq C_l\ |v-v'|\langle v\rangle^{l-3}\frac{\langle v_*\rangle^2 + |v-v_*|^2}{\langle v_*\rangle^2 + |v-v_*|^2 + |x|^2} \\ &\leq C_l \sin\left(\frac{\theta}{2}\right)\frac{W_l + W_{l-3}W_{3,*}}{\varphi(v_*, x)}. \end{aligned}$$

On the other hand

$$II \le C_l \frac{|v-v_*|^l \sin^l\left(\frac{\theta}{2}\right)}{1+(1-\sin^2\left(\frac{\theta}{2}\right))|v|^2+\frac{1}{2}\sin^2\left(\frac{\theta}{2}\right)|v_*|^2+|x|^2}$$
$$\le C_l \sin^{l-2}\left(\frac{\theta}{2}\right)\frac{W_l+W_{l,*}}{\varphi(v_*,x)}.$$

Since v and v' are symmetric, we get the first conclusion. The second one is a direct consequence of the first inequality of (5.1.3). □

Lemma 5.2. *Let $l \in \mathbb{N}$. If $0 < s < 1/2$, there exists $C > 0$ such that*

$$\Big|\Big((W_{\varphi,l}\ Q(f,\ g) - Q(f,\ W_{\varphi,l}\ g)),\ \ h\Big)_{L^2(\mathbb{R}^6)}\Big| \le C\|f\|_{L^\infty(\mathbb{R}^3_x;L^1_l(\mathbb{R}^3_v))}\|W_{\varphi,l}g\|_{L^2(\mathbb{R}^6)}\|h\|_{L^2(\mathbb{R}^6)}. \tag{5.1.4}$$

Moreover, if $l \ge 5$ then

$$\Big|\Big((W_{\varphi,l}\ Q(f,\ g) - Q(f,\ W_{\varphi,l}\ g)),\ \ h\Big)_{L^2(\mathbb{R}^6)}\Big| \le C\|W_{\varphi,l}f\|_{L^2(\mathbb{R}^6)}\|g\|_{L^\infty(\mathbb{R}^3_x;L^2_l(\mathbb{R}^3_v))}\|h\|_{L^2(\mathbb{R}^6)}. \tag{5.1.5}$$

Proof. It follows from (5.1.2) that

$$\begin{aligned}
&\Big|\Big((W_{\varphi,l}\ Q(f,\ g) - Q(f,\ W_{\varphi,l}\ g)),\ \ h\Big)_{L^2(\mathbb{R}^6)}\Big| \\
= &\ \Big|\iiiint b\ f'_*g'(W'_{\varphi,l}-W_{\varphi,l})\,h\,dvdv_*d\sigma dx\Big| \\
\le &\ C\iiiint b\,|\theta|\ |(W_lf)'_*|\ |(W_{\varphi,l}g)'|\ |h|\ dvdv_*d\sigma dx \\
= &\ C\iiiint b\,|\theta||(W_lf)_*|\ |(W_{\varphi,l}g)|\ |h'|\,dvdv_*d\sigma dx \\
\le &\ C\Big(\iiiint b\,|\theta|\ |(W_l\ f)_*|\ |(W_{\varphi,l}\,g)|^2dvdv_*d\sigma dx\Big)^{1/2} \\
&\times\Big(\iiiint b\,|\theta|\ |(W_l\ f)_*|\,|h'|^2\,dvdv_*d\sigma dx\Big)^{1/2} \\
= &\ CJ_1\times J_2.
\end{aligned}$$

Clearly, one has

$$J_1^2 \le C\|f\|_{L^\infty(\mathbb{R}^3_x;L^1_l(\mathbb{R}^3_v))}\|W_{\varphi,l}g\|^2_{L^2(\mathbb{R}^6)}\int_{\mathbb{S}^2} b(\cos\theta)\,|\theta|\,d\sigma \le C\|f\|_{L^\infty(\mathbb{R}^3_x;L^1_l(\mathbb{R}^3_v))}\|W_{\varphi,l}g\|^2_{L^2(\mathbb{R}^6)}.$$

Next, by the regular change of variables $v \to v'$, cf. [6, 16], we have

$$J_2^2 = \iiint D_0(v_*,v')|(W_lf)_*||h'|^2dv_*dv'dx,$$

where

$$D_0(v,v') = 2\int_{\mathbb{S}^2}\frac{\theta(v_*,v',\sigma)}{\cos^2(\theta(v_*,v',\sigma)/2)}b(\cos\theta(v_*,v',\sigma))d\sigma \le C\int_0^{\pi/4}\psi^{-1-2s}\sin\psi\ d\psi,$$

and

$$\cos\psi = \frac{v'-v_*}{|v'-v_*|}\cdot\sigma,\quad \psi = \theta/2,\qquad d\sigma = \sin\psi d\psi d\phi.$$

Thus,

$$J_2^2 \le C\|f\|_{L^\infty(\mathbb{R}^3_x;L^1_l(\mathbb{R}^3_v))}\|h\|^2_{L^2(\mathbb{R}^6)},$$

and this together with the estimate on J_1 give (5.1.4).

We now prove (5.1.5) by using (5.1.1) instead of (5.1.2). For this purpose, when $l \geq 5$, we write

$$\begin{aligned}
&\Big|\Big((W_{\varphi,l}\ Q(f,\ g) - Q(f,\ W_{\varphi,l}\ g)),\ \ h\Big)_{L^2(\mathbb{R}^6)}\Big| \\
\leq\ & C\Big\{\iiiint b\,|\theta|\ |(W_{\varphi,3}\,f)'_*|\ |(W_l g)'|\ |h|\ dvdv_*d\sigma dx \\
&+ \iiiint b\,|\theta|^{l-2}\ |(W_{\varphi,l}\,f)'_*|\ |g'|\ |h|\ dvdv_*d\sigma dx\Big\} \\
=\ & \mathcal{M}_1 + \mathcal{M}_2.
\end{aligned}$$

The estimation on $\mathcal{M}_1$ can be obtained following the proof of (5.1.4) except for the x variable. Indeed,

$$\begin{aligned}
\mathcal{M}_1 &\leq C\int\Big(\iiint b\,|\theta|\ |(W_{\varphi,3}\ f)_*|\ |(W_l\, g)|^2 dvdv_*d\sigma\Big)^{1/2} \\
&\qquad\times\Big(\iiint b\,|\theta|\ |(W_{\varphi,3}\ f)_*|\,|h'|^2\, dvdv_*d\sigma\Big)^{1/2} dx \\
&\leq C\|g\|_{L^\infty(\mathbb{R}^3_x;L^2_l(\mathbb{R}^3_v))}\int \|W_{\varphi,3}\ f\|_{L^1(\mathbb{R}^3_v)}\|h\|_{L^2(\mathbb{R}^3_v)}\, dx \\
&\leq C\|g\|_{L^\infty(\mathbb{R}^3_x;L^2_l(\mathbb{R}^3_v))}\Big(\int \|W_{\varphi,5}\ f\|^2_{L^2(\mathbb{R}^3_v)}dx\Big)^{1/2}\Big(\int \|h\|_{L^2(\mathbb{R}^3_v)}dx\Big)^{1/2} \\
&\leq C\|W_{\varphi,5}\ f\|_{L^2(\mathbb{R}^6)}\|g\|_{L^\infty(\mathbb{R}^3_x;L^2_l(\mathbb{R}^3_v))}\|h\|_{L^2(\mathbb{R}^6)}.
\end{aligned}$$

$\mathcal{M}_2$ can be estimated as follows. Firstly, we have

$$\begin{aligned}
\mathcal{M}_2^2 =&C^2\Big(\iiiint b\,|\theta|^{l-2}|(W_{\varphi,l}\,f)_*||g|\ |h'|\ dvdv_*d\sigma dx\Big)^2 \\
\leq&C^2\iiiint b\,|\theta|^{\,l-2-\frac{3}{2}}|g||(W_{\varphi,l}\,f)_*|^2 dvdv_*d\sigma dx \\
&\times\iiiint b\,|\theta|^{\,l-2+\frac{3}{2}}|g||h'|^2 dvdv_*d\sigma dx \\
=&\mathcal{M}_{2,1}\times\mathcal{M}_{2,2}.
\end{aligned}$$

Then, if $l-2-\frac{3}{2}-2s-1>-1$, that is, $l>2s+\frac{3}{2}+2$, we have

$$\mathcal{M}_{2,1} \leq C\|g\|_{L^\infty(\mathbb{R}^3_x;L^1(\mathbb{R}^3_v))}\|W_{\varphi,l}\,f\|^2_{L^2(\mathbb{R}^6)}.$$

On the other hand, for $\mathcal{M}_{2,2}$ we need to apply the singular change of variables $v_* \to v'$. The Jacobian of this transform is

$$\Big|\frac{\partial v_*}{\partial v'}\Big| = \frac{8}{\Big|I-\boldsymbol{k}\otimes\sigma\Big|} = \frac{8}{|1-\boldsymbol{k}\cdot\sigma|} = \frac{4}{\sin^2(\theta/2)} \leq 16\theta^{-2},\ \ \theta\in[0,\pi/2].$$

Notice that this gives rise to an additional singularity in the angle θ around 0. Actually, the situation is even worse in the following sense. Recall that θ is no longer a legitimate polar angle. In this case, the best choice of the pole is $\boldsymbol{k}'' = (v'-v)/|v'-v|$ for which polar angle ψ defined by $\cos\psi = \boldsymbol{k}''\cdot\sigma$ satisfies (cf. [6, Fig. 1])

$$\psi = \frac{\pi-\theta}{2},\qquad d\sigma = \sin\psi d\psi d\phi,\qquad \psi\in[\frac{\pi}{4},\frac{\pi}{2}].$$

This measure does not cancel any of the singularity of $b(\cos\theta)$, unlike the case in the usual polar coordinates. Nevertheless, this singular change of variables yields

$$\begin{aligned}\mathcal{M}_{1,2} &= C\iiiint b(\cos\theta)\,|\theta|^{l+\frac{3}{2}}|(g)|\ |h'|^2\ dvdv_*d\sigma dx\\ &\le C\iiint D_1(v,v')|(g)|\ |h'|^2 dvdv'dx,\end{aligned}$$

when $l-2>\frac{3}{2}+2s$ because

$$D_1(v,v')=\int_{\mathbb{S}^2}\theta^{l-2+\frac{3}{2}-2}b(\cos\theta)d\sigma\le C\int_{\pi/4}^{\pi/2}(\frac{\pi}{2}-\psi)^{-2-2s+l-2+\frac{3}{2}-2}d\psi\le C.$$

Therefore,

$$\mathcal{M}_{2,2}\le C\|g\|_{L^\infty(\mathbb{R}^3_x;L^1(\mathbb{R}^3_v))}\|h\|^2_{L^2(\mathbb{R}^6)}.$$

Now the proof of (5.1.1) is completed by using the imbedding estimate for $l>\frac{3}{2}$,

$$\|g\|_{L^1(\mathbb{R}^3_v)}\le C\|g\|_{L^2_l(\mathbb{R}^3_v)}.$$

And this completes the proof of the lemma. □

We are now ready to conclude the proof of the uniqueness theorem.

Proof of Theorem 1.2 : Set $F=f_1-f_2$. Then we have

$$\left\{\begin{array}{l} F_t+v\cdot\nabla_x F=Q(f_1,\ F)+Q(F,\ f_2)\,,\\ F|_{t=0}=0.\end{array}\right. \tag{5.1.6}$$

Let $S(\tau)\in C_0^\infty(\mathbb{R})$ satisfy $0\le S\le 1$ and

$$S(\tau)=1,\ \ |\tau|\le 1\ \ ;\ \ S(\tau)=0,\ \ |\tau|\ge 2.$$

Set $S_N(D_x)=S(2^{-2N}|D_x|^2)$ and multiply $W_{\varphi,l}S_N(D_x)^2W_{\varphi,l}F$ to (5.1.6). Integrating and letting $N\to\infty$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|W_{\varphi,l}F(t)\|^2_{L^2(\mathbb{R}^6)} &= \Big(W_{\varphi,l}\,Q(f_1,\ F)+W_{\varphi,l}\,Q(F,\ f_2)\,,W_{\varphi,l}F\Big)_{L^2(\mathbb{R}^6)}\\ &\quad-(v\cdot\nabla_x(\varphi^{-1})W_lF,W_{\varphi,l}F)_{L^2(\mathbb{R}^6)},\end{aligned}$$

because $(v\cdot\nabla_x S_N(D_x)W_{\varphi,l}F,S_N(D_x)W_{\varphi,l}F)_{L^2(\mathbb{R}^6)}=0$. The second term on the right hand side is estimated by $\|W_{\varphi,l}F\|^2_{L^2(\mathbb{R}^6)}$. Since $f_1\ge 0$, from the coercivity of $-(Q(f_1,g),g)$ it follows that

$$\Big(Q(f_1,\ W_{\varphi,l}F)\,,W_{\varphi,l}F\Big)_{L^2(\mathbb{R}^6)}\le C\|f_1(t)\|_{L^\infty(\mathbb{R}^3_x;\,L^1(\mathbb{R}^3_v))}\|W_{\varphi,l}F(t)\|^2_{L^2(\mathbb{R}^6_{x,v})}.$$

By Lemma 5.2, we have

$$\begin{aligned}&\Big|\Big((W_{\varphi,l}\ Q(f_1,\ F)-Q(f_1,\ W_{\varphi,l}\ F)),\ \ W_{\varphi,l}\ F\Big)_{L^2(\mathbb{R}^6)}\Big|\\ &\le C\|f_1\|_{L^\infty(\mathbb{R}^3_x;L^1_l(\mathbb{R}^3_v))}\|W_{\varphi,l}F\|^2_{L^2(\mathbb{R}^6)},\end{aligned}$$

and

$$\begin{aligned}&\Big|\Big((W_{\varphi,l}\ Q(F,\ f_2)-Q(F,\ W_{\varphi,l}\ f_2)),\ \ W_{\varphi,l}F\Big)_{L^2(\mathbb{R}^6)}\Big|\\ &\le C\|f_2\|_{L^\infty(\mathbb{R}^3_x;L^2_l(\mathbb{R}^3_v))}\|W_{\varphi,l}F\|^2_{L^2(\mathbb{R}^6)}\,.\end{aligned}$$

Finally, for $l > 7/2 + 2s$, we have

$$\begin{aligned}&\left|\Big(Q(F,\ W_{\varphi,l} f_2)\,,W_{\varphi,l}F\Big)_{L^2(\mathbb{R}^6)}\right| \leq C\|Q(F,\ W_{\varphi,l} f_2)\|_{L^2(\mathbb{R}^6)}\|W_{\varphi,l}F(t)\|_{L^2(\mathbb{R}^6)}\\&\leq C\|W_{\varphi,l}F(t)\|_{L^2(\mathbb{R}^6)}\left(\int_{\mathbb{R}^3_x}\|\frac{F(t,x,\cdot)}{\langle x\rangle^2}\|^2_{L^1_{2s}(\mathbb{R}^3_v)}\|\frac{\langle x\rangle^2}{\varphi} f_2(t,x,\cdot)\|^2_{H^{2s}_{l+2s}(\mathbb{R}^3_v)}dx\right)^{1/2}\\&\leq C\|W_{\varphi,l}F(t)\|^2_{L^2(\mathbb{R}^6)}\|f_2(t)\|_{L^\infty(\mathbb{R}^3_x;\ H^{2s}_{l+2s}(\mathbb{R}^3_v))},\end{aligned}$$

because $\langle x\rangle^{-2} \leq W_{\varphi,2}$ and $\langle x\rangle^2/\varphi$ is a bounded operator on H^{2s} uniformly with respect to x. Thus, we have, for any $0 < t < T$

$$\begin{aligned}&\frac{d}{dt}\|W_{\varphi,l}F(t)\|^2_{L^2_l(\mathbb{R}^6)}\\&\leq C\left(\|f_1\|_{L^\infty(]0,T[\times\mathbb{R}^3_x;\ L^1_l(\mathbb{R}^3_v))}+\|f_2\|_{L^\infty(]0,T[\times\mathbb{R}^3_x;\ H^{2s}_{l+2s}(\mathbb{R}^6_v))}\right)\|W_{\varphi,l}F(t)\|^2_{L^2_l(\mathbb{R}^6)}.\end{aligned}$$

Therefore, $\|W_{\varphi,l}F(0)\|_{L^2(\mathbb{R}^6)} = 0$ which implies $\|W_{\varphi,l}F(t)\|_{L^2(\mathbb{R}^6)} = 0$ for all $t \in [0,T[$. And this gives $f_1 = f_2$, and thus completes the proof of Theorem 1.2.

Remark 5.3. *For the function space considered in Theorem 1.1, the uniqueness of solutions is a direct consequence of Theorem 1.2 if there exists a non-negative solution. It is because $g_0 \in H^k_\ell$ and $g \in L^\infty(]0,T[\times H^k_\ell)$ with $k,\ell \geq 3$ imply $f_0 = \mu + \sqrt{\mu}g_0 \in L^\infty(\mathbb{R}^3_x; H^{2s}_m)$ and $f = \mu + \sqrt{\mu}g \in L^\infty(]0,T[\times\mathbb{R}^3_x; H^{2s}_m)$ for any m, respectively, and $k > 3/2 + 2s$.*

5.2. **Non-negativity.** In this subsection, we will prove the non-negativity of the solution obtained in Theorem 1.1.

Theorem 5.4. *Let $N \geq 3, \ell \geq 3$. There exists $\epsilon_1 > 0$ such that if $g_0 \in H^N_\ell(\mathbb{R}^6)$ with $\mu + \mu^{1/2}g_0 \geq 0$ and $\|g_0\|_{H^N_\ell(\mathbb{R}^6)} \leq \epsilon_1$, and $g \in L^\infty([0,T];\ H^N_\ell(\mathbb{R}^6))$ is a solution of Cauchy problem (1.3), then we have $\mu + \mu^{1/2}g \geq 0$ on $[0,T]\times\mathbb{R}^6$.*

Proof. By applying the Remark 5.3 on the uniqueness to the Cauchy problem (1.3), it is enough to prove the non-negativity of the approximate solutions given by Proposition 4.5, that is,

$$f^n = \mu + \mu^{1/2}g^n \geq 0\,, \qquad n \in \mathbb{N}. \tag{5.2.1}$$

Again, this will be proved by induction. It is clearly true for $n = 0$ by taking $g^0 = g_0$, and so we now assume that it is true for some n and will prove that (5.2.1) is true for $n+1$.

From (4.3.1), $f^{n+1} = \mu + \mu^{1/2}g^{n+1}$ is the solution of the following Cauchy problem :

$$\begin{cases}\partial_t f^{n+1} + v\,\cdot\,\nabla_x f^{n+1} = Q(f^n,\, f^{n+1}),\\ f^{n+1}|_{t=0} = f_0 = \mu + \mu^{1/2}g_0 \geq 0.\end{cases} \tag{5.2.2}$$

Take the convex function

$$\beta(s) = \frac{1}{2}(s^-)^2 = \frac{1}{2}s\,(s^-)$$

with $s^- = \min\{s,0\}$, and notice that

$$\beta_s(s) = \frac{d\beta(s)}{ds} = s^-.$$

Setting $\phi(x,v) = (1+|x|^2+|v|^2)^{-2}$ and noticing that

$$\beta_s(f^{n+1})\phi(x,v) = \min\{f^{n+1},0\}\phi(x,v) \in L^\infty([0,T];L^1(\mathbb{R}^3_x;L^2(\mathbb{R}^3_v)),$$

we have by (5.2.2),

$$\frac{d}{dt}\int_{\mathbb{R}^6}\beta(f^{n+1})\phi dxdv = \int_{\mathbb{R}^6} Q(f^n,\ f^{n+1})\ \beta_s(f^{n+1})\phi\ dxdv$$
$$-\int_{\mathbb{R}^6} v\cdot\nabla_x\ (\beta(f^{n+1})\phi)dxdv - \int_{\mathbb{R}^6}(\phi^{-1}\ v\cdot\nabla_x\phi)\ \beta(f^{n+1})\phi dxdv,$$

where the first term on the right hand side is well defined by Theorem 2.1, because f^n belongs to $L^\infty([0,T]\times\mathbb{R}^3_x; L^1_{2s}\cap H^{2s}_{2s}(\mathbb{R}^3_v))$. Since the second term vanishes and $|v\cdot\nabla_x\phi|\le C\phi$, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^6}\beta(f^{n+1})\phi dxdv \le \int_{\mathbb{R}^6} Q(f^n, f^{n+1})\beta_s(f^{n+1})\phi dxdv + C\int_{\mathbb{R}^6}\beta(f^{n+1})\phi dxdv.$$

For the first term on the right hand side, we have

$$\begin{aligned}&\int_{\mathbb{R}^6} Q(f^n, f^{n+1})\beta_s(f^{n+1})\phi dxdv\\ &= \int_{\mathbb{R}^6_{x,v}}\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} b(\cos\theta)\Big(f^{n\prime}_* f^{n+1\prime} - f^n_* f^{n+1}\Big)\beta_s(f^{n+1})\phi\\ &= \int_{\mathbb{R}^6_{x,v}}\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} b(\cos\theta)f^{n\prime}_*\Big(f^{n+1\prime} - f^{n+1}\Big)\beta_s(f^{n+1})\phi\\ &\qquad + \int_{\mathbb{R}^6_{x,v}}\beta_s(f^{n+1})f^{n+1}\phi\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} b(\cos\theta)\Big(f^{n\prime}_* - f^n_*\Big)\\ &= I + II\ .\end{aligned}$$

From (4.3.2), we have, for any $n\in\mathbb{N}$,

$$\begin{aligned}\|f^n\|_{L^\infty([0,T]\times\mathbb{R}^3_x;\,L^1(\mathbb{R}^3_v))} &\le 1 + \|\sqrt{\mu}\ g^n\|_{L^\infty([0,T]\times\mathbb{R}^3_x;\,L^1(\mathbb{R}^3_v))}\\ &\le 1 + C\|g^n\|_{L^\infty([0,T]\times\mathbb{R}^3_x;\,L^2(\mathbb{R}^3_v))} \le 1 + C\epsilon_0\,,\end{aligned}$$

so that the cancellation lemma from [6] implies that

$$\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} b(\cos\theta)\Big(f^{n\prime}_* - f^n_*\Big) = C\int_{\mathbb{R}^3_v} f^n(t,x,v)dv \le C\|f^n\|_{L^\infty([0,T]\times\mathbb{R}^3_x;\,L^1(\mathbb{R}^3_v))} \le C\,,$$

while $\beta_s(s)s = 2\beta(s)$ implies that

$$|II| \le C\int_{\mathbb{R}^6}\beta(f^{n+1})\phi dxdv\,.$$

On the other hand, by the convexity of β, that is,

$$\beta_s(a)(b-a) \le \beta(b) - \beta(a)\,,$$

and the assumption that $f^{n\prime}_* \geq 0$, we get

$$
\begin{aligned}
I &= \int_{\mathbb{R}^6_{x,v}} \int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} b(\cos\theta) f^{n'}_* \Big(f^{n+1'} - f^{n+1}\Big)\beta_s(f^{n+1})\phi \\
&\leq \int_{\mathbb{R}^6_{x,v}} \int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} b(\cos\theta) f^{n'}_* \Big(\beta(f^{n+1'}) - \beta(f^{n+1})\Big)\phi \\
&\leq \int_{\mathbb{R}^6_{x,v}} \int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} b(\cos\theta) \Big(f^{n'}_* \beta(f^{n+1'}) - f^n_*\beta(f^{n+1})\Big)\phi \\
&\qquad - \int_{\mathbb{R}^6_{x,v}} \int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} b(\cos\theta)\beta(f^{n+1})\Big(f^{n'}_* - f^n_*\Big)\phi \\
&\leq \int_{\mathbb{R}^6_{x,v}} \int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} b(\cos\theta) f^n_*\beta(f^{n+1})\Big(\phi' - \phi\Big) + C\int_{\mathbb{R}^6} \beta(f^{n+1})\phi dxdv \\
&= I_1 + I_2\,.
\end{aligned}
$$

Applying Taylor's formula to the first term gives

$$
I_1 = \int_0^1 d\tau \int_{\mathbb{R}^6_{x,v}} \int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} b(\cos\theta) f^n_*\beta(f^{n+1})\,(v'-v)\,\cdot\,\nabla_v\phi\Big(v+\tau(v'-v)\Big)\,.
$$

Since

$$
|v'-v| = |v-v_*|\sin\Big(\frac{\theta}{2}\Big) \leq \langle v\rangle\,\langle v_*\rangle \sin\Big(\frac{\theta}{2}\Big), \tag{5.2.3}
$$

by setting $v_\tau = v + \tau(v'-v)$, $0<\tau<1$, $0\leq\theta\leq\pi/2$, we have

$$
|v| \leq |v_\tau| + |v'-v| \leq |v_\tau| + \sin\Big(\frac{\theta}{2}\Big)\Big(|v|+|v_*|\Big) \leq |v_\tau| + \frac{\sqrt{2}}{2}|v| + |v_*|.
$$

Then

$$
(1+|x|^2+|v|^2) \leq C(1+|x|^2+|v_\tau|^2)(1+|v_*|^2),
$$

which implies

$$
|\nabla_v\phi(x,\,v_\tau)| \leq (1+|x|^2+|v_\tau|^2)^{-5/2} \leq C\phi(x,v)\frac{\langle v_*\rangle^5}{\langle v\rangle}\ .
$$

So we obtain

$$
|I_1| \leq C\|f^n\|_{L^\infty([0,T]\times\mathbb{R}^3_x;\,L^1_6(\mathbb{R}^3_v))}\int_{\mathbb{R}^6}\beta(f^{n+1})\phi dxdv.
$$

Again from (4.3.2), we have, for any $n\in\mathbb{N}$,

$$
\|f^n\|_{L^\infty([0,T]\times\mathbb{R}^3_x;\,L^1_6(\mathbb{R}^3_v))} \leq \|\mu^{1/2}\|_{L^1_6(\mathbb{R}^3)} + \|\mu^{1/2}g^n\|_{L^\infty([0,T]\times\mathbb{R}^3_x;\,L^1_6(\mathbb{R}^3_v))} \leq C(1+\epsilon_0).
$$

Finally, we have obtained, for $0<t<T$,

$$
\frac{d}{dt}\int_{\mathbb{R}^6}\beta(f^{n+1})\phi\,dx\,dv \leq C\int_{\mathbb{R}^6}\beta(f^{n+1})\phi\,dx\,dv, \qquad \beta(f^{n+1})|_{t=0} = 0.
$$

Therefore, for $0<t<T$, and by continuity,

$$
\int_{\mathbb{R}^6}\beta(f^{n+1}(t))\phi\,dx\,dv = 0
$$

which implies that, $f^{n+1}(t,x,v)\geq 0$ for $(t,x,v)\in[0,T]\times\mathbb{R}^6_{x,v}$. Therefore, the proof of Theorem 5.4 is completed. □

Remark 5.5. *Note that the above analysis can be extended to the strong singularity case. Indeed, by writing*

$$\begin{aligned}I_1 &= \int_{\mathbb{R}^6_{x,v}} \int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} b(\cos\theta) f^n_* \beta(f^{n+1})\,(v'-v)\,\cdot\,\nabla_v\phi(v)\\ &+ \frac{1}{2}\int_0^1 d\tau \int_{\mathbb{R}^6_{x,v}} \int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} b(\cos\theta) f^n_* \beta(f^{n+1})\,(v'-v)^2\;\nabla^2_v\phi\Big(v+\tau(v'-v)\big)\Big)\\ &= I_{11}+I_{12},\end{aligned}$$

since we have

$$|\nabla^2_v\phi(x,\,v_\tau)| \le (1+|x|^2+|v_\tau|^2)^{-3} \le C\phi(x,v)\frac{\langle v_*\rangle^6}{\langle v\rangle^2}\;,$$

it follows from (5.2.3) *that*

$$|I_{12}| \le C\|f^n\|_{L^\infty([0,T]\times\mathbb{R}^3_x;\,L^1_8(\mathbb{R}^3_v))}\int_{\mathbb{R}^6}\beta(f^{n+1})\phi dxdv.$$

On the other hand, setting $\mathbf{k}=\frac{v-v_*}{|v-v_*|}$ *and writing*

$$v'-v=\frac{1}{2}|v-v_*|\Big(\sigma-(\sigma\cdot\mathbf{k})\mathbf{k}\Big)+\frac{1}{2}((\sigma\cdot\mathbf{k})-1)(v-v_*),$$

we have

$$I_{11}=\frac{1}{2}\int_{\mathbb{R}^6_{x,v}}\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} b(\cos\theta) f^n_*\beta(f^{n+1})\,(\,\cos\theta-1)(v-v_*)\cdot\nabla_v\phi(v),$$

where we have used the symmetry that $\int_{\mathbb{S}^2} b(\sigma\cdot\mathbf{k})\Big(\sigma-(\sigma\cdot\mathbf{k})\mathbf{k}\Big)d\sigma=0$*. Therefore, we have*

$$|I_{11}| \le C\|f^n\|_{L^\infty([0,T]\times\mathbb{R}^3_x;\,L^1_6(\mathbb{R}^3_v))}\int_{\mathbb{R}^6}\beta(f^{n+1})\phi dxdv,$$

and the same estimation holds also in the strong singularity case.

6. Full regularity

We now prove the smoothness effect of the Cauchy problem for the non-cutoff Boltzmann equation. More precisely, the main result of this section is given by

Theorem 6.1. *Assume that* $0<s<1/2$*. There exists* $\epsilon_1>0$ *such that if* $g_0\in H^3_3(\mathbb{R}^6)$ *with* $\mu+\mu^{1/2}g_0\ge 0$, $\|g_0\|_{H^3_3(\mathbb{R}^6)}\le\epsilon_1$ *, and* $g\in L^\infty([0,T];\,H^3_3(\mathbb{R}^6))$ *is the solution of Cauchy problem (1.3), then we have* $g\in C^\infty(]0,T[\times\mathbb{R}^6)$*.*

Let us recall that $H^k_\ell(\mathbb{R}^7_{t,x,})$, $H^k_\ell(\mathbb{R}^6_{x,v})$ and $H^k_\ell(\mathbb{R}^3_v)$ denote some weighted Sobolev spaces with the weight defined in the variable v. Since the regularity result to be proved is local in space and time, for convenience, we define the corresponding local version of weighted Sobolev spaces. For $0\le T_1<T_2<+\infty$, and any given open domain $\Omega\subset\mathbb{R}^3_x$, define

$$\begin{aligned}\mathcal{H}^m_l(]T_1,\,T_2[\times\Omega\times\mathbb{R}^3_v) \;\;=\;\; &\Big\{f\in\mathcal{D}'(]T_1,\,T_2[\times\Omega\times\mathbb{R}^3_v);\\ &\varphi(t)\psi(x)f\in H^m_l(\mathbb{R}^7)\,,\;\forall\,\varphi\in C^\infty_0(]T_1,\,T_2[),\;\psi\in C^\infty_0(\Omega)\Big\}.\end{aligned}$$

The proof of Theorem 6.1 will be divided into several steps.

6.1. **Formulation of the problem.** First of all, we recall the main result in [12] given below.

Theorem 6.2. *Assume that* $0 < s < 1$, $0 \leq T_1 < T_2 < +\infty$, $\Omega \subset \mathbb{R}^3_x$ *is an open domain. Let* f *be a non-negative solution of the Boltzmann equation (1.1) satisfying* $f \in \mathcal{H}^5_\ell(]T_1, T_2[\times\Omega\times\mathbb{R}^3_v)$ *for all* $\ell \in \mathbb{N}$. *Moreover, assume that* f *satisfies the non-vacuum condition*

$$\|f(t, x, \cdot)\|_{L^1(\mathbb{R}^3_v)} > 0, \tag{6.1.1}$$

for all $(t, x) \in]T_1, T_2[\times\Omega$. *Then we have*

$$f \in \mathcal{H}^{+\infty}_\ell(]T_1, T_2[\times\Omega\times\mathbb{R}^3_v) \subset C^{+\infty}(]T_1, T_2[\times\Omega\times\mathbb{R}^3_v),$$

for any $\ell \in \mathbb{N}$.

To apply this result, let us firstly note that, by Theorem 4.3 and Theorem 5.4, under the assumption of Theorem 6.1, the unique solution of the Cauchy problem (1.3) satisfies

$$\|g\|_{L^\infty([0,T];\,H^3_3(\mathbb{R}^6))} \leq \epsilon_0, \qquad \mu + \sqrt{\mu}\,g \geq 0.$$

Therefore, $f = \mu + \sqrt{\mu}\,g \geq 0$ is a solution of Boltzmann equation (1.1). On the other hand, we can choose $\epsilon_0 > 0$ small enough such that

$$\|\sqrt{\mu}\,g\|_{L^\infty([0,T]\times\mathbb{R}^3_x;\,L^1(\mathbb{R}^3_v))} \leq C\|g\|_{L^\infty([0,T];\,H^2(\mathbb{R}^6))} \leq C\epsilon_0 < 1$$

where C is the Sobolev constant of the imbedding $H^2(\mathbb{R}^3_x) \subset L^\infty(\mathbb{R}^3_x)$. Thus, for any $(t, x) \in]0, T[\times\mathbb{R}^3_x$,

$$\int_{\mathbb{R}^3_v} f(t, x, v)dv = 1 + \int_{\mathbb{R}^3_v} \sqrt{\mu}\; g(t, x, v)dv \geq 1 - \|\sqrt{\mu}\,g\|_{L^\infty([0,T]\times\mathbb{R}^3_x;\,L^1(\mathbb{R}^3_v))} > 0\,, \tag{6.1.2}$$

so that $f = \mu + \sqrt{\mu}\,g$ satisfies the condition (6.1.1).

By using equation (1.1) and Remark 4.4, we have also, for any $\ell \in \mathbb{N}$, $0 < T_1 < T_2 < T$ and bounded open domain $\Omega \subset \mathbb{R}^3_x$,

$$f = \mu + \sqrt{\mu}\,g \in \mathcal{H}^3_\ell(]T_1, T_2[\times\Omega\times\mathbb{R}^3_v).$$

Note that we can not apply directly Theorem 6.2 because we now only know that f has regularity just in $\mathcal{H}^3_\ell(]T_1, T_2[\times\Omega\times\mathbb{R}^3_v)$. To overcome this, we prove the following theorem.

Theorem 6.3. *Under the assumptions of Theorem 6.1, we have, for any* $0 < T_1 < T_2 < T$ *and bounded open domain* $\Omega \subset \mathbb{R}^3_x$,

$$f = \mu + \sqrt{\mu}\,g \in \mathcal{H}^5_l(]T_1, T_2[\times\Omega\times\mathbb{R}^3_v),$$

holds for all $\ell \in \mathbb{N}$.

The proof of this theorem is similar but easier than that of Theorem 6.2 which was proved in [12]. In fact, since we have

$$g^n \to g,$$

by mollifying the initial data and using the uniqueness of solution, we do not need at all to mollify the solution as in [12]. It follows that to obtain the above regularization result, we only need to prove the à priori estimate on smooth solution, which can deduced from [12].

To make the paper self-contained, we give a proof here. Let us recall that here we consider the Maxwellian molecule type cross-sections with the mild singularity.

Here and below, ϕ denotes a cutoff function satisfying $\phi \in C^\infty_0$ and $0 \leq \phi \leq 1$. Notation $\phi_1 \subset\subset \phi_2$ stands for two cutoff functions such that $\phi_2 = 1$ on the support of ϕ_1.

Take the smooth cutoff functions $\varphi,\ \varphi_2, \varphi_3 \in C_0^\infty(]T_1, T_2[)$ and $\psi,\ \psi_2, \psi_3 \in C_0^\infty(\Omega)$ such that $\varphi \subset\subset \varphi_2 \subset\subset \varphi_3$ and $\psi \subset\subset \psi_2 \subset\subset \psi_3$. Set $f_1 = \varphi(t)\psi(x) f,\ f_2 = \varphi_2(t)\psi_2(x) f$ and $f_3 = \varphi_3(t)\psi_3(x) f$. For $\alpha \in \mathbb{N}^6, |\alpha| \le 3$, define

$$g = \partial^\alpha_{x,v}(\varphi(t)\psi(x) f) \in L^\infty(]T_1, T_2[;\ L^2_\ell(\mathbb{R}^6)).$$

Then the Leibniz formula yields the following equation :

$$g_t + v\ \cdot\ \partial_x g = Q(f_2,\ g) + G,\ \ (t, x, v) \in \mathbb{R}^7, \tag{6.1.3}$$

where

$$\begin{aligned} G \ &= \sum_{\alpha_1+\alpha_2=\alpha,\ 1\le|\alpha_1|} C^{\alpha_1}_{\alpha_2} Q\Big(\partial^{\alpha_1} f_2,\ \partial^{\alpha_2} f_1\Big) \\ &+\ \ \partial^\alpha\Big(\varphi_t\psi(x) f + v\ \cdot\ \psi_x(x)\varphi(t) f\Big) + [\partial^\alpha,\ \ v\ \cdot\ \partial_x](\varphi(t)\psi(x) f) \\ &\equiv\ \ A + B + C. \end{aligned} \tag{6.1.4}$$

Then we can take $W^{2\ell} g$ as a test function for equation (6.1.3). It follows by integration by parts on $\mathbb{R}^7 = \mathbb{R}^1_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v$ that

$$0 = \Big(W^\ell\ Q(f_2, g),\ W^\ell\ g\Big)_{L^2(\mathbb{R}^7)} + \Big(G,\ W^{2\ell}\ g\Big)_{L^2(\mathbb{R}^7)},$$

which is sufficient for obtaining the required initial regularity.

6.2. **Gain of regularity in velocity variable.** The next step is to show the gain of regularity in the velocity variable by using the coercivity of the collision operator.

Proposition 6.4. *Under the assumption of Theorem 6.1, for any* $0 < T_1 < T_2 < T$ *and bounded open domain* $\Omega \subset \mathbb{R}^3_x$*, one has,*

$$\Lambda^s_v f_1 \in L^2(\mathbb{R}_t;\ H^3_\ell(\mathbb{R}^6)),$$

for any $\ell \in \mathbb{N}$*, where* $\Lambda_v = (1 - \Delta_v)^{\frac{1}{2}},\ f_1 = \varphi(t)\psi(x) f$ *with* $\varphi \in C_0^\infty(]T_1, T_2[), \psi \in C^\infty(\Omega)$.

Proof. Firstly, the local positive lower bound (6.1.2) implies that

$$\inf_{(t,x)\in \mathrm{Supp}\ \varphi\times \mathrm{Supp}\ \psi_1} \| f_2(t, x, \cdot)\|_{L^1(\mathbb{R}^3_v)} = c_0 > 0.$$

Thus, the coercivity estimate (2.1.1) gives

$$\begin{aligned} &-\Big(Q(f_2,\ W^\ell g),\ \ W^\ell\ g\Big)_{L^2(\mathbb{R}^7)} = -\int_{t\in \mathrm{Supp}\ \varphi} \int_{x\in \mathrm{Supp}\ \psi_1} \Big(Q(f_2,\ W^\ell g),\ \ W^\ell\ g\Big)_{L^2(\mathbb{R}^3_v)} dxdt \\ &\ge \int_{\mathbb{R}_t}\int_{\mathbb{R}^3_x} \Big(C_0\|W^\ell\ g(t, x, \cdot)\|^2_{H^s(\mathbb{R}^3_v)} - C\| f_2(t, x, \cdot)\|_{L^1(\mathbb{R}^3_v)}\|W^\ell\ g(t, x, \cdot)\|^2_{L^2(\mathbb{R}^3_v)}\Big) dxdt \\ &\ge C_0\|\Lambda^s_v W^\ell\ g\|^2_{L^2(\mathbb{R}^7)} - CT\| f_2\|_{L^\infty(\mathbb{R}^4_{t,x};\ L^1(\mathbb{R}^3_v))}\|W^\ell\ g\|^2_{L^\infty([0,T];\, L^2(\mathbb{R}^6))}. \end{aligned}$$

Since

$$\| f_2\|_{L^\infty(\mathbb{R}^4_{t,x};\ L^1(\mathbb{R}^3_v))} \le C\| f_2\|_{L^\infty([0,T];\ H^{3/2+\epsilon}_{3/2+\epsilon}(\mathbb{R}^6))},$$

and

$$\|W^\ell\ g\|^2_{L^\infty([0,T];\, L^2(\mathbb{R}^6))} \le C\| f_1\|^2_{L^\infty([0,T];\, H^3_\ell(\mathbb{R}^6))},$$

for $l > 3/2$, we have

$$\begin{aligned} \|\Lambda^s_v W^\ell\ g\|^2_{L^2(\mathbb{R}^7)} \ \ &\le\ \ C\| f_2\|^2_{L^\infty([0,T];\ H^3_\ell(\mathbb{R}^6))} + \Big|(G,\ W^{2\ell}\ g)_{L^2(\mathbb{R}^7)}\Big| \\ &\quad + \Big|(W^\ell\ Q(f_2, g) - Q(f_2,\ W^\ell\ g),\ \ W^\ell\ g)_{L^2(\mathbb{R}^7)}\Big|. \end{aligned} \tag{6.2.1}$$

By applying Lemma 3.2, the third term on the right hand side of (6.2.1) can be estimated as follows:

$$\begin{aligned}&\Big|\Big((W^\ell\, Q(f_2,\ g) - Q(f_2,\ W^\ell\, g)),\ W^\ell\, g\Big)_{L^2(\mathbb{R}^7)}\Big|\\ &\leq CT\|f_2\|_{L^\infty([0,T];\, H^{3/2+\epsilon}_{\ell+3}(\mathbb{R}^6))}\|g\|_{L^\infty([0,T];\, L^2_\ell(\mathbb{R}^6))}\|g\|_{L^2_\ell(\mathbb{R}^7)}\\ &\leq CT^2\|f_2\|_{L^\infty([0,T];\, H^{3/2+\epsilon}_{\ell+3}(\mathbb{R}^6))}\|f_1\|^2_{L^\infty([0,T];\, H^3_\ell(\mathbb{R}^6))}.\end{aligned}$$

For the second term in (6.2.1), we shall prove the following claim:

For $0 < s < 1/2$, one has

$$\Big|\Big(G,\ W^{2\ell}\, g\Big)_{L^2(\mathbb{R}^7)}\Big| \leq C\Big(\|f_3\|_{L^\infty([0,T];\ H^3_\ell(\mathbb{R}^6))} + \|f_2\|^2_{L^\infty([0,T];\ H^3_\ell(\mathbb{R}^6))}\Big)\|\Lambda^s_v W^\ell\, g\|_{L^2(\mathbb{R}^7_{t,x,v})}. \tag{6.2.2}$$

In fact, recalling the expression (6.1.4), it is easy to get

$$\|B\|^2_{L^2_\ell(\mathbb{R}^7)} + \|C\|^2_{L^2_\ell(\mathbb{R}^7)} \leq CT\|f_3\|^2_{L^\infty([0,T];\ H^3_{\ell+1}(\mathbb{R}^6))}.$$

For the term A, firstly recall that $\alpha_1 + \alpha_2 = \alpha$, $|\alpha| \leq 3$ and $|\alpha_2| \leq 2$. In the following, we will apply Theorem 2.1 with $m = -s$. We separate the discussion on A into two cases.

Case 1. If $|\alpha_1| = 1$, we have

$$\begin{aligned}&\int_{\mathbb{R}_t}\int_{\mathbb{R}^3_x}\|Q(\partial^{\alpha_1} f_2,\ \partial^{\alpha_2} f_1)(t,x,\cdot)\|^2_{H^s_\ell(\mathbb{R}^3_v)}dxdt\\ \leq\ & C\int_{\mathbb{R}_t}\int_{\mathbb{R}^3_x}\|\partial^{\alpha_1} f_2(t,x,\cdot)\|^2_{L^1_{\ell+2s}(\mathbb{R}^3_v)}\|\partial^{\alpha_2} f_1(t,x,\cdot)\|^2_{H^s_{\ell+2s}(\mathbb{R}^3_v)}dxdt\\ \leq\ & C\|\partial^{\alpha_1} f_2\|^2_{L^\infty(\mathbb{R}^4_{t,x};\, L^1_{\ell+2s}(\mathbb{R}^3_v))}\int_{\mathbb{R}_t}\int_{\mathbb{R}^3_x}\|\partial^{\alpha_2} f_1(t,x,\cdot)\|^2_{H^s_{\ell+2s}(\mathbb{R}^3_v)}dxdt\\ \leq\ & CT\|f_2\|^2_{L^\infty([0,T];\ H^3_{\ell+3}(\mathbb{R}^6))}\|f_1\|^2_{L^\infty([0,T];\ H^3_{\ell+2s}(\mathbb{R}^6))}.\end{aligned}$$

Case 2. If $|\alpha_1| \geq 2$, then $|\alpha_2| \leq 1$, it follows that

$$\begin{aligned}&\int_{\mathbb{R}_t}\int_{\mathbb{R}^3_x}\|\partial^{\alpha_1} f_2(t,x,\cdot)\|^2_{L^1_{\ell+2s}(\mathbb{R}^3_v)}\|\partial^{\alpha_2} f_1(t,x,\cdot)\|^2_{H^s_{l+2s}(\mathbb{R}^3_v)}dxdt\\ \leq\ & C\|\partial^{\alpha_2} f_1\|^2_{L^\infty(\mathbb{R}^4_{t,x};\, H^s_{l+2s}(\mathbb{R}^3_v))}\int_{\mathbb{R}_t}\int_{\mathbb{R}^3_x}\|\partial^{\alpha_1} f_2(t,x,\cdot)\|^2_{L^2_{l+3/2+\delta+2s}(\mathbb{R}^3_v)}dxdt\\ \leq\ & CT\|f_1\|^2_{L^\infty([0,T];\ H^{1+3/2+\epsilon}_{\ell+2s}(\mathbb{R}^6))}\|f_2\|^2_{L^\infty([0,T];\ H^3_{\ell+3}(\mathbb{R}^6))}.\end{aligned}$$

By combining these two cases, we have

$$\Big|\Big(G,\ W^{2\ell}\, g\Big)_{L^2(\mathbb{R}^7)}\Big| \leq CT\Big(\|f_2\|^2_{L^\infty([0,T];\ H^3_{\ell+1}(\mathbb{R}^6))} + \|f_2\|^2_{L^\infty([0,T];\ H^3_{\ell+3}(\mathbb{R}^6))}\|\Lambda^s_v W^\ell\, g\|_{L^2(\mathbb{R}^7_{t,x,v})}\Big).$$

Therefore, we obtain

$$\|\Lambda^s_v W^\ell\, g\|^2_{L^2(\mathbb{R}^7)} \leq C\Big(1 + \|f_3\|_{L^\infty([0,T];\ H^3_\ell(\mathbb{R}^6))}\Big)^4.$$

The proof of the proposition is then completed. □

6.3. **Gain of regularity in space variable.** With the gain of regularity in the variable v given in the above subsection, we now want to prove the gain of regularity in the variable x. Here, the hypo-elliptic nature of the equation will be used.

For this purpose, we introduce a more general framework. First of all, let us consider a transport equation in the form of

$$f_t + v\cdot\nabla_x f = g \in \mathcal{D}'(\mathbb{R}^7), \tag{6.3.1}$$

where $(t, x, v) \in \mathbb{R}^7$.

In [9], by using a generalized uncertainty principle, we proved the following hypo-elliptic estimate.

Lemma 6.5. *Assume that $g \in H^{-s'}(\mathbb{R}^7)$, for some $0 \le s' < 1$. Let $f \in L^2(\mathbb{R}^7)$ be a weak solution of the transport equation (6.3.1), such that $\Lambda_v^s f \in L^2(\mathbb{R}^7)$ for some $0 < s \le 1$. Then it follows that*

$$\Lambda_x^{s(1-s')/(s+1)} f \in L^2_{-\frac{ss'}{s+1}}(\mathbb{R}^7), \quad \Lambda_t^{s(1-s')/(s+1)} f \in L^2_{-\frac{s}{s+1}}(\mathbb{R}^7),$$

where $\Lambda_\bullet = (1-\Delta_\bullet)^{1/2}$.

Of course g is typically linked with the Boltzmann collision operator. Through the above uncertainty principle, we have the following result on the gain of regularity in the variable x

Proposition 6.6. *Under the hypothesis of Theorem 6.1, one has*

$$\Lambda_x^{s_0} f_1 \in L^2(\mathbb{R}_t;\, H^3_\ell(\mathbb{R}^6)), \tag{6.3.2}$$

for any $\ell \in \mathbb{N}$ and $0 < s_0 = \frac{s(1-s)}{(s+1)}$.

Proof. For any $\ell \in \mathbb{N}$, it follows from Proposition 6.4 that $\Lambda_v^s W^\ell g \in L^2(\mathbb{R}^7)$, while using the upper bound estimation given by Theorem 2.1 with $m = -s$, we get

$$W^\ell Q(f_2,\ g) \in L^2(\mathbb{R}^4_{t,x};\ H^{-s}(\mathbb{R}^3_v)).$$

On the other hand, (6.2.2) gives

$$W^\ell G \in L^2(\mathbb{R}^4_{t,x};\ H^{-s}(\mathbb{R}^3_v)).$$

By using equation (6.1.3), it follows that

$$\partial_t(W_l g) + v\cdot\partial_x(W_l g) = W_l Q(f_2,\ g) + W_l G \in H^{-s}(\mathbb{R}^7).$$

Finally, by using Lemma 6.5 with $s' = s$, we can conclude (6.3.2) and this completes the proof of the proposition. $\square$

Therefore, under the hypothesis $f \in L^\infty([0,T];\, H^3_\ell(\mathbb{R}^6))$ for all $\ell \in \mathbb{N}$, it follows that for any $\ell \in \mathbb{N}$ we have

$$\Lambda_v^s(\varphi(t)\psi(x)f) \in L^2(\mathbb{R}_t;\, H^3_\ell(\mathbb{R}^6)), \qquad \Lambda_x^{s_0}(\varphi(t)\psi(x)f) \in L^2(\mathbb{R}_t;\, H^3_\ell(\mathbb{R}^6))\,. \tag{6.3.3}$$

We now improve this partial regularity in the variable x. Since fractional derivatives will be involved, it is not surprising that a Leibniz type formula for fractional derivatives in the variable x is needed. We shall use the following one, taken from [12]. Let $0 < \lambda < 1$. Then there exists a positive constant $C_\lambda \neq 0$ such that

$$\begin{aligned} |D_x|^\lambda Q(f,\ g) = Q(|D_x|^\lambda f,\ g) + Q(f,\ |D_x|^\lambda g) \\ + C_\lambda \int_{\mathbb{R}^3} |h|^{-3-\lambda} Q(f_h,\ g_h)dh\,, \end{aligned} \tag{6.3.4}$$

with

$$f_h(t,x,v) = f(t,x,v) - f(t,x+h,v), \quad h \in \mathbb{R}^3_x\,.$$

With this preparation, we need a preliminary step, given by

Proposition 6.7. *Let* $0<\lambda<1$ *and* $f\in L^\infty([0,T];\,\mathcal{H}^3_\ell(\mathbb{R}^6))$ *be a solution of (1.1). Assume that, for all* $\ell\in\mathbb{N}$*, we have*

$$\Lambda^s_v f_1\in H^3_\ell(\mathbb{R}^7),\quad \Lambda^\lambda_x f_1\in H^3_\ell(\mathbb{R}^7).$$

Then, one has for any $\ell\in\mathbb{N}$*,*

$$\begin{aligned}\|\Lambda^s_v\Lambda^\lambda_x f_1\|_{L^2(\mathbb{R}_t;\,H^3_\ell(\mathbb{R}^6))}&\le C\|f_2\|_{L^\infty([0,T];\,H^3_{\ell+3}(\mathbb{R}^6))}\\ &\times\left(\|\Lambda^s_v f_1\|_{H^3_{\ell+2s}(\mathbb{R}^7)}+\|\Lambda^\lambda_x f_1\|_{H^3_{\ell+2s}(\mathbb{R}^7)}\right).\end{aligned}$$

Proof. Set $g=\partial^\alpha_{x,v}f_1$ and $\alpha\in\mathbb{N}^6,|\alpha|\le 3$. We choose $W^\ell\,|D_x|^\lambda\,\psi_2^2(x)\,|D_x|^\lambda\,W^\ell g$ as a test function for equation (6.1.3). Then

$$\begin{aligned}&\Big((v\cdot\partial_x\psi_2(x))|D_x|^\lambda\,W^\ell g,\ \psi_2(x)\,|D_x|^\lambda\,W^\ell g\Big)_{L^2(\mathbb{R}^7)}\\ =&\Big(\psi_2(x)\,|D_x|^\lambda\,W^\ell\,Q(f_2,\ g),\ \psi_2(x)\,|D_x|^\lambda\,W^\ell g\Big)_{L^2(\mathbb{R}^7)}\\ &+\Big(\psi_2(x)\,|D_x|^\lambda\,W^\ell\,G,\ \psi_2(x)\,|D_x|^\lambda\,W^\ell g\Big)_{L^2(\mathbb{R}^7)}.\end{aligned}$$

One has

$$\left|\Big((v\cdot\partial_x\psi_2(x))|D_x|^\lambda\,W^\ell g,\ \psi_2(x)\,|D_x|^\lambda\,W^\ell g\Big)_{L^2(\mathbb{R}^7)}\right|\le C\|\,\Lambda^\lambda_x\partial^\alpha f_1\|_{L^2_{\ell+1}(\mathbb{R}^7)}\,,$$

and the same estimation for the linear term of G in (6.1.4)

$$\left|\Big(\psi_2(x)|D_x|^\lambda\,W^\ell\,(B+C),\ \psi_2(x)\,|D_x|^\lambda\,W^\ell g\Big)_{L^2(\mathbb{R}^7)}\right|\le C\|\,\Lambda^\lambda_x\partial^\alpha f_1\|_{L^2_{\ell+1}(\mathbb{R}^7)}\,.$$

For the nonlinear terms of G in (6.1.4), we shall use the Leibniz formula (6.3.4). First of all, the coercivity estimate (2.1.1) gives, as in (6.2.1),

$$\begin{aligned}&-\Big(Q(f_2,\,\psi_2(x)|D_x|^\lambda W^\ell g),\ \psi_2(x)|D_x|^\lambda W^\ell g\Big)_{L^2(\mathbb{R}^7)}\\ &\ge C_0\|\Lambda^s_v\psi_2(x)|D_x|^\lambda W^\ell g\|^2_{L^2(\mathbb{R}^7)}-CT\|f_2\|_{L^\infty(\mathbb{R}^4_{t,x};\,L^1(\mathbb{R}^3_v))}\|\psi_2(x)|D_x|^\lambda W^\ell g\|^2_{L^\infty([0,T];\,L^2(\mathbb{R}^6))}\,.\end{aligned}$$

On the other hand, the upper bound estimate of Theorem 2.1 with $m=-s$ gives,

$$\begin{aligned}&\left|\Big(Q(|D_x|^\lambda f_2,\,\psi_1(x)W^\ell g),\ \psi_1(x)|D_x|^\lambda W^\ell g\Big)_{L^2(\mathbb{R}^7)}\right|\\ \le\ & C\||D_x|^\lambda f_2\|_{L^\infty(\mathbb{R}^4_{t,x},\,L^1_{2s}(\mathbb{R}^3_v))}\|\psi_2(x)\Lambda^s_v W^\ell g\|_{L^2_{2s}(\mathbb{R}^7)}\|\psi_2(x)|D_x|^\lambda\Lambda^s_v W^\ell g\|_{L^2(\mathbb{R}^7)}\\ \le\ & C\|f_2\|_{L^\infty([0,T];\,H^{3/2+\lambda+\epsilon}_{3/2+2s+\epsilon}(\mathbb{R}^6))}\|\psi_2(x)\Lambda^s_v W^\ell g\|_{L^2_{2s}(\mathbb{R}^7)}\|\psi_2(x)|D_x|^\lambda\Lambda^s_v W^\ell g\|_{L^2(\mathbb{R}^7)}\\ \le\ & \delta\|\psi_2(x)|D_x|^\lambda\Lambda^s_v W^\ell g\|^2_{L^2(\mathbb{R}^7)}+C_\delta\|f_2\|^2_{L^\infty([0,T];\,H^3_3(\mathbb{R}^6))}\|\Lambda^s_v g\|^2_{L^2_{\ell+2s}(\mathbb{R}^7)}.\end{aligned}$$

For the cross term coming from the decomposition (6.3.4), by using again Theorem 2.1 with $m=-s$, we get

$$\begin{aligned}&\int_{\mathbb{R}^3}|h|^{-3-\lambda}\left|\Big(Q((f_2)_h,\,(W^\ell g)_h),\ \psi_2^2(x)|D_x|^\lambda W^\ell g\Big)_{L^2(\mathbb{R}^7)}dh\right|\\ \le\ & |C_\lambda|\|\psi_2(x)|D_x|^\lambda\Lambda^s_v W^\ell g\|_{L^2(\mathbb{R}^7)}\\ &\quad\times\int_{\mathbb{R}^3}|h|^{-3-\lambda}\|(f_2)_h\|_{L^\infty(\mathbb{R}^4_{t,x},\,L^1_{2s}(\mathbb{R}^3_v))}\|\Lambda^s_v(W^\ell g)_h\|_{L^2_{2s}(\mathbb{R}^7)}dh\\ \le\ & \delta\|\psi_1(x)|D_x|^\lambda\Lambda^s_v W^\ell g\|^2_{L^2(\mathbb{R}^7)}+C_\delta\|f_2\|^2_{L^\infty([0,T];\,H^3_3(\mathbb{R}^6))}\|\Lambda^s_v g\|^2_{L^2_{\ell+2s}(\mathbb{R}^7)}.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\left|\Big(|D_x|^\lambda Q(f_2,\,\psi_2(x)W^\ell g)-Q(|D_x|^\lambda f_2,\,\psi_2(x)W^\ell g),\ \psi_2(x)|D_x|^\lambda W^\ell g\Big)_{L^2(\mathbb{R}^7)}\right|\\ \le\ & \delta\|\psi_2(x)|D_x|^\lambda\Lambda^s_v W^\ell g\|^2_{L^2(\mathbb{R}^7)}+C_\delta\|f_2\|^2_{L^\infty([0,T];\,H^3_3(\mathbb{R}^6))}\|\Lambda^s_v g\|^2_{L^2_{\ell+2s}(\mathbb{R}^7)}.\end{aligned}$$

In conclusion, we get

$$
\begin{aligned}
& \|\psi_2(x)|D_x|^\lambda \Lambda_v^s W^\ell g\|^2_{L^2(\mathbb{R}^7)} \\
\leq\ & C\|f_2\|^2_{L^\infty([0,T];\ H^3_3(\mathbb{R}^6))}\Big(\|\ |D_x|^\lambda g\|^2_{L^2_{\ell+2s}(\mathbb{R}^7)} + \|\Lambda_v^s g\|^2_{L^2_{\ell+2s}(\mathbb{R}^7)}\Big) \\
+\ & \Big|\Big(|D_x|^\lambda\Big(W^\ell Q(f_2,\ g) - Q(f_2,\ W^\ell g)\Big),\ \psi_2^2(x)|D_x|^\lambda W^\ell g\Big)_{L^2(\mathbb{R}^7)}\Big| \\
+\ & \Big|\Big(|D_x|^\lambda W^\ell\ A,\ \psi_2^2(x)|D_x|^\lambda\ W^\ell g\Big)_{L^2(\mathbb{R}^7)}\Big| \\
=\ & \mathrm{I} + \mathrm{II} + \mathrm{III}\,.
\end{aligned}
$$

For the term II, again, formula (6.3.4) yields,

$$
\begin{aligned}
& \Big(|D_x|^\lambda\Big(W^\ell Q(f_2,\ g) - Q(f_2,\ W^\ell g)\Big),\ \psi_2^2(x)|D_x|^\lambda W^\ell g\Big)_{L^2(\mathbb{R}^7)} \\
& = \Big((W^\ell Q(|D_x|^\lambda f_2,\ g) - Q(|D_x|^\lambda f_2,\ W^\ell g)),\ \psi_2^2(x)|D_x|^\lambda W^\ell g\Big)_{L^2(\mathbb{R}^7)} \\
& + \Big((W^\ell Q(f_2,\ |D_{t,x}|^\lambda g) - Q(f_2,\ W^\ell |D_x|^\lambda g)),\ \psi_2^2(x)|D_x|^\lambda W^\ell g\Big)_{L^2(\mathbb{R}^7)} \\
& + C_\lambda \int_{\mathbb{R}^3} |h|^{-3-\lambda}\Big((W^\ell Q((f_2)_h,\ g_h) - Q((f_2)_h,\ W^\ell g_h)),\ \psi_2^2(x)|D_x|^\lambda W^\ell g\Big)_{L^2(\mathbb{R}^7)} dh.
\end{aligned}
$$

Since $0 < s < 1/2$, Lemma 3.2 implies

$$
\begin{aligned}
& \Big|\Big((W^\ell Q(|D_x|^\lambda f_2,\ g) - Q(|D_x|^\lambda f_2,\ W^\ell g)),\ \psi_2^2(x)|D_x|^\lambda W^\ell g\Big)_{L^2(\mathbb{R}^7)}\Big| \\
& \leq C\|\ |D_x|^\lambda f_2\|_{L^\infty(\mathbb{R}^4_{t,x},\ L^1_\ell(\mathbb{R}^3_v))}\|g\|_{L^2(\mathbb{R}^4_{t,x},\ L^2_\ell(\mathbb{R}^3_v))}\|\ |D_x|^\lambda g\|_{L^2_\ell(\mathbb{R}^7)} \\
& \leq C\|f_2\|_{L^\infty([0,T];\,H^{3/2+\lambda+\epsilon}_{\ell+3/2+\epsilon}(\mathbb{R}^6))}\|g\|_{L^2_\ell(\mathbb{R}^3)}\|\ |D_x|^\lambda g\|_{L^2_\ell(\mathbb{R}^7)},
\end{aligned}
$$

and

$$
\begin{aligned}
& \Big|\Big((W^\ell Q(f_2,\ |D_x|^\lambda g) - Q(f_2,\ W^\ell |D_x|^\lambda g)),\ \psi_2^2(x)|D_x|^\lambda W^\ell g\Big)_{L^2(\mathbb{R}^7)}\Big| \\
& \leq C\|f_2\|_{L^\infty(\mathbb{R}^4_{t,x},\ L^1_\ell(\mathbb{R}^3_v))}\||D_x|^\lambda g\|_{L^2(\mathbb{R}^4_{t,x},\ L^2_\ell(\mathbb{R}^3_v))}\|\ |D_x|^\lambda g\|_{L^2_\ell(\mathbb{R}^7)} \\
& \leq C\|f_2\|_{L^\infty([0,T];\,H^{1+3/2+\epsilon}_{\ell+3/2+\epsilon}(\mathbb{R}^6))}\||D_x|^\lambda g\|^2_{L^2_\ell(\mathbb{R}^3)}\,.
\end{aligned}
$$

For the cross term,

$$
\begin{aligned}
& \Big|\int_{\mathbb{R}^3} |h|^{-3-\lambda}\Big((W^\ell Q((f_2)_h,\ g_h) - Q((f_2)_h,\ W^\ell g_h)),\ \psi_2^2(x)|D_x|^\lambda W^\ell g\Big)_{L^2(\mathbb{R}^7)} dh\Big| \\
& \leq C\|f_2\|_{L^\infty([0,T];\,H^{1+3/2+\epsilon}_{\ell+3/2+\epsilon}(\mathbb{R}^6))}\|g\|_{L^2_\ell(\mathbb{R}^3)}\|\ |D_x|^\lambda g\|_{L^2_\ell(\mathbb{R}^7)}.
\end{aligned}
$$

Thus, we have

$$
\mathrm{II} \leq C\|f_2\|_{L^\infty([0,T];\,H^3_{\ell+2}(\mathbb{R}^6))}\||D_x|^\lambda g\|^2_{L^2_\ell(\mathbb{R}^3)}\,.
$$

We now consider the last term III. Recall that A stands for the nonlinear terms from G

$$
A = \sum_{\alpha_1+\alpha_2=\alpha\ \alpha_1\neq 0} C^{\alpha_1}_{\alpha_2} Q\Big(\partial^{\alpha_1} f_2,\ \partial^{\alpha_2} f_1\Big).
$$

We have

$$
\begin{aligned}
&\Big|\Big(|D_x|^\lambda \left(Q(\partial^{\alpha_1} f_2,\ \partial^{\alpha_2} f_1)\right),\ W^\ell\, \psi_2^2(x)|D_x|^\lambda\, W^\ell\, g\Big)_{L^2(\mathbb{R}^7)}\Big| \\
&\le C\|\Lambda_v^s \psi_1(x)|D_x|^\lambda W^\ell\, g\|_{L^2(\mathbb{R}^7)}\times \\
&\times\Big\{\big\|Q(|D_{t,x}|^\lambda\partial^{\alpha_1} f_2,\ \partial^{\alpha_2} f_1)\big\|_{L^2(\mathbb{R}^4_{t,x};H^{-s}_\ell(\mathbb{R}^3_v))}+\big\|Q(\partial^{\alpha_1} f_2,\ |D_x|^\lambda\partial^{\alpha_2} f_1)\big\|_{L^2(\mathbb{R}^4_{t,x};H^{-s}_\ell(\mathbb{R}^3_v))} \\
&+\Big\|\int h^{-3-\lambda} Q(\partial^{\alpha_1}(f_2)_h,\ \partial^{\alpha_2}(f_1)_h)dh\Big\|_{L^2(\mathbb{R}^4_{t,x};H^m_{l+|\gamma|/2}(\mathbb{R}^3_v))}\Big\}.
\end{aligned}
$$

We divide the discussion into two cases.

Case 1. $|\alpha_1| = 1$ (then $|\alpha_2| \le 2$). Applying (2.1.2) with $m = -s$. We have

$$
\begin{aligned}
&\Big\|Q\Big(|D_x|^\lambda\partial^{\alpha_1} f_2,\ \partial^{\alpha_2} f_1\Big)\Big\|_{L^2(\mathbb{R}^4_{t,x};H^{-s}_\ell(\mathbb{R}^3_v))} \\
&\le C\|\,\Lambda_x^\lambda\partial^{\alpha_1} f_2\|_{L^2(\mathbb{R}_t;\,(L^\infty(\mathbb{R}^3_x;L^1_{\ell+2s}(\mathbb{R}^3_v))))}\|\,\Lambda_v^s\partial^{\alpha_2} f_1\|_{L^\infty(\mathbb{R}_t;\,L^2_{\ell+2s}(\mathbb{R}^6))} \\
&\le C\|\,\Lambda_x^\lambda f_2\|_{H^{1+3/2+\epsilon}_{\ell+3/2+\epsilon+2s}(\mathbb{R}^7)}\|\Lambda_v^s f_1\|_{H^{2+1/2+\epsilon}_{\ell+2s}(\mathbb{R}^7)}, \\
&\Big\|Q\Big(\partial^{\alpha_1} f_2,\ |D_x|^\lambda\partial^{\alpha_2} f_1\Big)\Big\|_{L^2(\mathbb{R}^4_{t,x};H^{-s}_\ell(\mathbb{R}^3_v))} \\
&\le C\|\partial^{\alpha_1} f_2\|_{L^\infty(\mathbb{R}^4_{t,x};L^1_{\ell+2s}(\mathbb{R}^3_v))}\|\,\Lambda_v^s\Lambda_x^\lambda\partial^{\alpha_2} f_1\|_{L^2_{\ell+2s}(\mathbb{R}^7)} \\
&\le C\|f_2\|_{L^\infty([0,T];\,H^{1+3/2+\epsilon}_{\ell+3/2+\epsilon+2s}(\mathbb{R}^6))}\|\Lambda_v^s f_1\|_{H^3_{\ell+2s}(\mathbb{R}^7)},
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big\|\int_{\mathbb{R}^3} h^{-3-\lambda} Q\Big(\partial^{\alpha_1}(f_2)_h,\ \partial^{\alpha_2}(f_1)_h\Big)dh\Big\|_{L^2(\mathbb{R}^4_{t,x};H^{-s}_\ell(\mathbb{R}^3_v))} \\
&\le C\int |h|^{-3-\lambda}\|\,\partial^{\alpha_1}(f_2)_h\|_{L^\infty(\mathbb{R}^4_{t,x};\,L^1_{\ell+2s}(\mathbb{R}^3_v))}\|\,\Lambda_v^s\partial^{\alpha_2}(f_1)_h\|_{L^2_{\ell+2s}(\mathbb{R}^7)}dh \\
&\le C\|\,\partial^{\alpha_1} f_2\|_{L^\infty(\mathbb{R}^4_{t,x};L^1_{\ell+2s}(\mathbb{R}^3_v))}\|\,\Lambda_v^s\partial^{\alpha_2}\nabla_x f_1\|_{L^2_{\ell+2s}(\mathbb{R}^7)} \\
&\le C\|f_2\|_{L^\infty([0,T];\,H^{1+3/2+\epsilon}_{\ell+3/2+\epsilon+2s}(\mathbb{R}^6))}\|\Lambda_v^s f_1\|_{H^3_{\ell+2s}(\mathbb{R}^7)}\,.
\end{aligned}
$$

Case 2. $|\alpha_1| \ge 2$. By the same argument as above, one has

$$
\begin{aligned}
&\Big\|Q\Big(|D_x|^\lambda\partial^{\alpha_1} f_2,\ \partial^{\alpha_2} f_1\Big)\Big\|_{L^2(\mathbb{R}^4_{t,x};H^{-s}_\ell(\mathbb{R}^3_v))}+\Big\|Q\Big(\partial^{\alpha_1} f_2,\ |D_x|^\lambda\partial^{\alpha_2} f_1\Big)\Big\|_{L^2(\mathbb{R}^4_{t,x};H^{-s}_\ell(\mathbb{R}^3_v))} \\
&\le C\|f_2\|_{L^\infty([0,T];\,H^{1+3/2+\epsilon}_{\ell+3/2+\epsilon+2s}(\mathbb{R}^6))}\Big(\|\Lambda_v^s f_1\|_{H^3_{\ell+2s}(\mathbb{R}^7)}+\|\Lambda_x^\lambda f_1\|_{H^3_{\ell+2s}(\mathbb{R}^7)}\Big)\,.
\end{aligned}
$$

When $|\alpha_1| = 2$, we have

$$
\begin{aligned}
&\Big\|\int_{\mathbb{R}^3} h^{-3-\lambda} Q\Big(\partial^{\alpha_1}(f_2)_h,\ \partial^{\alpha_2}(f_1)_h\Big)dh\Big\|_{L^2(\mathbb{R}^4_{t,x};\,H^{-s}_\ell(\mathbb{R}^3_v))} \\
&\le C\int |h|^{-3-\lambda}\|\,\partial^{\alpha_1}(f_2)_h\|_{L^\infty(\mathbb{R}_t;\,(L^2(\mathbb{R}^3_x;\,L^1_{\ell+2s}(\mathbb{R}^3_v))))}\|\,\Lambda_v^s\partial^{\alpha_2}(f_1)_h\|_{L^2(\mathbb{R}_t;\,(L^\infty(\mathbb{R}^3_x;\,L^2_{\ell+2s}(\mathbb{R}^3_v))))}dh \\
&\le C\|\,\nabla_x\partial^{\alpha_1} f_2\|_{L^\infty(\mathbb{R}_t;\,(L^2(\mathbb{R}^3_x;\,L^1_{\ell+2s}(\mathbb{R}^3_v))))}\|\,\Lambda_v^s\partial^{\alpha_2} f_1\|_{L^2(\mathbb{R}_t;\,(L^\infty(\mathbb{R}^3_x;\,L^2_{\ell+2s}(\mathbb{R}^3_v))))} \\
&\le C\|f_2\|_{L^\infty([0,T];\,H^3_{\ell+3/2+\epsilon+2s}(\mathbb{R}^6))}\|\,\Lambda_v^s f_1\|_{H^3_{\ell+2s}(\mathbb{R}^7)},
\end{aligned}
$$

while when $|\alpha_1| = |\alpha| = 3$, we have

$$\begin{aligned}&\Big\| \int_{\mathbb{R}^3} h^{-3-\lambda} Q\Big(\partial^\alpha (f_2)_h,\ (f_1)_h\Big) dh \Big\|_{L^2(\mathbb{R}^4_{t,x}; H^{-s}_\ell(\mathbb{R}^3_v))}\\
&\le C \int |h|^{-3-\lambda} \|\partial^\alpha (f_2)_h\|_{L^2(\mathbb{R}^4_{t,x}; L^1_{\ell+2s}(\mathbb{R}^3_v))} \|\Lambda^s_v (f_1)_h\|_{L^\infty(\mathbb{R}^4_{t,x};\, L^2_{\ell+2s}(\mathbb{R}^3_v))} dh\\
&\le C\|\partial^\alpha f_2\|_{L^2(\mathbb{R}^4_{t,x};\, L^1_{\ell+2s}(\mathbb{R}^3_v))} \|\Lambda^s_v \nabla_x f_1\|_{L^\infty(\mathbb{R}^4_{t,x};\, L^2_{\ell+2s}(\mathbb{R}^3_v))}\\
&\le C\|f_2\|^2_{L^\infty([0,T];\, H^3_{\ell+3/2+\epsilon+2s}(\mathbb{R}^6))}.\end{aligned}$$

Thus, by the Cauchy-Schwarz inequality, we obtain

$$\mathrm{III} \le \delta\|\Lambda^s_v \psi_1(x) |D_x|^\lambda W^\ell g\|^2_{L^2(\mathbb{R}^7)} + C_\delta \|f_2\|^2_{L^\infty([0,T];\, H^3_{\ell+3}(\mathbb{R}^6))} \Big(\|\Lambda^s_v f_1\|^2_{H^3_{\ell+2s}(\mathbb{R}^7)} + \|\Lambda^\lambda_x f_1\|^2_{H^3_{\ell+2s}(\mathbb{R}^7)}\Big).$$

Finally, we get

$$\|\Lambda^s_v \psi_1(x) |D_x|^\lambda W^\ell g\|^2_{L^2(\mathbb{R}^7)} \le C_\delta \|f_2\|^2_{L^\infty([0,T];\, H^3_{\ell+3}(\mathbb{R}^6))} \Big(\|\Lambda^s_v f_1\|^2_{H^3_{\ell+2s}(\mathbb{R}^7)} + \|\Lambda^\lambda_x f_1\|^2_{H^3_{\ell+2s}(\mathbb{R}^7)}\Big),$$

and the proof of the proposition is completed. □

We are now ready to show that the gain of at least order 1 regularity in the variable x.

Proposition 6.8. *Under the hypothesis of Theorem 6.1, one has*

$$\Lambda^{1+\varepsilon}_x (\varphi(t)\psi(x) f) \in H^3_\ell(\mathbb{R}^7), \tag{6.3.5}$$

for any $\ell \in \mathbb{N}$ and some $\varepsilon > 0$.

Proof. By fixing $s_0 = \frac{s(1-s)}{(s+1)}$, then (6.3.3) and Proposition 6.7 with $\lambda = s_0$ imply

$$\Lambda^s_v \Lambda^{s_0}_x g \in H^3_\ell(\mathbb{R}^7).$$

It follows that,

$$(\Lambda^{s_0}_x g)_t + v \cdot \partial_x(\Lambda^{s_0}_x g) = \Lambda^{s_0}_x Q(f_2,\ g) + \Lambda^{s_0}_x G \in H^{-s}_\ell(\mathbb{R}^7).$$

By applying Lemma 6.5 with $s' = s$, we can deduce that

$$\Lambda^{s_0+s_0}_x (\varphi(t)\psi(x) f) \in H^3_\ell(\mathbb{R}^7),$$

for any $\ell \in \mathbb{N}$. If $2s_0 < 1$, by using again Proposition 6.7 with $\lambda = 2s_0$ and Lemma 6.5 with $s' = s$, we have

$$\Lambda^s_v(\varphi(t)\psi(x) f),\ \Lambda^{2s_0}_x(\varphi(t)\psi(x) f) \in H^3_\ell(\mathbb{R}^7)\ \Rightarrow\ \Lambda^{3s_0}_x(\varphi(t)\psi(x) f) \in H^3_\ell(\mathbb{R}^7).$$

Choose $k_0 \in \mathbb{N}$ such that

$$k_0 s_0 < 1, \qquad (k_0+1)s_0 = 1+\varepsilon > 1.$$

Finally, (6.3.5) follows from Proposition 6.7 with $\lambda = k_0 s_0$ by induction. And this completes the proof of the proposition. □

6.4. **Higher order regularity.** The proof of Theorem 6.3 will now be concluded with the above preparation.

From Proposition 6.4, Proposition 6.8 and equation (1.1), it follows that for any $\ell \in \mathbb{N}$,

$$\Lambda^s_v (\varphi(t)\psi(x) f), \qquad \nabla_x (\varphi(t)\psi(x) f) \in H^3_\ell(\mathbb{R}^7).$$

This fact will be used to show higher order regularity in the variable v by using the following

Proposition 6.9. *Let $0 < \lambda < 1$. Assume that, for any cutoff functions $\varphi \in C_0^\infty(]T_1, T_2[)$, $\psi \in C_0^\infty(\Omega)$ and all $\ell \in \mathbb{N}$,*

$$\Lambda_v^\lambda (\varphi(t)\psi(x) f), \quad \nabla_x (\varphi(t)\psi(x) f) \in H^3_\ell(\mathbb{R}^7).$$

Then, for any cutoff function and any $\ell \in \mathbb{N}$,

$$\Lambda_v^{\lambda+s} (\varphi(t)\psi(x) f) \in H^3_\ell(\mathbb{R}^7).$$

For the proof, we choose $W^\ell \Lambda_v^{2\lambda} W^\ell g$ as a test function for (6.1.3), and then proceed as in the proof of Proposition 6.7. The only difference is in the estimation on the commutator with the convection:

$$\left|\Big([\Lambda_v^\lambda,\ v] \cdot \partial_x W^\ell g,\ \Lambda_v^\lambda W^\ell g\Big)_{L^2(\mathbb{R}^7)}\right| \le C \|\Lambda_v^\lambda g\|_{L^2_\ell(\mathbb{R}^7)} \|\nabla_x g\|_{L^2_\ell(\mathbb{R}^7)},$$

since

$$[\Lambda_v^\lambda,\ v] \cdot \partial_x = \lambda \Lambda_v^{\lambda-2} \partial_v \cdot \partial_x,$$

and $\Lambda_v^{\lambda-2} \partial_v$ are bounded operators in L^2. This is the reason why we need in the first step the gain of the regularity of order 1 in the variable x.

To complete the proof of the full regularization result, firstly, exactly as Proposition 6.3.5, we can get

$$\Lambda_v^{1+\varepsilon}(\varphi(t)\psi(x) f) \in H^3_\ell(\mathbb{R}^7),$$

for any $\ell \in \mathbb{N}$ and some $\varepsilon > 0$.

Therefore, we obtain that there exists $\epsilon > 0$ such that for any $\ell \in \mathbb{N}$, and any cutoff functions $\varphi(t)$ and $\psi(x)$,

$$\varphi(t)\psi(x) f \in H^{4+\epsilon}_l(\mathbb{R}^7).$$

Notice that the estimate in Proposition 6.7 can be modified as follows. In fact, we can obtain

$$\begin{aligned}\|\Lambda_v^s \Lambda_x^\lambda f_1\|_{L^2(\mathbb{R}_t; H^4_\ell(\mathbb{R}^6))} &\le C \|f_2\|_{H^{4+\epsilon}_{\ell+3}(\mathbb{R}^7)} \\ &\quad \times \Big(\|\Lambda_v^s f_1\|_{H^4_{\ell+2s}(\mathbb{R}^7)} + \|\Lambda_x^\lambda f_1\|_{H^4_{\ell+2s}(\mathbb{R}^7)}\Big).\end{aligned}$$

By using this, the proof of

$$f \in \mathcal{H}^{4+\epsilon}_\ell(]T_1, T_2[\times\Omega\times\mathbb{R}^3_v),\ \forall \ell \in \mathbb{N} \implies f \in \mathcal{H}^5_\ell(]T_1, T_2[\times\Omega\times\mathbb{R}^3_v),\ \forall \ell \in \mathbb{N},$$

is direct and this completes the proof of Theorem 6.3 by the bootstrapping argument.

7. Global existence

We shall establish a global energy estimate for the Cauchy problem (1.3). For ease of exposition, and unless necessary, generic constants will be dropped out from the estimates in this section. Finally, all in all, we shall follow and adapt the method initiated by Guo [46] on the estimation on the macroscopic components. Here we point out that his method works also for the non-cutoff case but only when the estimations of the nonlinear and related terms are carried out in terms of the non-isotorpic norms (2.2.1). We also note that his method has been generalized to various directions. Among them, a few are the external force case [35, 37], the Vlassov-Maxwell-Boltzmann equation [65], the soft potential case [66, 74] and the Landau equation [45, 74]. Independently of his method which is based on the macro-micro decomposition near a global Maxwellian, the energy method based on the macro-micro decomposition around a local Maxwellian has also been developed with application to the classical fluid dynamical equations [52, 53, 54]. Further references are found in these paper.

Introduce the macro-micro decomposition near the absolute Maxwellian μ:

$$g = \mathbf{P}g + (\mathbf{I} - \mathbf{P})g = g_1 + g_2, \quad \mathbf{P}g = g_1 = (a + b \cdot v + c|v|^2)\mu^{1/2},$$

In this section, the following result on the energy estimate will be proved.

Theorem 7.1. *For $N, \ell \geq 3$, let $T > 0$ and suppose that g is a classical solution to the Cauchy problem* (1.3) *on $[0, T]$. There exist constants $M_0, M_1 > 0$ such that if*

$$\max_{0 \leq t \leq T} \mathcal{E}(t) \leq M_0,$$

then g enjoys the estimate

$$\mathcal{E}(t) + \int_0^t \mathcal{D}(\tau) d\tau \leq M_1 \mathcal{E}(0),$$

for any $t \in [0, T]$, where

$$\mathcal{E} = \|g\|^2_{H^N_\ell(\mathbb{R}^6)} \sim \|(a, b, c)\|^2_{H^N(\mathbb{R}^3_x)} + \|g_2\|^2_{H^N_\ell(\mathbb{R}^6)},$$

is the instant energy functional, and

$$\mathcal{D} = \|\nabla_x(a, b, c)\|^2_{H^{N-1}(\mathbb{R}^3_x)} + |||g_2|||^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)}$$

the total dissipation rate.

The proof of this theorem is divided into two parts, that is, the estimation on the macroscopic component and the microscopic component respectively.

7.1. **Macroscopic Energy Estimate.** By the macro-micro decomposition, the equation in (1.3) is reduced to

$$\begin{aligned}\partial_t\Big\{a+b \cdot v + c|v|^2\Big\}\mu^{1/2} + v \cdot \nabla_x\Big\{a + b \cdot v + |v|^2 c\Big\}\mu^{1/2}\\ = -(\partial_t + v \cdot \nabla_x)g_2 + \mathcal{L}g_2 + \Gamma(g, g).\end{aligned}$$

Using

$$v \cdot \nabla_x b \cdot v = \sum_{i,j} v_i v_j \partial_i b_j = \sum_i v_i^2 \partial_i b_i + \sum_{i>j} v_i v_j (\partial_i b_j + \partial_j b_i),$$

we deduce

$$\text{(7.1.1)} \qquad \begin{cases} \text{(i)} & v_i|v|^2\mu^{1/2}: & \nabla_x c & = -\partial_t r_c + l_c + h_c, \\ \text{(ii)} & v_i^2\mu^{1/2}: & \partial_t c + \partial_i b_i & = -\partial_t r_i + l_i + h_i, \\ \text{(iii)} & v_i v_j \mu^{1/2}: & \partial_i b_j + \partial_j b_i & = -\partial_t r_{ij} + l_{ij} + h_{ij}, \quad i \neq j, \\ \text{(iv)} & v_i\mu^{1/2}: & \partial_t b_i + \partial_i a & = -\partial_t r_{bi} + l_{bi} + h_{bi}, \\ \text{(v)} & \mu^{1/2}: & \partial_t a & = -\partial_t r_a + l_a + h_a, \end{cases}$$

where

$$r = (g_2, e)_{L^2_v}, \qquad l = -(v \cdot \nabla_x g_2 + \mathcal{L}g_2, e)_{L^2_v}, \qquad h = (\Gamma(g, g), e)_{L^2(\mathbb{R}^3_v)},$$

stand for $r_c, \cdots, h_a$, while

$$e \in \text{span}\{v_i|v|^2\mu^{1/2}, v_i^2\mu^{1/2}, v_i v_j \mu^{1/2}, v_i\mu^{1/2}, \mu^{1/2}\}.$$

Lemma 7.2. *Let $\partial^\alpha = \partial^\alpha_x$, $\alpha \in \mathbb{N}^3, |\alpha| \leq m, m \geq 3$. Then,*

$$\|\partial^\alpha(a, b, c)^2\|_{L^2(\mathbb{R}^3_x)} \leq \|\nabla_x(a, b, c)\|_{H^{m-1}(\mathbb{R}^3_x)} \|(a, b, c)\|_{H^{m-1}(\mathbb{R}^3_x)}.$$

Proof. Let $k = |\alpha|$. Then, one has for $k = 0$ that

$$\|(a,b,c)^2\|_{L^2(\mathbb{R}^3_x)} \le \|(a,b,c)\|_{L^6(\mathbb{R}^3_x)}\|(a,b,c)\|_{L^3(\mathbb{R}^3_x)} \le \|\nabla_x(a,b,c)\|_{L^2(\mathbb{R}^3_x)}\|(a,b,c)\|_{H^2(\mathbb{R}^3_x)},$$

since

$$\|ab\|_{L^2(\mathbb{R}^3_x)} \le \|a\|_{L^6(\mathbb{R}^3_x)}\|b\|_{L^3(\mathbb{R}^3_x)} \le \|\nabla_x a\|_{L^2(\mathbb{R}^3_x)}\|b\|^{1/3}_{L^\infty(\mathbb{R}^3_x)}\|b\|^{2/3}_{L^2(\mathbb{R}^3_x)} \le \|\nabla_x a\|_{L^2(\mathbb{R}^3_x)}\|b\|_{H^2(\mathbb{R}^3_x)}.$$

Also for $k = 1$, we have

$$\begin{aligned}\|\partial(a,b,c)^2\|_{L^2(\mathbb{R}^3_x)} &\le \|(a,b,c)\partial(a,b,c)\|_{L^2(\mathbb{R}^3_x)} \le \|(a,b,c)\|_{L^\infty(\mathbb{R}^3_x)}\|\nabla_x(a,b,c)\|_{L^2(\mathbb{R}^3_x)}\\ &\le \|(a,b,c)\|_{H^{m-1}(\mathbb{R}^3_x)}\|\nabla_x(a,b,c)\|_{L^2(\mathbb{R}^3_x)},\end{aligned}$$

and for $2 \le k \le m$,

$$\begin{aligned}&\|\partial^\alpha(a,b,c)^2\|_{L^2(\mathbb{R}^3_x)} \le \sum_{k'\le \frac{k}{2}} \|\partial^{k'}_x(a,b,c)\partial^{k-k'}_x(a,b,c)\|_{L^2(\mathbb{R}^3_x)}\\ &\le \sum \|\partial^{k'}_x(a,b,c)\|_{L^\infty(\mathbb{R}^3_x)}\|\partial^{k-k'}_x(a,b,c)\|_{L^2(\mathbb{R}^3_x)} \le \|(a,b,c)\|_{H^{m-1}(\mathbb{R}^3_x)}\|\nabla_x(a,b,c)\|_{H^{m-1}(\mathbb{R}^3_x)}.\end{aligned}$$

And this completes the proof of the lemma. □

Lemma 7.3. (Estimate of r, l, h).
Let $\partial^\alpha = \partial^\alpha_x, \partial_i = \partial_{x_i}$, $|\alpha| \le N-1, N \ge 3$. Then, one has

$$\|\partial_i\partial^\alpha r\|_{L^2(\mathbb{R}^3_x)} + \|\partial^\alpha l\|_{L^2_x} \le \|g_2\|_{H^N(\mathbb{R}^3_x,\, L^2(\mathbb{R}^3_v))} \equiv A_1, \tag{7.1.2}$$

$$\begin{aligned}\|\partial^\alpha h\|_{L^2(\mathbb{R}^3_x)} &\le \|\nabla_x(a,b,c)\|_{H^{N-2}(\mathbb{R}^3_x)}\|(a,b,c)\|_{H^{N-1}(\mathbb{R}^3_x)}\\ &+ \|(a,b,c)\|_{H^{N-1}(\mathbb{R}^3_x)}\|g_2\|_{H^{N-1}(\mathbb{R}^3_x,\, L^2(\mathbb{R}^3_v))} + \|g_2\|^2_{H^{N-1}(\mathbb{R}^3_x,\, L^2(\mathbb{R}^3_v))} \equiv A_2.\end{aligned} \tag{7.1.3}$$

Proof. (7.1.2) follows from

$$\|\partial_i\partial^\alpha r\|_{L^2(\mathbb{R}^3_x)} \le \|(\partial_i\partial^\alpha g_2, e)_{L^2(\mathbb{R}^3_v)}\|_{L^2(\mathbb{R}^3_x)} \le \|\partial_i\partial^\alpha g_2\|_{L^2(\mathbb{R}^6_{x,v})},$$

and

$$\|\partial^\alpha l\|_{L^2_x} \le \|(\nabla_x\partial^\alpha g_2, ve)_{L^2(\mathbb{R}^3_v)} + (\partial^\alpha g_2, \mathcal{L}^* e)_{L^2(\mathbb{R}^3_v)}\|_{L^2(\mathbb{R}^3_x)} \le \|\partial^\alpha_x g_2\|_{H^1(\mathbb{R}^3_x,\, L^2(\mathbb{R}^3_v))}.$$

We prove (7.1.3) as follows.

$$\begin{aligned}h &= \iiint b(\cos\theta)\mu_*^{1/2}(g'_*g' - g_*g)e\,dvdv_*d\sigma\\ &= \iiint b(\cos\theta)gg_*\Big((\mu^{1/2})'_*e' - \mu_*^{1/2}e\Big)dvdv_*d\sigma\\ &= \iiint b(\cos\theta)(\mu^{1/2}g)(\mu^{1/2}g)_*\Big(q(v') - q(v)\Big)dvdv_*d\sigma\\ &\equiv \Phi(g,g) = \sum_{i,j=1}^{2}\Phi(g_i,g_j) = \sum_{i,j=1}^{2}\Phi^{(ij)},\end{aligned}$$

where $q = q(v)$ is some polynomial of v. Firstly, we have

$$\Phi^{(11)} = \sum_{\eta_j,\eta_k\in\{a,b,c\}} \eta_j\eta_k\Phi(\psi_j,\psi_k),$$

where $\psi_j(v) \in \mathcal{N}$. Clearly, $|\Phi(\psi_j,\psi_k)| < \infty$, so that by virtue of Lemma 7.2,

$$\|\partial^\alpha\Phi^{(11)}\|_{L^2(\mathbb{R}^3_x)} \le \|\partial^\alpha(a,b,c)^2\|_{L^2(\mathbb{R}^3_x)} \le \|\nabla_x(a,b,c)\|_{H^{N-2}(\mathbb{R}^3_x)}\|(a,b,c)\|_{H^{N-1}(\mathbb{R}^3_x)}.$$

On the other hand,

$$\|\Phi(g, f)\|_{L^2(\mathbb{R}^3_x)} \le \|\mu^{1/2} g\|_{L^2(\mathbb{R}^3_x; L^1_3(\mathbb{R}^3_v))} \|\mu^{1/2} f\|_{L^2(\mathbb{R}^3_x; L^1_3(\mathbb{R}^3_v))} \le \|g\|_{L^2(\mathbb{R}^6_{x,v})} \|f\|_{L^2(\mathbb{R}^6_{x,v})}$$

which yields for $|\alpha| \le N-1$,

$$\|\partial^\alpha \Phi^{(12)}\|_{L^2(\mathbb{R}^3_x)} \le \|\partial^\alpha (a,b,c)\|_{L^2(\mathbb{R}^3_x)} \|\partial^\alpha g_2\|_{L^2(\mathbb{R}^6_{x,v})} \le \|(a,b,c)\|_{H^{N-1}(\mathbb{R}^3_x)} \|g_2\|_{H^{N-1}(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))},$$

$$\|\partial^\alpha \Phi^{(21)}\|_{L^2(\mathbb{R}^3_x)} \le \|\partial^\alpha g_2\|_{L^2(\mathbb{R}^6_{x,v})} \|\partial^\alpha (a,b,c)\|_{L^2(\mathbb{R}^3_x)} \le \|g_2\|_{H^{N-1}(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))} \|(a,b,c)\|_{H^{N-1}(\mathbb{R}^3_x)},$$

$$\|\partial^\alpha \Phi^{(22)}\|_{L^2(\mathbb{R}^3_x)} \le \|\partial^\alpha g_2\|^2_{L^2(\mathbb{R}^6_{x,v})} \le \|g_2\|^2_{H^{N-1}(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))}.$$

Now the proof of (7.1.3) is completed. □

Lemma 7.4. (Macro-energy estimate) *Let* $|\alpha| \le N-1$.

$$\|\nabla_x \partial^\alpha (a,b,c)\|^2_{L^2(\mathbb{R}^3_x)} \le -\frac{d}{dt}\Big\{(\partial^\alpha r, \nabla_x \partial^\alpha (a,-b,c))_{L^2(\mathbb{R}^3_x)} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}^3_x)}\Big\} + \|g_2\|^2_{H^N(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))} + \mathcal{D}_1 \mathcal{E}_1\,, \tag{7.1.4}$$

where

$$\mathcal{D}_1 = \|\nabla_x (a,b,c)\|^2_{H^{N-1}(\mathbb{R}^3_x)} + \|g_2\|^2_{H^N(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))}$$

is a dissipation rate and

$$\mathcal{E}_1 = \|(a,b,c)\|^2_{H^{N-1}(\mathbb{R}^3_x)} + \|g_2\|^2_{H^{N-1}(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))} = \|g\|^2_{H^{N-1}(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))}$$

is an instant energy functional.

Proof. (a) Estimate of $\nabla_x \partial^\alpha a$. Let A_1, A_2 be as in Lemma 7.3. From (7.1.1) (iv),

$$\begin{aligned} \|\nabla_x \partial^\alpha a\|^2_{L^2(\mathbb{R}^3_x)} &= (\nabla_x \partial^\alpha a, \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}^3_x)} \\ &= (\partial^\alpha(-\partial_t b - \partial_t r + l + h), \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}^3_x)} \\ &\le R_1 + C_\eta (A_1^2 + A_2^2) + \eta \|\nabla_x \partial^\alpha a\|^2_{L^2(\mathbb{R}^3_x)}. \end{aligned}$$

$$\begin{aligned} R_1 &= -(\partial^\alpha \partial_t b + \partial^\alpha \partial_t r, \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}^3_x)} \\ &= -\frac{d}{dt}(\partial^\alpha (b+r), \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}^3_x)} + (\nabla_x \partial^\alpha (b+r), \partial_t \partial^\alpha a)_{L^2(\mathbb{R}^3_x)} \\ &\le -\frac{d}{dt}(\partial^\alpha (b+r), \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}^3_x)} + C_\eta (\|\nabla_x \partial^\alpha b\|^2_{L^2(\mathbb{R}^3_x)} + A_1^2) + \eta \|\partial_t \partial^\alpha a\|^2_{L^2(\mathbb{R}^3_x)}, \end{aligned}$$

(b) Estimate of $\nabla_x \partial^\alpha b$. From (7.1.1) (iii) and (ii),

$$\begin{aligned} \Delta_x \partial^\alpha b_i + \partial_i^2 \partial^\alpha b_i &= \sum_{j\ne i} \partial_j \partial^\alpha (\partial_j b_i + \partial_i b_j) + \partial_i \partial^\alpha (2\partial_i b_i - \sum_{j \ne i} \partial_i b_j) \\ &= \partial_i \partial^\alpha (-\partial_t r + l + h), \end{aligned}$$

$$\|\nabla_x \partial^\alpha b\|^2_{L^2(\mathbb{R}^3_x)} + \|\partial_i \partial^\alpha b\|^2_{L^2(\mathbb{R}^3_x)} = -(\Delta_x \partial^\alpha b_i + \partial_i^2 \partial^\alpha b_i, \partial^\alpha b)_{L^2(\mathbb{R}^3_x)} = R_2 + R_3 + R_4,$$

where

$$\begin{aligned} R_2 &= (\partial_t \partial^\alpha r, \partial_i \partial^\alpha b)_{L^2(\mathbb{R}^3_x)} = \frac{d}{dt}(\partial^\alpha r, \partial_i \partial^\alpha b)_{L^2(\mathbb{R}^3_x)} + (\partial_i \partial^\alpha r, \partial_t \partial^\alpha b)_{L^2(\mathbb{R}^3_x)} \\ &\le \frac{d}{dt}(\partial^\alpha r, \partial_i \partial^\alpha b)_{L^2(\mathbb{R}^3_x)} + C_\eta A_1^2 + \eta \|\partial_t \partial^\alpha b\|^2_{L^2(\mathbb{R}^3_x)}, \\ R_3 &= -(\partial^\alpha l, \partial_i \partial^\alpha b)_{L^2(\mathbb{R}^3_x)} \le C_\eta A_1^2 + \eta \|\partial_i \partial^\alpha b\|^2_{L^2(\mathbb{R}^3_x)}, \\ R_4 &= -(\partial^\alpha h, \partial_i \partial^\alpha b)_{L^2(\mathbb{R}^3_x)} \le C_\eta A_2^2 + \eta \|\partial_i \partial^\alpha b\|^2_{L^2(\mathbb{R}^3_x)}. \end{aligned}$$

(c) Estimate of $\nabla_x \partial^\alpha c$. From (7.1.1) (i),

$$\begin{aligned}\|\nabla_x \partial^\alpha c\|^2_{L^2(\mathbb{R}^3_x)} &= (\nabla_x \partial^\alpha c, \nabla_x \partial^\alpha c)_{L^2(\mathbb{R}^3_x)} = (\partial^\alpha(-\partial_t r + l + h), \nabla_x \partial^\alpha c)_{L^2(\mathbb{R}^3_x)}\\ &\le R_5 + C_\eta(A_1^2 + A_2^2) + \eta\|\nabla_x \partial^\alpha c\|^2_{L^2(\mathbb{R}^3_x)}\end{aligned}$$

where

$$\begin{aligned}R_5 = -(\partial^\alpha \partial_t r, \nabla_x \partial^\alpha c)_{L^2(\mathbb{R}^3_x)} &= -\frac{d}{dt}(\partial^\alpha r, \nabla_x \partial^\alpha c)_{L^2(\mathbb{R}^3_x)} + (\nabla_x \partial^\alpha r, \partial_t \partial^\alpha c)_{L^2(\mathbb{R}^3_x)}\\ &\le -\frac{d}{dt}(\partial^\alpha r, \nabla_x \partial^\alpha c)_{L^2(\mathbb{R}^3_x)} + C_\eta A_1^2 + \eta\|\partial_t \partial^\alpha c\|^2_{L^2(\mathbb{R}^3_x)},\end{aligned}$$

(d) Estimate of $\partial_t \partial^\alpha(a, b, c)$.

$$\begin{aligned}\|\partial_t \partial^\alpha(a, b, c)\|_{L^2(\mathbb{R}^3_x)} &= \|\partial^\alpha \partial_t \mathbf{P} g\|_{L^2(\mathbb{R}^6_{x,v})}\\ &= \|\partial^\alpha \mathbf{P}\Big(- v\cdot\nabla_x g - \mathcal{L} g + \Gamma(g, g)\Big)\|_{L^2(\mathbb{R}^6_{x,v})}\\ &= \|\partial^\alpha \mathbf{P}(v\cdot\nabla_x g)\|_{L^2(\mathbb{R}^6_{x,v})} \le \|\nabla_x \partial^\alpha(a, b, c)\|_{L^2(\mathbb{R}^3_x)} + \|\nabla_x \partial^\alpha g_2\|_{L^2(\mathbb{R}^6_{x,v})}.\end{aligned} \tag{7.1.5}$$

Combining all the above estimates and taking $\eta > 0$ sufficiently small, we deduce

$$\begin{aligned}\|\nabla_x \partial^\alpha(a, b, c)\|^2_{L^2(\mathbb{R}^3_x)} \le -\frac{d}{dt}\Big\{&(\partial^\alpha r, \nabla_x \partial^\alpha(a, -b, c))_{L^2(\mathbb{R}^3_x)} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}^3_x)}\Big\}\\ &+ A_1^2 + A_2^2 + \eta\|\nabla_x \partial^\alpha g_2\|^2_{L^2(\mathbb{R}^6_{x,v})}.\end{aligned}$$

Finally, choosing $|\alpha| \le N - 1$, and using Lemma (7.3), we obtain

$$A_1^2 + A_2^2 + \eta\|\nabla_x \partial^\alpha g_2\|^2_{L^2(\mathbb{R}^6_{x,v})} \le \mathcal{D}_1 \mathcal{E}_1 + \eta\|g_2\|^2_{H^N_x(L^2(\mathbb{R}^3_v))},$$

which completes the proof of Lemma 7.1.4. □

7.2. **Microscopic Energy Estimate.** In this subsection, We shall use Lemma 2.6 and Proposition 3.5 to estimate the microscopic component.

Step 1. Let $\alpha \in \mathbb{N}^3$, $|\alpha| \le N$, and apply $\partial^\alpha = \partial^\alpha_x$ to (1.3) to have,

$$\partial_t(\partial^\alpha g) + v\cdot\nabla_x(\partial^\alpha g) + \mathcal{L}(\partial^\alpha g) = \partial^\alpha \Gamma(g,\, g),$$

and take the $L^2(\mathbb{R}^6_{x,v})$ inner product with $\partial^\alpha g$. By Lemma 2.6, we have

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha g\|^2_{L^2(\mathbb{R}^6_{x,v})} + D_1 \le J, \tag{7.2.1}$$

where D_1 is a dissipation rate

$$D_1 = \int_{\mathbb{R}^3} |||\partial^\alpha g_2|||^2 dx = |||\partial^\alpha g_2|||^2_{\mathcal{B}^0_0(\mathbb{R}^6)},$$

and J is given by

$$\begin{aligned}J = (\partial^\alpha \Gamma(g,\, g), \partial^\alpha g)_{L^2(\mathbb{R}^6)} &= \sum_{i,j=1}^{2} (\partial^\alpha \Gamma(g_i,\, g_j), \partial^\alpha g_2)_{L^2(\mathbb{R}^6)}\\ &= \sum_{i,j=1}^{2} J^{(ij)}.\end{aligned}$$

First, consider $J^{(11)}$. We have, with $\psi_j \in \mathcal{N}$,

$$
\begin{aligned}
(7.2.2)\qquad |J^{(11)}| &\le \|\partial^\alpha \Gamma(g_1,\, g_1)\|_{L^2(\mathbb{R}^6)} \|\partial^\alpha g_2\|_{L^2(\mathbb{R}^6)} \\
&\le \|\Big(\|\partial^\alpha \Gamma(g_1,\, g_1)\|_{L^2(\mathbb{R}^3_v)}\Big)\|_{L^2(\mathbb{R}^3_x)} \|\partial^\alpha g_2\|_{L^2(\mathbb{R}^6)},\\
\|\partial^\alpha \Gamma(g_1,\, g_1)\|_{L^2(\mathbb{R}^3_v)} &\le \sum_{\eta_j,\eta_k \in \{a,b,c\}} |\partial^\alpha(\eta_j \eta_k)| \|\Gamma(\psi_j, \psi_k)\|_{L^2(\mathbb{R}^3_v)},\\
\|\Gamma(\psi_j,\psi_k)\|^2_{L^2(\mathbb{R}^3_v)} &= \int \Big(\int\int b(\cos\theta) \mu_*^{1/2} \{(\psi_j)'_* \psi'_k - (\psi_j)_* \psi_k\} dv_* d\sigma \Big)^2 dv\\
&= \int \mu \Big(\int\int b(\cos\theta) \mu_* \{(p_j)'_* p'_k - (p_j)_* p_k\} dv_* d\sigma \Big)^2 dv < \infty,
\end{aligned}
$$

where $p_j \in \{1, v, |v|^2\}$. Consequently, by virtue of Lemma 7.2,

$$
\begin{aligned}
|J^{(11)}| &\le \|\partial^\alpha (a,b,c)^2\|_{L^2(\mathbb{R}^3_x)} \|\partial^\alpha g_2\|_{L^2(\mathbb{R}^6)}\\
&\le \|\nabla_x (a,b,c)\|_{H^{N-2}(\mathbb{R}^3_x)} \|(a,b,c)\|_{H^{N-1}(\mathbb{R}^3_x)} \|g_2\|_{H^N(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))}\\
&\le \|(a,b,c)\|_{H^{N-1}(\mathbb{R}^3_x)} \Big(\|\nabla_x (a,b,c)\|^2_{H^{N-2}(\mathbb{R}^3_x)} + |||g_2|||^2_{\mathcal{B}^N_0(\mathbb{R}^6)} \Big).
\end{aligned}
$$

On the other hand, using Proposition 3.5 gives

$$
\begin{aligned}
|J^{(12)}| &\le \|g_1\|_{H^N_0(\mathbb{R}^6)} |||g_2|||_{\mathcal{B}^N_0(\mathbb{R}^6)} |||g_2|||_{\mathcal{B}^N_0(\mathbb{R}^6)}\\
&\le \|(a,b,c)\|_{H^N(\mathbb{R}^3_x)}\ |||g_2|||^2_{\mathcal{B}^N_0(\mathbb{R}^6)},\\
|J^{(21)}| &\le \|g_2\|_{H^N_0(\mathbb{R}^6)} |||g_1|||_{\mathcal{B}^N_0(\mathbb{R}^6)} |||g_2|||_{\mathcal{B}^N_0(\mathbb{R}^6)}\\
&\le \|g_2\|_{H^N_0(\mathbb{R}^6)} \|(a,b,c)\|_{H^N(\mathbb{R}^3_x)}\ |||g_2|||_{\mathcal{B}^N_0(\mathbb{R}^6)},\\
|J^{(22)}| &\le \|g_2\|_{H^N_0(\mathbb{R}^6)} |||g_2|||_{\mathcal{B}^N_0(\mathbb{R}^6)} |||g_2|||_{\mathcal{B}^N_0(\mathbb{R}^6)}\\
&\le \|g_2\|_{H^N_0(\mathbb{R}^6)}\ |||g_2|||^2_{\mathcal{B}^N_0(\mathbb{R}^6)}.
\end{aligned}
$$

Taking the summation of (7.2.1) for $|\alpha| \le N, N \ge 3$, we have

Lemma 7.5. *Let $N \ge 3$. Then,*

$$
(7.2.3)\qquad \frac{d}{dt}\|g\|^2_{H^N(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))} + |||g_2|||^2_{\mathcal{B}^N_0(\mathbb{R}^6)} \le \mathcal{D}_2 \mathcal{E}_2^{1/2},
$$

where

$$
\begin{aligned}
\mathcal{D}_2 &= \|\nabla_x (a,b,c)\|^2_{H^{N-1}(\mathbb{R}^3_x)} + |||g_2|||^2_{\mathcal{B}^N_0(\mathbb{R}^6)},\\
\mathcal{E}_2 &= \|g\|^2_{H^N(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))} \sim \|(a,b,c)\|^2_{H^N(\mathbb{R}^3_x)} + \|g_2\|^2_{H^N(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))}.
\end{aligned}
$$

Step 2. Let $\partial^\alpha = \partial^\alpha_x, 1 \le |\alpha| \le N, N \ge 3$, and apply $W^\ell \partial^\alpha$ to (1.3). We have

$$
(7.2.4)\qquad \partial_t(W^\ell \partial^\alpha g) + v\cdot\nabla_x (W^\ell \partial^\alpha g) + \mathcal{L}(W^\ell \partial^\alpha g) = S_1 + S_2,
$$

where

$$
S_1 = W^\ell \partial^\alpha \Gamma(g,\, g), \qquad S_2 = [\mathcal{L},\, W^\ell](\partial^\alpha g).
$$

Take the $L^2(\mathbb{R}^6_{x,v})$ inner product with $W^\ell \partial^\alpha g$ to deduce

$$
\frac{1}{2}\frac{d}{dt}\|W^\ell \partial^\alpha g\|^2_{L^2(\mathbb{R}^6)} + D_2 \le G,
$$

where D_2 is a dissipation rate

$$\begin{aligned} D_2 &= \int_{\mathbb{R}^3} |||(\mathbf{I}-\mathbf{P})W^\ell\partial^\alpha g|||^2 dx \\ &\geq \frac{1}{2}|||\partial^\alpha g|||^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)} - (\|\nabla_x(a,b,c)\|_{H^{N-1}(\mathbb{R}^3_x)} + \|g_2\|^2_{H^N(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))}). \end{aligned}$$

Here we have used for $1 \leq |\alpha| \leq N$,

$$\begin{aligned} \int_{\mathbb{R}^3} |||\mathbf{P}W^\ell\partial^\alpha g|||^2 dx &\leq \|\partial^\alpha g\|^2_{L^2(\mathbb{R}^6_{x,v})} \\ &\leq \|\nabla_x(a,b,c)\|^2_{H^{N-1}(\mathbb{R}^3_x)} + \|g_2\|^2_{H^N(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))}. \end{aligned}$$

On the other hand, G is defined by

$$G = G_1 + G_2, \quad G_i = (S_i, W^\ell\partial^\alpha g)_{L^2(\mathbb{R}^6)}, \qquad i = 1, 2.$$

The estimation on G_1 and G_2 will be given in the following lemmas.

Lemma 7.6. *Let $N \geq 3, \ell \geq 3$. Then, for $\mathcal{E}$ and $\mathcal{D}$ defined in Theorem 7.1, we have*

$$G_1 \leq \mathcal{E}^{1/2}\mathcal{D},$$

Proof. First, write

$$G_1 = \sum_{i,j=1,2} (W^\ell\partial^\alpha\Gamma(g_i,\, g_j), W^\ell\partial^\alpha g)_{L^2(\mathbb{R}^6)} = \sum_{i,j=1,2} G_1^{(ij)}.$$

Proceeding as in (7.2.2),

$$\begin{aligned} (7.2.5)\quad |G^{(11)}| &\leq \|W^{2\ell}\partial^\alpha\Gamma(g_1,\, g_1)\|_{L^2(\mathbb{R}^6)}\|\partial^\alpha g\|_{L^2(\mathbb{R}^6)} \\ &\leq \|\Big(\|W^{2\ell}\partial^\alpha\Gamma(g_1,\, g_1)\|_{L^2(\mathbb{R}^3_v)}\Big)\|_{L^2(\mathbb{R}^3_x)}\|\partial^\alpha g\|_{L^2(\mathbb{R}^6)}, \\ &\sim \|\partial^\alpha\{(a,b,c)^2\}\|_{L^2(\mathbb{R}^3_x)}\|W^{2\ell}\Gamma(\psi_j,\psi_k)\|_{L^2(\mathbb{R}^3_v)}\|\partial^\alpha g\|_{L^2(\mathbb{R}^6)}, \end{aligned}$$

$$\begin{aligned} \|W^{2\ell}\Gamma(\psi_j,\psi_k)\|^2_{L^2(\mathbb{R}^3_v)} &= \int \Big(W^{2\ell}\int\int b(\cos\theta)\mu_*^{1/2}\{(\psi_j)'_*\psi'_k - (\psi_j)_*\psi_k\}dv_*d\sigma\Big)^2 dv \\ &= \int \mu W^{4\ell}\Big(\int\int b(\cos\theta)\mu_*\{(p_j)'_*p'_k - (p_j)_*p_k\}dv_*d\sigma\Big)^2 dv < \infty. \end{aligned}$$

Since $1 \leq |\alpha| \leq N$,

$$\begin{aligned} |G^{(11)}| &\leq \|\nabla_x(a,b,c)\|_{H^{N-1}(\mathbb{R}^3_x)}\|(a,b,c)\|_{H^{N-1}(\mathbb{R}^3_x)} \\ &\quad\times (\|\nabla_x(a,b,c)\|_{H^{N-1}(\mathbb{R}^3_x)} + \|g_2\|_{H^N(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))}). \end{aligned}$$

On the other hand, we have

$$|||g_1|||^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)} = \sum_{|\beta|\leq N}\int_{\mathbb{R}^3_x} |||W^\ell\partial^\beta_{x,v}g_1(x,\,\cdot)|||^2 dx \leq \|(a,b,c)\|^2_{H^N(\mathbb{R}^3_x)}.$$

This fact and Proposition 3.5 yield

$$\begin{aligned} |G_1^{(12)}| &\leq \|g_1\|_{H^N_\ell(\mathbb{R}^6)}\ |||g_2|||_{\mathcal{B}^N_\ell(\mathbb{R}^6)}\ |||W^\ell\partial^\alpha g|||_{\mathcal{B}^0_0(\mathbb{R}^6)} \\ &\leq \|(a,b,c)\|_{H^N(\mathbb{R}^3_x)}\ |||g_2|||_{\mathcal{B}^N_\ell(\mathbb{R}^6)}|||\partial^\alpha g|||_{\mathcal{B}^0_\ell(\mathbb{R}^6)}, \\ |G_1^{(21)}| &\leq \|g_2\|_{H^N_\ell(\mathbb{R}^6)}\ |||g_1|||_{\mathcal{B}^N_\ell(\mathbb{R}^6)}\ |||W^\ell\partial^\alpha g|||_{\mathcal{B}^0_0(\mathbb{R}^6)} \\ &\leq \|g_2\|_{H^N_\ell(\mathbb{R}^6)}\|(a,b,c)\|_{H^N(\mathbb{R}^3_x)}\ |||\partial^\alpha g|||_{\mathcal{B}^0_\ell(\mathbb{R}^6)}, \\ |G_1^{(22)}| &\leq \|g_2\|_{H^N_\ell(\mathbb{R}^6)}\ |||g_2|||_{\mathcal{B}^N_\ell(\mathbb{R}^6)}\ |||W^\ell\partial^\alpha g|||_{\mathcal{B}^0_0(\mathbb{R}^6)} \\ &\leq \|g_2\|_{H^N_\ell(\mathbb{R}^6)}|||g_2|||_{\mathcal{B}^N_\ell(\mathbb{R}^6)}|||\partial^\alpha g|||_{\mathcal{B}^0_\ell(\mathbb{R}^6)}. \end{aligned}$$

Noticing that for $1 \le |\alpha| \le N$,

$$
\begin{aligned}
|||\partial^\alpha g|||^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)} &\le \int_{\mathbb{R}^3_x} |||W^\ell \partial^\alpha g_1(x,\,\cdot)|||^2 dx + \int_{\mathbb{R}^3_x} |||W^\ell \partial^\alpha g_2(x,\,\cdot)|||^2 dx \\
&\le \|\nabla_x(a,b,c)\|^2_{H^{N-1}(\mathbb{R}^3_x)} + |||g_2|||_{\mathcal{B}^N_\ell(\mathbb{R}^6)},
\end{aligned}
$$

we now conclude the proof of the lemma. □

We shall evaluate G_2. In view of Proposition 3.8,

$$
\begin{aligned}
|G_2| \le & \left|\left([\mathcal{L}_1, W^\ell\,]\partial^\alpha g, W^\ell\,\partial^\alpha g\right)_{L^2(\mathbb{R}^6)}\right| \\
&+ \left|\left(W^\ell\,\mathcal{L}_2(\partial^\alpha g),\;W^\ell\partial^\alpha g\right)_{L^2(\mathbb{R}^6)}\right| + \left|\left(\mathcal{L}_2(W^\ell\,\partial^\alpha g),\;W^\ell\partial^\alpha g\right)_{L^2(\mathbb{R}^6)}\right| \\
\le & \|\partial^\alpha g\|_{L^2_\ell(\mathbb{R}^6)} |||\partial^\alpha g|||_{\mathcal{B}^0_\ell(\mathbb{R}^6)} \\
\le & (\|\nabla_x(a,b,c)\|_{H^{N-1}(\mathbb{R}^3_x)} + \|\partial^\alpha g_2\|_{L^2_\ell(\mathbb{R}^6)})|||\partial^\alpha g|||_{\mathcal{B}^0_\ell(\mathbb{R}^6)} \\
\le & \|\nabla_x(a,b,c)\|^2_{H^{N-1}(\mathbb{R}^3_x)} + C_\eta\|\partial^\alpha g_2\|^2_{L^2(\mathbb{R}^6)} + \eta|||\partial^\alpha g|||^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)}, \quad (\eta > 0).
\end{aligned}
$$

Thus, we have established

Lemma 7.7. *Let* $1 \le |\alpha| \le N, N \ge 3$. *Then,*

$$
\begin{aligned}
&\frac{d}{dt}\|\partial^\alpha g\|^2_{L^2_\ell(\mathbb{R}^6)} + |||\partial^\alpha g|||^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)} \\
&\le \mathcal{E}^{1/2}\mathcal{D} + \|\partial^\alpha g_2\|^2_{L^2(\mathbb{R}^6)} + \|\nabla_x(a,b,c)\|^2_{H^{N-1}(\mathbb{R}^3_x)}.
\end{aligned}
\tag{7.2.6}
$$

Step 3 . We need also to estimate $W^\ell g_2$. Apply $W^\ell(\mathbf{I}-\mathbf{P})$ to the equation in (1.3) to have

$$
\begin{aligned}
&\partial_t(W^\ell g_2) + v\cdot\nabla_x(W^\ell g_2) + \mathcal{L}(W^\ell g_2) \\
&\qquad = W^\ell\Gamma(g,\,g) + W^\ell[v\cdot\nabla_x,\,\mathbf{P}]g + [\mathcal{L},\,W^\ell]g_2.
\end{aligned}
$$

And then take the inner product with $W^\ell g_2$ to get

$$
\frac{d}{dt}\|W^\ell g_2\|^2_{L^2(\mathbb{R}^6)} + D_3 \le H,
$$

where

$$
\begin{aligned}
D_3 &= \int_{\mathbb{R}^3} |||(\mathbf{I}-\mathbf{P})W^\ell g_2|||^2 dx \\
&\ge \frac{1}{2}|||g_2|||^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)} - C\|g_2\|_{L^2(\mathbb{R}^3)},
\end{aligned}
$$

while

$$
\begin{aligned}
H =& H_1 + H_2 + H_3, \\
&H_1 = (W^\ell\Gamma(g,g), W^\ell g_2)_{L^2(\mathbb{R}^6)}, \\
&H_2 = (W^\ell[v\cdot\nabla_x,\,\mathbf{P}]g, W^\ell g_2)_{L^2(\mathbb{R}^6)}, \\
&H_3 = ([\mathcal{L},\,W^\ell]g_2, W^\ell g_2)_{L^2(\mathbb{R}^6)}.
\end{aligned}
$$

$$
H_1 = \sum_{i,j=2}(W^\ell\Gamma(g_i,\,g_j), W^\ell g_2)_{L^2(\mathbb{R}^6)} = \sum_{i,j=2} H_1^{(ij)}.
$$

Proceeding as in (7.2.5),

$$
\begin{aligned}
|H^{(11)}| &\le \|W^{2\ell}\Gamma(g_1,\, g_1)\|_{L^2(\mathbb{R}^6)}\|g_2\|_{L^2(\mathbb{R}^6)}\\
&\le \|\Big(\|W^{2\ell}\Gamma(g_1,\, g_1)\|_{L^2(\mathbb{R}^3_v)}\Big)\|_{L^2(\mathbb{R}^3_x)}\|g_2\|_{L^2(\mathbb{R}^6)}\\
&\sim \|(a,b,c)^2\|_{L^2(\mathbb{R}^3_x)}\|W^{2\ell}\Gamma(\psi_j,\psi_k)\|_{L^2(\mathbb{R}^3_v)}\|g_2\|_{L^2(\mathbb{R}^6)},
\end{aligned}
$$

$$
\begin{aligned}
\|W^{2\ell}\Gamma(\psi_j,\psi_k)\|^2_{L^2(\mathbb{R}^3_v)} &= \int \Big(W^{2\ell}\int\int b(\cos\theta)\mu_*^{1/2}\{(\psi_j)'_*\psi'_k-(\psi_j)_*\psi_k\}dv_*d\sigma\Big)^2 dv\\
&= \int \mu W^{4\ell}\Big(\int\int b(\cos\theta)\mu_*\{(p_j)'_*p'_k-(p_j)_*p_k\}dv_*d\sigma\Big)^2 dv < \infty.
\end{aligned}
$$

Then we have, by using Lemma 7.2,

$$
|H^{(11)}| \le \|\nabla_x(a,b,c)\|_{L^2(\mathbb{R}^3_x)}\|(a,b,c)\|_{H^1(\mathbb{R}^3_x)}\|g_2\|_{L^2(\mathbb{R}^6)}.
$$

On the other hand, we have

$$
|||g_1|||^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)} = \int_{\mathbb{R}^3_x}|||W^\ell g_1(x,\,\cdot)|||^2dx \le \|(a,b,c)\|^2_{L^2(\mathbb{R}^3_x)}.
$$

This fact and Proposition 3.5 yield

$$
\begin{aligned}
|H_1^{(12)}| &\le \|g_1\|_{H^N_\ell(\mathbb{R}^6)}\ |||g_2|||_{\mathcal{B}^N_\ell(\mathbb{R}^6)}\ |||W^\ell g_2|||_{\mathcal{B}^0_0(\mathbb{R}^6)}\\
&\le \|(a,b,c)\|_{H^N(\mathbb{R}^3_x)}\ |||g_2|||^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)},\\
|H_1^{(21)}| &\le \|g_2\|_{H^N_\ell(\mathbb{R}^6)}\ |||g_1|||_{\mathcal{B}^N_\ell(\mathbb{R}^6)}\ |||W^\ell g_2|||_{\mathcal{B}^0_0(\mathbb{R}^6)}\\
&\le \|g_2\|_{H^N_\ell(\mathbb{R}^6)}\|(a,b,c)\|_{H^N(\mathbb{R}^3_x)}\ |||g_2|||_{\mathcal{B}^0_\ell(\mathbb{R}^6)},\\
|H_1^{(22)}| &\le \|g_2\|_{H^N_\ell(\mathbb{R}^6)}\ |||g_2|||_{\mathcal{B}^N_\ell(\mathbb{R}^6)}\ |||W^\ell g_2|||_{\mathcal{B}^0_0(\mathbb{R}^6)}\\
&\le \|g_2\|_{H^N_\ell(\mathbb{R}^6)}|||g_2|||^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)}.
\end{aligned}
$$

And H_2 is evaluated as follows

$$
\begin{aligned}
|H_2| &\le |(W^{2\ell}[v\cdot\nabla_x,\,\mathbf{P}]g, g_2)_{L^2(\mathbb{R}^6)}| \le \|\nabla_x g\|_{L^2(\mathbb{R}^6)}\|g_2\|_{L^2(\mathbb{R}^6)}\\
&\le \|\nabla_x(a,b,c)\|^2_{L^2(\mathbb{R}^3)} + \|g_2\|^2_{H^1(\mathbb{R}^3_x;L^2(\mathbb{R}^3_v))}.
\end{aligned}
$$

Finally, in view of Proposition 3.8 ,

$$
\begin{aligned}
|H_3| \le &\Big|\Big([\mathcal{L}_1, W^\ell\,]g_2, W^\ell g_2)_{L^2(\mathbb{R}^6)}\Big|\\
&+\Big|\Big(W^\ell\,\mathcal{L}_2(g_2),\ W^\ell g_2\Big)_{L^2(\mathbb{R}^6)}\Big| + \Big|\Big(\mathcal{L}_2(W^\ell g_2),\ W^\ell g_2\Big)_{L^2(\mathbb{R}^6)}\Big|\\
\le &\|g_2\|_{L^2_\ell(\mathbb{R}^6)}|||g_2|||_{\mathcal{B}^0_\ell(\mathbb{R}^6)}\ .
\end{aligned}
$$

Since it holds by interpolation inequality that

$$
\|g_2\|_{L^2_\ell(\mathbb{R}^6)} \le \eta\|g_2\|_{L^2_{\ell+s}(\mathbb{R}^6)} + C_\eta\|g_2\|_{L^2(\mathbb{R}^6)} \le \eta|||g_2|||_{\mathcal{B}^0_\ell(\mathbb{R}^6)} + C_\eta\|g_2\|_{L^2(\mathbb{R}^6)}, \tag{7.2.7}
$$

for any small enough $\eta > 0$, we have established

Lemma 7.8.

$$
\begin{aligned}
&\frac{d}{dt}\|g_2\|^2_{L^2_\ell(\mathbb{R}^6)} + |||g_2|||^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)}\\
&\le \mathcal{E}^{1/2}\mathcal{D} + \|g_2\|^2_{L^2(\mathbb{R}^6)} + \|\nabla_x(a,b,c)\|^2_{L^2(\mathbb{R}^3_x)}.
\end{aligned} \tag{7.2.8}
$$

Step 4 . Let

$$\partial^\beta = \partial^\beta_{x,v} = \partial^\alpha \partial^\gamma, \quad \partial^\alpha = \partial^\alpha_x, \quad \partial^\gamma = \partial^\gamma_v, \qquad |\beta| = |\alpha| + |\gamma| \leq N, \gamma \neq 0,\ N \geq 3,$$

and apply $W^\ell \partial^\beta (\mathbf{I} - \mathbf{P})$ to (7.2.4) to have

$$\begin{aligned}\partial_t (W^\ell \partial^\beta g_2) &+ v \cdot \nabla_x (W^\ell \partial^\beta g_2) + \mathcal{L}(W^\ell \partial^\beta g_2)\\ =&W^\ell \partial^\beta \Gamma(g,\ g) + [v \cdot \nabla_x, W^\ell \partial^\beta] g_2\\ &- W^\ell \partial^\beta [\mathbf{P},\ v \cdot \nabla] g\\ &+ [\mathcal{L},\ W^\ell \partial^\beta] g_2 + W^\ell \partial^\beta (\partial_t + v \cdot \nabla_x) g_1.\end{aligned}$$

And then take the $L^2(\mathbb{R}^6_{x,v})$ inner product with $W^\ell \partial^\beta g_2$ to get

$$\frac{1}{2}\frac{d}{dt} \|\partial^\beta g_2\|^2_{L^2_\ell(\mathbb{R}^6)} + D_4 \leq K.$$

Here D_4 is a dissipation rate given by

$$\begin{aligned}D_4 &= \int_{\mathbb{R}^3} |||(\mathbf{I} - \mathbf{P}) W^\ell \partial^\beta g_2|||^2 dx\\ &\geq \frac{1}{2} |||\partial^\beta g_2|||^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)} - C\|\partial^\alpha g_2\|^2_{L^2(\mathbb{R}^6)}.\end{aligned}$$

where we used, with $\psi \in \mathcal{N}$ and $\tilde{\psi} = (-1)^{|\gamma|} \partial^\gamma (W^\ell \psi)$,

$$|||\mathbf{P} W^\ell \partial^\beta g_2|||^2 = |||\partial^\alpha (\psi, W^\ell \partial^\gamma g_2)_{L^2(\mathbb{R}^3_v)} \psi|||^2 = \|(\tilde{\psi}, \partial^\alpha g_2)_{L^2_v}\|^2_{L^2(\mathbb{R}^3_x)} |||\psi|||^2 \leq \|\partial^\alpha g_2\|^2_{L^2(\mathbb{R}^6_{x,v})}.$$

On the other hand, K is given by

$$\begin{aligned}K =&(W^\ell \partial^\beta \Gamma(g,\ g), W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)}\\ &+ ([v \cdot \nabla_x,\ W^\ell \partial^\beta] g_2, W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)} - (W^\ell \partial^\beta [\mathbf{P},\ v \cdot \nabla] g, W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)}\\ &+ ([\mathcal{L},\ W^\ell \partial^\beta] g_2, W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)}\\ &+ (W^\ell \partial^\beta (\partial_t + v \cdot \nabla_x) g_1, W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)}\\ =&K_1 + K_2 + K_3 + K_4 + K_5.\end{aligned}$$

Lemma 7.9. *Let $N \geq 3$. Then $|K_1| \leq \mathcal{E}^{1/2} \mathcal{D}$.*

Proof. First, write

$$K_1 = \sum_{i,j=2} (W^\ell \partial^\beta \Gamma(g_i,\ g_j), W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)} = \sum_{i,j=2} K_1^{(ij)}.$$

In view of Lemma 7.2,

$$\begin{aligned}|K_1^{(11)}| &= \left|(W^\ell \partial^\beta \Gamma(g_1,\ g_1), W^\ell \partial^\beta g_2)_{L^2(\mathbb{R}^6)}\right|\\ &\leq B_2 \|\partial^\alpha (a,b,c)^2\|_{L^2(\mathbb{R}^3_x)} \|W^\ell \partial^\beta g_2\|_{L^2(\mathbb{R}^6)}\\ &\leq \|\nabla_x (a,b,c)\|_{H^{N-1}_x} \|(a,b,c)\|_{H^{N-1}_x} \|W^\ell \partial^\beta g_2\|_{L^2(\mathbb{R}^6)},\end{aligned}$$

where

$$
\begin{aligned}
B_2^2 \sim & \|W^\ell \partial_v^\gamma \Gamma(\psi_j, \psi_k)\|^2_{L^2(\mathbb{R}^3_v)} \\
= & \int (W^\ell \int\int b(\cos\theta)(\partial_v^{\gamma_1}\mu)_*^{1/2} \\
& \times \{(\partial_v^{\gamma_2}\psi_j)'_*(\partial_v^{\gamma_3}\psi_k)' - (\partial_v^{\gamma_2}\psi_j)_*(\partial_v^{\gamma_3}\psi_k)\} dv_* d\sigma)^2 dv \\
= & \int \mu W^{2\ell}\Big(\int\int b(\cos\theta)\mu_* q_* \{(q_j)'_* q'_k - (q_j)_* q_k\} dv_* d\sigma\Big)^2 dv < \infty.
\end{aligned}
$$

Here, q, q_j, q_k are polynomials of v.

On the other hand, we have

$$
|||g_1|||^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)} = \sum_{|\beta|\le N} \int_{\mathbb{R}^3_x} |||W^\ell\, \partial^\beta g_1(x,\,\cdot)|||^2 dx \le \|(a,b,c)\|^2_{H^N(\mathbb{R}^3_x)}.
$$

This point and Proposition 3.5 yield

$$
\begin{aligned}
|K_1^{(12)}| \le & \|g_1\|_{H^N_\ell(\mathbb{R}^6)}\ |||g_2|||_{\mathcal{B}^N_\ell(\mathbb{R}^6)}\ |||W^\ell\partial^\beta g_2|||_{\mathcal{B}^0_0(\mathbb{R}^6)} \\
& \le \|(a,b,c)\|_{H^N(\mathbb{R}^3_x)}\ |||g_2|||^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)}, \\
|K_1^{(21)}| = & \|g_2\|_{H^N_\ell(\mathbb{R}^6)}\ |||g_1|||_{\mathcal{B}^N_\ell(\mathbb{R}^6)}\ |||W^\ell\partial^\beta g_2|||_{\mathcal{B}^0_0(\mathbb{R}^6)} \\
& \le \|g_2\|_{H^N_\ell(\mathbb{R}^6)}\|(a,b,c)\|_{H^N(\mathbb{R}^3_x)}\ |||g_2|||_{\mathcal{B}^N_\ell(\mathbb{R}^6)}, \\
|K_1^{(22)}| = & \|g_2\|_{H^N_\ell(\mathbb{R}^6)}\ |||g_2|||_{\mathcal{B}^N_\ell(\mathbb{R}^6)}\ |||W^\ell\partial^\beta g_2|||_{\mathcal{B}^0_0(\mathbb{R}^6)} \\
& \le \|g_2\|_{H^N_\ell(\mathbb{R}^6)}|||g_2|||^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)}.
\end{aligned}
$$

Now the proof of the lemma is completed. □

K_2, K_4, K_5 are estimated as follows. We have, for $|\beta| = |\alpha| + |\gamma| \le N, \gamma \ne 0$,

$$
\begin{aligned}
|K_2| = & \Big|([v\cdot\nabla_x, W^\ell\partial^\beta]g_2, W^\ell\partial^\beta g_2)_{L^2(\mathbb{R}^6)}\Big| \\
& \le \|W^\ell\partial_x^{\alpha+1}\partial_v^{\gamma-1}g_2\|_{L^2(\mathbb{R}^6)}\|W^\ell\partial^\beta g_2\|_{L^2(\mathbb{R}^6)} \\
& \le C_\eta\|W^\ell\partial_x^{\alpha+1}\partial_v^{\gamma-1}g_2\|^2_{L^2(\mathbb{R}^6)} + \eta\|W^\ell\partial^\beta g_2\|^2_{L^2(\mathbb{R}^6)}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
|K_3| = & |(W^\ell\partial^\beta[\mathbf{P},\ v\cdot\nabla]g,\ W^\ell\partial^\beta g_2)_{L^2(\mathbb{R}^6)}, W^\ell\partial^\beta g_2)_{L^2(\mathbb{R}^6)}| \\
& \le |(\partial^\gamma\Big\{W^{2\ell}\partial^\beta[\mathbf{P},\ v\cdot\nabla]g\Big\}, \partial^\alpha g_2)_{L^2(\mathbb{R}^6)}| \\
& \le \Big(\|\nabla_x\partial^\alpha(a,b,c)\|_{L^2(\mathbb{R}^3_x)} + \|\nabla_x\partial^\alpha g_2\|_{L^2(\mathbb{R}^6_{x,v})}\Big)\|\partial^\alpha g_2\|_{L^2(\mathbb{R}^6_{x,v})} \\
& \le \|\nabla_x(a,b,c)\|^2_{H^{N-1}(\mathbb{R}^3_x)} + \|g_2\|^2_{H^N(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))}.
\end{aligned}
$$

In view of Proposition 3.8,

$$
\begin{aligned}
|K_4| = & \Big|([\mathcal{L},\ W^\ell\partial^\beta]g_2, W^\ell\partial^\beta g_2)_{L^2(\mathbb{R}^6)}\Big| \\
\le & \Big|\Big([\mathcal{L}_1,\ W^\ell\ \partial^\beta_{x,v}]g_2, W^\ell\ \partial^\beta g_2)_{L^2(\mathbb{R}^6)}\Big| \\
& + \Big|\Big(W^\ell\ \partial^\beta_{x,v}\mathcal{L}_2(g_2),\ W^\ell\partial^\beta g_2\Big)_{L^2(\mathbb{R}^6)}\Big| + \Big|\Big(\mathcal{L}_2(W^\ell\ \partial^\beta_{x,v}g_2),\ W^\ell\partial^\beta g_2\Big)_{L^2(\mathbb{R}^6)}\Big| \\
\le & \Big(\|g_2\|_{H^{|\beta|}_\ell(\mathbb{R}^6)} + |||g_2|||_{\mathcal{B}^{|\beta|-1}_\ell(\mathbb{R}^6)}\Big)\ |||W^\ell\partial^\beta g_2|||_{\mathcal{B}^0_0(\mathbb{R}^6)}
\end{aligned}
$$

Hence

$$|K_4| \le C_\eta\Big(\|g_2\|^2_{H^{|\beta|}_\ell(\mathbb{R}^6)} + |||g_2|||^2_{\mathcal{B}^{|\beta|-1}_\ell(\mathbb{R}^6)}\Big) + \eta|||W^\ell\partial^\beta g_2|||^2_{\mathcal{B}^0_0(\mathbb{R}^6)}$$

Finally, recalling (7.1.5),

$$\begin{aligned}|K_5| =&\Big|(W^\ell\partial^\beta(\partial_t + v\cdot\nabla_x)g_1, W^\ell\partial^\beta g_2)_{L^2(\mathbb{R}^6)}\Big|\\ &\le |(\partial^\gamma\Big\{W^{2\ell}\partial^\beta(\partial_t + v\cdot\nabla_x)g_1\Big\}, \partial^\alpha g_2)_{L^2(\mathbb{R}^6)}|\\ &\le \|(\partial^\alpha(\partial_t + \nabla_x)(a,b,c)\|_{L^2(\mathbb{R}^3_x)}\|\partial^\alpha g_2\|_{L^2(\mathbb{R}^6)}, \quad (|\alpha|\le N-1, |\gamma|\ge 1)\\ &\le \Big(\|\nabla_x\partial^\alpha(a,b,c)\|_{L^2(\mathbb{R}^3_x)} + \|\nabla_x\partial^\alpha g_2\|_{L^2(\mathbb{R}^6_{x,v})}\Big)\|\partial^\alpha g_2\|_{L^2(\mathbb{R}^6_{x,v})}\\ &\le \|\nabla_x(a,b,c)\|^2_{H^{N-1}_x} + \|g_2\|^2_{H^N(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))}.\end{aligned}$$

Now using (7.2.7) we conclude the

Lemma 7.10. *Let* $|\beta| = |\alpha+\gamma| \le N, |\alpha| \le N-1, |\gamma| \ge 1, N \ge 3$. *Then,*

$$\begin{aligned}&\frac{d}{dt}\|\partial^\beta g_2\|^2_{L^2_\ell(\mathbb{R}^6)} + |||\partial^\beta g_2|||^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)}\\ &\le \mathcal{E}^{1/2}\mathcal{D} + \|g_2\|^2_{H^{|\beta|}_\ell(\mathbb{R}^6)}\\ &\quad + |||g_2|||^2_{\mathcal{B}^{|\alpha|+|\gamma|-1}_\ell(\mathbb{R}^6)} + \|\nabla_x(a,b,c)\|^2_{H^{N-1}(\mathbb{R}^3_x)} + \|g_2\|^2_{H^N(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))}.\end{aligned} \tag{7.2.9}$$

7.3. **A Priori Estimate.** We take the linear combination

$$\begin{aligned}&\sum_{|\alpha|\le N-1} C^{(1)}_\alpha(7.1.4)_\alpha + \sum_{|\alpha|\le N} C^{(2)}_\alpha(7.2.3)_\alpha + \sum_{1\le|\alpha|\le N} C^{(3)}_\alpha(7.2.6)_\alpha\\ &+C^{(4)}(7.2.8) + \sum_{|\beta|=|\alpha+\gamma|\le N, |\alpha|\le N-1, |\gamma|\ge 1} C^{(3)}_{\alpha,\gamma}(7.2.9)_{\alpha,\gamma}.\end{aligned}$$

With a suitable choice of the coefficients $C^{(1)}_\alpha, C^{(2)}_\alpha, C^{(3)}_\alpha, C^{(4)}, C^{(5)}_{\alpha,\gamma}$, we get

$$\frac{d}{dt}\tilde{\mathcal{E}} + \tilde{\mathcal{D}} \le \mathcal{H}, \tag{7.3.1}$$

where

$$\begin{aligned}\tilde{\mathcal{E}} =& -\sum_{|\alpha|\le N-1} C^{(1)}_\alpha\Big[(\partial^\alpha r, \nabla_x\partial^\alpha(a,-b,c))_{L^2(\mathbb{R}^3_x)} + (\partial^\alpha b, \nabla_x\partial^\alpha a)_{L^2(\mathbb{R}^3_x)}\Big]\\ &+ \sum_{|\alpha|\le N} C^{(2)}_\alpha\|\partial^\alpha_x g\|^2_{L^2(\mathbb{R}^6)} + \sum_{1\le|\alpha|\le N} C^{(3)}_\alpha\|\partial^\alpha_x g\|^2_{L^2_\ell(\mathbb{R}^6)}\\ &+ C^{(4)}\|g_2\|^2_{L^2_\ell(\mathbb{R}^6)} + \sum_{|\beta|=|\alpha+\gamma|\le N, |\alpha|\le N-1, |\gamma|\ge 1} C^{(5)}_{\alpha,\gamma}\|W^\ell\partial^\beta_{x,v}g_2\|^2_{L^2(\mathbb{R}^6)},\\ \tilde{\mathcal{D}} =& \sum_{|\alpha|\le N-1} C^{(1)}_\alpha\|\nabla_x\partial^\alpha_x(a,b,c)\|^2_{L^2(\mathbb{R}^3_x)} + \sum_{|\alpha|\le N} C^{(2)}_\alpha|||\partial^\alpha_x g_2|||^2_{\mathcal{B}^0_0(\mathbb{R}^6)}\\ &+ \sum_{1\le|\alpha|\le N} C^{(3)}_\alpha|||\partial^\alpha_x g_2|||^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)} + C^{(4)}|||g_2|||^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)}\\ &+ \sum_{|\beta|=|\alpha+\gamma|\le N, |\alpha|\le N-1, |\gamma|\ge 1} C^{(3)}_{\alpha,\gamma}|||\partial^\beta_{x,v}g_2|||^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)},\\ \mathcal{H} =& \mathcal{D}_1\mathcal{E}_1 + \mathcal{D}_2\mathcal{E}_2 + \mathcal{D}\mathcal{E}.\end{aligned}$$

Clearly, it holds that

$$\mathcal{E} \sim \tilde{\mathcal{E}}, \quad \mathcal{D} \sim \tilde{\mathcal{D}},$$

and

$$\mathcal{H} \le \mathcal{D}\mathcal{E}.$$

Now (7.3.1) gives

$$\mathcal{E}(t) + \Big[1 - C \sup_{0\le\tau\le t} \mathcal{E}(\tau)\Big] \int_0^t \mathcal{D}(\tau) d\tau \le C\mathcal{E}(0),$$

which leads to the closure of à priori estimate and then completes the proof of Theorem 7.1.

Now, the proof of Theorem 1.1 can be completed by the usual continuation argument based on Theorem 4.3 and Theorem 7.1.

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