RICCI CURVATURE AND CONVERGENCE OF LIPSCHITZ FUNCTIONS

Shouhei Honda

Abstract

We give the definition of a convergence of the differentials of Lipschitz functions with respect to the measured Gromov-Hausdorff topology, and several properties of the convergence.

1 Introduction

Let $\{(M_i, m_i)\}_{i\in\mathbf{N}}$ be a sequence of pointed n-dimensional complete Riemannian manifolds ($n \geq 2$) with $\mathrm{Ric}_{M_i} \geq -(n-1)$, and (Y, y, υ) a pointed proper metric space (i.e. every bounded subset of Y is relatively compact) with a Radon measure υ on Y satisfying that $(M_i, m_i, \underline{\mathrm{vol}})$ converges to (Y, y, υ) with respect to the measured Gromov-Hausdorff topology. Here $\underline{\mathrm{vol}}$ is the renormalized Riemannian volume of (M_i, m_i): $\underline{\mathrm{vol}} = \mathrm{vol}/\mathrm{vol}\, B_1(m_i)$. Fix $R > 0$, a sequence $\{f\}_{1\leq i<\infty}$ of Lipschitz functions f_i on $B_R(m_i) = \{w \in M_i; \overline{w, m_i} < R\}$, and a Lipschitz function f_∞ on $B_R(y)$ with $\sup_i \mathbf{Lip} f_i < \infty$. Here $\overline{w, m_i}$ is the distance between w and m_i, $\mathbf{Lip} f_i$ is the Lipschitz constant of f_i. Then we say that f_i *converges to* f_∞ on $B_R(y)$ if $f_i(x_i) \to f_\infty(x_\infty)$ for every $x_i \in B_R(m_i)$ and every $x_\infty \in B_R(y)$ satisfying that x_i converges to x_∞. See section 2 for these precise definitions. Assume that f_i converges to f_∞ on $B_R(y)$, below.

The purpose of this paper is to give a definition: *the differentials df_i of f_i converges to the differential df_∞ of f_∞* in this setting. In order to give the definition below, we shall recall celebrated works on limit spaces of Riemannian manifolds by Cheeger-Colding. By [1] and [6], it is known that the cotangent bundle T^*Y of Y exists. We remark that each fiber T^*_wY is a finite dimensional real vector space with canonical inner product $\langle \cdot, \cdot \rangle(w)$

2000 *Mathematics Subject Classification.* Primary 53C20; Secondary 53C43.

Key words and phrases. Gromov-Hausdorff convergence, geometric measure theory, Ricci curvature, Lipschitz functions, harmonic functions.

for a.e. $w \in Y$, and that every Lipschitz function g on $B_R(y)$ has the canonical differential section: $dg(w) \in T^*_w Y$ for a.e. $w \in B_R(y)$. See section 4 in [1], and section 6 in [6] for the details.

We shall give the definition of a convergence of the differentials of Lipschitz functions (see Definition 4.15):

DEFINITION 1.1 (Convergence of the differentials of Lipschitz functions). We say that *df_i converges to df_∞ on $B_R(y)$* if for every $\epsilon > 0$, every $x_\infty \in B_R(y)$, every $z_\infty \in Y$, every sequence $\{x_i\}_{1 \le i < \infty}$ of points $x_i \in B_R(m_i)$ satisfying that x_i converges to x_∞, and every sequence $\{z_i\}_{1 \le i < \infty}$ of points $z_i \in M_i$ satisfying that z_i converges to z_∞, there exists $r > 0$ such that

$$\limsup_{i \to \infty} \left| \frac{1}{\underline{\mathrm{vol}}\, B_t(x_i)} \int_{B_t(x_i)} \langle dr_{z_i}, df_i \rangle d\underline{\mathrm{vol}} - \frac{1}{\upsilon(B_t(x_\infty))} \int_{B_t(x_\infty)} \langle dr_{z_\infty}, df_\infty \rangle d\upsilon \right| < \epsilon$$

and

$$\limsup_{i \to \infty} \frac{1}{\underline{\mathrm{vol}}\, B_t(x_i)} \int_{B_t(x_i)} |df_i|^2 d\underline{\mathrm{vol}} \le \frac{1}{\upsilon(B_t(x_\infty))} \int_{B_t(x_\infty)} |df_\infty|^2 d\upsilon + \epsilon$$

for every $0 < t < r$. Here r_{z_i} is the distance function from z_i: $r_{z_i}(w) = \overline{z_i, w}$.

Roughly speaking, this convergence: $df_i \to df_\infty$, implies *"$H_{1,2}$ (or $H_{1,p}$)-convergence with respect to the measured Gromov-Hausdorff topology"*. See Theorem 1.2 and Remark 4.23. If df_i converges to df_∞ on $B_R(y)$, then we denote it by $(f_i, df_i) \to (f_\infty, df_\infty)$ on $B_R(y)$. Assume $(f_i, df_i) \to (f_\infty, df_\infty)$ and $(g_i, dg_i) \to (g_\infty, dg_\infty)$ on $B_R(y)$ below.

In the paper, we will study several properties of the convergence and give their applications. For example, we will show the following in section 4:

THEOREM 1.2. *Let $\{F_i\}_{1 \le i \le \infty}$ be a sequence of continuous functions on $\mathbf{R}$. Assume that F_i converges to F_∞ with respect to the compact uniformly topology. Then, we have*

$$\lim_{i \to \infty} \int_{B_R(m_i)} F_i(\langle df_i, dg_i \rangle) d\underline{\mathrm{vol}} = \int_{B_R(y)} F_\infty(\langle df_\infty, dg_\infty \rangle) d\upsilon.$$

Especially, if $f_\infty = g_\infty$, then

$$\lim_{i \to \infty} \int_{B_R(m_i)} F_i(|df_i - dg_i|) d\underline{\mathrm{vol}} = F_\infty(0)\upsilon(B_R(y)).$$

See Corollary 4.20 for the proof. We will also show the following in section 4:

THEOREM 1.3. *Let $\{h_i\}_{1 \le i < \infty}$ be a sequence of harmonic functions h_i on $B_R(m_i)$, and h_∞ a Lipschitz function on $B_R(y)$. Assume that $\sup_i \mathbf{Lip}\, h_i < \infty$ and that h_i converges to h_∞ on $B_R(y)$. Then we have $(h_i, dh_i) \to (h_\infty, dh_\infty)$ on $B_R(y)$.*

We remark that in Theorem 1.3, h_∞ is a harmonic function on $B_R(y)$, proved in [11] by Ding. We will give an alternative proof of it in section 4. See Corollary 4.34.

The organization of this paper is as follows:

In the next section, we will recall several important notions and propeties of metric spaces, Riemannian manifolds and their limit spaces. Most of statements in section 2 do not have the proof, we will give a reference for them only.

In section 3, we will show several results about rectifiability of limit spaces of Riemannian manifolds. See Theorem 3.16 and Theorem 3.49. It is important that their functions in these theorems which give a rectifiability of limit spaces, are *distance functions*. As a corollary of them, we will give an explicit geometric formula for the radial derivative of Lipschitz functions from a given point. See Theorem 3.30. These results are used in section 4 essentially.

In section 4, we will give two-definitions of pointwise convergence of L^∞-functions with respect to the measured Gromov-Hausdorff topology, and give the definition of a convergence of the differentials of Lipschitz functions again via the definitions of convergence of L^∞-functions. We will also give several properties of the convergence. The main properties are Theorem 4.17, Theorem 4.24 and Corollary 4.32.

Finally, we shall introduce several applications of this paper. In [24], we will give an application of this section 4 to a study of harmonic functions with polynomial growth on asymptotic cones of non-negatively Ricci curved manifolds having Euclidean volume growth. For example, we will show that a space of harmonic functions on asymptotic cones with polynomial growth of a fixed rate is a finite dimensional vector space. We can regard it as *asymptotic cones version* of the conjecture [9, Conjecture 0.1] by Yau. Moreover, in [24], we will give "Laplacian comparison theorems on limit spaces of Riemannian manifolds" by using several results given in section 4, and show a stability of lower bounds on Ricci curavture with respect to the Gromov-Hausdorff topology as a corollary of them. In [25], we will also give a geometric application by using several results in this section 4, to limit spaces of Riemannian manifolds with Ricci curvature bounded below.

Acknowledgments. The author would like to express his deep gratitude to Professor Kenji Fukaya and Professor Tobias Holck Colding for warm encouragement and their numerous suggestions and advice. He is grateful to Professor Takashi Shioya for giving many valuable suggestions. He wishes to thank the referees for valuable suggestions, comments and for pointing out a valuable reference [27]. This work was done during the stay at MIT, he also thanks to them and all members of Informal Geometry Seminar in MIT for warm hospitality and for giving nice environment. He was supported by Grant-in-Aid for Research Activity Start-up 22840027 from JSPS. He was also supported by GCOE 'Fostering top leaders in mathematics', Kyoto University.

2 Background

Our aim in this section is to give several notation, important notions and properties for metric measure spaces and manifolds. For a positive number $\epsilon > 0$ and real numbers a, b, we use the following notations:

$$a = b \pm \epsilon \Longleftrightarrow |a - b| < \epsilon.$$

We denote by $\Psi(\epsilon_1, \epsilon_2, \ldots, \epsilon_k; c_1, c_2, \ldots, c_l)$ (more simply, Ψ) some positive function on $\mathbf{R}^k_{>0} \times \mathbf{R}^l$ satisfying

$$\lim_{\epsilon_1, \epsilon_2, \ldots, \epsilon_k \to 0} \Psi(\epsilon_1, \epsilon_2, \ldots, \epsilon_k; c_1, c_2, \ldots, c_l) = 0$$

for each fixed real numbers $c_1, c_2, \ldots, c_l$. We often denote by $C(c_1, c_2, \ldots, c_l)$ some positive constant depending only on fixed real numbers $c_1, c_2, \ldots, c_l$.

2.1 Metric measure spaces

For a metric space Z, a point $z \in Z$ and positive numbers r, R with $r < R$, we use the following notations: $B_r(z) = \{x \in Z; \overline{z, x} < r\}, \overline{B}_r(z) = \{x \in Z; \overline{z, x} \le r\}, \partial B_r(z) = \{x \in Z; \overline{z, x} = r\}$. Here $\overline{y, x}$ is the distance between y and x, we often denote the distance by $d_Z(y, x)$. For every subset A of Z, we also put $B_r(A) = \{x \in Z; \overline{A, w} < r\}$ and $\overline{B}_r(A) = \{x \in Z; \overline{A, x} \le r\}$. For $z \in Z$, we define an 1-Lipschitz function r_z on Z by $r_z(w) = \overline{z, w}$. For a Lipschitz function f on Z and a point $z \in Z$ which is not isolated in Z, we put

$$lip f(z) = \liminf_{r \to 0} \left(\sup_{x \in B_r(z) \setminus \{z\}} \frac{|f(x) - f(z)|}{\overline{x, z}} \right), \mathrm{Lip} f(z) = \limsup_{r \to 0} \left(\sup_{x \in B_r(z) \setminus \{z\}} \frac{|f(x) - f(z)|}{\overline{x, z}} \right).$$

If z is an isolated point in Z, then we put $lip f(z) = \mathrm{Lip} f(z) = 0$. We also denote the Lipschitz constant of f by $\mathbf{Lip} f$. We remark that for every subset A of Z and every Lipschitz function f on A, there exists a Lipschitz function f^* on Z such that $f^*|_A = f$ and $\mathbf{Lip} f^* = \mathbf{Lip} f$. See for instance (8.2) in [2].

We say that Z is *proper* if every bounded subset of Z is relatively compact. We also say that Z is a *geodesic space* if for every $x_1, x_2 \in Z$, there exists an isometric embedding γ from $[0, \overline{x_1, x_2}]$ to Z such that $\gamma(0) = x_1, \gamma(\overline{x_1, x_2}) = x_2$. γ is called a *minimal geodesic from* x_1 *to* x_2. For a proper geodesic space W and a point w in W, we put $C_w = \{z \in W; \overline{w, z} + \overline{z, x} > \overline{w, x}$ for every $x \in W \setminus \{z\}\}$ (if W is a single point, then we put $C_w = \emptyset$), and call it *the cut locus of* W *at* w.

For a proper metric space Z and a Radon measure υ on Z, we say that the pair (Z, υ) is a *metric measure space* in this paper. For a metric measure space (Z, υ), a point z in

Z and a nonnegative integer k, we say that υ *is Ahlfors k-regular at z* if there exist $r > 0$ and $C \geq 1$ such that $C^{-1} \leq \upsilon(B_t(z))/t^k \leq C$ for every $0 < t < r$. We shall introduce the notion of *υ-rectifiability* for metric measure spaces by Cheeger-Colding. See [6, Definition 5.3] and [6, Theorem 5.7]. For metric spaces X_1, X_2, a positive number δ with $\delta < 1$, and a bijection map f from X_1 to X_2, we say that f *is $(1 \pm \delta)$-bi-Lipschitz to* X_2 if f and f^{-1} are $(1+\delta)$-Lipschitz maps.

Definition 2.1 (Rectifiability for a Borel subset of metric measure spaces). For a metric measure space (Z, υ) and a Borel subset A of Z, we say that A *is υ-rectifiable* if there exist a positive integer m, a collection of Borel subsets $\{C_{k,i}\}_{1 \leq k \leq m, i \in \mathbf{N}}$ of A, and a collection of bi-Lipschitz embedding maps $\{\phi_{k,i} : C_{k,i} \to \mathbf{R}^k\}_{k,i}$ such that the following properties hold:

1. $\upsilon(A \setminus \bigcup_{k,i} C_{k,i}) = 0$
2. υ is Ahlfors k-regular at each $x \in C_{k,i}$.
3. For every k, $x \in \bigcup_{i \in \mathbf{N}} C_{k,i}$ and every $0 < \delta < 1$, there exists $C_{k,i}$ such that $x \in C_{k,i}$ and that the map $\phi_{k,i}$ is $(1 \pm \delta)$-bi-Lipschitz to the image $\phi_{k,i}(C_{k,i})$.

Remark 2.2. The third $(1 \pm \delta)$-bi-Lipschitz condition in the above definition is important. Actually, the existence of the canonical inner product of the cotangent bundle of Ricci limit spaces follows from this property. See condition *iii*) of page 60 of [6] and section 6 in [6].

2.2 Gromov-Hausdorff convergence

For compact metric spaces X_1, X_2, we denote *the Gromov-Hausdorff distance between* X_1 *and* X_2 by $d_{GH}(X_1, X_2)$. See [17] for the definition. On the other hand, for compact metric spaces X_1, X_2, a positive number $\epsilon > 0$ and a map ϕ from X_1 to X_2, we say that ϕ is *an ϵ-Gromov-Hausdorff approximation* if $X_2 = B_\epsilon(\mathrm{Image}\phi)$ and $|\overline{x, y} - \overline{\phi(x), \phi(y)}| < \epsilon$ for every $x, y \in X_1$. For a sequence of compact metric spaces $\{X_i\}_{1 \leq i \leq \infty}$, we say that X_i *converges to* X_∞ if $d_{GH}(X_i, X_\infty)$ converges to 0. Then we denote it by $X_i \to X_\infty$. Similarly, for pointed compact metric spaces $(X_1, x_1), (X_2, x_2)$, we can define the *pointed Gromov-Hausdorff distance* $d_{GH}((X_1, x_1), (X_2, x_2))$. Moreover, for a sequence of pointed proper geodesic spaces $\{(Z_i, z_i)\}_{1 \leq i \leq \infty}$, we say that (Z_i, z_i) *converges to* (Z_∞, z_∞) if there exist sequences $\{\epsilon_i\}_i$, $\{R_i\}_i$ of positive numbers, and $\{\phi_i\}_i$ of Borel maps ϕ_i from $(B_{R_i}(z_i), z_i)$ to $(B_{R_i}(z_\infty), z_\infty)$ such that $\epsilon_i \to 0$, $R_i \to \infty$ as $i \to \infty$, $B_{R_i}(z_\infty) \subset B_{\epsilon_i}(\mathrm{Image}\phi_i)$ and $|\overline{\alpha, \beta} - \overline{\phi_i(\alpha), \phi_i(\beta)}| \leq \epsilon_i$ for every $\alpha, \beta \in B_{R_i}(x_i)$. We denote it by $(Z_i, z_i) \overset{(\phi_1, R_i, \epsilon_i)}{\to} (Z_\infty, z_\infty)$, or more simply, $(Z_i, z_i) \to (Z_\infty, z_\infty)$. It is easy to check that $(Z_i, z_i) \to (Z_\infty, z_\infty)$ if and only if $d_{GH}((\overline{B}_R(z_i), z_i), (\overline{B}_R(z_\infty), z_\infty)) \to 0$ for every $R > 0$. For a sequence $\{x_i\}_{1 \le i \le \infty}$ of points $x_i \in Z_i$, we say that x_i *converges to* x_∞ if $x_i \in B_{R_i}(z_i)$ and $\overline{\phi_i(x_i), x_\infty} \to 0$. Then, we denote it by $x_i \to x_\infty$.

Let $(Z_i, z_i) \to (Z_\infty, z_\infty)$. For a sequence $\{A_i\}_{1 \le i \le \infty}$ of subsets A_i of Z_i with $\sup_i \overline{z_i, A_i} < \infty$, we say that A_i *is included by* A_∞ *asymptotically* if for every $\epsilon > 0$, there exists i_0 such that $\phi_i(A_i) \subset B_\epsilon(A_\infty)$ for every $i \ge i_0$. Then we denote it by $\limsup^{GH}_{i \to \infty} A_i \subset A_\infty$ (if $A_\infty = \emptyset$, then $\limsup^{GH}_{i \to \infty} A_i \subset A_\infty$ implies $A_i = \emptyset$ for every sufficiently large i). Similarly, we also say that A_∞ *is included by* A_i *asymptotically* if for every $\epsilon > 0$, there exists i_0 such that $A_\infty \subset B_\epsilon(\phi_i(A_i))$ for every $i \ge i_0$. Then we denote it by $A_\infty \subset \liminf^{GH}_{i \to \infty} A_i$. Let $C_\infty \subset \liminf^{GH}_{i \to \infty} C_i$. For a sequence $\{f_i\}_{1 \le i \le \infty}$ of Lipschitz functions f_i on C_i with $\sup_i \mathbf{Lip}\, f_i < \infty$, we say that f_∞ *is a restriction of* f_i *asymptotically* if $\lim_{i \to \infty} f_{n(i)}(w_{n(i)}) = f_\infty(w)$ for every $w \in C_\infty$, every subsequence $\{n(i)\}_i$ of $\mathbf{N}$, and every $w_{n(i)} \in C_{n(i)}$ with $\overline{\phi_{n(i)}(w_{n(i)}), w} \to 0$. Let $\limsup_{i \to \infty} D_i \subset D_\infty$ and assume that D_∞ is compact. For a sequence $\{g_i\}_{1 \le i \le \infty}$ of Lipschitz function g_i on D_i with $\sup_i \mathbf{Lip}\, g_i < \infty$, we say that g_∞ *is an extension of* g_i *asymptotically* if $\lim_{i \to \infty} g_{n(i)}(w_{n(i)}) = g_\infty(w)$ for every $w \in D_\infty$, every subsequence $\{n(i)\}_i$ of $\mathbf{N}$, and every $w_{n(i)} \in D_{n(i)}$ with $\overline{\phi_{n(i)}(w_{n(i)}), w} \to 0$.

For a sequence $\{K_i\}_{1 \le i \le \infty}$ of compact subsets K_i of Z_i, we say that (Z_i, z_i, K_i) *converges to* $(Z_\infty, z_\infty, K_\infty)$ if $\limsup^{GH}_{i \to \infty} K_i \subset K_\infty$ and $K_\infty \subset \liminf^{GH}_{i \to \infty} K_i$ hold. Then we denote it by $(Z_i, z_i, K_i) \stackrel{(\phi_i, R_i, \epsilon_i)}{\to} (Z_\infty, z_\infty, K_\infty)$, or more simply, $(Z_i, z_i, K_i) \to (Z_\infty, z_\infty, K_\infty)$, or $K_i \to K_\infty$.

Let $(Z_i, z_i, K_i) \to (Z_\infty, z_\infty, K_\infty)$. For sequences $\{f_i^1\}_{1 \le i \le \infty}, \ldots, \{f_i^k\}_{1 \le i \le \infty}$ of Lipschitz functions f_i^l on K_i with $\sup_{i,l}(\mathbf{Lip} f_i^l + |f_i^l|_{L^\infty}) < \infty$, we say that $(Z_i, z_i, K_i, f_i^1, \ldots, f_i^k)$ *converges to* $(Z_\infty, z_\infty, K_\infty, f_\infty^1, \ldots, f_\infty^k)$ if f_∞^l is an extension of $\{f_i^l\}_i$ asymptotically for every l. We denote it by $(Z_i, z_i, K_i, f_i^1, \ldots, f_i^k) \to (Z_\infty, z_\infty, K_\infty, f_\infty^1, \ldots, f_\infty^k)$, or more simply, $f_i^l \to f_\infty^l$ for every l. Then it is easy to check that $\lim_{i \to \infty} |f_i^l - f_\infty^l \circ \phi_i|_{L^\infty(K_i)} = 0$.

It is not difficult to check the following proposition:

Proposition 2.3. *Let $\{(Z_i, z_i)\}_{1 \le i \le \infty}$ be a sequence of pointed proper geodesic spaces, Λ a set and $\{A_i^\lambda\}_{\lambda \in \Lambda}$ a collection of bounded subsets of Z_i for every $1 \le i \le \infty$. Assume that (Z_i, z_i) converges to (Z_∞, z_∞), A_∞^λ is compact for every $\lambda \in \Lambda$ and that $\limsup^{GH}_{i \to \infty} A_i^\lambda \subset A_\infty^\lambda$ for every $\lambda \in \Lambda$. Then, we have $\limsup^{GH}_{i \to \infty} \bigcap_{\lambda \in \Lambda} A_i^\lambda \subset \bigcap_{\lambda \in \Lambda} A_\infty^\lambda$ and $\limsup^{GH}_{i \to \infty}(A_i \setminus B_r(x_i)) \subset A_\infty \setminus B_r(x_\infty)$ for every $r > 0$ and every sequence $\{x_i\}_i$ of points x_i in Z_i with $x_i \to x_\infty$.*

We shall recall a fundamental covering lemma for proper metric spaces. See chapter 1 in [38] for the proof.

Proposition 2.4. *Let X be a proper metric space, A a subset of X, Λ a set, $\{x_\lambda\}_{\lambda \in \Lambda}$ a collection of points in X and $\{r_\lambda\}_{\lambda \in \Lambda}$ a collection of positive numbers. Assume that for* *every $x \in A$ and every $\epsilon > 0$, there exists $\lambda \in \Lambda$ such that $x \in \overline{B}_{r_\lambda}(x_\lambda)$ and* $\operatorname{diam} \overline{B}_{r_\lambda}(x_\lambda) < \epsilon$*. Then, there exists a countable subset Λ_1 of Λ such that the following properties hold:*

1. *$\{\overline{B}_{r_{\lambda_1}}(x_{\lambda_1})\}_{\lambda_1 \in \Lambda_1}$ are pairwise disjoint collection.*
2. *We have*
$$A \setminus \bigcup_{\lambda_2 \in \Lambda_2} \overline{B}_{r_{\lambda_2}}(x_{\lambda_2}) \subset \bigcup_{\lambda \in \Lambda_1 \setminus \Lambda_2} \overline{B}_{5 r_\lambda}(x_\lambda)$$
for every finite subset Λ_2 of Λ_1.

We shall recall the definition of measured Gromov-Hausdorff convergence. Let $(Z_i, z_i) \to (Z_\infty, z_\infty)$. For a sequence $\{\upsilon_i\}_{1 \le i \le \infty}$ of Radon measures υ_i on Z_i, we say that (Z_i, z_i, υ_i) *converges to $(Z_\infty, z_\infty, \upsilon_\infty)$ with respect to the measured Gromov-Hausdorff topology* if $\lim_{i \to \infty} \upsilon_i(B_r(x_i)) = \upsilon_\infty(B_r(x_\infty))$ for every $r > 0$ and every sequence $\{x_i\}_i$ of points x_i in Z_i with $x_i \to x_\infty$. Then we denote it by $(Z_i, z_i, \upsilon_i) \to (Z_\infty, z_\infty, \upsilon_\infty)$. The next proposition is used many times in this paper. We skip the proof because it is not difficult to check it by using Proposition 2.4.

Proposition 2.5. *Let $\{(Z_i, z_i, \upsilon_i)\}_{1 \le i \le \infty}$ be a sequence of pointed proper geodesic spaces with Radon measures, and $\{A_i\}_{1 \le i \le \infty}$ a sequence of Borel subsets A_i of Z_i. Assume that $\upsilon_i(B_1(z_i)) = 1$, A_∞ is compact, $(Z_i, z_i, \upsilon_i) \to (Z_\infty, z_\infty, \upsilon_\infty)$, $\limsup^{GH}_{i \to \infty} A_i \subset A_\infty$ and that for every $R > 0$ there exists $\kappa = \kappa(R) \ge 1$ such that $\upsilon_i(B_{2r}(x_i)) \le 2^\kappa \upsilon_i(B_r(x_i))$ for every $0 < r < R$, every $1 \le i \le \infty$ and every $x_i \in Z_i$. Then we have*
$$\limsup_{i \to \infty} \upsilon_i(A_i) \le \upsilon_\infty(A_\infty).$$

We shall give a proof of the following proposition:

Proposition 2.6. *Let $\{(Z_i, z_i, \upsilon_i)\}_{1 \le i \le \infty}$ be a sequence of pointed proper geodesic spaces with Radon measures. Assume that $\upsilon_i(B_1(z_i)) = 1$ for every i, $\operatorname{diam} Z_\infty > 0$, $(Z_i, z_i, \upsilon_i) \stackrel{(\phi_i, R_i, \epsilon_i)}{\to} (Z_\infty, z_\infty, \upsilon_\infty)$, and that for every $R > 0$, there exists $\kappa = \kappa(R) \ge 1$ such that $\upsilon_i(B_{2r}(x_i)) \le 2^\kappa \upsilon_i(B_r(x_i))$ for every $0 < r < R$, every $1 \le i \le \infty$ and every $x_i \in Z_i$. Then, we have*
$$\lim_{i \to \infty} \sup_{x_i \in B_R(z_i), 0 < r < R} |\upsilon_i(B_r(x_i)) - \upsilon_\infty(B_r(\phi_i(x_i)))| = 0$$
for every $R \ge 1$.

Proof. It is easy to check that $\operatorname{rad} Z_\infty > 0$. Here $\operatorname{rad} X = \inf_{x_2 \in X}(\sup_{x_1 \in X} \overline{x_1, x_2})$ for a metric space X. Put $\kappa = \kappa(100R)$. Let $\tau > 0$ with $\tau << \operatorname{rad} Z_\infty$. Then, there exists N such that for every $N \le i \le \infty$ and every $w \in Z_i$, there exists $\hat{w} \in Z_i$ such that $\overline{w, \hat{w}} = \tau$. Since $B_\delta(w) \subset B_{\tau+\delta}(\hat{w}) \setminus B_{\tau-\delta}(\hat{w})$ for every $0 < \delta < \tau$, by [10, Lemma 3.3], there exists $\hat{\tau} << \tau$ such that $\upsilon_i(B_t(w)) \le \Psi(t; \kappa, R)\upsilon_i(B_{10\tau}(w))$ for every $N \le i \le \infty$, every $w \in Z_i$ and every $0 < t < \hat{\tau}$. Fix $\epsilon > 0$. Then, there exist $N_1 \in \mathbf{N}$ and $0 < r_1 << \min\{R, \hat{\tau}, \epsilon, 1\}$ such that $\upsilon_i(B_s(z)) \le \epsilon$ for every $N_1 \le i \le \infty$, every $0 < s < r_1$ and every $z \in B_R(z_i)$. Let $\{x_j\}_{1 \le j \le l} \subset B_R(z_\infty)$ and $\{t_j\}_{1 \le j \le \hat{l}} \subset [0, R]$ satisfying that $B_R(z_\infty) \subset \bigcup_{j=1}^{l} B_{\epsilon r_1}(x_j)$ and $[0, R] \subset \bigcup_{j=1}^{\hat{l}} B_{\epsilon r_1}(t_j)$. Let $x_j(i) \in B_R(z_i)$ with $x_j(i) \to x_j$. There exists $N_2 \ge N_1$ such that $|\upsilon_i(B_{t_{\hat{j}}}(x_j(i))) - \upsilon_\infty(B_{t_{\hat{j}}}(x_j))| < \epsilon$ for every $i \ge N_2$, every $1 \le j \le l$ and every $1 \le \hat{j} \le \hat{l}$. Fix $z \in B_R(z_\infty)$ and $s \in [r_1, R]$. Let $j \in \{1, \ldots, l\}$ and $\hat{j} \in \{1, \ldots, \hat{l}\}$ satisfying that $\overline{z, x_j} < \epsilon r_1$ and $|s - t_{\hat{j}}| < \epsilon r_1$. Then, by [10, Lemma 3.3], we have $|\upsilon_\infty(B_s(z)) - \upsilon_\infty(B_{t_{\hat{j}}}(x_j))| \le \upsilon_\infty(B_{s+5\epsilon r_1}(z)) - \upsilon_\infty(B_{s-5\epsilon r_1}(z)) \le \Psi(\epsilon; \kappa, R, \tau)\upsilon_\infty(B_R(z_\infty))$ On the other hand, for a sequence $\{z(i)\}_i$ of points $z(i)$ in $B_R(z_i)$ with $z(i) \to z$, $|\upsilon_i(B_s(z(i))) - \upsilon_i(B_{t_{\hat{j}}}(x_j(i)))| \le \upsilon_i(B_{s+10\epsilon r_1}(z(i))) - \upsilon_i(B_{s-10\epsilon r_1}(z(i))) \le \Psi(\epsilon; \kappa, R, \tau)\upsilon_i(B_R(z_i)) \le \Psi(\epsilon; \kappa, R, \tau)\upsilon_\infty(B_R(z_\infty))$ for every $i \ge N_2$. Thus, we have $|\upsilon_i(B_s(z(i))) - \upsilon_\infty(B_s(z))| < \Psi(\epsilon; \kappa, R, \tau)\upsilon_\infty(B_R(z_\infty))$ for every $i \ge N_2$. Therefore, we have the assertion. $\square$

2.3 Riemannian manifolds and their limit spaces

For a real number K and a pointed proper geodesic space (Y, y), in this paper, we say that (Y, y) *is a* (n, K)*-Ricci limit space* if there exist sequences of real numbers $\{K_i\}_i$, and of pointed n-dimensional complete Riemannian manifolds $\{(M_i, m_i)\}_i$ with $\mathrm{Ric}_{M_i} \ge K_i(n-1)$ such that $K_i \to K$ and $(M_i, m_i) \to (Y, y)$. Similarly, for a pointed proper geodesic space with Radon measure (Y, y, υ), we also say that (Y, y, υ) is *a* (n, K)*-Ricci limit space (of* $\{(M_i, m_i, \underline{\mathrm{vol}})\}_i$*)* if $(M_i, m_i, \underline{\mathrm{vol}}) \to (Y, y, \upsilon)$ as above. More simply, for a $(n, -1)$-Ricci limit space (Y, y) (or (Y, y, υ)), we say that (Y, y) *is a Ricci limit space.* See for instance section 4.1 in [34]. We shall fix a Ricci limit space (Y, y, υ) in this subsection and give a very short review of structure theory of Ricci limit spaces developed by Cheeger-Colding, below. See [4, 5, 6] for the details.

For pointed proper geodesic spaces (Z, z) and (X, x), we say that (Z, z) *is a tangent cone of* X *at* x if there exists a sequence of positive numbers $\{r_i\}_i$ such that $r_i \to 0$ and $(X, x, r_i^{-1}d_X) \to (Z, z)$. For $k \ge 1$, we put $\mathcal{R}_k(Y) = \{x \in Y;$ All tangent cones at x are isometric to $\mathbf{R}^k\}$ and call it *the* k*-dimensional regular set.* More simply, we shall denote it by $\mathcal{R}_k$. We also put $\mathcal{R} = \bigcup_{1 \le k \le n} \mathcal{R}_k$ and call it *the regular set.* Then we have $\upsilon(Y \setminus \mathcal{R}) = 0$. See [4, Theorem 2.1] for the proof. For $\delta, r > 0$ and $0 < \alpha < 1$, we put $(\mathcal{R}_k)_{\delta,r} = \{x \in Y; d_{GH}((\overline{B}_s(x), x), (\overline{B}_s(0_k), 0_k)) \le \delta s$ for every $0 < s \le r\}$ and $(\mathcal{R}_{k;\alpha})_r = \{x \in Y; d_{GH}((\overline{B}_s(x), x), (\overline{B}_s(0_k), 0_k)) \le s^{1+\alpha}$ for every $0 < s \le r\}$. Here $0_k \in \mathbf{R}^k$. We remark that $(\mathcal{R}_k)_{\delta,r}$ and $(\mathcal{R}_{k;\alpha})_r$ are closed, $\bigcap_{\delta>0} \left(\bigcup_{r>0}(\mathcal{R}_k)_{\delta,r}\right) = \mathcal{R}_k$. We also put $\mathcal{R}_{k;\alpha} = \bigcup_{r>0}(\mathcal{R}_{k;\alpha})_r$. By [4, Theorem 3.23] and [4, Theorem 4.6], there exists

$0 < \alpha(n) < 1$ such that $\upsilon(\mathcal{R}_k \setminus \mathcal{R}_{k;\alpha(n)}) = 0$ and that υ is Ahlfors k-regular at every point in $\mathcal{R}_{k;\alpha(n)}$ for every k.

On the other hand, it is known that Y is υ-rectifiable. See [6, Theorem 5.5] and [6, Theorem 5.7]. Thus, by section 6 in [6] or section 4 in [2], the cotangent bundle T^*Y of Y exists. We will give several fundamental properties of the cotangent bundle only:

1. T^*Y is a topological space.
2. There exists a Borel map $\pi : T^*Y \to Y$ such that $\upsilon(Y \setminus \pi(T^*Y)) = 0$.
3. $\pi^{-1}(w)$ is a finite dimensional real vector space with canonical inner product $\langle \cdot, \cdot \rangle(w)$ for every $w \in \pi(T^*Y)$.
4. For every open subset U of Y and every Lipschitz function f on U, there exist a Borel subset V of U, and a Borel map df (called *the differential section of f* or *the differential of f*) from V to T^*Y such that $\upsilon(U \setminus V) = 0$ and that $\pi \circ df(w) = w$, $|df|(w) = \mathrm{Lip}f(w) = lipf(w)$ for every $w \in V$, where $|v|(w) = \sqrt{\langle v, v \rangle(w)}$.

We call $\{\langle \cdot, \cdot \rangle(w)\}_{w \in \pi(T^*Y)}$ *the Riemannian metric of Y* and denote it by $\langle \cdot, \cdot \rangle$. Finally, we remark that $\upsilon(C_x) = 0$ for every $x \in Y$. See [22, Theorem 3.2]. These results above are used in section 3, essentially.

3 Rectifiability on limit spaces

In this section, we shall study a rectifiability of Ricci limit spaces. These results given in this section are used in section 4, essentially.

3.1 Radial rectifiability

The main result in this subsection is Theorem 3.16.

Lemma 3.1. *Let Z be a proper geodesic space, z a point in Z, s, δ positive numbers, υ a Radon measure on Z and F a nonnegative valued Borel function on $B_s(m)$. Assume that*

$$\frac{1}{\upsilon(B_s(z))}\int_{B_s(z)} F d\upsilon \le \delta$$

and that there exists $\kappa \ge 1$ such that $0 < \upsilon(B_{2t}(w)) \le 2^\kappa \upsilon(B_t(w))$ for every $w \in B_s(z)$ and every $0 < t \le s$. Then, there exists a compact subset K of $\overline{B}_{s/10^2}(z)$ such that $\upsilon(K)/\upsilon(B_{s/10^2}(z)) \ge 1 - \Psi(\delta; \kappa)$ and

$$\frac{1}{\upsilon(B_t(x))}\int_{B_t(x)} F d\upsilon \le \Psi(\delta; \kappa)$$

for every $x \in K$ *and every* $0 < t \leq s/10^2$.

Proof. Without loss of generality, we can assume that F is a nonnegative valued Borel function on Z by defining $F \equiv 0$ on $Z \setminus B_s(z)$. Fix $C > 0$ and put $A_1(C) = \{w \in B_s(z); \int_{B_{s/10^2}(w)} F dv \geq C v(B_{s/10^2}(w))\}$. Let $\{x_j^1\}_{1 \leq j \leq k_1}$ be an $s/10$-maximal separated subset of $A_1(C)$. Put $A_2(C) = \{w \in B_s(m) \setminus \bigcup_{i=1}^{k_1} B_s(x_i^1); \int_{B_{s/10^3}(w)} F dv \geq C v(B_{s/10^3}(w))\}$. Let $\{x_j^2\}_{1 \leq j \leq k_2}$ be an $s/10^2$-maximal separated subset of $A_2(C)$. By iterating this argument, put $A_l(C) = \{w \in B_s(m) \setminus \bigcup_{1 \leq j \leq l-1,\ 1 \leq i \leq k_j} B_{s/10^{l-2}}(x_i^{l-1}); \int_{B_{s/10^{l+1}}(w)} F dv \geq C v(B_{s/10^{l+1}}(w))\}$. Let $\{x_j^l\}_{1 \leq j \leq k_l}$ be an $s/10^l$-maximal separated subset of $A_l(C)$.

Claim 3.2. *The collection* $\{\overline{B}_{s/10^{l+1}}(x_i^l)\}_{i,l}$ *are pairwise disjoint.*

Let $w \in \overline{B}_{s/10^{\hat{l}+1}}(x_{\hat{i}}^{\hat{l}}) \cap \overline{B}_{s/10^{l+1}}(x_i^l)$. Assume that $l < \hat{l}$. Then, by the construction, we have $x_{\hat{i}}^{\hat{l}} \in M \setminus \bigcup_{j=1}^{k_l} B_{s/10^{l-1}}(x_j^l)$. Especially, we have $\overline{x_{\hat{i}}^{\hat{l}}, x_i^l} \geq s/10^{l-1}$. Therefore, we have $\overline{B}_{s/10^{\hat{l}+1}}(x_{\hat{i}}^{\hat{l}}) \cap \overline{B}_{s/10^{l+1}}(x_i^l) = \emptyset$. This is a contradiction. Therefore, we have $l = \hat{l}$. By the definition, we have $i = \hat{i}$. Thus, we have Claim 3.2.

It is easy to check the following claim:

Claim 3.3. *We have* $\bigcup_{i \in \mathbf{N}} A_i(C) \subset \bigcup_{l \in \mathbf{N}, 1 \leq i \leq k_l} \overline{B}_{s/10^{l-2}}(x_i^l)$

We have

$$\sum_{l \in \mathbf{N}, 1 \leq i \leq k_l} \int_{B_{\frac{s}{10^{l+1}}}(x_i^l)} F dv \geq C \sum_{l \in \mathbf{N}, 1 \leq i \leq k_l} v(B_{\frac{s}{10^{l+1}}}(x_i^l))$$

$$\geq C C(\kappa) \sum_{l \in \mathbf{N}, 1 \leq i \leq k_l} v(B_{\frac{s}{10^{l-2}}}(x_i^l)) \geq C C(\kappa) v\left(\bigcup_{l \in \mathbf{N}, 1 \leq i \leq k_l} B_{\frac{s}{10^{l-2}}}(x_i^l)\right).$$

On the other hand, we have

$$\sum_{l \in \mathbf{N}, 1 \leq i \leq k_l} \int_{B_{\frac{s}{10^{l+1}}}(x_i^l)} F dv = \int_{\bigcup_{l \in \mathbf{N}, 1 \leq i \leq k_l} B_{\frac{s}{10^{l+1}}}(x_i^l)} F dv \leq \int_{B_s(z)} F dv \leq C(\kappa) v(B_s(z)) \delta.$$

Therefore, we have

$$\frac{v\left(\bigcup_{l \in \mathbf{N}, 1 \leq i \leq k_l} B_{\frac{s}{10^{l-2}}}(x_i^l)\right)}{v(B_s(m))} \leq \frac{\delta}{C} C(\kappa).$$

By letting $C = \sqrt{\delta}$ and $K = \overline{B}_{s/10^2}(z) \setminus \bigcup_{l \in \mathbf{N}, 1 \leq i \leq k_l} B_{\frac{s}{10^{l-2}}}(x_i^l)$, we have the assertion. □

Let (Y, y) be a Ricci limit space, k an integer with $k \leq n$, and r, δ positive numbers with $r < 1$, $\delta < 1$. Let $(\mathcal{R}_k)_{\delta,r}^y$ be the set of points w in Y satisfying that for every $0 < s \leq r$, there exists a map Φ from $\overline{B}_s(w)$ to $\mathbf{R}^k$ such that $\pi_1 \circ \Phi = r_y$ and that Φ is an δs-Gromov-Hausdorff approximation to $\overline{B}_s(\Phi(w))$ Here, π_1 is the projection from $\mathbf{R}^k = \mathbf{R} \times \mathbf{R}^{k-1}$ to $\mathbf{R}$ defined by $\pi_1(x_1, \ldots, x_k) = x_1$.

LEMMA 3.4. *We have*

$$\bigcap_{\delta>0}\left(\bigcup_{r>0}((\mathcal{R}_k)^x_{\delta,r}\setminus C_x)\right)=\mathcal{R}_k\setminus C_x.$$

PROOF. It is easy to check that

$$\bigcap_{\delta>0}\left(\bigcup_{r>0}((\mathcal{R}_k)^x_{\delta,r}\setminus C_x)\right)\subset\mathcal{R}_k\setminus C_x.$$

Let $w\in\mathcal{R}_k\setminus C_x$. For every $\delta>0$, there exists $r>0$ such that for every $0<s<r$, there exists an δs-Gromov-Hausdorff approximation from $(\overline{B}_s(0_k),0_k)$ to $(\overline{B}_s(w),w)$. Here, $0_k\in\mathbf{R}^k$. On the other hand, by the splitting theorem on limit spaces [2, Theorem 9.27], there exist a pointed proper geodesic space (W_s,w_s) and a map $\hat{\Phi}$ from $(\overline{B}_s(w),w)$ to $(\overline{B}_s(0,w_s),(0,w_s))$ such that $\pi_{\mathbf{R}}\circ\hat{\Phi}=r_x-\overline{x,w}$ and that $\hat{\Phi}$ is an δs-Gromov-Hausdorff approximation. Here, $\overline{B}_s(0,w_s)\subset\mathbf{R}\times W_s$ with the product metric $\sqrt{d_{\mathbf{R}}^2+d_{W_s}^2}$, $\pi_{\mathbf{R}}$ is the projection from $\mathbf{R}\times W_s$ to $\mathbf{R}$. By rescaling $s^{-1}d_{\mathbf{R}^k}$ and [21, Claim 4.4], there exists an $\Psi(\delta;n)s$-Gromov-Hausdorff approximation f from $(\overline{B}_s(w_s),w_s)$ to $(\overline{B}_s(0_{k-1}),0_{k-1})$. Define a map g from $\overline{B}_s(w)$ to $\mathbf{R}^k$ by $g(z)=(\overline{x,z},f\circ\hat{\Phi})$. Let π_s be the canonical retraction from $\mathbf{R}^k$ to $\overline{B}_s(g(w))$. Put $\hat{g}=\pi_s\circ g$. Then, it is easy to check that $\hat{g}$ is an $\Psi(\delta;n)s$-Gromov-Hausdorff approximation to $(\overline{B}_s(\hat{g}(w)),\overline{g}(w))$. Since δ is arbitrary, we have the assertion. $\square$

Put $\mathcal{D}^\tau_x=\{w\in X;$ There exists $\alpha\in X$ such that $\overline{\alpha,w}\geq\tau$ and $\overline{x,w}+\overline{w,\alpha}=\overline{x,\alpha}\}$ for a proper geodesic space X, a point $x\in X$ and a positive number $\tau>0$. It is easy to check that $\mathcal{D}^\tau_x$ is closed. By the definition, we have $\bigcup_{\tau>0}\mathcal{D}^\tau_x=X\setminus C_x$. Let $\mathrm{Leb}\,A=\{a\in A;\lim_{r\to 0}\upsilon(B_r(a)\cap A)/\upsilon(B_r(a))=1\}$ for a metric measure space (X,υ) and a Borel subset A of X.

We shall give a fundamental result about rectifiability of limit spaces by distance functions. The essential idea of the proof is to replace harmonic functions giving rectifiability in [6, Theorem 3.26] with suitable distance functions via the Poincaré inequality.

LEMMA 3.5. *Let (Y,y,υ) be a Ricci limit space, k a positive integer satisfying $k\leq n$, δ,r positive numbers satisfying $\delta<1,r<1$, x a point in Y and w a point in $(\mathcal{R}_k)^x_{\delta,r}\cap\mathrm{Leb}((\mathcal{R}_k)_{\delta,r})\setminus(C_x\cup\{x\})$. Then, there exists $\eta(w)>0$ such that the following property holds: For every $0<s\leq\eta(w)$, there exist a compact subset L of $\overline{B}_s(w)\cap(\mathcal{R}_k)_{\delta,r}$ and a collection of points $\{x_j\}_{2\leq j\leq k}$ in Y such that $\upsilon(L)/\upsilon(B_s(w))\geq 1-\Psi(\delta;n)$ and that the map $\Phi=(r_x,r_{x_2},\dots,r_{x_k})$ from L to $\mathbf{R}^k$, is an $(1\pm\Psi(\delta;n))$-bi-Lipschitz equivalent to the image $\Phi(L)$.*

Proof. There exists $0 < \tau < r$ such that $w \in \mathcal{D}_x^\tau \setminus B_\tau(x)$ and $\upsilon(B_s(w) \cap (\mathcal{R}_k)_{\delta,r})/\upsilon(B_s(w)) \geq 1-\delta$ for every $0 < s < \tau$. Let $(M_i, m_i, \underline{\mathrm{vol}}) \to (Y, y, \upsilon)$, and let $\{x_i\}_i, \{w_i\}_i$ be sequences of points x_i, w_i in M_i satisfying that $w_i \to w$ and $x_i \to x$. Fix $0 < s << \min\{\delta, \tau\}$. Then, for every sufficiently large i, there exists an δs-Gromov-Hausdorff approximation $\Phi^i = (\Phi^i_1, \ldots, \Phi^i_k)$ from $(\overline{B}_s(w_i), w_i)$ to $(\overline{B}_s(0_k), 0_k)$ such that $\Phi^i_1 = r_{x_i} - r_{x_i}(w_i)$. Put $s_0 = \sqrt{\delta}s$. For convenience, we shall use the following notations for rescaled metrics $s_0^{-1}d_{M_i}, s_0^{-1}d_Y$: $\hat{\mathrm{vol}} = \mathrm{vol}^{s_0^{-1}d_{M_i}}$, $\hat{r}_w(\alpha) = s_0^{-1}r_w(\alpha)$, $\hat{B}_t(\alpha) = B_t^{s_0^{-1}d_{M_i}}(\alpha) = B_{s_0 t}(\alpha)$, $\hat{\upsilon} = \upsilon/\upsilon(B_{s_0}(y))$, $\hat{g} = s_0^{-1}g$ for a Lipschitz function g and so on. We also denote the differential section of g as rescaled manifolds $(M_i, s_0^{-1}d_{M_i})$ by $\hat{d}g : M_i \to T^*M_i$ and denote the Riemannian metric of $(M_i, s_0^{-1}d_{M_i})$ by $\langle\cdot,\cdot\rangle_{s_0} = s_0^{-2}\langle\cdot,\cdot\rangle$. We remark that $(M_i, m_i, s_0^{-1}d_{M_i}, \underline{\mathrm{vol}}^{s_0^{-1}d_{M_i}}) \to (Y, y, s_0^{-1}d_Y, \hat{\upsilon})$. The following claim follows from the proof of the splitting theorem on limit spaces (see for instance [2, Lemma 9.8], [2, Lemma 9.10] and [2, Lemma 9.13]).

Claim 3.6. *For every sufficiently large i, there exist collections of harmonic functions $\{\hat{\mathbf{b}}^i_j\}_{1\leq j\leq k}$ on $\hat{B}_{100^2}(w_i)$, and of points $\{x^i_j\}_{2\leq j\leq k}$ in $\hat{B}_{\sqrt{\delta}^{-1}}(w_i)$ such that $|\hat{\mathbf{b}}^i_j - \hat{r}_{x^i_j}|_{L^\infty(\hat{B}_{100^2}(w_i))} \leq \Psi(\delta; n)$,*

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{100^2}(w_i)}\int_{\hat{B}_{100^2}(w_i)} \left(|\hat{d}\hat{\mathbf{b}}^i_j - \hat{d}\hat{r}_{x^i_j}|^2_{s_0} + |\mathrm{Hess}_{\hat{\mathbf{b}}^i_j}|^2_{s_0}\right) d\hat{\mathrm{vol}} \leq \Psi(\delta; n),$$

and

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{100^2}(w_i)}\int_{\hat{B}_{100^2}(w_i)} |\langle \hat{d}\hat{\mathbf{b}}^i_j, \hat{\mathbf{b}}^i_l\rangle_{s_0}| d\hat{\mathrm{vol}} = \delta_{jl} \pm \Psi(\delta; n)$$

for every $1 \leq j \leq l \leq k$, where $x = x^i_1$ for every i.

Define a nonnegative valued Borel function F_i on $\hat{B}_{100^2}(w_i)$ by

$$F_i = \sum_{l=1}^k \hat{\mathrm{Lip}}(\hat{\mathbf{b}}^i_l - \hat{r}_{x^i_l})^2 + \sum_{l\neq j} |\langle \hat{d}\hat{\mathbf{b}}^i_l, \hat{d}\hat{\mathbf{b}}^i_j\rangle_{s_0}| + \sum_{l=1}^k |\mathrm{Hess}_{\hat{\mathbf{b}}^i_l}|^2_{s_0}.$$

By Lemma 3.1, for every sufficiently large i, there exists a compact subset K_i of $\overline{\hat{B}}_{100}(w_i)$ such that $\hat{\mathrm{vol}}\,K_i/\hat{\mathrm{vol}}\,\hat{B}_{100}(w_i) \geq 1 - \Psi(\delta; n)$ and

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)}\int_{\hat{B}_t(\alpha)} F_i d\hat{\mathrm{vol}} \leq \Psi(\delta; n)$$

for every $\alpha \in K_i$ and every $0 < t < 100$.

Claim 3.7. *For every sufficiently large i, every $\alpha \in K_i \cap \hat{B}_{50}(w_i)$, every $1 \leq j \leq k$, and every $0 < t < 50$, there exists a constant C^i_j such that $\hat{\mathbf{b}}^i_j = \hat{r}_{x^i_j} + C^i_j \pm \Psi(\delta; n)t$ on $\hat{B}_t(\alpha)$.*

The proof is as follows. By the Poincaré inequality, we have

$$\begin{aligned}&\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)}\int_{\hat{B}_t(\alpha)}\left|(\hat{\mathbf{b}}^i_j-\hat{r}_{x^i_j})-\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)}\int_{\hat{B}_t(\alpha)}(\hat{\mathbf{b}}^i_j-\hat{r}_{x^i_j})d\hat{\mathrm{vol}}\right|d\hat{\mathrm{vol}}\\ &\leq tC(n)\sqrt{\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)}\int_{\hat{B}_t(\alpha)}(\hat{\mathrm{Lip}}(\hat{\mathbf{b}}^i_1-\hat{r}_{x_i}))^2 d\hat{\mathrm{vol}}}\\ &\leq t\Psi(\delta;n).\end{aligned}$$

For $C>0$, let $A_j(C)$ be the set of points $\beta\in\hat{B}_t(\alpha)$ satisfying that

$$\left|(\hat{\mathbf{b}}^i_j(\beta)-\hat{r}_{x^i_j}(\beta))-\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)}\int_{\hat{B}_t(\alpha)}(\hat{\mathbf{b}}^i_j-\hat{r}_{x^i_j})d\hat{\mathrm{vol}}\right|\geq C.$$

Then, we have

$$\Psi(\delta;n)t\geq\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)}\int_{\hat{B}_t(\alpha)}\left|(\hat{\mathbf{b}}^i_j-\hat{r}_{x^i_j})-\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)}\int_{\hat{B}_t(\alpha)}(\hat{\mathbf{b}}^i_j-\hat{r}_{x^i_j})d\hat{\mathrm{vol}}\right|d\hat{\mathrm{vol}}\geq C\frac{\hat{\mathrm{vol}}\,A_j(C)}{\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)}.$$

Put $C=\sqrt{\Psi(\delta;n)}t$ for $\Psi(\delta;n)$ as above. Then we have $\hat{\mathrm{vol}}\,A_j(C)/\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)\leq\sqrt{\Psi(\delta;n)}$.

Assume that there exist $\beta\in\hat{B}_t(\alpha)$ and $\epsilon>0$ such that $\hat{B}_{\epsilon t}(\beta)\subset A_j(C)$. Then, by Bishop-Gromov volume comparison theorem, we have $C(n)\epsilon^n\leq\hat{\mathrm{vol}}\,B_{\epsilon t}(\beta)/\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)\leq\hat{\mathrm{vol}}\,A_j(C)/\hat{\mathrm{vol}}\,\hat{B}_t(\alpha)\leq\sqrt{\Psi(\delta;n)}$. Therefore, by letting $\epsilon=\left(2C(n)^{-1}\sqrt{\Psi(\delta;n)}\right)^{1/n}$, we have a contradiction.

Put $\epsilon=\left(2C(n)^{-1}\sqrt{\Psi(\delta;n)}\right)^{1/n}$. Let $\beta\in\hat{B}_t(\alpha)$ and $\hat{\beta}\in\hat{B}_{(1-\epsilon)t}(\alpha)$ with $\hat{r}_\beta(\hat{\beta})<\epsilon t$. Then, there exists $\gamma\in\hat{B}_{\epsilon t}(\hat{\beta})\setminus A_j(C)$. Especially, we have $\gamma\in\hat{B}_t(\alpha)$. By the definition of $A_j(C)$, we have

$$\hat{\mathbf{b}}^i_j(\gamma)=\hat{r}_{x^i_j}(\gamma)+\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{100}(\alpha)}\int_{\hat{B}_{100}(\alpha)}(\hat{\mathbf{b}}^i_j-\hat{r}_{x^i_j})d\hat{\mathrm{vol}}\pm\sqrt{\Psi(\delta;n)}t.$$

By Cheng-Yau's gradient estimate (see [7]), we have $|\hat{\nabla}\hat{\mathbf{b}}^i_j|_{s_0}\leq C(n)$. Thus, we have

$$\hat{\mathbf{b}}^i_j(\beta)=\hat{r}_{x^i_j}(\beta)+\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{100}(\alpha)}\int_{\hat{B}_{100}(\alpha)}(\hat{\mathbf{b}}^i_j-\hat{r}_{x^i_j})d\hat{\mathrm{vol}}\pm\Psi(\epsilon;n)t.$$

Therefore we have Claim 3.7.

By an argument similar to the proof of [6, Theorem 3.3], we have the following:

Claim 3.8. *For every sufficiently large i, every $\alpha\in K_i\cap\hat{B}_{50}(w_i)$ and every $0<t\leq 10^{-5}$, there exist a compact subset Z_t of M_i, a point z_t in Z_t and a map ϕ from $(\overline{\hat{B}}_t(\alpha),\alpha)$ to $(\overline{\hat{B}}_t(z_t),z_t)$ such that the map $\Phi=(\hat{\mathbf{b}}^i_1,\ldots,\hat{\mathbf{b}}^i_k,\phi)$ from $\overline{\hat{B}}_t(\alpha)$ to $\overline{\hat{B}}_{t+\Psi(\delta;n)t}(\Phi(\alpha))\subset\left(\mathbf{R}^k\times Z_t,\sqrt{d^2_{\mathbf{R}^k}+({s_0}^{-1}d_{M_i})^2}\right)$, is an $\Psi(\delta;n)t$-Gromov-Hausdorff approximation.*

Put $\hat{K}_i = K_i \cap \hat{\overline{B}}_{40}(w_i)$. Then, we have $\hat{\mathrm{vol}}\, K_i / \hat{\mathrm{vol}}\, \hat{B}_{40}(w_i) \geq 1 - \Psi(\delta; n)$. By Gromov's compactness theorem, without loss of generality, we can assume that there exist a compact subset K_∞ of $\hat{\overline{B}}_{40}(w)$ and a collection $\{x_j^\infty\}_{2 \leq j \leq k}$ of points in Y such that $x_j^i \to x_j^\infty$ and $K_i \to K_\infty$. By Proposition 2.5, we have $\hat{\upsilon}(K_\infty)/\hat{\upsilon}(\hat{B}_{40}(w)) \geq 1 - \Psi(\delta; n)$. On the other hand, by Claim 3.7 and 3.8, for every $\alpha \in K_\infty$ and every $0 < t \leq 10^{-5}$, there exist a compact metric space Z_∞, a point z_∞ in Z_∞, and a map ϕ from $(\hat{\overline{B}}_t(\alpha), \alpha)$ to $(\overline{B}_t(z_\infty), z_\infty)$ such that the map $\hat{\phi} = (\hat{r}_x, \hat{r}_{x_2^\infty}, \ldots, \hat{r}_{x_k^\infty}, \phi)$ from $\hat{\overline{B}}_t(\alpha)$ to $\hat{\overline{B}}_{t+\Psi(\delta;n)t}(\hat{\phi}(\alpha))$, is an $\Psi(\delta; n)t$-Gromov-Hausdorff approximation. Put $\hat{K_\infty} = K_\infty \cap (\mathcal{R}_k)_{\delta,r} \cap \overline{B}_{10^{-10}s_0}(w)$. Then, we have $\upsilon(\hat{K}_\infty)/\upsilon(\overline{B}_{10^{-10}s_0}(w)) \geq 1 - \Psi(\delta; n)$. On the other hand, for every $\alpha \in \hat{K}_\infty$ and every $0 < t \leq 10^{-5}$, let ϕ, Z_∞, z_∞ as above. Then, since $\alpha \in (\mathcal{R}_k)_{\delta,r}$, we have $\mathrm{diam}\, Z_\infty \leq \Psi(\delta; n)t$. Especially, the map $f = (\hat{r}_x, \hat{r}_{x_2^\infty}, \ldots, \hat{r}_{x_k^\infty})$ from $\hat{\overline{B}}_t(\alpha)$ to $\overline{B}_{t+\Psi(\delta;n)t}(f(\alpha))$, is an $\Psi(\delta; n)t$-Gromov-Hausdorff approximation. Especially, for every $\alpha, \beta \in \hat{K}_\infty$ with $\alpha \neq \beta$, by letting $t = \hat{r}_\alpha(\beta) (\leq 10^{-5})$, we have

$$\sqrt{(\overline{x, \alpha}^{s_0^{-1} d_Y} - \overline{x, \beta}^{s_0^{-1} d_Y})^2 + \sum_{l=2}^{k} (\overline{x_l^\infty, \alpha}^{s_0^{-1} d_Y} - \overline{x_l^\infty, \beta}^{s_0^{-1} d_Y})^2} = \overline{\alpha, \beta}^{s_0^{-1} d_Y} \pm \Psi(\delta; n)t$$
$$= (1 \pm \Psi(\delta; n)) \overline{\alpha, \beta}^{s_0^{-1} d_Y}.$$

Therefore, we have the assertion. □

LEMMA 3.9. *Let (Y, y, υ) be a Ricci limit space and x a point in Y. Then, there exist collections of compact subsets $\{C_{k,i}^x\}_{1 \leq k \leq n, i \in \mathbf{N}}$ of Y, and of points $\{x_{k,i}^l\}_{2 \leq l \leq k \leq n, i \in \mathbf{N}}$ in Y such that the following properties hold:*

1. *$\bigcup_{i \in \mathbf{N}} C_{k,i}^x \subset \mathcal{R}_k$ and $\upsilon(\mathcal{R}_k \setminus \bigcup_{i \in \mathbf{N}} C_{k,i}^x) = 0$ for every k.*
2. *For every $z \in \bigcup_{i \in \mathbf{N}} C_{k,i}^x$ and every $0 < \delta < 1$, there exists $C_{k,i}^x$ such that $z \in C_{k,i}^x$ and that the map $\Phi_{k,i}^x = (r_x, r_{x_{k,i}^2}, \ldots, r_{x_{k,i}^k})$ from $C_{k,i}^x$ to $\mathbf{R}^k$, is $(1 \pm \delta)$-bi-Lipschitz to the image $\Phi_{k,i}^x(C_{k,i}^x)$.*

PROOF. Put

$$A_k = \bigcap_{m_1 \in \mathbf{N}} \left(\bigcup_{m_2 \in \mathbf{N}} \left((\mathcal{R}_k)_{1/m_1, 1/m_2}^x \cap \mathrm{Leb}((\mathcal{R}_k)_{1/m_1, 1/m_2}) \setminus (C_x \cup \{x\}) \right) \right).$$

CLAIM 3.10. *We have $A_k \subset \mathcal{R}_k$ and $\upsilon(\mathcal{R}_k \setminus A_k) = 0$.*

The proof is as follows. Put

$$B_k = \bigcap_{m_1 \in \mathbf{N}} \left(\bigcup_{m_2 \in \mathbf{N}} \left((\mathcal{R}_k)_{1/m_1, 1/m_2}^x \cap (\mathcal{R}_k)_{1/m_1, 1/m_2} \setminus (C_x \cup \{x\}) \right) \right).$$

Then we have $A_k \subset B_k$ and $\upsilon(B_k \setminus A_k) = 0$. On the other hand, by Lemma 3.4, we have $B_k = \mathcal{R}_k \setminus (C_x \cup \{x\})$. Since $\upsilon(C_x) = 0$, we have Claim 3.10.

For every $z \in A_k$ and every $N \in \mathbf{N}$, there exists $m_2 = m_2(z, N)$ such that $z \in (\mathcal{R}_k)^x_{1/N,1/m_2} \cap \mathrm{Leb}((\mathcal{R}_k)_{1/N,1/m_2}) \setminus (C_x \cup \{x\})$. By Lemma 3.5, there exists $\eta(z, N) > 0$ such that for every $0 < s \le \eta(z, N)$, there exist a compact subset $L(z, s, N)$ of $\overline{B}_s(z) \cap (\mathcal{R}_k)_{1/N,1/m_2}$ and a collection of points $\{x_j(z, s, N)\}_{1 \le j \le k}$ in Y such that $\upsilon(L(z, s, N))/\upsilon(\overline{B}_s(z)) \ge 1 - \Psi(N^{-1}; n)$ and that the map $\Phi_{z,s,N} = (r_x, r_{x_2(z,s,N)} \ldots, r_{x_k(z,s,N)})$ from $L(z, s, N)$ to $\mathbf{R}^k$, is $(1 \pm \Psi(N^{-1}; n))$-bi-Lipschitz to the image. Fix $R > 1$ and $N \in \mathbf{N}$. By Lemma 2.4, there exists a pairwise disjoint collection $\{\overline{B}_{s_i^{N,R}}(z_i^{N,R})\}_{i \in \mathbf{N}}$ such that $z_i^{N,R} \in A_k \cap \overline{B}_R(y)$, $0 < s_i^{N,R} \le \eta(z_i^{N,R}, N)/100$ and $A_k \cap \overline{B}_R(y) \setminus \bigcup_{i=1}^m \overline{B}_{s_i^{N,R}}(z_i^{N,R}) \subset \bigcup_{i=m+1}^\infty \overline{B}_{5s_i^{N,R}}(z_i^{N,R})$ for every m. Put $\hat{L}(i, N, R) = L(z_i^{N,R}, 5s_i^{N,R}, N) \cap A_k \cap \overline{B}_R(y) \subset A_k \cap \overline{B}_R(y)$.

Claim 3.11. *$\upsilon\left(A_k \cap \overline{B}_R(y) \setminus \bigcup_{N \ge N_0, i \in \mathbf{N}} \hat{L}(i, N, R)\right) = 0$ for every $N_0 \in \mathbf{N}$.*

Because we have

$$\begin{aligned}
&\upsilon\left(A_k \cap \overline{B}_R(y) \setminus \bigcup_{i \in \mathbf{N}} \hat{L}(i, N, R)\right) \\
&\le \upsilon\left(\bigcup_{i \in \mathbf{N}} \left(\overline{B}_{5s_i^{N,R}}(z_i^{N,R}) \cap A_k \cap \overline{B}_R(y)\right) \setminus \bigcup_{i \in \mathbf{N}} \left(L(z_i^{N,R}, 5s_i^{N,R}, N) \cap A_k \cap \overline{B}_R(y)\right)\right) \\
&\le \sum_{i \in \mathbf{N}} \upsilon\left(\overline{B}_{5s_i^{N,R}}(z_i^{N,R}) \setminus L(z_i^{N,R}, 5s_i^{N,R}, N)\right) \\
&\le \Psi(N^{-1}; n) \sum_{i \in \mathbf{N}} \upsilon(\overline{B}_{5s_i^{N,R}}(z_i^{N,R})) \le \Psi(N^{-1}; n) \sum_{i \in \mathbf{N}} \upsilon(B_{s_i^{N,R}}(z_i^{N,R})) \le \Psi(N^{-1}; n) \upsilon(B_{2R}(y)).
\end{aligned}$$

for every $N \ge N_0$. Therefore, by letting $N \to \infty$, we have Claim 3.11.

By Claim 3.11, we have $\upsilon\left(A_k \cap \overline{B}_R(y) \setminus \bigcap_{N_0} \left(\bigcup_{N \ge N_0, i \in \mathbf{N}} \hat{L}(i, N, R)\right)\right) = 0$. Put $E(i, N, R) = \hat{L}(i, N, R) \cap \bigcap_{N_0 \in \mathbf{N}} \left(\bigcup_{N \ge N_0, j \in \mathbf{N}} \hat{L}(j, N, R)\right)$. Then, we have $\upsilon\big(A_k \cap \overline{B}_R(y) \setminus \bigcup_{i,N \in \mathbf{N}} E(i, N, R)\big) = 0$. Fix $z \in \bigcup_{i,N \in \mathbf{N}} E(i, N, R)$ and $0 < \delta < 1$. Then there exist i, N such that $z \in E(i, N, R)$. Let $N_0 \in \mathbf{N}$ with $N_0^{-1} << \delta$. Then there exist $\hat{N} \ge N_0$ and $\hat{i} \in \mathbf{N}$ such that $z \in \hat{L}(\hat{i}, \hat{N}, R)$. By the definition, the map $\phi = (r_x, r_{x_2\left(z_{\hat{i}}^{\hat{N},R}, s_{\hat{i}}^{\hat{N},R}\right)}, \ldots, r_{x_k\left(z_{\hat{i}}^{\hat{N},R}, s_{\hat{i}}^{\hat{N},R}\right)})$ from $L(z_{\hat{i}}^{\hat{N},R}, s_{\hat{i}}^{\hat{N},R}, \hat{N})$ to $\mathbf{R}^k$, is $\Psi(N^{-1}, n)$-bi-Lipschitz to the image. Especially, the map is $(1 \pm \delta)$-bi-Lipschitz to the image. We remark that $\hat{L}(\hat{i}, \hat{N}, R) \subset L(z_{\hat{i}}^{\hat{N},R}, s_{\hat{i}}^{\hat{N},R}, \hat{N})$ and $z \in \hat{L}(\hat{i}, \hat{N}, R) \cap \bigcap_{l \in \mathbf{N}} \left(\bigcup_{j \ge l, p \in \mathbf{N}} \hat{L}(p, j, R)\right) = E(\hat{i}, \hat{N}, R)$. Therefore, by letting $x_j(i, N, R) = x_j(z_i^{N,R}, s_i^{N,R}, R)$ for every $2 \le j \le k$, we have the following claim:

Claim 3.12. *For every* $z \in \bigcup_{i,N\in\mathbf{N}} E(i,N,R)$ *and every* $0<\delta<1$, *there exists* $E(i,N,R)$ *such that* $z \in E(i,N,R)$ *and that the map* $\phi = (r_x, r_{x_{2(i,N,R)}}, \dots, r_{x_{k(i,N,R)}})$ *from* $E(i,N,R)$ *to* $\mathbf{R}^k$, *is* $(1\pm\delta)$*-bi-Lipschitz to the image.*

By Claim 3.12, it is easy to check the assertion. □

Lemma 3.13. *With the same notaion as in Lemma 3.9, for every* k, i, *let* $\{\mathcal{F}^x_{k,i,j}\}_{j\in\mathbf{N}}$ *be a collection of Borel subsets of* $C^x_{k,i}$ *with* $\upsilon\left(C^x_{k,i}\setminus\bigcup_{j\in\mathbf{N}}\mathcal{F}^x_{k,i,j}\right)=0$. *Then, there exists a collection of Borel subsets* $\{\mathcal{E}^x_{k,i,j}\}_{k,i,j}$ *of* Y *such that* $\mathcal{E}^x_{k,i,j}\subset\mathcal{F}^x_{k,i,j}$, $\upsilon(\mathcal{F}^x_{k,i,j}\setminus\mathcal{E}^x_{k,i,j})=0$ *and that for every* k, *every* $z\in\bigcup_{i,j\in\mathbf{N}}\mathcal{E}^x_{k,i,j}$ *and every* $0<\delta<1$, *there exists* $\mathcal{E}^x_{k,i,j}$ *such that* $z\in\mathcal{E}^x_{k,i,j}$ *and that the map* $\Phi^x_{k,i,j}=(r_x, r_{x^2_{k,i}},\dots,r_{x^k_{k,i}})$ *from* $\mathcal{E}^x_{k,i,j}$ *to* $\mathbf{R}^k$, *is* $(1\pm\delta)$*-bi-Lipschitz to the image.*

Proof. Fix $1\le k\le n$. For every $M\in\mathbf{N}$, put $\mathcal{B}_M=\{i\in\mathbf{N};$ The map $\phi=(r_x, r_{x^2_{k,i}},\dots,r_{x^k_{k,i}})$ from $C^x_{k,i}$ to $\mathbf{R}^k$, is $(1\pm M^{-1})$-bi-Lipschitz to the image $\}$ and $\mathcal{E}^x_{k,i,j}=\mathcal{F}^x_{k,i,j}\cap\bigcap_{M\in\mathbf{N}}\left(\bigcup_{i\in\mathcal{B}_M, j\in\mathbf{N}}\mathcal{F}^x_{k,i,j}\right)$.

Claim 3.14. $\upsilon(\mathcal{F}^x_{k,i,j}\setminus\mathcal{E}^x_{k,i,j})=0$.

The proof is as follows. By Lemma 3.9, we have $\bigcup_{i\in\mathbf{N}}C^x_{k,i}\subset\bigcap_{M\in\mathbf{N}}\left(\bigcup_{i\in\mathcal{B}_M}C^x_{k,i}\right)$. On the other hand, it is easy to check that $\bigcap_{M\in\mathbf{N}}\left(\bigcup_{i\in\mathcal{B}_M}C^x_{k,i}\right)\subset\bigcup_{i\in\mathbf{N}}C^x_{k,i}$. Therefore, we have $\bigcap_{M\in\mathbf{N}}\left(\bigcup_{i\in\mathcal{B}_M}C^x_{k,i}\right)=\bigcup_{i\in\mathbf{N}}C^x_{k,i}$. Thus, $\upsilon(\mathcal{F}^x_{k,i,j}\setminus\mathcal{E}^x_{k,i,j})=\upsilon\left(\mathcal{F}^x_{k,i,j}\cap\bigcup_{l\in\mathbf{N}}C^x_{k,l}\setminus\mathcal{E}^x_{k,i,j}\right)=\upsilon\left(\mathcal{F}^x_{k,i,j}\cap\bigcap_{M\in\mathbf{N}}\left(\bigcup_{l\in\mathcal{B}_M}C^x_{k,l}\right)\setminus\mathcal{E}^x_{k,i,j}\right)=\upsilon\left(\mathcal{F}^x_{k,i,j}\cap\bigcap_{M\in\mathbf{N}}\left(\bigcup_{l\in\mathcal{B}_M, j\in\mathbf{N}}\mathcal{F}^x_{k,l,j}\right)\setminus\mathcal{E}^x_{k,i,j}\right)=0$. Therefore we have Claim 3.14.

Claim 3.15. *For every* $z\in\bigcup_{i,j\in\mathbf{N}}\mathcal{E}^x_{k,i,j}$ *and every* $0<\delta<1$, *there exists* $\mathcal{E}^x_{k,i,j}$ *such that* $z\in\mathcal{E}^x_{k,i,j}$ *and that the map* $\phi=(r_x, r_{x^2_{k,i}},\dots,r_{x^k_{k,i}})$ *from* $\mathcal{E}^x_{k,i,j}$ *to* $\mathbf{R}^k$, *is* $(1\pm\delta)$*-bi-Lipschitz to the image.*

The proof is as follows. Let M, i, j be positive integers with $M^{-1}<<\delta$, $z\in\mathcal{E}^x_{k,i,j}$. There exist $N_0\in\mathcal{B}_M$ and $N_1\in\mathbf{N}$ such that $z\in\mathcal{F}^x_{k,N_0,N_1}$. Therefore, we have $z\in\mathcal{F}^x_{k,N_0,N_1}\cap\bigcap_{\hat{M}\in\mathbf{N}}\left(\bigcup_{\hat{i}\in\mathcal{B}_{\hat{M}},\hat{j}\in\mathbf{N}}\mathcal{F}^x_{k,\hat{i},\hat{j}}\right)=\mathcal{E}^x_{k,N_0,N_1}$ and that the map $\phi=(r_x, r_{x^2_{k,j}},\dots,r_{x^k_{k,j}})$ from $\mathcal{E}^x_{k,N_0,N_1}$ to $\mathbf{R}^k$, is $(1\pm M^{-1})$-bi-Lipschitz to the image. Thus, we have Claim 3.15.

By Claim 3.14 and 3.15, we have the assertion. □

The following theorem is the main result in this subsection. See (2.2) in [5] or [22, Definition 4.1] for the definition of the measure υ_{-1}.

Theorem 3.16 (Radial rectifiability). *Let* (Y, y, υ) *be a Ricci limit space with* $Y\neq\{y\}$, *and* x *a point in* Y. *Then, there exist collections of Borel subsets* $\{C^x_{k,i}\}_{1\le k\le n, i\in\mathbf{N}}$ *of* Y, *of points* $\{x^l_{k,i}\}_{2\le l\le k\le n, i\in\mathbf{N}}$ *in* Y, *a positive number* $0<\alpha(n)<1$ *and a Borel subset* A *of* $[0,\mathrm{diam}Y)$ *such that the following properties hold:*

1. $\bigcup_{i\in\mathbf{N}} C^x_{k,i} \subset \mathcal{R}_{k,\alpha(n)} \setminus C_x$ *and* $\upsilon\left(\mathcal{R}_k \setminus \bigcup_{i\in\mathbf{N}} C^x_{k,i}\right) = 0$ *for every* k.

2. $\lim_{r\to 0} \upsilon(B_r(z)\cap C^x_{k,i})/\upsilon(B_r(z)) = 1$ *for every* $C^x_{k,i}$ *and every* $z \in C^x_{k,i}$.

3. *For every* $C^x_{k,i}$, *there exists* $A^x_{k,i} > 1$ *such that* $(A^x_{k,i})^{-1} \le \upsilon(B_r(z))/r^k \le A^x_{k,i}$ *for every* $z \in C^x_{k,i}$ *and every* $0 < r < 1$.

4. *The limit measure* υ *and the* k*-dimensional Hausdorff measure* H^k *are mutually absolutely continuous on* $C^x_{k,i}$.

5. *For every* $z \in \bigcup_{i\in\mathbf{N}} C^x_{k,i}$ *and every* $0 < \delta < 1$, *there exists* $C^x_{k,i}$ *such that* $z \in C^x_{k,i}$ *and that the map* $\Phi^x_{k,i} = (r_x, r_{x^2_{k,i}}, \dots, r_{x^k_{k,i}})$ *from* $C^x_{k,i}$ *to* $\mathbf{R}^k$, *is* $(1\pm\delta)$*-bi-Lipschitz to the image.*

6. $H^1([0, \mathrm{diam} Y)\setminus A) = 0$.

7. *For every* $R \in A$, *the collection* $\{\partial B_R(x)\cap C^x_{k,i}\}_{k,i} \subset \partial B_R(x)\setminus C_x$ *satisfies the following properties:*

   (a) $\upsilon_{-1}\left((\partial B_R(x)\setminus C_x)\setminus \bigcup_{1\le k\le n, i\in\mathbf{N}} C^x_{k,i}\right) = 0$.

   (b) *For every* $\partial B_R(x)\cap C^x_{k,i}$, *there exist* $B^x_{k,i} > 1$ *and* $\tau^x_{k,i} > 0$ *such that* $(B^x_{k,i})^{-1} \le \upsilon_{-1}(\partial B_R(x)\cap B_r(z)\setminus C_x)/r^{k-1} \le \upsilon_{-1}(\partial B_R(x)\cap \overline{B}_r(z))/r^{k-1} \le B^x_{k,i}$ *for every* $z \in \partial B_R(x)\cap C^x_{k,i}$ *and every* $0 < r < \tau^x_{k,i}$.

   (c) *For every* $z \in \bigcup_{i\in\mathbf{N}}(\partial B_R(x)\cap C^x_{k,i})$ *and every* $0<\delta<1$, *there exists* $\partial B_R(x)\cap C^x_{k,i}$ *such that* $z \in \partial B_R(x)\cap C^x_{k,i}$ *and that the map* $\hat{\Phi}^x_{k,i} = (r_{x^2_{k,i}}, \dots, r_{x^k_{k,i}})$ *from* $\partial B_R(x)\cap C^x_{k,i}$ *to* $\mathbf{R}^{k-1}$, *is* $(1\pm\delta)$*-bi-Lipschitz to the image.*

Especially, $\partial B_R(x)\setminus C_x$ *is* υ_{-1}*-rectifiable.*

Proof. First, we shall prove the following claim:

Claim 3.17. *We have* $\upsilon_{-1}(\partial B_{\overline{x,z}}(x)\cap \overline{B}_\epsilon(z)) \le C(n)\upsilon(B_\epsilon(z))/\epsilon$ *for every* $R > 0$, *every* $z \in \overline{B}_R(x)\setminus\{x\}$ *and every* $\epsilon > 0$ *with* $\epsilon < \min\{\overline{z,x}/100, 1\}$.

The proof is as follows. By [23, Corollary 5.7], we have

$$\frac{\upsilon_{-1}(\partial B_{\overline{x,z}}(x)\cap \overline{B}_\epsilon(z))}{\mathrm{vol}\,\partial B_{\overline{x,z}}(\underline{p})} \le C(n)\frac{\upsilon(C_x(\partial B_{\overline{x,z}}(x)\cap \overline{B}_\epsilon(z))\cap A_{\overline{x,z}-2\epsilon,\overline{x,z}}(x))}{\mathrm{vol}\,A_{\overline{x,z}-2\epsilon,\overline{x,z}}(\underline{p})}.$$

Here $C_x(A) = \{z \in Y;$ There exists $a \in A$ such that $\overline{x,z}+\overline{z,a} = \overline{z,a}\}$ for every subset A of Y, $\underline{p}$ is a point in the n-dimensional hyperbolic space form. On the other hand, by triangle inequality, we have $C_x(\partial B_{\overline{x,z}}(x) \cap \overline{B}_\epsilon(z)) \cap A_{\overline{x,z}-2\epsilon,\overline{x,z}}(x) \subset \overline{B}_{100\epsilon}(z)$. Thus, we have

$$\upsilon_{-1}(\partial B_{\overline{x,z}}(x) \cap \overline{B}_\epsilon(z)) \le \frac{\operatorname{vol} \partial B_{\overline{x,z}}(\underline{p})}{\operatorname{vol} A_{\overline{x,z}-2\epsilon,\overline{x,z}}(\underline{p})} \upsilon(B_{100\epsilon}(z)) C(n) \le C(n,R)\frac{1}{\epsilon}\upsilon(B_\epsilon(z)).$$

Therefore, we have Claim 3.17.

Let $\{C^x_{k,i}\}_{k,i}$ be a collection of Borel subsets of Y and $\{x^l_{k,i}\}_{k,i,l}$ a collection of points in Y as in Lemma 3.9. By Lemma 3.13, without loss of generality, we can assume that for every $C^x_{k,i}$, there exists $\tau > 0$ such that $C^x_{k,i} \subset \mathcal{D}^\tau_x \setminus B_\tau(x)$. Moreover, by [6, Theorem 3.23] and [6, Theorem 4.6], we can assume that for every $C^x_{k,i}$, there exists $A^x_{k,i} > 1$ such that $(A^x_{k,i})^{-1} \le \upsilon(B_r(z))/r^k \le A^x_{k,i}$ for every $0 < r < 1$ and every $z \in C^x_{k,i}$, and that $\lim_{r\to 0} \upsilon(B_r(z) \cap C^x_{k,i})/\upsilon(B_r(z)) = 1$ for every $C^x_{k,i}$ and every $z \in C^x_{k,i}$.

Claim 3.18. *Let (Y, y, υ) be a Ricci limit space, x a point in Y, τ, R positive numbers with $0 < \tau < 1 < R$, and z a point in $\mathcal{D}^\tau_x \cap B_R(x) \setminus B_\tau(x)$. Then, we have $\upsilon_{-1}(\partial B_{\overline{x,z}}(x) \cap B_\epsilon(z) \setminus C_x) \ge C(n,R)\upsilon(B_\epsilon(z))/\epsilon$ for every $0 < \epsilon < \tau/100$.*

The proof is as follows. Let $w \in Y$ with $\overline{z,w} = \epsilon/100$, $\overline{x,z} + \overline{z,w} = \overline{x,w}$. By [23, Theorem 4.6], we have

$$\frac{\upsilon(B_{\frac{\epsilon}{1000}}(w))}{\operatorname{vol} A_{\overline{x,z},\overline{x,z}+\epsilon}(\underline{p})} \le C(n) \frac{\upsilon_{-1}\left(C_x(B_{\frac{\epsilon}{1000}}(w)) \cap \partial B_{\overline{x,z}}(x)\right)}{\operatorname{vol} \partial B_{\overline{x,z}}(\underline{p})}.$$

By triangle inequality, we have $C_x(B_{\epsilon/1000}(w)) \cap \partial B_{\overline{x,z}}(x) \subset \partial B_{\overline{x,z}}(x) \cap B_\epsilon(z)$. Thus, by Bishop-Gromov volume comparison theorem for υ, we have

$$\begin{aligned}\upsilon_{-1}(\partial B_{\overline{x,z}}(x) \cap B_\epsilon(z) \setminus C_x) &\ge C(n) \frac{\operatorname{vol} \partial B_{\overline{x,z}}(\underline{p})}{\operatorname{vol} A_{\overline{x,z},\overline{x,z}+\epsilon}(\underline{p})} \upsilon(B_{\epsilon/1000}(w)) \\ &\ge C(n,R)\frac{1}{\epsilon}\upsilon(B_{\frac{\epsilon}{1000}}(w)) \ge C(n,R)\frac{1}{\epsilon}\upsilon(B_{5\epsilon}(w)) \ge C(n,R)\frac{\upsilon(B_\epsilon(z))}{\epsilon}.\end{aligned}$$

Therefore we have Claim 3.18.

By Claim 3.17 and 3.18, for every $C^x_{k,i}$, there exist $B^x_{k,i} > 1$ and $\tau^x_{k,i} > 0$ such that $(B^x_{k,i})^{-1} \le \upsilon_{-1}(\partial B_{\overline{x,z}}(x) \cap B_r(z) \setminus C_x)/r^k \le B^x_{k,i}$ for every $z \in C^x_{k,i}$ and every $0 < r < \tau^x_{k,i}$. Put $\hat{A} = \{t \in [0, \operatorname{diam} Y); \upsilon_{-1}\left(\partial B_t(x) \setminus \bigcup C^x_{k,i}\right) = 0\}$. Since $\upsilon\left(Y \setminus \bigcup C^x_{k,i}\right) = 0$, it follows from [23, Proposition 5.1] and [23, Theorem 5.2] that $\hat{A}$ is Lebesgue measurable and that $H^1([0, \operatorname{diam} Y) \setminus \hat{A}) = 0$. Since H^1 is a Radon measure on $\mathbf{R}$, we have the assertion. □

3.2 Calculation of radial derivatives of Lipschitz functions

The purpose in this subsection is to calculate the radial derivative from a given point x, of a given Lipschitz function f: $\langle dr_x, df\rangle$ explicitly. The main result in this subsection is Theorem 3.30.

LEMMA 3.19. *Let (Y, y) be a Ricci limit space with $Y \neq \{y\}$, z a point in $Y \setminus C_y$, f a Lipschitz function on Y, τ a positive number and γ_i an isometric embedding from $[0, \overline{y,z} + \tau]$ to Y satisfying $\gamma_i(0) = y$, $\gamma_i(\overline{y,z}) = z$ for every $i \in \{1, 2\}$. Put $f_i = f \circ \gamma_i$. Then, we have $lipf_1(\overline{y,z}) = lipf_2(\overline{y,z})$ and $\mathrm{Lip} f_1(\overline{y,z}) = \mathrm{Lip} f_2(\overline{y,z})$.*

PROOF. For every real number ϵ with $0 < |\epsilon| << \tau$, by the splitting theorem on limit space, we have $\overline{\gamma_1(\overline{x,z} + \epsilon), \gamma_2(\overline{x,z} + \epsilon)} \leq \Psi(|\epsilon|; n)|\epsilon|$. Therefore, we have

$$\frac{|f_1(\overline{x,z} + \epsilon) - fa_1(\overline{x,z})|}{|\epsilon|} \leq \frac{|f_2(\overline{x,z} + \epsilon) - f_2(\overline{x,z})|}{|\epsilon|} + \mathbf{Lip} f \Psi(|\epsilon|; n).$$

Thus, we have $\mathrm{Lip} f_1(\overline{y,z}) \leq \mathrm{Lip} f_2(\overline{y,z})$ and $lipf_1(\overline{y,z}) \leq lipf_2(\overline{y,z})$. Therefore we have $\mathrm{Lip} f_1(\overline{y,z}) = \mathrm{Lip} f_2(\overline{y,z})$ and $lipf_1(\overline{y,z}) = lipf_2(\overline{y,z})$. □

Let (Y, y) be a Ricci limit space, z a point in $Y \setminus C_y$, τ a positive number, γ an isometric embedding from $[0, \overline{y,z} + \tau]$ to Y satisfying $\gamma(0) = y$, $\gamma(\overline{y,z}) = z$. Put $F = f \circ \gamma$, $lip_y^{\mathrm{rad}} f(z) = lipF(\overline{y,z})$ and $\mathrm{Lip}_y^{\mathrm{rad}} f(z) = \mathrm{Lip} F(\overline{y,z})$. It is not difficult to check the following lemma:

LEMMA 3.20. *Let (Z, υ) be a metric measure space. Assume that the following properties hold:*

1. *$\upsilon(B_r(z)) > 0$ for every $z \in Z$ and every $r > 0$*
2. *There exist $r_0 > 0$ and $\kappa > 1$ such that $\upsilon(B_{2r}(z)) \leq 2^\kappa \upsilon(B_r(z))$ for every $z \in Z$ and every $0 < r < r_0$.*

Then, we have $\mathrm{Lip} f(a) = \mathrm{Lip}(f|_A)(a)$ and $lipf(a) = lip(f|_A)(a)$ for every $a \in \mathrm{Leb}(A)$, every Lipschitz function f on Z, and every Borel subset A of Z.

The following theorem implies that $\partial B_R(x) \perp \nabla r_x$ in some sense:

THEOREM 3.21. *Let (Y, y, υ) be a Ricci limit space, x a point in Y and f a Lipschitz function on Y. Then, we have the following:*

1. *$lipf(z)^2 = lip_x^{\mathrm{rad}} f(z)^2 + lip(f|_{\partial B_{\overline{x,z}}(x)})(z)^2$ for a.e. $z \in Y$.*
2. *$\mathrm{Lip} f(z)^2 = \mathrm{Lip}_x^{\mathrm{rad}} f(z)^2 + \mathrm{Lip}(f|_{\partial B_{\overline{x,z}}(x)})(z)^2$ for a.e. $z \in Y$.*
3. *$\mathrm{Lip}(f|_{\partial B_{\overline{x,z}}(x)})(z) = lip(f|_{\partial B_{\overline{x,z}}(x) \setminus C_x})(z)$ for a.e. $z \in Y \setminus C_x$.*

PROOF. First we shall remark the following:

CLAIM 3.22. *Let f be a Lipschitz function on $\mathbf{R}^k$. Then, we have $\mathrm{Lip} f(z)^2 = (\mathrm{Lip}(f|_{\mathbf{R} \times \{z_2, \ldots, z_k\}})(z))^2 + (\mathrm{Lip}(f|_{\{z_1\} \times \mathbf{R}^{k-1}})(z))^2 = (lip(f|_{\mathbf{R} \times \{z_2, \ldots, z_k\}})(z))^2 + (lip(f|_{\{z_1\} \times \mathbf{R}^{k-1}})(z))^2 = lipf(z)^2$ for a.e $z = (z_1, \ldots, z_k) \in \mathbf{R}^k$.*

Because, by Rademacher's theorem about differentiability of Lipschitz functions on $\mathbf{R}^k$, f is totally differentiable at a.e $z \in \mathbf{R}^k$. Therefore we have Claim 3.22.

The next claim is clear:

CLAIM 3.23. *Let $\{Z_i\}_{i=1,2}$ be metric spaces, δ a positive number with $0 < \delta < 1$, and Φ a map from Z_1 to Z_2 satisfying that $\Phi(Z_1) = Z_2$ and $(1-\delta)\overline{x_1, x_2} \leq \overline{\Phi(x_1), \Phi(x_2)} \leq (1+\delta)\overline{x_1, x_2}$ for every $x_1, x_2 \in Z_1$. Then, for every Lipschitz function f on Z_2, we have, $(1-\Psi(\delta))\mathrm{Lip}f(\Phi(z_1)) \leq \mathrm{Lip}(f \circ \Phi)(z_1) \leq (1+\Psi(\delta))\mathrm{Lip}f(z_1)$, $(1-\Psi(\delta))lipf(\Phi(z_1)) \leq lip(f \circ \Phi)(z_1) \leq (1+\Psi(\delta))lipf(\Phi(z_1))$ for every $z_1 \in Z_1$.*

We will give a proof of the following claim in Appendix:

CLAIM 3.24. *For every Lebesgue measurable subset A of $\mathbf{R}^k$, put $sl_1 - \mathrm{Leb}A = \{a = (a_1, \ldots, a_k) \in A; \lim_{r \to 0} H^{k-1}\left((\{a_1\} \times \overline{B}_r(a_2, \ldots, a_k)) \cap A\right) / H^{k-1}\left(\{a_1\} \times \overline{B}_r(a_2, \ldots, a_k)\right) = 1\}$. Then the following properties hold:*

1. *$sl_1 - \mathrm{Leb}A$ is a Lebesgue measurable set.*

2. *$H^{k-1}\left(A \cap (\{t\} \times \mathbf{R}^{k-1} \setminus sl_1 - \mathrm{Leb}A)\right) = 0$ for every $t \in \mathbf{R}$.*

3. *$H^k(A \setminus sl_1 - \mathrm{Leb}A) = 0$.*

Put $L = \mathbf{Lip}f$. Let $\{C^x_{k,i}\}_{1 \leq k \leq n, i \in \mathbf{N}}$ be a collection of Borel subsets of Y, and $\{x^l_{k,i}\}_{2 \leq k \leq n, i \in \mathbf{N}, 2 \leq l \leq k}$ a collection of points in Y as in Theorem 3.16. Fix a sufficiently small $\delta > 0$ and $C_{k,i}$ satisfying that the map $\Phi^x_{k,i} = (r_x, r_{x^2_{k,i}}, \ldots, r_{x^k_{k,i}})$ from $C^x_{k,i}$ to $\mathbf{R}^k$, is $(1 \pm \delta)$-bi-Lipschitz to the image. Put $f^x_{k,i} = f \circ (\Phi^x_{k,i})^{-1}$ on $\Phi^x_{k,i}(C^x_{k,i})$. Let $F^x_{k,i}$ be a Lipschitz function on $\mathbf{R}^k$ satisfying that $F^x_{k,i}|_{\Phi^x_{k,i}(C^x_{k,i})} = f^x_{k,i}$ and $\mathbf{Lip}F^x_{k,i} = \mathbf{Lip}f^x_{k,i}$.

CLAIM 3.25. *With the notation as above, we have the following:*

1. *$(1-\Psi(\delta; n))\mathrm{Lip}F^x_{k,i}(w) \leq \mathrm{Lip}f((\Phi^x_{k,i})^{-1}(w)) \leq (1+\Psi(\delta; n))\mathrm{Lip}F^x_{k,i}(w)$ for a.e $w \in \Phi^x_{k,i}(C^x_{k,i})$.*

2. *$(1-\Psi(\delta; n))lipF^x_{k,i}(w) \leq lipf((\Phi^x_{k,i})^{-1}(w)) \leq (1+\Psi(\delta; n))lipF^x_{k,i}(w)$ for a.e $w \in \Phi^x_{k,i}(C^x_{k,i})$.*

3. *$\mathrm{Lip}(F^x_{k,i}|_{\mathbf{R} \times \{w_2, \ldots, w_k\}})(w) - L\Psi(\delta; n) \leq \mathrm{Lip}^{\mathrm{rad}}_x f((\Phi^x_{k,i})^{-1}(w)) \leq \mathrm{Lip}(F^x_{k,i}|_{\mathbf{R} \times \{w_2, \ldots, w_k\}})(w) + L\Psi(\delta; n)$ for a.e $w = (w_1, \ldots, w_k) \in \Phi^x_{k,i}(C^x_{k,i})$.*

4. *$lip(F^x_{k,i}|_{\mathbf{R} \times \{w_2, \ldots, w_k\}})(w) - L\Psi(\delta; n) \leq lip^{\mathrm{rad}}_x f((\Phi^x_{k,i})^{-1}(w)) \leq lip(F^x_{k,i}|_{\mathbf{R} \times \{w_2, \ldots, w_k\}})(w) + L\Psi(\delta; n)$ for a.e $w = (w_1, \ldots, w_k) \in \Phi^x_{k,i}(C^x_{k,i})$.*

5. *$(1-\Psi(\delta; n))\mathrm{Lip}(F^x_{k,i}|_{\{w_1\} \times \mathbf{R}^{k-1}})(w) \leq \mathrm{Lip}(f|_{\partial B_{\overline{x, (\Phi^x_{k,i})^{-1}(w)}}(x) \cap C^x_{k,i}})((\Phi^x_{k,i})^{-1}(w)) \leq (1+\Psi(\delta; n))\mathrm{Lip}(F^x_{k,i}|_{\{w_1\} \times \mathbf{R}^{k-1}})(w)$ for a.e. $w = (w_1, \ldots, w_k) \in \Phi^x_{k,i}(C^x_{k,i})$.*

6. $(1-\Psi(\delta;n))lip(F^x_{k,i}|_{\{w_1\}\times\mathbf{R}^{k-1}})(w) \leq lip(f|_{\partial B_{\overline{x,(\Phi^x_{k,i})^{-1}(w)}}(x)\cap C^x_{k,i}})((\Phi^x_{k,i})^{-1}(w)) \leq (1+\Psi(\delta;n))lip(F^x_{k,i}|_{\{w_1\}\times\mathbf{R}^{k-1}})(w)$ *for a.e.* $w=(w_1,\ldots,w_k)\in\Phi^x_{k,i}(C^x_{k,i})$.

The proof is as follows. First, we shall give a proof of the statement 1. Put $\mathbf{C}^x_{k,i} = \mathrm{Leb}(\Phi^x_{k,i}(C^x_{k,i}))\cap\Phi^x_{k,i}(\mathrm{Leb}C^x_{k,i})$. Then, we have $H^k(\Phi^x_{k,i}(C^x_{k,i})\setminus\mathbf{C}^x_{k,i})=0$. By Lemma 3.20 and Claim 3.23, we have $(1-\Psi(\delta))\mathrm{Lip}(F^x_{k,i}|_{\Phi_{k,i}(C^x_{k,i})})(w)\leq\mathrm{Lip}(f|_{C^x_{k,i}})((\Phi^x_{k,i})^{-1}(w))\leq(1+\Psi(\delta))\mathrm{Lip}(F^x_{k,i}|_{\Phi^x_{k,i}(C^x_{k,i})})(w)$, $\mathrm{Lip}(F^x_{k,i}|_{\Phi^x_{k,i}(C^x_{k,i})})(w)=\mathrm{Lip}F^x_{k,i}(w)$ and $\mathrm{Lip}(f|_{C^x_{k,i}})((\Phi^x_{k,i})^{-1}(w)) = \mathrm{Lip}f((\Phi^x_{k,i})^{-1}(w))$ for every $w\in\mathbf{C}^x_{k,i}$. Therefore we have the statement 1. Similarly, we have the statement 2.

Next, we shall give a proof of the statement 3. Put $\mathbf{C}^{x,f}_{k,i} = sl_1-\mathrm{Leb}\mathbf{C}^x_{k,i}\cap\{w\in\mathbf{R}^k; F^x_{k,i}$ is totally differentiable at $w.\}$. Then, by Claim 3.24, we have $H^k(\mathbf{C}^x_{k,i}\setminus\mathbf{C}^{x,f}_{k,i})=0$. Fix $w\in\mathbf{C}^{x,f}_{k,i}$ and put $w_\epsilon = w+(\epsilon,0,\ldots,0)$ for every $\epsilon>0$. Since $w\in sl_1-\mathrm{Leb}\mathbf{C}^x_{k,i}$, for every $\epsilon>0$, there exist $\hat{w}_\epsilon\in\mathbf{C}^x_{k,i}$ and $a(\epsilon)>0$ such that $\overline{w_\epsilon,\hat{w}_\epsilon}\leq a(\epsilon)\epsilon$ and $a(\tau)\to0$ as $\tau\to0$. It is clear that $(1-\delta)(\epsilon-a(\epsilon)\epsilon)\leq(1-\delta)\overline{w,\hat{w}_\epsilon}\leq\overline{(\Phi^x_{k,i})^{-1}(w),(\Phi^x_{k,i})^{-1}(\hat{w}_\epsilon)}\leq(1+\delta)\overline{w,\hat{w}_\epsilon}\leq(1+\delta)(\epsilon+a(\epsilon)\epsilon)$. Let π_1 be the projection from $\mathbf{R}^k$ to $\mathbf{R}$ defined by $\pi_1(w)=w_1$. Then we have $\overline{x,(\Phi^x_{k,i})^{-1}(\hat{w}_\epsilon)}=\pi_1(\hat{w}_\epsilon)=\pi_1(w_\epsilon)\pm a(\epsilon)\epsilon=\pi_1(w)+\epsilon\pm a(\epsilon)\epsilon=\overline{x,(\Phi^x_{k,i})^{-1}(w)}+\overline{(\Phi^x_{k,i})^{-1}(w),(\Phi^x_{k,i})^{-1}(\hat{w}_\epsilon)}\pm(\delta+a(\epsilon))\epsilon$. By Lemma 3.13, without loss of generality, we can assume that there exists $\tau_0>0$ such that $C_{k,i}\subset\mathcal{D}^{\tau_0}_x$. Fix an isometric embedding γ from $[0,\overline{x,(\Phi^x_{k,i})^{-1}(w)}+\tau_0]$ to Y with $\gamma(0)=x$, $\gamma(\overline{x,(\Phi^x_{k,i})^{-1}(w)})=(\Phi^x_{k,i})^{-1}(w)$. Then, by rescaling $\epsilon^{-1}d_Y$ and the splitting theorem on limit spaces, we have $\overline{(\Phi^x_{k,i})^{-1}(\hat{w}_\epsilon),\gamma(\overline{x,(\Phi^x_{k,i})^{-1}(w)}+\epsilon)}\leq\Psi(a(\epsilon),\delta;n)\epsilon$. Thus we have

$$\begin{aligned}\frac{|F^x_{k,i}(w)-F^x_{k,i}(w_\epsilon)|}{\epsilon}&\leq\frac{|F^x_{k,i}(w)-F^x_{k,i}(\hat{w}_\epsilon)|}{\epsilon}+La(\epsilon)\\&\leq\frac{|f((\Phi^x_{k,i})^{-1}(w))-f(\gamma(\overline{x,(\Phi^x_{k,i})^{-1}(w)}+\epsilon))|}{\epsilon}+L\Psi(a(\epsilon),\delta;n)\end{aligned}$$

for every $\epsilon>0$ with $\epsilon<<\tau_0$. By letting $\epsilon\to0$, we have $\mathrm{Lip}(F^x_{k,i}|_{\mathbf{R}\times\{w_2,\ldots,w_k\}})(w)\leq\mathrm{Lip}^{\mathrm{rad}}_xf((\Phi^x_{k,i})^{-1}(w))+L\Psi(\delta;n)$. Let $\{\epsilon_i\}_i$ be a sequence of real numbers such that $\epsilon_j\to0$ and

$$\lim_{j\to\infty}\frac{|f\circ(\Phi^x_{k,i})^{-1}(w)-f(\gamma(\overline{x,(\Phi^x_{k,i})^{-1}(w)}+\epsilon_j))|}{|\epsilon_j|}=\mathrm{Lip}^{\mathrm{rad}}_xf((\Phi^x_{k,i})^{-1}(w)).$$

Since $(\Phi^x_{k,i})^{-1}(w)\in\mathrm{Leb}\,C^x_{k,i}$, there exist sequences $\{\hat{w}(j)\}_j\subset C^x_{k,i}$, $\{\tau_j\}_j\subset\mathbf{R}_{>0}$ such that $\overline{\hat{w}(j),\gamma(\overline{x,(\Phi^x_{k,i})^{-1}(w)}+\epsilon_j)}\leq\tau_j\epsilon_j$ and $\tau_j\to0$ as $j\to\infty$. Fix $j\in\mathbf{N}$. Assume that $\epsilon_j>0$. Then, we have

$$\begin{aligned}\pi_1(\hat{w}(j))-\pi_1(w)&=\overline{x,\hat{w}(j)}-\overline{x,(\Phi^x_{k,i})^{-1}(w)}\\&=\overline{x,\gamma(\overline{x,(\Phi^x_{k,i})^{-1}(w)}+\epsilon_j)}\pm\tau_j\epsilon_j\\&=\epsilon_j\pm\tau_j\epsilon_j\\&=\overline{\gamma(\overline{x,(\Phi^x_{k,i})^{-1}(w)}+\epsilon_j),(\Phi^x_{k,i})^{-1}(w)}\pm\tau_j\epsilon_j\geq(1-\delta)\overline{\Phi^x_{k,i}(\hat{w}(j)),w}-\tau_j\epsilon_j.\end{aligned}$$

On the other hand, since $\overline{\Phi^x_{k,i}(\hat{w}(j)), w} \leq (1+\delta)\epsilon_j + \tau_j\epsilon_j$, we have $\overline{w + (\epsilon_j, 0, \ldots, , 0), \Phi^x_{k,i}(\hat{w}(j))} \leq \Psi(|\epsilon_j|, \delta; n)|\epsilon_j|$. Similarly, we have $\overline{w + (\epsilon_j, 0, \ldots, , 0), \Phi^x_{k,i}(\hat{w}(j))} \leq \Psi(|\epsilon_j|, \delta; n)|\epsilon_j|$ in the case $\epsilon_j < 0$. Put $w(j) = w + (\epsilon_j, 0, \ldots, 0)$. Then, we have

$$\begin{aligned} \frac{|f\left((\Phi^x_{k,i})^{-1}(w)\right) - f\left(\gamma(\overline{x, (\Phi^x_{k,i})^{-1}(w)} + \epsilon_j)\right)}{|\epsilon_j|} &\leq \frac{|F^x_{k,i}(w) - F^x_{k,i}\left(\Phi^x_{k,i}(\hat{w}(j))\right)|}{|\epsilon_j|} + L\tau_j \\ &\leq \frac{|F^x_{k,i}(w) - F^x_{k,i}(w(j))|}{|\epsilon_j|} + L\Psi(|\epsilon_j|, \tau_j, \delta; n). \end{aligned}$$

By letting $j \to \infty$, we have the statement 3. Similarly, we have the statement 4.

We shall give a proof of the statement 5. Fix $w \in \mathbf{C}^{x,f}_{k,i}$. By Claim 3.23, we have

$$\begin{aligned} (1 - \Psi(\delta))\mathrm{Lip}(F^x_{k,i}|_{(\{w_1\}\times\mathbf{R}^{k-1})\cap\mathbf{C}^x_{k,i}})(w) &\leq \mathrm{Lip}(f|_{(\Phi^x_{k,i})^{-1}\left((\{w_1\}\times\mathbf{R}^{k-1})\cap\mathbf{C}^x_{k,i}\right)})\left((\Phi^x_{k,i})^{-1}(w)\right) \\ &\leq (1 + \Psi(\delta))\mathrm{Lip}(F^x_{k,i}|_{(\{w_1\}\times\mathbf{R}^{k-1})\cap\mathbf{C}^x_{k,i}})(w). \end{aligned}$$

We remark that $(\Phi^x_{k,i})^{-1}\left((\{w_1\} \times \mathbf{R}^{k-1}) \cap \mathbf{C}^x_{k,i}\right) = \partial B_{\overline{x, (\Phi^x_{k,i})^{-1}(w)}}(x) \cap (\Phi^x_{k,i})^{-1}(\mathbf{C}^x_{k,i})$. By Proposition 3.20, we have $\mathrm{Lip}(F^x_{k,i}|_{\{w_1\}\times\mathbf{R}^{k-1}\cap\mathbf{C}^x_{k,i}})(w) = \mathrm{Lip}(F^x_{k,i}|_{\{w_1\}\times\mathbf{R}^{k-1}})(w)$. Therefore, by Claim 3.23, we have

$$\begin{aligned} (1 - \Psi(\delta))\mathrm{Lip}(F^x_{k,i}|_{\{w_1\}\times\mathbf{R}^{k-1}})(w) &\leq \mathrm{Lip}(f|_{\partial B_{\overline{x,(\Phi^x_{k,i})^{-1}(w)}}(x)\cap(\Phi^x_{k,i})^{-1}(\mathbf{C}^x_{k,i})})\left((\Phi^x_{k,i})^{-1}(w)\right) \\ &\leq \mathrm{Lip}(f|_{\partial B_{\overline{x,(\Phi^x_{k,i})^{-1}(w)}}(x)\cap C^x_{k,i}})\left((\Phi^x_{k,i})^{-1}(w)\right) \\ &\leq (1 + \Psi(\delta))\mathrm{Lip}(F^x_{k,i}|_{(\{w_1\}\times\mathbf{R}^{k-1})\cap\Phi^x_{k,i}(C^x_{k,i})})(w) \\ &\leq (1 + \Psi(\delta))\mathrm{Lip}(F^x_{k,i}|_{\{w_1\}\times\mathbf{R}^{k-1}})(w). \end{aligned}$$

Thus we have the statement 5. Similarly, we have the statement 6.

Therefore we have Claim 3.25.

Claim 3.26. *With the same notation as in Claim 3.25, we have*

$$lip(f|_{\partial B_{\overline{x,(\Phi^x_{k,i})^{-1}(w)}}(x)\cap C^x_{k,i}})((\Phi^x_{k,i})^{-1}(w)) \geq \mathrm{Lip}(f|_{\partial B_{\overline{x,(\Phi^x_{k,i})^{-1}(w)}}(x)})((\Phi^x_{k,i})^{-1}(w)) - \Psi(\delta; n, L)$$

for a.e $w \in \Phi^x_{k,i}(C^x_{k,i})$.

The proof is as follows. We shall use the same notaion as in the proof of Claim 3.25. Fix $w \in \Phi^x_{k,i}(\mathrm{Leb}(\Phi^x_{k,i})^{-1}(\mathbf{C}^{x,f}_{k,i}))$ and put $z = (\Phi^x_{k,i})^{-1}(w)$.

First, assume $k \geq 2$. Then we shall prove that z is not an isolated point in $\partial B_{\overline{x,z}}(x) \setminus C_x$. Because, by the definition of $sl_1 - \mathrm{Leb}(\mathbf{C}^x_{k,i})$, there exists a sequence of points $\{\beta(j)\}_j$ in $\mathbf{C}^x_{k,i}$ such that $\pi_1(\beta(j)) = \pi_1(w)$, $\beta(j) \neq w$ for every j, and $\beta(j) \to w$. Then, we have $(\Phi^x_{k,i})^{-1}(\beta(j)) \neq z$, $(\Phi^x_{k,i})^{-1}(\beta(j)) \in \partial B_{\overline{x,z}}(x) \setminus C_x$ and $(\Phi^x_{k,i})^{-1}(\beta(j)) \to z$. Especially, z is not an isolated point in $\partial B_{\overline{x,z}}(x) \setminus C_x$. Let $\{z(j)\}_j \subset \partial B_{\overline{x,z}}(x) \setminus \{z\}$ with

$z(j) \to z$, $|f(z(j)) - f(z)|/\overline{z(j), z} \to \mathrm{Lip}(f|_{\partial B_{\overline{x,z}}(x)})(z)$. Put $\eta_j = \overline{z(j), z} > 0$. Since $z \in \mathrm{Leb}(\Phi^x_{k,i})^{-1}(\mathbf{C}^{x,f}_{k,i})$, there exist sequences $\{\hat{z}(j)\}_j \subset (\Phi^x_{k,i})^{-1}(\mathbf{C}^{x,f}_{k,i})$ and $\{\hat{\tau}_j\}_j \mathbf{R}_{>0}$ such that $\overline{z(j), \hat{z}(j)} \le \hat{\tau}_j \eta_j$ and $\hat{\tau}_j \to 0$ as $j \to \infty$. Put $\alpha(j) = \Phi^x_{k,i}(\hat{z}(j))$. Then we have $|\pi_1(\alpha(j)) - \pi_1(w)| \le (1+\delta)\hat{\tau}_j \eta_j$. Therefore, there exists $\hat{\alpha}(j) \in \{w_1\} \times \mathbf{R}^{k-1}$ such that $\overline{w(j), \hat{\alpha}(j)} \le \Psi(\hat{\tau}_j; n)\eta_j$. Then, we have

$$\begin{aligned}\frac{|f(z(j)) - f(z)|}{\overline{z(j), z}} &\le \frac{|f(\hat{z}(j)) - f(z)|}{\eta_j} + L\hat{\tau}_j \\ &\le \frac{|F^x_{k,i}(w(j)) - F^x_{k,i}(w)|}{\eta_j} + \Psi(\hat{\tau}_j; n, L) \le \frac{|F^x_{k,i}(\hat{\alpha}(j)) - F^x_{k,i}(w)|}{\overline{\hat{\alpha}(j), w}} \frac{\overline{\hat{\alpha}(j), w}}{\eta_j} + L\Psi(\hat{\tau}_j; n, L).\end{aligned}$$

By letting $j \to \infty$, we have Claim 3.26 for the case $k \ge 2$.

Next, assume $k = 1$. It suffices to check that z is an isolated point in $\partial B_{\overline{x,z}}(x)$. The proof is done by a contradiction. Assume that z is not an isolated point in $\partial B_{\overline{x,z}}(x)$. Then, there exists a sequence $\{z(i)\}_i$ of points in $\partial B_{\overline{x,z}}(x) \setminus \{z\}$ such that $z(i) \to z$. On the other hand, there exist $\tau_0 > 0$ and an isometric embedding γ from $[0, \overline{x,z} + \tau_0]$ to Y such that $\gamma(0) = x$ and $\gamma(\overline{x,z}) = z$. Put $\epsilon(i) = \overline{z, z(i)}$. Then we have $\overline{z(i), \gamma(\overline{x,z} - \epsilon_i)} \ge \overline{x, z(i)} - \overline{x, \gamma(\overline{x,z} - \epsilon_i)} = \epsilon_i$ and $\overline{z(i), \gamma(\overline{x,z} + \epsilon_i)} \ge \overline{x, \gamma(\overline{x,z} + \epsilon_i)} - \overline{x, z(i)} = \epsilon_i$. By Gromov's compactness theorem, without loss of generality, we can assume that $(Y, \epsilon_i^{-1} d_Y, z)$ converges to a tangent cone $(T_z Y, 0_z)$ at z. By the argument above and the splitting theorem on limit spaces, there exists a pointed proper geodesic space (W, w) such that $T_z Y = \mathbf{R} \times W$ and $W \neq \{w\}$. However, since $z \in C_{1,i} \subset \mathcal{R}_1$, this is a contradiction. Therefore we have the Claim 3.26.

By Claim 3.22, 3.25 and 3.26, for every $N \in \mathbf{N}$, we have $\mathrm{Lip} f(z)^2 = \mathrm{Lip}^{\mathrm{rad}}_x f(z)^2 + \mathrm{Lip}(f|_{\partial B_{\overline{x,z}}(x)})(z)^2 \pm N^{-1} = lip^{\mathrm{rad}}_x f(z)^2 + lip(f|_{\partial B_{\overline{x,z}}(x) \setminus C_x})(z)^2 \pm N^{-1} = lip f(z)^2 \pm N^{-1}$ for a.e. $z \in Y \setminus C_x$. Therefore, we have the assertion. $\square$

Remark 3.27. For every Ricci limit space (Y, y, υ) and every Lipschitz function f on Y, we have $lip f(x) = \mathrm{Lip} f(x)$ for a.e. $x \in Y$. See [2, Corollary 6.36]

By an argument similar to the proof of Lemma 3.19, we have the following:

Lemma 3.28. *Let (Y, y) be a Ricci limit space with $Y \neq \{y\}$, z a point in $Y \setminus C_y$, f a Lipschitz function on Y, τ a positive number and $\{\gamma_i\}_{i=1,2}$ isometric embeddings from $[0, \overline{y,z} + \tau]$ to Y with $\gamma_i(0) = y$, $\gamma_i(\overline{y,z}) = z$. Then, we have* $\liminf_{r \to 0} |f \circ \gamma_1(\overline{y,z} + r) - f(z)|/|r| = \liminf_{r \to 0} |f \circ \gamma_2(\overline{y,z} + r) - f(z)|/|r|$. *Moreover, if the limit* $\lim_{r \to 0}(f \circ \gamma_1(\overline{y,z} + r) - f(z))/r$ *exists, then, we have* $\lim_{r \to 0}(f \circ \gamma_2(\overline{y,z} + r) - f(z))/r = \lim_{r \to 0}(f \circ \gamma_1(\overline{y,z} + r) - f(z))/r$.

With the same notaion as in Lemma 3.28, put $\underline{\mathrm{Lip}}^{\mathrm{rad}}_x f(z) = \liminf_{r \to 0} |f \circ \gamma_1(\overline{y,z} + r) - f(z)|/|r|$. Let (Y, y) be a Ricci limit space with $Y \neq \{y\}$, and f a Lipschitz function on Y. Put

$$A_y = \left\{ x \in Y \setminus C_y ; \text{The limit } \lim_{r \to 0} \frac{f \circ \gamma(\overline{x,y} + r) - f(x)}{r} \text{ exists} \right\}.$$

Here γ is an isometric embedding from $[0, \overline{y,x}+\tau]$ $(\tau > 0)$ to Y with $\gamma(0) = y$, $\gamma(\overline{y,x}) = x$. Put

$$\frac{df}{dr_y}(x) = \lim_{r \to 0} \frac{f \circ \gamma(\overline{x,y} + r) - f(x)}{r}$$

for every $x \in A_y$.

LEMMA 3.29. *Let (Y, y, υ) be a Ricci limit space, x a point in Y and f a Lipschitz function on Y. Then, we have $\underline{\mathrm{Lip}}_x^{\mathrm{rad}} f(z) = \mathrm{Lip}_x^{\mathrm{rad}} f(z)$ for a.e. $z \in Y$.*

PROOF. We will use the same notaion as in the proof of Claim 3.25. Put $L = \mathbf{Lip} f$. Let δ be a sufficiently small positive number and $C_{k,i}^x$ a Borel subset of Y satisfying that the map $\Phi_{k,i}^x = (r_x, r_{x_{k,i}^2}, \ldots, r_{x_{k,i}^k})$ from $C_{k,i}^x$ to $\mathbf{R}^k$, is $(1 \pm \delta)$-bi-Lipschitz to the image. Fix $w \in \mathbf{C}_{k,i}^{x,f}$ and put $z = (\Phi_{k,i}^x)^{-1}(w)$. There exist a positive number τ and an isometric embedding γ from $[0, \overline{x,z} + \tau]$ to Y such that $\gamma(0) = x$ and $\gamma(\overline{x,z}) = z$. Let $\{\epsilon_i\}_i$ be a sequence of real numbers satisfying that $\epsilon_i \to 0$ and $\lim_{i \to \infty} |f \circ \gamma(\overline{x,z} + \epsilon_i) - f(z)|/|\epsilon_i| = \underline{\mathrm{Lip}}_x^{\mathrm{rad}} f(z)$. By an argument similar to the proof of Claim 3.12, there exist sequences $\{\hat{w}(j)\}_j \subset C_{k,i}^x$ and $\{\tau_j\}_j \subset \mathbf{R}_{>0}$ such that $\overline{\hat{w}(j), \gamma(\overline{x,z} + \epsilon_j)} \le \tau_j |\epsilon_j|$, $\tau_j \to 0$ as $j \to \infty$, and

$$\frac{|f(z) - f(\gamma(\overline{x,z} + \epsilon_j))|}{|\epsilon_j|} = \frac{|F_{k,i}^x(w) - F_{k,i}^x(\Phi_{k,i}^x(\hat{w}(j)))|}{|\epsilon_j|} - 2L\tau_j \ge \frac{|F_{k,i}^x(w) - F_{k,i}^x(w_j)|}{|\epsilon_j|} - \Psi(\tau_j, \delta; n, L).$$

By letting $j \to \infty$, we have $\underline{\mathrm{Lip}}_x^{\mathrm{rad}} f(z) \ge \mathrm{Lip}(F_{k,i}^x|_{\mathbf{R} \times \{w_2, \ldots, w_k\}})(w) - \Psi(\delta; n, L) \ge \mathrm{Lip}_x^{\mathrm{rad}} f(z) - \Psi(\delta; n, L)$. Therefore, we have the assertion. $\square$

We shall state the main theorem in this subsection:

THEOREM 3.30 (Radial derivatives of Lipschitz functions). *Let (Y, y, υ) be a Ricci limit space with $Y \neq \{y\}$, x a point in Y and f a Lipschitz function on Y. Then, we have $\upsilon(Y \setminus A_x) = 0$ and*

$$\frac{df}{dr_x}(z) = \langle df, dr_x \rangle (z)$$

for a.e. $z \in A_x$.

PROOF. For every $w \in Y \setminus C_x$, there exist $\tau > 0$ and an isometric embedding γ from $[0, \overline{x,z}+\tau]$ to Y such that $\gamma(0) = x$ and $\gamma(\overline{x,w}) = w$. Then, by Theorem 3.21 and Lemma

3.29, for a.e. $w \in Y \setminus C_x$, we have

$$\begin{aligned}
\langle dr_x, df\rangle(w) &= \frac{1}{2}(\mathrm{Lip}(r_x+f)(w)^2 - \mathrm{Lip}f(w)^2 - \mathrm{Lip}r_x(w)^2)\\
&= \frac{1}{2}(\mathrm{Lip}_x^{\mathrm{rad}}(r_x+f)(w)^2 + \mathrm{Lip}((r_x+f)|_{\partial B_{\overline{x,z}}(x)\setminus C_x})(w)^2\\
&\quad - \mathrm{Lip}_x^{\mathrm{rad}}f(w)^2 - \mathrm{Lip}(f|_{\partial B_{\overline{x,z}}\setminus C_x})(w)^2 - 1)\\
&= \frac{1}{2}(\mathrm{Lip}_x^{\mathrm{rad}}(r_x+f)(w)^2 + \mathrm{Lip}(f|_{\partial B_{\overline{x,z}}(x)\setminus C_x})(w)^2\\
&\quad - \mathrm{Lip}_x^{\mathrm{rad}}f(w)^2 - \mathrm{Lip}(f|_{\partial B_{\overline{x,z}}\setminus C_x})(w)^2 - 1)\\
&= \frac{1}{2}(\mathrm{Lip}_x^{\mathrm{rad}}(r_x+f)(w)^2 - \mathrm{Lip}_x^{\mathrm{rad}}f(w)^2 - 1)\\
&= \frac{1}{2}\left(\lim_{h\to 0}\frac{|(r_x+f)\circ\gamma(\overline{x,w}+h)-(r_x+f)(w)|^2}{|h|^2} - \lim_{h\to 0}\frac{|f\circ\gamma(\overline{x,w}+h)-f(w)|^2}{|h|^2} - 1\right)\\
&= \frac{1}{2}\left(\lim_{h\to 0}\left|1+\frac{f\circ\gamma(\overline{x,w}+h)-f(w)}{h}\right|^2 - \lim_{h\to 0}\frac{|f\circ\gamma(\overline{x,w}+h)-f(w)|^2}{|h|^2} - 1\right)\\
&\left(\text{Here, we have the existence of the limit } \lim_{h\to 0}\frac{f\circ\gamma(\overline{x,w}+h)-f(w)}{h}.\right)\\
&= \frac{1}{2}\left(1+2\lim_{h\to 0}\frac{f\circ\gamma(\overline{x,w}+h)-f(w)}{h} + \lim_{h\to 0}\frac{|f\circ\gamma(\overline{x,w}+h)-f(w)|^2}{|h|^2}\right.\\
&\quad \left. - \lim_{h\to 0}\frac{|f\circ\gamma(\overline{x,w}+h)-f(w)|^2}{|h|^2} - 1\right)\\
&= \lim_{h\to 0}\frac{f\circ\gamma(\overline{x,w}+h)-f(w)}{h} = \frac{df}{dr_x}(w).
\end{aligned}$$

□

3.3 Rectifiability associated with Lipschitz functions

In this section, we will give a generalization of Theorem 3.16. The main result in this subsection is Theorem 3.49.

LEMMA 3.31. *Let δ be a positive number, $\{(M_i, m_i)\}_i$ a sequence of n-dimensional complete Riemannian manifolds with $\mathrm{Ric}_{M_i} \geq -\delta(n-1)$, (Y, y, υ) a $(n, -\delta)$-Ricci limit space of $\{(M_i, m_i, \underline{\mathrm{vol}})\}_i$, x, x_1, x_2 points in Y, $x(i), x_1(i), x_2(i)$ points in M_i for every $i < \infty$, $\mathbf{b}_1^i$ a harmonic function on $B_{100}(x(i))$ for every $i < \infty$, and $\mathbf{b}_1^\infty$ a Lipschitz function on $B_{100}(x)$. Assume that $\overline{x,x_1} \geq \delta^{-1}$, $\overline{x,x_2} \geq \delta^{-1}$, $\overline{x,x_1} + \overline{x,x_2} - \overline{x_1,x_2} \leq \delta$, $x(i) \to x$, $x_j(i) \to x_j(i)$ for every $j \in \{1,2\}$, $\sup_i \mathbf{Lip}\mathbf{b}_1^i < \infty$, $\mathbf{b}_1^i \to \mathbf{b}_1^\infty$ on $B_{100}(x)$, $|\mathbf{b}_1^i - r_{x_1(i)}|_{L^\infty(B_{100}(x(i)))} \leq \delta$ and*

$$\frac{1}{\mathrm{vol}\, B_{100}(x(i))}\int_{B_{100}(x(i))}\left(|\nabla \mathbf{b}_1^i - \nabla r_{x_1(i)}|^2 + |\mathrm{Hess}_{\mathbf{b}_1^i}|^2\right) d\mathrm{vol} \leq \delta$$

Then, we have

$$\frac{1}{\upsilon(B_1(x))}\int_{B_1(x)}|d\mathbf{b}_1^{\infty}-dr_{x_1}|^2 d\upsilon < \Psi(\delta;n).$$

We remark that Lemma 3.31 does *not* follow from [2, Lemma 9.10] directly. We shall give a proof of Lemma 3.31 in the proof of the following Lemma 3.32.

LEMMA 3.32. *Let δ be a positive number, $\{(M_i,m_i)\}_i$ a sequence of n-dimensional complete Riemannian manifolds with $\mathrm{Ric}_{M_i}\geq -\delta(n-1)$, (Y,y,υ) a $(n,-\delta)$-Ricci limit space of $\{(M_i,m_i,\underline{\mathrm{vol}})\}_i$, x a point in Y, $\{x_j\}_{1\leq j\leq 4}$ a collection of points in Y, and $\{x(i)\}\cup\{x_j(i)\}_{1\leq j\leq 4}$ of points in M_i for every i. Assume that $x(i)\to x$, $x_j(i)\to x_j$ for every j, $\overline{x,x_j}\geq \delta^{-1}$ for every j, $\overline{x,x_1}+\overline{x,x_2}-\overline{x_1,x_2}\leq\delta$ and $\overline{x,x_3}+\overline{x,x_4}-\overline{x_3,x_4}\leq\delta$. Then, we have*

$$\frac{1}{\upsilon(B_1(x))}\int_{B_1(x)}\left|\langle dr_{x_1},dr_{x_3}\rangle d\upsilon-\frac{1}{\mathrm{vol}\,B_1(x(i))}\int_{B_1(x(i))}\langle dr_{x_1(i)},dr_{x_3(i)}\rangle d\mathrm{vol}\right|d\upsilon<\Psi(\delta;n)$$

and

$$\frac{1}{\mathrm{vol}\,B_1(x(i))}\int_{B_1(x(i))}\left|\langle dr_{x_1(i)},dr_{x_3(i)}\rangle-\frac{1}{\upsilon(B_1(x))}\int_{B_1(x)}\langle dr_{x_1},dr_{x_3}\rangle d\upsilon\right|d\mathrm{vol}<\Psi(\delta;n)$$

for every sufficiently large i.

PROOF. First, we remark the following claim:

CLAIM 3.33. *For every sufficiently large i, there exist harmonic functions $\mathbf{b}_1^i,\mathbf{b}_3^i$ on $B_{100}(x(i))$ such that $\mathbf{Lip}\mathbf{b}_j^i\leq C(n)$, $|\mathbf{b}_j^i-r_{x_j(i)}|_{L^\infty(B_{100}(x(i)))}\leq\Psi(\delta;n)$ and*

$$\frac{1}{\mathrm{vol}\,B_{100}(x(i))}\int_{B_{100}(x(i))}\left(|d\mathbf{b}_j^i-dr_{x_j(i)}|^2+|\mathrm{Hess}_{\mathbf{b}_j^i}|^2\right)d\mathrm{vol}\leq\Psi(\delta;n)$$

for every $j\in\{1,3\}$.

See for instance [2, Lemma 9.8], [2, Lemma 9.10], [2, Lemma 9.13] for a proof of Claim 3.33.

Since $C(n)(|\mathrm{Hess}_{\mathbf{b}_1^i}|^2+|\mathrm{Hess}_{\mathbf{b}_3^i}|^2)$ is an upper gradient of $\langle d\mathbf{b}_1^i,d\mathbf{b}_3^i\rangle$, by the Poincaré inequality, we have

$$\frac{1}{\mathrm{vol}\,B_{100}(x(i))}\int_{B_{100}(x(i))}\left|\langle d\mathbf{b}_1^i,d\mathbf{b}_3^i\rangle-\frac{1}{\mathrm{vol}\,B_{100}(x(i))}\int_{B_{100}(x(i))}\langle d\mathbf{b}_1^i,d\mathbf{b}_3^i\rangle d\mathrm{vol}\right|d\mathrm{vol}$$
$$\leq C(n)\sqrt{\frac{1}{\mathrm{vol}\,B_{100}(x(i))}\int_{B_{100}(x(i))}\left(|\mathrm{Hess}_{\mathbf{b}_1^i}|^2+|\mathrm{Hess}_{\mathbf{b}_3^i}|^2\right)d\mathrm{vol}}\leq\Psi(\delta;n).$$

Therefore, we have

$$\frac{1}{\mathrm{vol}\,B_{100}(x(i))}\int_{B_{100}(x(i))}\left|\langle d\mathbf{b}_3^i,dr_{x_1(i)}\rangle-\frac{1}{\mathrm{vol}\,B_{100}(x(i))}\int_{B_{100}(x(i))}\langle d\mathbf{b}_3^i,dr_{x_1(i)}\rangle d\mathrm{vol}\right|d\mathrm{vol}\leq\Psi(\delta;n).$$

Without loss of generality, we can assume that there exist Lipschitz functions $\mathbf{b}_1^\infty, \mathbf{b}_3^\infty$ on $B_{100}(x)$ such that $\mathbf{b}_j^i \to \mathbf{b}_j^\infty$ on $B_{100}(x)$. By Theorem 3.30, there exists a Borel subset A of $B_{100}(x) \setminus C_{x_1}$ such that $\upsilon(B_{100}(x) \setminus A) = 0$ and $\lim_{h\to 0}(f \circ \gamma(\overline{x_1, a} + h) - f(a))/h = \langle dr_{x_1}, d\mathbf{b}_3^\infty\rangle(a)$ for every $a \in A$ and every minimal geodesic γ from x_1 to a. By Lusin's theorem, there exists a Borel subset $A(\delta)$ of A such that $\upsilon(A \setminus A(\delta)) < \delta\upsilon(B_1(x))$ and that the function $\langle dr_{x_1}, df\rangle$ is continuous on $A(\delta)$. Define a function f_η^δ on $A(\delta) \setminus B_{2\delta}(x)$ by

$$f_\eta^\delta(z) = \sup_{w \in C_z(\{x_1\}) \cap \overline{B}_\eta(z)} \left| \frac{f(z) - f(w)}{\overline{z, w}} - \langle dr_{x_1}, df\rangle(z) \right|$$

for every $0 < \eta < \delta$. It is easy to check that f_η^δ is an upper semi-continuous function. Especially, f_η^δ is a Borel function. We also have $\lim_{\eta\to 0} f_\eta^\delta(a) = 0$ for every $a \in A$. Thus, by Egoroff's theorem, there exists a Borel subset $X = X(\delta)$ of $A(\delta)$ such that $\upsilon(A(\delta) \setminus X(\delta)) < \delta\upsilon(B_1(x))$ and $\lim_{\eta\to 0}(\sup_{a\in X} f_\eta^\delta(a)) = 0$. Let $\eta = \eta(\delta)$ be a positive number satisfying that $\eta << \delta$, and $\sup_{a\in X} f_{\eta_0}^\delta(a) < \delta$ for every $\eta_0 \leq \eta$. For every i, let X_i be the set of points $w \in B_1(x(i))$ satisfying that

$$\left| \langle d\mathbf{b}_3^i, dr_{x_1(i)}\rangle(w) - \frac{1}{\operatorname{vol} B_{100}(x(i))} \int_{B_{100}(x(i))} \langle d\mathbf{b}_3^i, dr_{x_1(i)}\rangle d\operatorname{vol} \right| \leq \Psi(\delta; n).$$

Then, we have $\operatorname{vol}(B_1(x(i)) \setminus X_i)/\operatorname{vol} B_1(x(i)) \leq \Psi(\delta; n)$ for every sufficiently large i. Define a Borel function F_i on $B_{100}(x(i)) \setminus C_{x_1(i)}$ by

$$F_i(w) = \frac{\mathbf{b}_3^i\left(\gamma(\overline{x_1(i), w} - \eta^2)\right) - \mathbf{b}_3^i(w)}{-\eta^2}$$

for every i, where γ is the minimal geodesic from $x_1(i)$ to w.

Claim 3.34. *We have*

$$\frac{1}{\operatorname{vol} B_{10}(x(i))} \int_{B_{10}(x(i)) \setminus C_{x_1(i)}} |\langle d\mathbf{b}_3^i, dr_{x_1(i)}\rangle - F_i(w)| d\operatorname{vol} \leq \Psi(\delta; n)$$

for every sufficiently large i.

The proof is as follows. It is easy to check that

$$f(t) = f(c) + f'(t)(t - c) - \int_c^t (s - c) f''(s) ds$$

for every $a < b$, every C^2-function f on (a, b), and every $c \in (a, b)$. Therefore, we have

$$\frac{\mathbf{b}_3^i(\gamma(\overline{x_1(i), w} - \eta^2)) - \mathbf{b}_3^i(w)}{-\eta^2} = \frac{d\mathbf{b}_3^i}{dr_{x_1(i)}}(w) - \frac{1}{\eta^2} \int_{\overline{x_1(i), w} - \eta^2}^{\overline{x_1(i), w}} \left(s - (\overline{x_1(i), w} - \eta^2)\right) \frac{d^2\mathbf{b}_3^i}{dr_{x_1(i)}^2}(\gamma(s)) ds.$$

Thus, by an argument similar to the proof of [21, Estimate 2.6], we have

$$\begin{aligned}
&\frac{1}{\mathrm{vol}\, B_{10}(x(i))}\int_{B_{10}(x(i))\setminus C_{x_1(i)}}\left|\langle d\mathbf{b}^i_3, dr_{x_1(i)}\rangle - F_i(w)\right| d\mathrm{vol}\\
&\le \frac{1}{\eta^2}\frac{1}{\mathrm{vol}\, B_{10}(x(i))}\int_{B_{10}(x(i))}\int_{\overline{x_1(i),w}-\eta^2}^{\overline{x_1(i),w}}\eta^2|\mathrm{Hess}_{\mathbf{b}^i_3}|(\gamma(s))ds d\underline{\mathrm{vol}}\\
&\le \eta^2 C(n)\frac{1}{\mathrm{vol}\, B_{100}(x(i))}\int_{B_{100}(x(i))}|\mathrm{Hess}_{\mathbf{b}^i_3}|d\mathrm{vol}\\
&\le \eta^2 C(n)\sqrt{\frac{1}{\mathrm{vol}\, B_{100}(x(i))}\int_{B_{100}(x(i))}|\mathrm{Hess}_{\mathbf{b}^i_3}|^2 d\mathrm{vol}} \le \eta^2 C(n)\Psi(\delta;n).
\end{aligned}$$

Therefore, we have Claim 3.34

Claim 3.35. *We have*

$$\frac{1}{\upsilon(B_1(x))}\int_{B_1(x)}\left|\langle d\mathbf{b}^\infty_3, dr_{x_1}\rangle - \frac{1}{\mathrm{vol}\, B_1(x(i))}\int_{B_1(x(i))}\langle d\mathbf{b}^i_3, dr_{x_1(i)}\rangle d\mathrm{vol}\right| d\upsilon \le \Psi(\delta;n)$$

for sufficiently large i.

The proof is as follows. Let $Y_i = \{w \in \overline{B}_1(x(i)) \setminus C_{x_1(i)}; |\langle d\mathbf{b}^i_3, dr_{x_1(i)}\rangle(w) - F_i(w)| \le \Psi(\delta;n)\}$. By Claim 3.34, we have $\mathrm{vol}\,(\overline{B}_1(x(i)) \setminus Y_i)/\mathrm{vol}\,\overline{B}_1(x(i)) \le \Psi(\delta;n)$ for every sufficiently large i. Put $Z_i = X_i \cap Y_i$. There exists a compact subset W_i of Z_i such that $\mathrm{vol}(Z_i \setminus W_i)/\mathrm{vol}\,\overline{B}_1(x(i)) \le \Psi(\delta;n)$. Then, we have $\mathrm{vol}(\overline{B}_1(x(i)) \setminus W_i)/\mathrm{vol}\,\overline{B}_1(x(i)) \le \Psi(\delta;n)$ for every sufficiently large i. Without loss of generality, we can assume that there exists a compact subset W_∞ of $\overline{B}_1(x)$ such that $W_j \to W_\infty$. By Lemma 2.5, we have $\upsilon(W_\infty)/\upsilon(\overline{B}_1(x)) \ge 1 - \Psi(\delta;n)$. Put $E = W_\infty \cap X$. Then we have $\upsilon(\overline{B}_1(x) \setminus E) \le \Psi(\delta;n)\upsilon(\overline{B}_1(x))$. For every $w_i \in W_i$ and every $w \in E$, let γ_{w_i} be the minimal geodesic from $x_1(i)$ to w_i, and γ_w a minimal geodesic from x_1 to w. Then, there exists i_0 such that $\epsilon_i << \eta$,

$$\left|\langle d\mathbf{b}^i_3, dr_{x_1(i)}\rangle(w) - \frac{\mathbf{b}^i_3(\gamma_i(\overline{x_1(i), w_i} - \eta^2)) - \mathbf{b}^i_3(w_i)}{-\eta^2}\right| \le \Psi(\delta;n)$$

and

$$\left|\langle d\mathbf{b}^i_3, dr_{x_1(i)}\rangle(w_i) - \frac{1}{\mathrm{vol}\, B_{100}(x(i))}\int_{B_{100}(x(i))}\langle d\mathbf{b}^i_3, dr_{x_1(i)}\rangle d\mathrm{vol}\right| \le \Psi(\delta;n)$$

for every $i \ge i_0$, every $w \in E$ and every $w_i \in W_i$ with $w_i \to w$. Now, we shall consider the rescaled metric $\eta^{-2}d_Y$. Since

$$\overline{x_1, \phi_i(\gamma_i(\overline{x_1(i), w_i} - \eta^2))}^{\eta^{-2}d_Y} \ge \eta^{-1},\ \overline{\phi_i(\gamma_i(\overline{x_1(i), w_i} - \eta^2)), w}^{\eta^{-2}d_Y} \ge \eta^{-1}$$

and

$$\overline{x_1, \phi_i(\gamma_i(\overline{x_1(i), w_i} - \eta^2))}^{\eta^{-2}d_Y} + \overline{\phi_i(\gamma_i(\overline{x_1(i), w_i} - \eta^2)), w}^{\eta^{-2}d_Y} - \overline{x_1, w}^{\eta^{-2}d_Y} \le \eta,$$

by the splitting theorem on limit spaces, we have

$$\overline{\phi_i(\gamma_i(\overline{x_1(i),w_i}-\eta^2)),\gamma(\overline{x_1,w}-\eta^2)}^{\eta^{-2}d_Y}\leq\Psi(\delta;n).$$

Therefore, we have

$$\left|\frac{\mathbf{b}_3^i(\gamma_i(\overline{x_1(i),w_i}-\eta^2))-\mathbf{b}_3^i(w_i)}{-\eta^2}-\frac{\mathbf{b}_3^\infty(\gamma(\overline{x_1,w}-\eta^2))-\mathbf{b}_3^\infty(w)}{-\eta^2}\right|\leq\Psi(\delta;n).$$

Thus, for every $i\geq i_0$, we have

$$\left|\langle d\mathbf{b}_3^\infty,dr_{x_1}\rangle(w)-\frac{1}{\operatorname{vol}B_{100}(x(i))}\int_{B_{100}(x(i))}\langle d\mathbf{b}_3^i,dr_{x_1(i)}\rangle d\operatorname{vol}\right|\leq\Psi(\delta;n).$$

Let

$$C_i=\frac{1}{\operatorname{vol}B_{100}(x(i))}\int_{B_{100}(x(i))}\langle d\mathbf{b}_3^i,dr_{x_1(i)}\rangle d\operatorname{vol}.$$

Then

$$\begin{aligned}&\frac{1}{\upsilon(B_1(x))}\int_{B_1(x)}|\langle d\mathbf{b}_3^\infty,dr_{x_1}\rangle-C_i|\,d\upsilon\\&=\frac{1}{\upsilon(B_1(x))}\int_{B_1(x)\setminus E}|\langle d\mathbf{b}_3^\infty,dr_{x_1}\rangle-C_i|\,d\upsilon+\frac{1}{\upsilon(B_1(x))}\int_E|\langle d\mathbf{b}_3^\infty,dr_{x_1}\rangle-C_i|\,d\upsilon\\&\leq\frac{C(n)\upsilon(B_1(x)\setminus E)}{\upsilon(B_1(x))}+\frac{\upsilon(E)}{\upsilon(B_1(x))}\Psi(\delta;n)\leq\Psi(\delta;n).\end{aligned}$$

Therefore, we have Claim 3.35.

CLAIM 3.36. *We have*

$$\frac{1}{\upsilon(B_1(x))}\int_{B_1(x)}|d\mathbf{b}_3^\infty|^2d\upsilon\leq1+\Psi(\delta;n).$$

This proof is as follows. Since

$$\frac{1}{\operatorname{vol}B_1(x(i))}\int_{B_1(x(i))}||d\mathbf{b}_3^i|-1|d\operatorname{vol}\leq\Psi(\delta;n)$$

for every sufficiently large i, by [1, Lemma 16.2], there exists a compact subset K_i of $\overline{B}_1(x(i))$ such that $\underline{\operatorname{vol}}(B_1(x(i))\setminus K_i)/\underline{\operatorname{vol}}\,B_1(x(i))\leq\Psi(\delta;n)$ and $\mathbf{Lip}(\mathbf{b}_3^i|_{K_i})\leq1+\Psi(\delta;n)$. Without loss of generality, we can assume that there exists a compact subset K_∞ of $\overline{B}_1(x)$ such that $K_i\to K_\infty$. By Lemma 2.5, we have $\upsilon(K_\infty)/\upsilon(B_1(x))\geq1-\Psi(\delta;n)$. Then, we have $\mathbf{Lip}(\mathbf{b}_3^\infty|_{K_\infty}) \le 1 + \Psi(\delta; n)$. Put $\hat{K}_\infty = \mathrm{Leb}K_\infty$. Then by Lemma 3.20, we have

$$\begin{aligned}\frac{1}{\upsilon(B_1(x))}\int_{B_1(x)} |d\mathbf{b}_3^\infty|^2 d\upsilon &= \frac{1}{\upsilon(B_1(x))}\int_{\hat{K}_\infty} |d\mathbf{b}_3^\infty|^2 d\upsilon + \frac{1}{\upsilon(B_1(x))}\int_{B_1(x)\setminus K_\infty} |d\mathbf{b}_3^\infty|^2 d\upsilon \\ &\le \frac{1}{\upsilon(B_1(x))}\int_{\hat{K}_\infty} (\mathrm{Lip}\mathbf{b}_3^\infty)^2 d\upsilon + C(n)\frac{\upsilon(B_1(x)\setminus K_\infty)}{\upsilon(B_1(x))} \\ &\le \frac{1}{\upsilon(B_1(x))}\int_{\hat{K}_\infty} (\mathrm{Lip}(\mathbf{b}_3^\infty|_{K_\infty}))^2 d\upsilon + \Psi(\delta; n) \\ &\le \frac{1}{\upsilon(B_1(x))}\int_{\hat{K}_\infty} (1 + \Psi(\delta; n)) d\upsilon + \Psi(\delta; n) \le 1 + \Psi(\delta; n).\end{aligned}$$

Therefore, we have Claim 3.36.

Assume that $x_1 = x_3$ and $x_2 = x_4$. Then, by Claim 3.33, 3.35 and 3.36, we have

$$\begin{aligned}&\frac{1}{\upsilon(B_1(x))}\int_{B_1(x)} |d\mathbf{b}_3^\infty - dr_{x_3}|^2 d\upsilon \\ &= \frac{1}{\upsilon(B_1(x))}\int_{B_1(x)} |d\mathbf{b}_3^\infty|^2 d\upsilon - 2\frac{1}{\upsilon(B_1(x))}\int_{B_1(x)} \langle d\mathbf{b}_3^\infty, dr_{x_3}\rangle d\upsilon + \frac{1}{\upsilon(B_1(x))}\int_{B_1(x)} |dr_{x_3}|^2 d\upsilon \\ &\le 1 + \Psi(\delta; n) - 2(1 - \Psi(\delta; n)) + 1 \le \Psi(\delta; n)\end{aligned}$$

for every sufficiently large i. Therefore, we have Lemma 3.31. On the other hand, Lemma 3.32 follows from Lemma 3.31 and Claim 3.35, directly. □

COROLLARY 3.37. *Let $\{(M_i, m_i)\}_i$ be a sequence of n-dimensional complete Riemannian manifolds with $\mathrm{Ric}_{M_i} \ge -(n-1)$, (Y, y, υ) a Ricci limit space of $\{(M_i, m_i, \underline{\mathrm{vol}})\}_i$, τ a positive number, x, x_1, x_2 points in Y, $\{x(i)\}_i, \{x_1(i)\}_i, \{x_2(i)\}_i$ sequences of points $x(i), x_1(i), x_2(i)$ in M_i. Assume that $x \in \bigcap_{j=1,2}(\mathcal{D}_{x_j}^\tau \setminus B_\tau(x_j))$, $x(i) \to x$, and $x_j(i) \to x_j$ for every j. Then, we have*

$$\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)} \left| \langle dr_{x_1}, dr_{x_2}\rangle - \frac{1}{\mathrm{vol}\, B_r(x(i))}\int_{B_r(x)} \langle dr_{x_1(i)}, dr_{x_2(i)}\rangle d\mathrm{vol} \right| d\upsilon \le \Psi\left(r, \frac{r}{\tau}; n\right)$$

and

$$\frac{1}{\mathrm{vol}\, B_r(x(i))}\int_{B_r(x(i))} \left| \langle dr_{x_1(i)}, dr_{x_2(i)}\rangle - \frac{1}{\upsilon(B_r(x))}\int_{B_r(x)} \langle dr_{x_1}, dr_{x_2}\rangle d\upsilon \right| d\mathrm{vol} \le \Psi\left(r, \frac{r}{\tau}; n\right)$$

for every sufficiently large i.

PROOF. By rescaling $r^{-1}d_Y$, and Lemma 3.32, it is easy to check the assertion. □

LEMMA 3.38. *Let $\{(M_i, m_i)\}_i$ be a sequence of n-dimensional complete Riemannian manifolds with $\mathrm{Ric}_{M_i} \ge -(n-1)$, (Y, y, υ) a Ricci limit space of $\{(M_i, m_i, \underline{\mathrm{vol}})\}_i$, l a positive integer, r, ϵ, τ, L positive real numbers, x a point in Y, $\{x(i)\}_i$ a sequence of* *points x_i in M_i, $\{k_\alpha\}_{1\le\alpha\le l}$ a collection of positive integers, $\{x^s_t\}_{1\le s\le l, 1\le t\le k_s}$ of points in Y, $\{x^s_t(i)\}_{1\le s\le l, 1\le t\le k_s}$ of points in M_i for every $i<\infty$, and $\{a^s_t\}_{1\le s\le l,1\le t\le k_s}$ of real numbers. Let $f_j=\sum_{m=1}^{k_j} a^j_m r_{x^j_m}$ and $f^i_j=\sum_{m=1}^{k_j} a^j_m r_{x^j_m(i)}$. Assume that $l\le n$, $k_i\le n$ for every $1\le i\le l$, $x\in\bigcap^k_{1\le i\le l,1\le j\le k_i}(\mathcal{D}^\tau_{x^i_j}\setminus B_\tau(x^i_j))$, $x(i)\to x$, $x^s_t(i)\to x^s_t$, $\sum_{i,j}(a^i_j)^2\le L$ and*

$$\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\langle df_j, df_i\rangle d\upsilon=\delta_{ij}\pm\epsilon.$$

Then, for every sufficiently large i, there exists a compact subset K^i_r of $\overline{B}_{r/10}(x(i))$ such that the following properties hold:

1. $\mathrm{vol}(B_{r/10}(x(i))\setminus K^i_r)/\mathrm{vol}\,B_{r/10}(x(i))\le\Psi(r,r/\tau,\epsilon;n,L)$.
2. *For every $w\in K^i_r$ and every $0<s<r/10^6$, there exist a compact subset Z of $\overline{B}_s(w)$, a point z in Z, and a map ϕ from $(\overline{B}_s(w),w)$ to (Z,z) such that the map $\Phi=(f^i_1,f^i_2,\dots,f^i_l,\phi)$ from $\overline{B}_s(w)$ to $\overline{B}_{s+\Psi(r,r/\tau,\epsilon;n,L)s}(f^i_1(w),\dots,f^i_l(w),\phi(w))$, is an $\Psi(r,r/\tau,\epsilon;n,L)s$-Gromov-Hausdorff approximation.*
3. *We have*
$$\frac{1}{\mathrm{vol}\,B_s(w)}\int_{B_s(w)}|\langle df^i_\alpha, df^i_\beta\rangle-\delta_{\alpha\beta}|d\underline{\mathrm{vol}}<\Psi\left(r,\frac{r}{\tau},\epsilon;n,L\right)$$
for every $w\in K^i_r$ and every $0<s<r/10^6$.

Proof. By Corollary 3.37, we have

$$\frac{1}{\mathrm{vol}\,B_r(x(i))}\int_{B_r(x(i))}|\langle df^i_j, df^i_{\hat{l}}\rangle-\delta_{j,\hat{l}}|d\mathrm{vol}\le\Psi\left(r,\frac{r}{\tau},\epsilon;n,L\right)$$

for every sufficiently large i. We shall consider rescaled distances $r^{-1}d_Y$ and $r^{-1}d_{M_i}$ below. For convenience, we shall use the following notations: $\hat{\mathrm{vol}}=\mathrm{vol}^{r^{-1}d_{M_i}}$, $\hat{\upsilon}=\upsilon/\upsilon(B_r(y))$, $\hat{r}_z(w)=r^{-1}\overline{w,z}^{d_Y}$, $\hat{B}_s(w)=B^{r^{-1}d_Y}_s(w)=B_{sr}(w)$, $\hat{g}=r^{-1}g$ for a Lipschitz function g and so on. We remark that $(M_i,m_i,r^{-1}d_{M_i},\underline{\mathrm{vol}}^{r^{-1}d_{M_i}})\to(Y,y,r^{-1}d_Y,\hat{\upsilon})$. We also denote the differential of a Lipschitz function f on Y as a metric measure space $(Y,\hat{\upsilon})$ by $\hat{d}f:Y\to T^*Y$, and the Riemannian metric of rescaled Ricci limit space $(Y,y,r^{-1}d_Y,\hat{\upsilon})$ by $\langle\cdot,\cdot\rangle_r$. Thus, we have $\langle\cdot,\cdot\rangle_r=r^{-2}\langle\cdot,\cdot\rangle$. Then we have

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_1(x(i))}\int_{\hat{B}_1(x(i))}|\langle\hat{d}\hat{f}^i_j,\hat{d}\hat{f}^i_{\hat{l}}\rangle_r-\delta_{j,\hat{l}}|d\hat{\mathrm{vol}}\le\Psi\left(r,\frac{r}{\tau},\epsilon;n,L\right)$$

for every sufficiently large i. On the other hand, by [2, Lemma 9.8], [2, Lemma 9.10], [2, Lemma 9.13], for every sufficiently large i, there exists a collection of harmonic functions $\{\hat{\mathbf{b}}^{m,i}_j\}_{1\le m\le l,1\le j\le k_m}$ on $\hat{B}_{100}(x(i))$ such that $|\hat{\mathbf{b}}^{m,i}_j-\hat{r}_{x^m_j(i)}|_{L^\infty(\hat{B}_{100}(x(i)))}\le\Psi(r,r/\tau;n)$ and

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{100}(x(i))}\int_{\hat{B}_{100}(x(i))}\left(|\hat{d}\hat{\mathbf{b}}^{m,i}_j-\hat{d}\hat{r}_{x^m_j(i)}|^2_r+|\mathrm{Hess}_{\hat{\mathbf{b}}^{m,i}_j}|^2_r\right)d\hat{\mathrm{vol}}\le\Psi\left(r,\frac{r}{\tau};n\right).$$

Let $\hat{\mathbf{b}}^i_j = \sum_{m=1}^{k_j} a^j_m \hat{\mathbf{b}}^{m,i}_j$.

$$\hat{F}_i = \sum_{j=1}^{l} |\hat{d}\hat{\mathbf{b}}^i_j - \hat{d}\hat{f}^i_j|^2_r + \sum_{j=1}^{l} ||\hat{d}\hat{\mathbf{b}}^i_j|^2_r - 1| + \sum_{j<\hat{l}} |\langle \hat{d}\mathbf{b}^i_j, \hat{d}\mathbf{b}^i_{\hat{l}}\rangle_r| + \sum_{j=1}^{l} |\mathrm{Hess}_{\hat{\mathbf{b}}^i_j}|^2_r.$$

The next claim follows from Lemma 3.1, directly:

Claim 3.39. *For every sufficiently large i, there exists a compact subset K^i_r of $\hat{B}_{1/10}(x(i))$ such that $\hat{\mathrm{vol}}(\hat{B}_{\frac{1}{10}}(x(i)) \setminus K^i_r)/\hat{\mathrm{vol}}\, \hat{B}_{\frac{1}{10}}(x(i)) \leq \Psi(r, r/\tau, \epsilon; n, L)$ and*

$$\frac{1}{\hat{\mathrm{vol}}\, \hat{B}_{5s}(w)} \int_{\hat{B}_{5s}(w)} \hat{F}_i d\hat{\mathrm{vol}} \leq \Psi\left(r, \frac{r}{\tau}, \epsilon; n, L\right)$$

for every $w \in K^i_r$ and every $0 < s < 1/10$.

Fix $w \in K^i_r$ and $0 < s \leq 1/10$. By an argument same to the proof of [6, Theorem 3.3], we have the following:

Claim 3.40. *There exist a compact subset Z of $\hat{\overline{B}}_s(w)$, a point z in Z and a map ϕ from $\hat{\overline{B}}_{s/10^5}(w)$ to Z such that the map $\Phi(\alpha) = (\hat{\mathbf{b}}^i_1(\alpha), \ldots, \hat{\mathbf{b}}^i_l(\alpha), \phi(\alpha))$ from $\hat{\overline{B}}_{s/10^5}(w)$ to $\overline{B}_{s/10^5+\Psi s}(\hat{\mathbf{b}}^i_1(w), \ldots, \hat{\mathbf{b}}^i_l(w), \phi(w)) \subset \mathbf{R}^k \times Z$, is an Ψs-Gromov-Hausdorff approximation. Here, $\Psi = \Psi(r, r/\tau, \epsilon; n, L)$.*

Since

$$\frac{1}{\hat{\mathrm{vol}}\, \hat{B}_{5s}(w)} \int_{\hat{B}_{5s}(w)} |\hat{d}\hat{\mathbf{b}}^i_j - \hat{d}\hat{f}^i_j|^2_r d\hat{\mathrm{vol}} \leq \Psi\left(r, \frac{r}{\tau}, \epsilon; n, L\right),$$

by the segment inequality on manifolds [6, Theorem 2.15], for every $z_1 \in \hat{\overline{B}}_s(w)$, there exist $\hat{z}_1 \in \hat{\overline{B}}_{5s}(w)$, $\hat{w} \in \hat{\overline{B}}_{5s}(w)$ and a minimal geodesic γ from $\hat{z}_1$ to $\hat{w}$ such that $\overline{z_1, \hat{z}_1} \leq \Psi(r, r/\tau, \epsilon; n, L)$, $\overline{w, \hat{w}} \leq \Psi(r, r/\tau, \epsilon; n, L)$ and

$$\int_0^{\overline{\hat{z}_1, \hat{w}}} \hat{\mathrm{Lip}}(\hat{\mathbf{b}}^i_j - \hat{f}^i_j)(\gamma(t))dt \leq \Psi\left(r, \frac{r}{\tau}, \epsilon; n, L\right) s.$$

Therefore, we have

$$|\hat{\mathbf{b}}^i_j(\hat{z}_1) - \hat{f}^i_j(\hat{z}_1) - (\hat{\mathbf{b}}^i_j(\hat{w}) - \hat{f}^i_j(\hat{w}))| \leq \int_0^{\overline{\hat{z}_1, \hat{w}}} \hat{\mathrm{Lip}}(\hat{\mathbf{b}}^i_j - \hat{f}^i_j)(\gamma(t))dt \leq \Psi\left(r, \frac{r}{\tau}, \epsilon; n, L\right) s.$$

By Cheng-Yau's gradient estimate, we have $\hat{\mathbf{Lip}}(\hat{\mathbf{b}}^i_j|_{\hat{B}_{2s}(w)}) \leq C(n, L)$. Thus, we have $|\hat{\mathbf{b}}^i_j(z_1) - \hat{f}^i_j(z_1) - (\hat{\mathbf{b}}^i_j(w) - \hat{f}^i_j(w))| \leq \Psi(r, r/\tau, \epsilon; n, L)s$. Let $C = \hat{\mathbf{b}}^i_j(w) - \hat{f}^i_j(w)$. Then we have that $\hat{\mathbf{b}}^i_j = \hat{f}^i_j + C \pm \Psi(r, r/\tau, \epsilon; n, L)\, s$ on $\hat{\overline{B}}_s(w)$.

Thus, the map $\hat{\Phi}(\alpha) = (\hat{f}^i_1(\alpha), \ldots, \hat{f}^i_l(\alpha), \phi(\alpha))$ from $\hat{\overline{B}}_{s/10^5}(w)$ to $\overline{B}_{s/10^5+\Psi s}(\hat{f}^i_1(w), \ldots, \hat{f}^i_l(w), \phi(w))$, is an Ψs-Gromov-Hausdorff approximation. Therefore we have the assertion. □

Lemma 3.41. *Let (Y, y, υ) be a Ricci limit space, $\tau, \epsilon, \delta, L$ positive numbers, l, m positive integers, x a point in Y, $\{k_s\}_{1 \le s \le l}$ a collection of positive integers, $\{a^s_t\}_{1 \le s \le l, 1 \le t \le k_s}$ of real numbers, and $\{x^s_t\}_{1 \le s \le l, 1 \le t \le k_s}$ of points in Y. Let $f_j = \sum_{p=1}^{k_j} a^j_p r_{x^j_p}$. Assume that $x \in \mathrm{Leb}\left(\bigcap_{1 \le i \le l, 1 \le j \le k_i} (\mathcal{D}^\tau_{x^i_j} \setminus \{x^i_j\}) \cap (\mathcal{R}_m)_{\delta,\tau}\right)$, $\sum_{i,j}(a^i_j)^2 \le L$ and*

$$\limsup_{r \to 0} \frac{1}{\upsilon(B_r(x))} \int_{B_r(x)} |\langle df_j, df_i \rangle - \delta_{ij}| d\upsilon \le \epsilon.$$

Then, for every sufficiently small $s > 0$, there exists a compact subset K_s of $\overline{B}_s(x)$ such that the following properties hold:

1. *$\upsilon(K_s)/\upsilon(B_s(x)) \ge 1 - \Psi(\epsilon, \delta; n, L)$.*
2. *For every $\alpha \in K_s$ and every sufficiently small $t > 0$, there exist a collection of points $\{w^t_j(\alpha)\}_{1 \le j \le m-l}$ in Y, and a compact subset U_t of $\overline{B}_t(\alpha)$ such that $\upsilon(U_t)/\upsilon(B_t(\alpha)) \ge 1 - \Psi(\epsilon, \delta; n, L)$ and that the map $\Phi_t = (f_1, \ldots, f_l, r_{w^t_1(\alpha)}, \ldots, r_{w^t_{m-l}(\alpha)})$ from U_t to $\mathbf{R}^m$, is $(1 \pm \Psi(\epsilon, \delta; n, L))$-bi-Lipschitz to the image.*

Proof. Let $(M_i, m_i, x^s_t(i), \underline{\mathrm{vol}}) \to (Y, y, x^s_t, \upsilon)$ and $f^i_j = \sum_{p=1}^{k_j} a^j_p r_{x^j_p(i)}$. There exists $s_1 > 0$ such that $s_1 << \tau$ and

$$\frac{1}{\upsilon(B_{10^{10}s}(x))} \int_{B_{10^{10}s}(x)} |\langle df_j, df_i \rangle - \delta_{ij}| d\upsilon + \frac{\upsilon\left(B_{10^{10}s}(x) \cap \bigcap_{1 \le i \le l, 1 \le j \le k_i} (\mathcal{D}^\tau_{x^i_j} \cap (\mathcal{R}_m)_{\delta,r})\right)}{\upsilon(B_{10^{10}s}(x))} \le 3\epsilon$$

for every $0 < s < s_1$. By Proposition 2.5 and Lemma 3.38, for every $0 < s < s_1$, there exists a compact subset K_s of $\overline{B}_{10^9 s}(x)$ such that the following properties hold:

1. $\upsilon(K_s)/\upsilon(B_{10^9 s}(x)) \ge 1 - \Psi(\epsilon; n, L)$.
2. For every $w \in K_s$ and every $0 < t < 10^4 s$, there exist a compact subset Z^w_t of $\overline{B}_t(w)$ and a map ϕ^w_t from $\overline{B}_t(w)$ to Z^w_t such that the map $\Phi^w_t = (f_1, \ldots, f_l, \phi^w_t)$ from $\overline{B}_t(w)$ to $\overline{B}_{10^9(t + \Psi t)}(f_1(w), \ldots, f_l(w), \phi^w_t(w))$, is an Ψt-Gromov-Hausdorff approximation. Here $\Psi = \Psi(\epsilon; n, L)$.
3. We have
$$\frac{1}{\upsilon(B_t(w))} \int_{B_t(w)} |\langle df_j, df_i \rangle - \delta_{ij}| d\upsilon \le \Psi(\epsilon; n, L)$$
for every $w \in K_s$ and every $0 < t < 10^4 s$.

Here, with the same notation as in Lemma 3.38, we applied Proposition 4.12 to get

$$\lim_{k \to \infty} \frac{1}{\underline{\mathrm{vol}}\, B_t(w(k))} \int_{B_t(w(k))} |\langle df^k_j, df^k_i \rangle - \delta_{ij}| d\underline{\mathrm{vol}} = \frac{1}{\upsilon(B_t(w))} \int_{B_t(w)} |\langle df_j, df_i \rangle - \delta_{ij}| d\upsilon$$

for every sequence $w(k) \to w$. Fix $0 < s < s_1$, $w \in K_s \cap \mathrm{Leb}(\bigcap_{1\le i\le l, 1\le j\le k_i}(\mathcal{D}^\tau_{x^i_j} \setminus \{x^i_j\}) \cap (\mathcal{R}_m)_{\delta,r})$, $0 < t < 10^4 s$, Z^w_t, ϕ^w_t, and Φ^w_t as above. We remark that $\upsilon(K_s \cap \mathrm{Leb}(\bigcap_{1\le i\le l,1\le j\le k_i}(\mathcal{D}^\tau_{x^i_j} \setminus \{x^i_j\}) \cap (\mathcal{R}_m)_{\delta,r}))/\upsilon(B_{10^9 s}(x)) \ge 1 - \Psi(\epsilon; n, L)$. Assume that t is sufficiently small and

$$\frac{\upsilon\left(B_{\hat{t}}(w) \cap \bigcap_{1\le i\le l,1\le j\le k_i}(\mathcal{D}^\tau_{x^i_j} \setminus \{x^i_j\}) \cap (\mathcal{R}_m)_{\delta,r}\right)}{\upsilon(B_{\hat{t}}(w))} \ge 1 - \epsilon$$

for every $0 < \hat{t} \le t$, below. Then, for every $1 \le j \le l$, there exist points $y^+_j, y^-_j \in \overline{B}_t(w)$ such that $\overline{\Phi^w_t(y^+_j), (\underbrace{0,\dots,0,t}_{j},0,\dots,0,\phi^w_t(w))} \le \Psi t$ and $\overline{\Phi^w_t(y^-_j), (\underbrace{0,\dots,0,-t}_{j},0,\dots,0,\phi^w_t(w))} \le \Psi t$. Let $\hat{\Phi}^w_t$ be an Ψt-Gromov-Hausdorff approximation from $\overline{B}_{10^9(t+\Psi t)}(f_1(w),\dots,f_l(w),\phi^w_t(w))$ to $\overline{B}_t(w)$ satisfying that $\overline{\Phi^w_t \circ \hat{\Phi}^w_t(\alpha), \alpha} \le \Psi t$ for every $\alpha \in \overline{B}_{10^9(t+\Psi t)}(f_1(w),\dots,f_l(w),\phi^w_t(w))$, and that $\overline{\hat{\Phi}^w_t \circ \Phi^w_t(\beta), \beta} \le \Psi t$ for every $\beta \in \overline{B}_t(w)$. On the other hand, there exist δt-Gromov-Hausdorff approximations ψ^w_t from $(\overline{B}_t(w), w)$ to $(\overline{B}_t(0_m), 0_m)$, and $\hat{\psi}^w_t$ from $(\overline{B}_t(0_m), 0_m)$ to $(\overline{B}_t(w), w)$ such that $\overline{\psi^w_t \circ \hat{\psi}^w_t(\alpha), \alpha} \le 5\delta t$ for every $\alpha \in \overline{B}_t(0_m)$, and that $\overline{\hat{\psi}^w_t \circ \psi^w_t(\beta), \beta} \le 5\delta t$ for every $\beta \in \overline{B}_t(w)$. Especially, there exists an Ψt-Gromov-Hausdorff approximation $\hat{h}^w_t$ from $(\overline{B}_t(0_{m-l}), 0_{m-l})$ to $(Z^w_t, \phi^w_t(w))$ such that $\overline{(0,\dots,0,\alpha), \psi^w_t \circ \hat{\Phi}^w_t(f_1(w),\dots,f_l(w),\hat{h}^w_t(\alpha))} \le \Psi t$ for every $\alpha \in Z^w_t$, where $\Psi = \Psi(\epsilon, \delta; n, L)$. Without loss of generality, we can assume that $\overline{\psi^w_t(y^+_i), (\underbrace{0,\dots,0,t}_{i},0,\dots,0)} \le \Psi t$. Then, for every $i \in \{l+1,\dots,m\}$, there exist points $z^+_i, z^-_i \in \overline{B}_t(w)$ such that $\overline{\psi^w_t(z^+_i), (\underbrace{0,\dots,0,t}_{i},0,\dots,0)} \le \Psi t$ and $\overline{\psi^w_t(z^-_i), (\underbrace{0,\dots,0,-t}_{i},0,\dots,0)} \le \Psi t$. Let $F_i = f_i - f_i(w)$ and $G_i = F_i \circ \psi^w_t$ on $(\overline{B}_t(0_m), 0_m)$. Since $\overline{\pi_{\mathbf{R}^{m-l}}(\psi^w_t \circ \hat{\Phi}^w_t(f_1(w),\dots,f_l(w),\hat{h}^w_t(\alpha))), \alpha} \le \Psi t$ for every $\alpha \in \overline{B}_t(0_{m-l})$, we have that the map $G = (G_1,\dots,G_l,\pi_{l+1},\dots,\pi_m)$ from $(\overline{B}_t(0_m), 0_m)$ to $(\overline{B}_{t+\Psi t}(0_m), 0_m)$, satisfies $\overline{G((\underbrace{0,\dots,0,\pm t}_{i},0,\dots,0), (\underbrace{0,\dots,0,\pm t}_{i},0,\dots,0)} \le \Psi t$ for every i, and that it is an Ψt-Gromov-Hausdorff approximation, where $\pi_{\mathbf{R}^{m-l}}$ is the canonical projection $\mathbf{R}^m = \mathbf{R}^l \times \mathbf{R}^{m-l}$ to $\mathbf{R}^{m-l}$, π_i is the i-th projection from $\mathbf{R}^m$ to $\mathbf{R}$. Thus, we have $\overline{\alpha, G(\alpha)} \le \Psi t$ for every $\alpha \in \overline{B}_t(0_m)$. Especially, we have the following claim:

Claim 3.42. *We have* $|G_i - \pi_i| \le \Psi(\epsilon, \delta; n, L)t$ *on* $B_t(0_m)$.

Fix $0 < \hat{t} < t$. By rescaling $\hat{t}^{-1}d_Y$, $\hat{t}^{-1}d_{\mathbf{R}^m}$, Claim 3.42 and the definition of Busemann function, we have the following:

Claim 3.43. *We have*

$$|F_i(\alpha) - (r_{y^-_i}(\alpha) - r_{y^-_i}(w))| \le \Psi\left(\epsilon, \delta, \frac{\hat{t}}{t}, \frac{\Psi(\epsilon,\delta;n,L)t}{\hat{t}}; n, L\right)\hat{t}$$

for every $\alpha \in \overline{B}_{\hat{t}}(w)$.

Let $y_j^-(k), z_j^-(k), w(k)$ be points in M_k satisfying that $y_j^-(k) \to y_j^-, z_j^-(k) \to z_j^-$ and $w(k) \to w$. Put $r = \sqrt{\Psi} t$ for $\Psi = \Psi(\epsilon, \delta; n, L)$ as in Claim 3.43. For convenience, for rescaled distances $r^{-1}d_Y$ and $r^{-1}d_{M_i}$, we shall use the same notation as in the proof of Lemma 3.38: $\hat{f}_i^k, \hat{d}f, \hat{\mathrm{vol}}$, and so on.

Claim 3.44. *We have*

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{100}(w(k))} \int_{\hat{B}_{100}(w(k))} |\hat{d}\hat{f}_i^k - \hat{d}\hat{r}_{y_i^-(k)}|_r^2 d\hat{\mathrm{vol}} \le \Psi(\epsilon, \delta; n, L)$$

for every sufficiently large k.

The proof is as follows. By the assumption and Proposition 4.12, we have

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{1000}(x(k))} \int_{\hat{B}_{1000}(x(k))} ||\hat{d}\hat{f}_i^k|_r^2 - 1| d\hat{\mathrm{vol}} \le \Psi(\epsilon, \delta; n, L)$$

for every sufficiently large k. By an argument similar to the proof of Lemma 3.38, for every sufficiently large k, there exists a harmonic function $\hat{\mathbf{b}}_i^k$ on $\hat{B}_{100}(w(k))$ such that $\mathbf{Lip}\,\hat{\mathbf{b}}_i^k \le C(n)$, $|\hat{\mathbf{b}}_i^k - \hat{f}_i^k|_{L^\infty(\hat{B}_{100}(w(k)))} \le \Psi(r, r/\tau; n, L)$ and

$$\frac{1}{\hat{\mathrm{vol}}\,\hat{B}_{1000}(w(k))} \int_{\hat{B}_{1000}(w(k))} \left(|\hat{d}\hat{\mathbf{b}}_i^k - \hat{d}\hat{f}_i^k|_r^2 + |\mathrm{Hess}_{\hat{\mathbf{b}}_i^k}|_r^2\right) d\hat{\mathrm{vol}} \le \Psi(r, r/\tau; n, L).$$

For every $\alpha \in \hat{B}_{1000}(w(k)) \setminus C_{y_i^-(k)}$, let γ_i^α be the minimal geodesic from $y_i^-(k)$ to α on $(M_i, r^{-1}d_{M_i})$. Fix $0 < h < 1$. By Claim 3.43, there exists k_0 such that

$$\begin{aligned}
&\frac{\hat{\mathbf{b}}_i^k(\alpha) - \hat{\mathbf{b}}_i^k\left(\gamma_i^\alpha\left(\overline{y_i^-(k), \alpha}^{r^{-1}d_{M_k}} - h\right)\right)}{h} \\
&= \frac{\hat{f}_i^k(\alpha) - \hat{f}_i^k\left(\gamma_i^\alpha\left(\overline{y_i^-(k), \alpha}^{r^{-1}d_{M_k}} - h\right)\right)}{h} \pm \frac{\Psi(\epsilon, \delta; n, L)}{h} \\
&= \frac{\hat{f}_i(\phi_k(\alpha)) - \hat{f}_i\left(\phi_k\left(\gamma_i^\alpha\left(\overline{y_i^-(k), \alpha}^{r^{-1}d_{M_k}} - h\right)\right)\right)}{h} \pm \frac{\Psi(\epsilon, \delta; n, L)}{h} \\
&= \frac{\overline{y_i^-, \phi_k(\alpha)}^{r^{-1}d_Y} - \overline{y_i^-, \phi_k\left(\gamma_i^\alpha\left(\overline{y_i^-(k), \alpha}^{r^{-1}d_{M_k}} - h\right)\right)}^{r^{-1}d_Y}}{h} \pm \frac{\Psi(\epsilon, \delta; n, L)}{h} \\
&= \frac{\overline{y_i^-(k), \alpha}^{r^{-1}d_{M_k}} - \overline{y_i^-(k), \gamma_i^\alpha\left(\overline{y_i^-(k), \alpha}^{r^{-1}d_{M_k}} - h\right)}^{r^{-1}d_{M_k}}}{h} \pm \frac{\Psi(\epsilon, \delta; n, L)}{h} \\
&= 1 \pm \frac{\Psi(\epsilon, \delta; n, L)}{h}
\end{aligned}$$

for every $k \geq k_0$ and every $\alpha \in \hat{B}_{1000}(w(k)) \setminus C_{y_i^-(k)}$. On the other hand, by an argument similar to the proof of Claim 3.34, we have

$$\left| \frac{1}{\hat{\mathrm{vol}}\, \hat{B}_{100}(w(k))} \int_{\hat{B}_{100}(w(k))} \frac{1}{h} \int_{\overline{y_i^-(k),\alpha}^{r^{-1}d_{M_k}}-h}^{\overline{y_i^-(k),\alpha}^{r^{-1}d_{M_k}}} \left(s - (\overline{y_i^-(k),\alpha}^{r^{-1}d_{M_k}} - h)\right) \frac{d^2 \hat{\mathbf{b}}_i^k \circ \gamma_i^\alpha}{ds^2} ds d\hat{\mathrm{vol}} \right|$$
$$\leq C(n) \frac{h}{\hat{\mathrm{vol}}\, \hat{B}_{1000}(w(k))} \int_{\hat{B}_{1000}(w(k))} |\mathrm{Hess}_{\hat{\mathbf{b}}_i^k}|_r d\hat{\mathrm{vol}} \leq \Psi(\epsilon, \delta; n, L).$$

Since

$$\hat{\mathbf{b}}_i^k(\alpha) = \hat{\mathbf{b}}_i^k(\gamma_i^\alpha(\overline{y_i^-(k),\alpha}^{r^{-1}d_{M_k}} - h)) + \frac{\hat{d}\hat{\mathbf{b}}_i^k}{\hat{d}\hat{r}_{y_i^-(k)}}(\alpha) h$$
$$- \int_{\overline{y_i^-(k),\alpha}^{r^{-1}d_{M_k}}-h}^{\overline{y_i^-(k),\alpha}^{r^{-1}d_{M_k}}} \left(s - (\overline{y_i^-(k),\alpha}^{r^{-1}d_{M_k}} - h)\right) \frac{d^2 \hat{\mathbf{b}}_i^k \circ \gamma_i^\alpha}{ds^2} ds$$

for every $\alpha \in \hat{B}_{100}(w(k)) \setminus C_{y_i^-(k)}$, we have

$$\frac{1}{\hat{\mathrm{vol}}\, \hat{B}_{100}(w(k))} \int_{\hat{B}_{100}(w(k))} \langle \hat{d}\hat{\mathbf{b}}_i^k, \hat{d}\hat{r}_{y_i^-(k)} \rangle_r d\hat{\mathrm{vol}} = 1 \pm \frac{\Psi(\epsilon, \delta; n, L)}{h}.$$

Therefore, we have

$$\frac{1}{\hat{\mathrm{vol}}\, \hat{B}_{100}(w(k))} \int_{\hat{B}_{100}(x(k))} |\hat{d}\hat{f}_i^k - \hat{d}\hat{r}_{y_i^-(k)}|_r^2 d\hat{\mathrm{vol}}$$
$$= \frac{1}{\hat{\mathrm{vol}}\, \hat{B}_{100}(w(k))} \int_{\hat{B}_{100}(w(k))} |\hat{d}\hat{f}_i^k|_r^2 d\hat{\mathrm{vol}} - \frac{2}{\hat{\mathrm{vol}}\, \hat{B}_{100}(w(k))} \int_{\hat{B}_{100}(w(k))} \langle \hat{d}\hat{f}_i^k, \hat{d}\hat{r}_{y_i^-(k)} \rangle_r d\hat{\mathrm{vol}} + 1$$
$$= 1 - 2\frac{1}{\hat{\mathrm{vol}}\, \hat{B}_{100}(w(k))} \int_{\hat{B}_{100}(w(k))} \langle \hat{d}\hat{\mathbf{b}}_i^k, \hat{d}\hat{r}_{y_i^-(k)} \rangle_r d\hat{\mathrm{vol}} + 1 \pm \Psi(\epsilon, \delta; n, L)$$

$$= 2 - 2\left(1 \pm \frac{\Psi(\epsilon, \delta; n, L)}{h}\right) \pm \Psi(\epsilon, \delta; n, L) = \frac{\Psi(\epsilon, \delta; n, L)}{h}.$$

Therefore, we have Claim 3.44.

Next claim follows from Claim 3.44 and [2, Theorem 9.29] directly:

Claim 3.45. *For every sufficiently large k, we have*

$$\frac{1}{\hat{\mathrm{vol}}\, \hat{B}_{100}(w(k))} \int_{\hat{B}_1(w(k))} |\langle \hat{d}\hat{f}_i^k, \hat{d}\hat{r}_{z_j^-(k)} \rangle_r| d\hat{\mathrm{vol}} \leq \Psi(\epsilon, \delta; n, L)$$

for every $1 \leq i \leq l$ and every $l+1 \leq j \leq m$. Moreover we have

$$\frac{1}{\hat{\mathrm{vol}}\, \hat{B}_{100}(w(k))} \int_{\hat{B}_1(w(k))} |\langle \hat{d}\hat{f}_i^k, \hat{d}\hat{f}_{\hat{i}}^k \rangle_r| d\hat{\mathrm{vol}} \leq \Psi(\epsilon, \delta; n, L)$$

for every $1 \leq i < \hat{i} \leq l$.

For every i with $l+1 \le i \le m$, and every sufficiently large k, there exists a harmonic function $\hat{\mathbf{b}}_i^k$ on $\hat{B}_{1000}(w(k))$ such that $|\hat{r}_{z_i^-} - \hat{\mathbf{b}}_i^k|_{L^\infty(\hat{B}_{1000}(w(k)))} \le \Psi(\epsilon, \delta; n, L)$ and

$$\frac{1}{\hat{\mathrm{vol}}\, \hat{B}_{1000}(w(k))} \int_{\hat{B}_{1000}(w(k))} \left(|\hat{d}\hat{\mathbf{b}}_i^k - \hat{d}\hat{r}_{z_i^-(k)}|_r^2 + |\mathrm{Hess}_{\hat{\mathbf{b}}_i^k}|_r^2 \right) d\hat{\mathrm{vol}} \le \Psi(\epsilon, \delta; n, L).$$

Let

$$\hat{F}_k = \sum_{1 \le i,j \le m} |\langle \hat{d}\hat{\mathbf{b}}_i^k, \hat{d}\hat{\mathbf{b}}_j^k \rangle_r - \delta_{i,j}| + \sum_{1 \le i \le m} |\mathrm{Hess}_{\hat{\mathbf{b}}_i^k}|_r^2 + \sum_{i=1}^{l} |\hat{d}\hat{\mathbf{b}}_i^k - \hat{d}\hat{f}_i^k|_r^2 + \sum_{i=l+1}^{m} |\hat{d}\hat{\mathbf{b}}_i^k - \hat{d}\hat{r}_{z_i^-}|_r^2.$$

Then, by Lemma 3.1, for every sufficiently large k, there exists a compact subset $Z(k)$ of $\overline{\hat{B}}_1(w(k))$ such that $\hat{\mathrm{vol}}(\hat{B}_1(w(k)) \setminus Z(k))/\hat{\mathrm{vol}}\, \hat{B}_1(w(k)) \le \Psi(\epsilon, \delta; n, L)$ and

$$\frac{1}{\hat{\mathrm{vol}}\, \hat{B}_{\hat{s}}(\alpha)} \int_{\hat{B}_{\hat{s}}(\alpha)} \hat{F}_k d\hat{\mathrm{vol}} \le \Psi(\epsilon, \delta; n, L)$$

for every $\alpha \in Z(k)$ and every $0 < \hat{s} < 10$. Thus, by an argument similar to the proof of [6, Theorem 3.3], for every $\alpha \in Z(k)$ and every $0 < \hat{s} < 1$, there exist a compact subset P_s^α of $\overline{\hat{B}}_{\hat{s}}(\alpha)$, a point $p_{\hat{s}}^\alpha \in P_{\hat{s}}^\alpha$, and a map $q_{\hat{s}}^\alpha$ from $(\overline{\hat{B}}_{\hat{s}}(\alpha), \alpha)$ to $(\overline{B}_{\hat{s}}(p_{\hat{s}}^\alpha), p_{\hat{s}}^\alpha)$ such that the map $Q_{\hat{s}}^\alpha = (\hat{\mathbf{b}}_1^k, \ldots, \hat{\mathbf{b}}_m^k, q_{\hat{s}}^\alpha)$ from $\overline{\hat{B}}_{\hat{s}}(\alpha)$ to $\overline{\hat{B}}_{\hat{s}+\Psi\hat{s}}(\hat{\mathbf{b}}_1^k(\alpha), \ldots, \hat{\mathbf{b}}_m^k(\alpha), p_{\hat{s}}^\alpha)$, is an $\Psi\hat{s}$-Gromov-Hausdorff approximation. By an argument similar to the proof of Claim 3.40, for every $\alpha \in Z(k)$ and every $0 < \hat{s} < 1$, we have that $\hat{\mathbf{b}}_i^k = \hat{f}_i^k + \text{constant} \pm \Psi\hat{s}$ on $\hat{B}_{\hat{s}}(\alpha)$ for every $1 \le i \le l$, and $\hat{\mathbf{b}}_i^k = \hat{r}_{z_i^-(k)} + \text{constant} \pm \Psi\hat{s}$ on $\hat{B}_{\hat{s}}(\alpha)$ for every $l+1 \le i \le m$. Therefore, the map $\hat{Q}_{\hat{s}}^\alpha = (\hat{f}_1^k, \ldots, \hat{f}_l^k, \hat{r}_{z_{l+1}^-(k)}, \ldots, \hat{r}_{z_m^-(k)}, q_{\hat{s}}^\alpha)$ from $\overline{\hat{B}}_{\hat{s}}(\alpha)$ to $\overline{\hat{B}}_{\hat{s}+\Psi\hat{s}}(\hat{f}_1^k(\alpha), \ldots, \hat{f}_l^k(\alpha), \hat{r}_{z_{l+1}^-(k)}(\alpha), \ldots, \hat{r}_{z_m^-(k)}(\alpha), p_{\hat{s}}^\alpha)$, is an $\Psi\hat{s}$-Gromov-Hausdorff approximation. Without loss of generality, we can assume that there exists a compact subset $Z(\infty)$ of $\overline{\hat{B}}_1(w)$ such that $Z(k) \to Z(\infty)$. Let $U = Z(\infty) \cap \bigcap_{1 \le i \le l, 1 \le j \le k_i} (\mathcal{D}_{x_j^i}^\tau \setminus \{x_j^i\}) \cap (\mathcal{R}_m)_{\delta,r}$. By Proposition 2.5, we have $\hat{\upsilon}(\hat{B}_1(w) \cap U)/\hat{\upsilon}(\hat{B}_1(w)) \ge 1 - \Psi$. Since $\alpha \in (\mathcal{R}_m)_{\tau,\delta}$, we have that the map $T_{\hat{s}}^\alpha = (\hat{f}_1, \ldots, \hat{f}_l, \hat{r}_{z_{l+1}^-}, \ldots, \hat{r}_{z_m^-})$ from $\overline{\hat{B}}_{\hat{s}}(\alpha)$ to $\overline{B}_{\hat{s}}(T_{\hat{s}}^\alpha(\alpha))$, is an $\Psi\hat{s}$-Gromov-Hausdorff approximation for every $\alpha \in U$ and every $0 < \hat{s} < 1$. Let α, β be points in $U \cap \hat{B}_{1/2}(w)$ with $\alpha \ne \beta$. Put $\hat{s} = \overline{\alpha, \beta}^{r^{-1}d_Y} < 1$. Then we have

$$\begin{aligned} &\overline{(\hat{f}_1(\alpha), \ldots, \hat{f}_l(\alpha), \hat{r}_{z_{l+1}^-}(\alpha), \ldots, \hat{r}_{z_m^-}(\alpha)), (\hat{f}_1(\beta), \ldots, \hat{f}_l(\beta), \hat{r}_{z_{l+1}^-}(\beta), \ldots, \hat{r}_{z_m^-}(\beta))} \\ &= \overline{\alpha, \beta}^{r^{-1}d_Y} \pm \Psi\hat{s} = (1 \pm \Psi)\overline{\alpha, \beta}^{r^{-1}d_Y}. \end{aligned}$$

Therefore we have the assertion. □

LEMMA 3.46. *Let (Y, y, υ) be a Ricci limit space, l, k, m positive integers with $1 \le l \le m \le n$, x a point in Y, $\{h_i\}_{1 \le i \le l}$ a collection of Lipschitz functions on Y, $\{x_i\}_{1 \le i \le k}$ of points in Y, and $\{a_i^j\}_{1 \le i \le k, 1 \le j \le l}$ of real numbers Let $f_j = \sum_{i=1}^k a_i^j r_{x_i}$. Assume that the following properties hold:*

1. *We have*
$$\lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}|df_j - dh_j|d\upsilon = 0$$
for every j.

2. *We have*
$$x \in \bigcup_{\tau>0}\left(\bigcap_{\delta>0}\left(\bigcup_{r>0}\mathrm{Leb}\left(\bigcap_i (\mathcal{D}^{\tau}_{x_i}\setminus\{x_i\})\cap(\mathcal{R}_m)_{\delta,r}\right)\right)\right).$$

3. *The limit*
$$\lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\langle dh_i, dh_j\rangle d\upsilon \in \mathbf{R}$$
exists for every i, j.

4. *We have*
$$\det\left(\lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\langle dh_i, dh_j\rangle d\upsilon\right)_{i,j} \neq 0.$$

Then, for every $0<\delta<1$*, there exists* $r_0>0$ *such that for every* $0<s<r_0$*, there exists compact subset* K_s *of* $\overline{B}_s(x)$ *such that the following properties hold:*

1. $\upsilon(K_s)/\upsilon(B_s(x)) \geq 1-\delta$.

2. *For every* $\alpha \in K_s$ *and every sufficiently small* $t>0$*, there exist a collection* $\{w^t_j(\alpha)\}_{1\leq j\leq m-l}$ *of points in* Y*, and a compact subset* U_t *of* $\overline{B}_t(\alpha)$ *such that* $\upsilon(U_t)/\upsilon(B_t(\alpha)) \geq 1-\delta$ *and that the map* $\Phi_t = ((h_1,\ldots,h_l)A, r_{w^t_1(\alpha)},\ldots,r_{w^t_{m-l}(\alpha)})$ *from* U_t *to* $\mathbf{R}^m$*, is an* $(1\pm\delta)$*-bi-Lipschitz to the image, where*
$$A = \sqrt{\left(\lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\langle dh_i, dh_j\rangle d\upsilon\right)_{i,j}^{-1}}.$$

Proof. Define a collection $\{g_i\}_{1\leq i\leq l}$ of Lipschitz functions g_i on Y by $(g_1,\ldots,g_l) = (h_1,\ldots,h_l)A$. By the definition, we have
$$\lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\langle g_i, g_j\rangle d\upsilon = \delta_{i,j}.$$
By the assumption and Corollary 3.37, we have
$$\lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}|\langle g_i, g_j\rangle - \delta_{i,j}|d\upsilon = 0.$$
Put $(F_1,\ldots,F_l) = \left(\sum_{i=1}^k b^1_i r_{x_i},\ldots,\sum_{i=1}^k b^l_i r_{x_i}\right) = \left(\sum_{i=1}^k a^1_i r_{x_i},\ldots,\sum_{i=1}^k a^l_i r_{x_i}\right)A$. Let $L\geq 1$ satisfying $|A| + \sum_{i,j}(b^j_i)^2 \leq L$. Fix $0<\delta<1$. By Lemma 3.41, we have the following claim:

CLAIM 3.47. *There exists $r_1 > 0$ such that for every $0 < s \le r_1$, there exist a compact subset K_s of $\overline{B}_s(x)$ such that the following properties hold:*

1. $\upsilon(K_s)/\upsilon(B_s(x)) \ge 1 - \delta$.
2. *For every $\alpha \in K_s$ and every sufficiently small $t > 0$, there exist a collection of points $\{w_j^t(\alpha)\}_{1 \le j \le m-l}$ in Y, and a compact subset E_t of $\overline{B}_t(\alpha)$ such that $\upsilon(E_t)/\upsilon(B_t(\alpha)) \ge 1 - \delta$ and that the map $\Phi_t^\alpha = (F_1, \ldots, F_l, r_{w_1^t(\alpha)}, \ldots, r_{w_{m-l}^t(\alpha)})$ from E_t to $\mathbf{R}^m$, is $(1 \pm \delta)$-bi-Lipschitz to the image.*

On the other hand, there exists $r_0 > 0$ such that

$$\frac{1}{\upsilon(B_s(x))}\int_{B_s(x)} \sum_j |dF_j - dg_j| d\upsilon \le \delta$$

for every $0 < s < r_0$. Thus, by Lemma 3.1, for every $0 < s < r_0/100$, there exists a compact subset X_s of $\overline{B}_s(x)$ such that $\upsilon(X_s)/\upsilon(\overline{B}_s(x)) \ge 1 - \Psi(\delta; n)$ and

$$\frac{1}{\upsilon(B_{5\hat{s}}(\alpha))}\int_{B_{5\hat{s}}(\alpha)} \sum_j |dF_j - dg_j| d\upsilon \le \Psi(\delta; n)$$

for every $\alpha \in X_s$ and every $0 < \hat{s} \le s$. Put $V_s = K_s \cap X_s$ for every $0 < s < \min\{r_0, r_1\}/1000$. Then we have $\upsilon(V_s)/\upsilon(B_s(x)) \ge 1 - \Psi(\delta; n)$. Fix $0 < s < \min\{r_0, r_1\}/1000$ and $\alpha \in V_s$. By an argument similar to the proof of Claim 3.40, for every sufficiently small $t > 0$, we have $F_j = f_j + \text{constant} \pm \Psi(\delta; n)t$ on $\overline{B}_t(\alpha)$. Fix such $t > 0$ and put $U_t = B_{t/2}(\alpha) \cap E_t$. Then we have $\upsilon(U_t)/\upsilon(B_{t/2}(\alpha)) \ge 1 - \Psi(\delta; n)$. Let p_1, p_2 be points in U_t with $p_1 \neq p_2$. Put $\hat{t} = \overline{p_1, p_2} > 0$. Then we have

$$\begin{aligned}
&\overline{(f_1(p_1), \ldots, f_l(p_1), r_{w_1^t(\alpha)}, \ldots, r_{w_{m-l}^t(\alpha)}(p_1)), (f_1(p_2), \ldots, f_l(p_2), r_{w_1^t(\alpha)}(p_2), \ldots, r_{w_{m-l}^t(\alpha)}(p_2))} \\
&= \overline{(F_1(p_1), \ldots, F_l(p_1), r_{w_1^t(\alpha)}(p_1), \ldots, r_{w_{m-l}^t(\alpha)}(p_1)), (F_1(p_2), \ldots, F_l(p_2), r_{w_1^t(\alpha)}(p_2), \ldots, r_{w_{m-l}^t(\alpha)}(p_2))} \pm \Psi\hat{t} \\
&= (1 \pm \delta)\overline{p_1, p_2} \pm \Psi\hat{t} = (1 \pm \Psi)\overline{p_1, p_2}.
\end{aligned}$$

Therefore we have the assertion. □

LEMMA 3.48. *Let (Y, y, υ) be a Ricci limit space, l a positive integer, $\{f_i\}_{1 \le i \le l}$ a collection of Lipschitz functions on Y, f a Lipschitz function on Y, and A a Borel subset of Y. Assume that $\operatorname{span}\{df_1(x), \ldots, df_l(x)\} = T_x^*Y$ for a.e. $x \in A$. Then, for a.e. $x \in A$, there exists a collection of real numbers $\{b_i(x)\}_{1 \le i \le l}$ such that*

$$\lim_{r \to 0} \frac{1}{\upsilon(B_r(x))}\int_{B_r(x)} \left| df - \sum_{i=1}^{l} b_i(x) df_i \right|^2 d\upsilon = 0.$$

PROOF. Without loss of generality, we can assume that $\{df_i(x)\}_i$ is a basis of T^*_xY for every $x \in A$. Put

$$(b_1(x), \ldots, b_l(x)) = (\langle df, df_1\rangle(x), \ldots, \langle df, df_l\rangle(x))\sqrt{(\langle df_i, df_j\rangle(x))_{i,j}}^{-1}$$

for every $x \in A$. Then, by Lebesgue's differentiation theorem, for a.e. $x \in A$, we have

$$\lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}|df|^2 d\upsilon = |df|^2(x),\quad \lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\langle df, df_i\rangle d\upsilon = \langle df, df_i\rangle(x)$$

and

$$\lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\langle df_i, df_j\rangle d\upsilon = \langle df_i, df_j\rangle(x)$$

for every i, j. Then, since it is easy to check that

$$\begin{aligned}\lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}|df|^2 d\upsilon &= \lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\left\langle df, \sum_{i=1}^{l} b_i(x)df_i\right\rangle d\upsilon \\ &= \lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\left|\sum_{i=1}^{l} b_i(a)df_i\right|^2 d\upsilon \\ &= \left|\sum_{i=1}^{l} b_i(x)df_i(x)\right|^2\end{aligned}$$

for a.e. $x \in A$, we have

$$\begin{aligned}&\lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\left|df - \sum_{i=1}^{l} b_i(x)df_i\right|^2 d\upsilon \\ &= \lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}|df|^2 d\upsilon - 2\lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\left\langle df, \sum_{i=1}^{l} b_i(x)df_i\right\rangle d\upsilon \\ &\quad + \lim_{r\to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\left|\sum_{i=1}^{l} b_i(a)df_i\right|^2 d\upsilon = 0\end{aligned}$$

for a.e. $x \in A$. □

THEOREM 3.49 (Rectifiability associated with Lipschitz functions). *Let (Y, y, υ) be a Ricci limit space, l a positive integer, $\{f_i\}_{1\le i\le l}$ a collection of Lipschitz functions on Y, A a Borel subset of Y. Assume that $\{df_i(x)\}_{1\le i\le l}$ are linearly independent in T^*_xY for a.e. $x \in A$. Then, there exist $0 < \alpha(n) < 1$, collections of compact subsets $\{C_{k,i}\}_{l\le k\le n, i\in\mathbf{N}}$ of A, of points $\{x_{k,i}\}_{k,i}$ in A, and of points $\{x^s_{k,i}\}_{k,i,1\le s\le k-l}$ in Y such that the following properties hold:*

1. *$C_{k,i} \subset \mathcal{R}_{k,\alpha(n)} \cap \bigcap_{j=1}^{k-l}(A \setminus (C_{x^j_{k,i}} \cup \{x^j_{k,i}\}))$ and $\upsilon\left(A \setminus \bigcup_{l\le k\le n, i\in\mathbf{N}} C_{k,i}\right) = 0$ for evrey k.*

2. *For every $l \le k \le n$, every $x \in \bigcup_{i \in \mathbf{N}} C_{k,i}$ and every $0 < \delta < 1$, there exists $C_{k,i}$ such that $x \in C_{k,i}$ and that the map $\phi_{k,i} = ((f_1, \dots, f_l)\sqrt{(\langle df_i, df_j\rangle(x_{k,i}))_{i,j}}^{-1}, r_{x^1_{k,i}}, \dots, r_{x^{k-l}_{k,i}})$ from $C_{k,i}$ to $\mathbf{R}^k$, is $(1 \pm \delta)$-bi-Lipschitz to the image.*

3. *The limit measure υ and the k-dimensional Hausdorff measure H^k are mutually absolutely continuous on $C_{k,i}$. Moreover, υ is Ahlfors k-regular at every $x \in C_{k,i}$.*

Proof. Let $\{C^y_{k,i}\}_{k,i}$ be a collection of Borel subset of Y, and $\{x^{\hat{l}}_{k,i}\}_{k,i}$ of points in Y as in Theorem 3.16, where $x^1_{k,i} = y$. By Lemma 3.13, without loss of generality, we can assume that $C_{k,i}$ is bounded for every i, k. By the construction of T^*Y, we have $\mathrm{span}\{dr_{x^1_{k,i}}(x), \dots, dr_{x^k_{k,i}}(x)\} = T^*_x Y$ for a.e. $x \in C^y_{k,i}$. Therefore, we have $\upsilon(A \cap C^y_{k,i}) = 0$ for every $k < l$. Since

$$\upsilon\left(\mathcal{R}_k \setminus \bigcup_{\tau>0}\left(\bigcap_{\delta>0}\left(\bigcup_{r>0}\mathrm{Leb}\left(\bigcap_{i,j}(\mathcal{D}^\tau_{x^j_i} \setminus \{x^j_i\}) \cap (\mathcal{R}_k)_{\delta,r}\right)\right)\right)\right) = 0,$$

the following claim follows from Lemma 3.46 and Lemma 3.48, directly:

Claim 3.50. *For every $k \ge l$ and every $i \in \mathbf{N}$, there exists a Borel subset $A_{k,i}$ of $A \cap C_{k,i}$ such that the following properties hold:*

1. $\upsilon(A \cap C_{k,i} \setminus A_{k,i}) = 0$.

2. *For every $x \in A_{k,i}$ and every $0 < \delta < 1$, there exists $r^\delta_x > 0$ such that for every $0 < s < r^\delta_x$, there exists a compact subset $K(x, \delta, s)$ of $\overline{B}_s(x)$ such that the following properties hold:*

   (a) $\upsilon(K(x, \delta, s))/\upsilon(B_s(x)) \ge 1 - \delta$.

   (b) *For every $\alpha \in K(x, \delta, s)$ and every sufficiently small $t > 0$, there exist a collection of points $\{w(i, x, \delta, s, \alpha, t)\}_{1 \le i \le k-l}$ in Y, and a compact subset $U(x, \delta, s, \alpha, t)$ of $\overline{B}_t(\alpha)$ such that the map $\Phi^{x,\delta,s,\alpha,t} = \big((f_1, \dots, f_l)A(x), r_{w(1,x,\delta,s,\alpha,t)}, \dots, r_{w(k-l,x,\delta,s,\alpha,t)}\big)$ from $U(x, \delta, s, \alpha, t)$ to $\mathbf{R}^k$, is $(1 \pm \delta)$-bi-Lipschitz to the image, where*

$$A(x) = \sqrt{\left(\lim_{r \to 0}\frac{1}{\upsilon(B_r(x))}\int_{B_r(x)}\langle df_s, df_t\rangle d\upsilon\right)_{s,t}}^{-1} = \sqrt{(\langle df_s, df_t\rangle(x))_{s,t}}^{-1}.$$

Put $\hat{A}_{k,i} = \mathrm{Leb}(A_{k,i})$. For every $N \in \mathbf{N}$ and every $x \in \hat{A}_{k,i}$, let s^N_x be a positive number satisfying that $0 < s^N_x < \min\{r_x^{1/N}, N^{-1}\}$ and $\upsilon(B_{s^N_x}(x) \cap A_{k,i})/\upsilon(B_{s^N_x}(x)) \ge 1 - N^{-1}$. Let $K(x, N^{-1}, s^N_x)$ be a compact subset as in Claim 3.50. Put $\hat{K}(x, N^{-1}, s^N_x) = K(x, N^{-1}, s^N_x) \cap \hat{A}_{k,i}$. Then, we have $\upsilon\left(B_{s^N_x}(x) \cap \hat{K}(x, N^{-1}, s^N_x)\right)/\upsilon(B_{s^N_x}(x)) \ge 1 - 100N^{-1}$. For every $\alpha \in \hat{K}(x, N^{-1}, s^N_x)$, there exists $0 < t = t(\alpha) < N^{-1}$ such that $\upsilon(B_{\hat{t}}(\alpha) \cap$

$A_{k,i})/\upsilon(B_{\hat{t}}(\alpha)) \geq 1-N^{-1}$ for every $0 < \hat{t} < t$. Take $w(i, x, N^{-1}, s_x^N, \alpha, \hat{t})$ and $U(x, N^{-1}, s_x^N, \alpha, \hat{t})$ as in Claim 3.50. Put $\hat{U}(x, N^{-1}, s_x^N, \alpha, \hat{t}) = U(x, N^{-1}, s_x^N, \alpha, \hat{t}) \cap \hat{A}_{k,i}$. Then we have $\upsilon\left(B_{\hat{t}}(\alpha) \cap \hat{U}(x, N^{-1}, s_x^N, \alpha, \hat{t})\right)/\upsilon(B_{\hat{t}}(\alpha)) \geq 1 - 1000N^{-1}$. By Lemma 2.4, it is not difficult to check that the following claim:

Claim 3.51. *There exist $x_j^N \in \hat{A}_{k,i}$, $\alpha_j^N \in \hat{K}(x_j^N, N^{-1}, s_{x_j^N}^N)$ and $0 < t_j^N < t(\alpha_j^N)$ such that*

$$\upsilon\left(A_{k,i} \setminus \bigcup_{j \in \mathbf{N}} \hat{U}(x_j^N, N^{-1}, s_{x_j^N}^N, \alpha_j^N, t_j^N)\right) \leq \Psi(N^{-1}; n)\upsilon(B_{10}(A_{k,i})).$$

Put $\hat{U}(j,N) = \hat{U}(x_j^N, N^{-1}, s_{x_j^N}^N, \alpha_j^N, t(\alpha_j^N))$, $w(i,j,N) = w(i, x_j^N, N^{-1}, s_{x_j^N}^N, \alpha_j^N, t(\alpha_j^N))$, $U(j) = \bigcap_{N_0 \in \mathbf{N}}\left(\bigcup_{N_1 \geq N_0} \hat{U}(j, N_1)\right)$ and $U(j,N) = \hat{U}(j,N) \cap U(j)$. Then we have $\upsilon\left(A_{k,i} \setminus \bigcup_{j \in \mathbf{N}} U(j)\right) = 0$ and $\bigcup_{N \in \mathbf{N}} U(j,N) = U(j)$. Fix j, $w \in \bigcup_{N \in \mathbf{N}} U(j,N)$ and $0 < \delta < 1$. There exists N_0 such that $w \in U(j, N_0)$. Let $N_1 \in \mathbf{N}$ with $N_1^{-1} << \delta$. Since $w \in \bigcup_{N_2 \geq N_1} \hat{U}(j, N_2)$, there exists $N_2 \geq N_1$ such that $w \in \hat{U}(j, N_2)$. Especially we have $w \in U(j, N_2)$. Thus the map $G_{j,N_2} = ((f_1, \ldots, f_l)A(x_j^{N_2}), r_{w(1,j,N_2)}, \ldots, r_{w(k-l,j,N_2)})$ from $U(j, N_2)$ to $\mathbf{R}^k$, is $(1 \pm N_2^{-1})$-bi-Lipschitz to the image. Especially, G_{j,N_2} is $(1 \pm \delta)$-bi-Lipschitz to the image. Therefore, we have the assertion. $\square$

Remark 3.52. The radial rectifiability theorem, Theorem 3.16, corresponds to Theorem 3.49 for the case: $l = 1, f_1 = r_x, A = Y$.

We will end this subsection by giving two corollaries of Theorem 3.49. For a metric space X, define a distance on $\mathbf{R}_{\geq 0} \times X/(\{0\} \times X)$ by $\overline{(t_1, x_1), (t_2, x_2)} = \sqrt{t_1^2 + t_2^2 - 2t_1t_2 \cos \min\{\overline{x_1, x_2}, \pi\}}$. Denote this metric space by $C(X)$, and put $p = [(0,x)] \in C(X)$. The next corollary is used in [24], essentially.

Corollary 3.53. *Let X be a compact geodesic space and l a nonnegative integer. Assume that $l \leq n$, $\dim_H X = n - l - 1$ and that $(\mathbf{R}^l \times C(X), (0_l, p))$ is a Ricci limit space. Then, X is H^{n-l-1}-rectifiable.*

Proof. Define a collection of 1-Lipschitz functions $\{g\} \cup \{\pi_j\}_{1 \leq j \leq l}$ on $\mathbf{R}^k \times C(X)$ by $\pi_j(t_1, \ldots, t_l, w) = t_j$ and $g(t_1, \ldots, t_l, w) = \overline{p, w}$. By Theorem 3.30 and [4, Theorem 5.9], we have $\langle d\pi_i, d\pi_j \rangle(\alpha) = \delta_{i,j}, \langle d\pi_i, dg \rangle(\alpha) = 0, |dg|(\alpha) = 1$ for a.e. $\alpha \in \mathbf{R}^l \times C(X)$ with respect to the n-dimensional Hausdorff measure H^n. Therefore, by applying Theorem 3.49 for a collection of Lipschitz functions $\{\pi_j\}_{1 \leq j \leq l} \cup \{g\}$ and $A = \mathbf{R}^l \times C(X)$, there exists a collection of Borel subsets $\{C_{k,i}\}_{i, l+1 \leq k \leq n}$ as in Theorem 3.49. Since the product measure $H^l \times H^{n-l}$ on $\mathbf{R}^l \times C(X)$ is equal to H^n (see appendix in [24]), by Fubini's theorem, we have

$$0 = H^n\left((\mathbf{R}^l \times C(X)) \setminus \bigcup_{k,i} C_{k,i}\right) = \int_{\mathbf{R}^l} H^{n-l}\left((\{t_1, \ldots, t_l\} \times C(X)) \setminus \bigcup_{k,i} C_{k,i}\right) dH^l.$$

Especially, there exists $(t_1, \ldots, t_l) \in \mathbf{R}^l$ such that $H^{n-l}\left((\{t_1, \ldots, t_l\} \times C(X)) \setminus \bigcup_{k,i} C_{k,i}\right) = 0$. Put $\hat{C}_{k,i} = (\{t_1, \ldots, t_l\} \times C(X)) \cap C_{k,i}$ and regard it as a subset of $C(X)$, canonically. Now, we remark that

$$\int_{C(X)} f dH^{n-l} = \int_0^{\infty} \int_{\partial B_t(p)} f dH^{n-l-1} dt$$

holds for every $f \in L^1(C(X))$ (this is the *co-area formula for the distance function from the pole in* $C(X)$. See for instance appendix in [24]). Thus, especially, we have $H^{n-l-1}\left(\partial B_t(p) \cap C(X) \setminus \bigcup_{k,i} \hat{C}_{k,i}\right) = 0$ for a.e. $t > 0$. Then it is not difficult to check the assertion. $\square$

REMARK 3.54. With the same notation as in Corollary 3.53, we have $0 < H^{n-l-1}(B_r(x)) < \infty$ for every $x \in X$ and every $r > 0$. It follows from [4, Theorem 5.9], [6, Theorem 4.6] and the above co-area formula for the distance function from the pole on $C(X)$. We skipped the proof because it is not difficult to check it.

Similarly, we have the following:

COROLLARY 3.55. *Let* (X, x) *be a pointed proper geodesic space,* l *a nonnegative integer. Assume that* $l \le n$, $\dim_H X = n - l$ *and that* $(\mathbf{R}^l \times X, (0_l, x))$ *is a Ricci limit space. Then,* X *is* H^{n-l}*-rectifiable.*

4 Convergence of L^{∞}-functions and of Lipschitz functions

In this section, we will give two-notions of convergence of a sequence of L^{∞}-functions with respect to the measured Gromov-Hausdorff topology. By using these notions, we will give the definition of a convergence of the differentials of Lipschitz functions (see Definition 4.15). Moreover, by combining with several results given in section 3, we will discuss convergence of harmonic functions. In [27], we can also find related important, interesting results to this section. Throughout the following subsections 4.1 and 4.2, we shall fix the following:

1. Let $\{(Z_i, z_i)\}_{1 \le i \le \infty}$ be a sequence of pointed proper geodesic spaces, $x_i \in Z_i$.
2. Let υ_i be a Radon measure on Z_i for every $1 \le i \le \infty$.
3. $\upsilon_i(B_1(z_i)) = 1$ holds for every i.
4. For every $R \ge 1$, there exists $\kappa = \kappa(R) \ge 1$ such that $\upsilon_i(B_{2s}(z)) \le 2^{\kappa} \upsilon_i(B_s(z))$ for every $1 \le i \le \infty$, every $z \in Z_i$ and every $0 < s \le R$.

5. $(Z_i, x_i, z_i, \upsilon_i) \overset{(\phi_i, R_i, \epsilon_i)}{\to} (Z_\infty, x_\infty, z_\infty, \upsilon_\infty)$.

4.1 Pointwise strong convergence of L^∞-functions

Our aims in this subsection are to give the following notion and several fundamental properties of it:

DEFINITION 4.1 (Pointwise strong convergence of L^∞-functions). Let R be a positive number, w_∞ a point in $B_R(x_\infty)$, and $\{f_i\}_{1\le i\le\infty}$ a sequence of L^∞-functions f_i on $B_R(x_i)$ with $\sup_i |f_i|_{L^\infty(B_R(x_i))} < \infty$. We say that f_i *converges strongly to* f_∞ *at* w_∞ if for every $\epsilon > 0$, there exists $r > 0$ such that

$$\limsup_{i\to\infty} \frac{1}{\upsilon_i(B_t(w_i))} \int_{B_t(w_i)} \left| f_i - \frac{1}{\upsilon_\infty(B_t(w_\infty))} \int_{B_t(w_\infty)} f_\infty d\upsilon_\infty \right| d\upsilon_i \le \epsilon$$

and

$$\limsup_{i\to\infty} \frac{1}{\upsilon_\infty(B_t(w_\infty))} \int_{B_t(w_\infty)} \left| f_\infty - \frac{1}{\upsilon_i(B_t(w_i))} \int_{B_t(w_i)} f_i d\upsilon_i \right| d\upsilon_\infty \le \epsilon$$

for every $0 < t < r$ and every $w_i \to w_\infty$.

EXAMPLE 4.2. Fix $f \in C^0(B_R(x_\infty))$ and put $f_i = f \circ \phi_i$. Then, it is easy to check that f_i converges strongly to f_∞ at every $w \in B_R(x_\infty)$.

We shall give a fundamental result about this convergence without the proof because it is not difficult to check it:

PROPOSITION 4.3. *Let k be a positive integer, R a positive number, $\{f_i^l\}_{1\le l\le k}$ a collection of L^∞-functions on $B_R(x_i)$ for every $1 \le i \le \infty$ with $\sup_{i,l} |f_i^l|_{L^\infty(B_R(x_i))} < \infty$, w_∞ a point in $B_R(x_\infty)$ and $\{F_i\}_{1\le i\le\infty}$ a sequence of continuous functions on $\mathbf{R}^k$. Assume that f_i^l converges strongly to f_∞^l at w_∞ for every l, and that F_i converges to F_∞ with respect to the compact uniformly topology. Then, $F_i(f_i^1, \ldots, f_i^k)$ converges strongly to $F_\infty(f_\infty^1, \ldots, f_\infty^k)$ at w_∞.*

REMARK 4.4. Let k be a positive integer, $\{f_i^l\}_{1\le l\le k}$ a collection of L^∞-functions f_i^l on $B_R(x_i)$ for every $1 \le i \le \infty$, w_∞ a point in $B_R(x_\infty)$, and $\{F_i\}_{1\le i\le\infty}$ a sequence of locally L^∞-functions on $\mathbf{R}^k$. Assume that the following properties hold:

1. $\sup_{i,l} |f_i^l|_{L^\infty(B_R(x_i))} < \infty$.
2. f_i^l converges strongly to f_∞^l at w_∞ for every l.
3. The limits
$$a^l = \lim_{r\to 0} \frac{1}{\upsilon_\infty(B_r(w_\infty))} \int_{B_r(w_\infty)} f_\infty^l d\upsilon_\infty \in \mathbf{R}$$
exist for every l.

4. There exists an open neighborhood U at $(a^1, \dots, a^k) \in \mathbf{R}^k$ such that F_i is continuous on U for every $1 \le i \le \infty$, and that F_i converges to F_∞ on U uniformly.

Then, we also have that $F_i(f_i^1, \dots, f_i^k)$ converges strongly to $F_\infty(f_\infty^1, \dots, f_\infty^k)$ at w_∞.

The following proposition is the main result in this subsection:

PROPOSITION 4.5. *Let $\{(M_i, m_i)\}_i$ be a sequence of pointed n-dimensional complete Riemannian manifolds with $\mathrm{Ric}_{M_i} \ge -(n-1)$, (Y, y, υ) a Ricci limit space of $\{(M_i, m_i, \underline{\mathrm{vol}})\}_i$, R a positive number, x_∞, z_∞ points in Y, x_i, z_i points in M_i for every $i < \infty$, f_i a C^2-function on $B_R(x_i)$ for every $i < \infty$, and f_∞ a Lipschitz function on $B_R(x_\infty)$. Assume that $\sup_i \mathbf{Lip} f_i < \infty$, $(M_i, m_i, x_i, z_i, f_i, \underline{\mathrm{vol}}) \stackrel{(\phi_i, R_i, \epsilon_i)}{\to} (Y, y, x_\infty, z_\infty, f_\infty, \upsilon)$ and*

$$\sup_i \int_{B_R(x_i)} |\mathrm{Hess}_{f_i}| d\underline{\mathrm{vol}} < \infty.$$

Then, $\langle dr_{z_i}, df_i \rangle$ converges strongly to $\langle dr_{z_\infty}, df_\infty \rangle$ at a.e. $w_\infty \in B_R(x_\infty)$.

PROOF. Fix $\epsilon > 0$ and let $L \ge 1$ with

$$\sup_i \left(\frac{1}{\underline{\mathrm{vol}}\, B_R(x_i)} \int_{B_R(x_i)} |\mathrm{Hess}_{f_i}| d\underline{\mathrm{vol}} + \mathbf{Lip} f_i \right) \le L.$$

By Theorem 3.30, there exist $0 < \eta << \epsilon$ and a Borel subset $X(\epsilon)$ of $B_R(x_\infty) \cap \mathcal{D}_z^\eta \setminus B_\eta(z_\infty)$ such that $\upsilon(B_R(x_\infty) \setminus X(\epsilon))/\upsilon(B_R(x_\infty)) \le \epsilon$ and

$$\left| \frac{f_\infty \circ \gamma(\overline{z, \alpha} + h) - f_\infty(\alpha)}{h} - \langle dr_{z_\infty}, df_\infty \rangle (\alpha) \right| \le \epsilon$$

for every $\alpha \in X(\epsilon)$, every real number h with $0 < |h| < \eta$, and every isometric embedding γ from $[0, \overline{z_\infty, \alpha} + \eta]$ to Y with $\gamma(0) = z_\infty$, $\gamma(\overline{z_\infty, \alpha}) = \alpha$. On the other hand, by Lebesgue's differentiation theorem, there exists a Borel subset $\hat{X}(\epsilon)$ of $X(\epsilon)$ such that $\upsilon(X(\epsilon) \setminus \hat{X}(\epsilon)) = 0$ and that for every $\alpha \in \hat{X}(\epsilon)$, there exists $r(\alpha) > 0$ such that

$$\frac{1}{\upsilon(B_t(\alpha))} \int_{B_t(\alpha)} |\langle dr_{z_\infty}, df_\infty \rangle - \langle dr_{z_\infty}, df_\infty \rangle (\alpha)| d\upsilon < \epsilon$$

for every $0 < t < r(\alpha)$. Put $l = \eta^{-1/4}$. By an argument similar to the proof of Proposition 3.1, for every $1 \le i < \infty$, there exists a compact subset K_i of $B_{R-\epsilon}(x_i)$ such that

$$\frac{\underline{\mathrm{vol}}(B_{R-\epsilon}(x_i) \setminus K_i)}{\underline{\mathrm{vol}}\, B_{R-\epsilon}(x_i)} \le \Psi(l^{-1}; n, R, L) \ \text{ and } \ \frac{1}{\underline{\mathrm{vol}}\, B_t(w)} \int_{B_t(w)} |\mathrm{Hess}_{f_i}| d\underline{\mathrm{vol}} \le l$$

for every $w \in K_i$ and every $0 < t < \epsilon/100$. Without loss of generality, we can assume that there exists a compact subset K_∞ of $\overline{B}_R(x_\infty)$ such that $K_i \to K_\infty$. Put $W(\epsilon) = K_\infty \cap X(\epsilon)$. By Proposition 2.5, we have $\upsilon(W(\epsilon))/\upsilon(B_R(x_\infty)) \ge 1 - \Psi(\epsilon; n, R, L)$. Fix $\alpha \in W(\epsilon)$,

$0 < t << \min\{\eta, r(\alpha)\}$ and an isometric embedding γ from $[0, \overline{z_\infty, \alpha} + \eta]$ to Y wtih $\gamma(0) = z_\infty$, $\gamma(\overline{z_\infty, \alpha}) = \alpha$. Let $\{\alpha_i\}_i$ be a sequence of points α_i in K_i with $\alpha_i \to \alpha$. Define a Borel function F_i on $B_t(\alpha_i) \setminus (C_{z_i} \cup \{z_i\})$ by $F_i(\beta) = (f_i \circ \gamma_\beta(\overline{z_i, \beta} - \eta^2) - f_i(\beta))/(-\eta^2)$, where γ_β is the minimal geodesic from z_i to β. By an argument similar to the proof of Claim 3.34, we have

$$\frac{1}{\underline{\mathrm{vol}}\, B_t(\alpha_i)} \int_{B_t(\alpha_i)} |\langle df_i, dr_{z_i}\rangle - F_i| d\underline{\mathrm{vol}} \le \eta^2 \frac{C(n)}{\underline{\mathrm{vol}}\, B_{10t}(\alpha_i)} \int_{B_{10t}(\alpha_i)} |\mathrm{Hess}_{f_i}| d\underline{\mathrm{vol}} \le \eta^2 C(n) l \le \Psi(\epsilon; n)$$

for every i. Fix i_0 with $\epsilon_i << t$ for every $i \ge i_0$. We remark that $\overline{\phi_i(\beta_i), \alpha} \le t + \epsilon_i \le \eta^3$ for every $i \ge i_0$ and every $\beta_i \in B_t(\alpha_i)$. Then, since

$$\overline{z, \phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2))}^{\eta^{-2} d_Y} + \overline{\phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2)), \phi_i(\beta_i)}^{\eta^{-2} d_Y} - \overline{z, \phi_i(\beta_i)}^{\eta^{-2} d_Y} < 3\epsilon_i,$$

we have

$$\overline{z, \phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2))}^{\eta^{-2} d_Y} + \overline{\phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2)), \alpha}^{\eta^{-2} d_Y} - \overline{z, \alpha}^{\eta^{-2} d_Y} < 5\eta.$$

Similarly, we have

$$\overline{z, \phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2))}^{\eta^{-2} d_Y} + \overline{\phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2)), \gamma(\overline{z, \alpha} + \eta)}^{\eta^{-2} d_Y} - \overline{z, \gamma(\overline{z, \alpha} + \eta)}^{\eta^{-2} d_Y} < 5\eta,$$

$$\overline{\phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2)), \gamma(\overline{z, \alpha} + \eta)}^{\eta^{-2} d_Y} \ge \eta^{-1} - \eta, \quad \overline{\phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2)), z}^{\eta^{-2} d_Y} \ge \eta^{-1} - \eta$$

and

$$\overline{\phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2)), \alpha}^{\eta^{-2} d_Y} = 1 \pm 5\eta.$$

Therefore, by the splitting theorem on limit spaces, we have

$$\overline{\phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2)), \gamma(\overline{z, \alpha} - \eta^2)}^{\eta^{-2} d_Y} \le \Psi(\eta; n).$$

Thus we have

$$\begin{aligned}
\frac{f_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2)) - f_i(\beta_i)}{-\eta^2} &= \frac{f_\infty(\phi_i(\gamma_{\beta_i}(\overline{z_i, \beta_i} - \eta^2))) - f_\infty(\phi_i(\beta_i))}{-\eta^2} \pm \frac{\epsilon_i}{\eta^2} \\
&= \frac{f_\infty(\gamma(\overline{z, \alpha} - \eta^2))) - f_\infty(\alpha)}{-\eta^2} \pm \Psi(\eta; n, L) \\
&= \langle dr_z, df_\infty \rangle(\alpha) \pm \Psi(\eta; n, L).
\end{aligned}$$

Especially, we have

$$\frac{1}{\underline{\mathrm{vol}}\, B_t(\alpha_i)} \int_{B_t(\alpha_i)} |F_i - \langle dr_z, df_\infty \rangle(\alpha)| d\underline{\mathrm{vol}} \le \Psi(\eta; n, L)$$

for every $i \ge i_0$. Put $W = \bigcap_{N_1 \in \mathbf{N}} \left(\bigcup_{N_2 \ge N_1} W(N_2^{-1}) \right)$. Then we have $\upsilon(B_R(x_\infty) \setminus W) = 0$. Moreover, by the argument above, $\langle dr_{z_i}, df_i \rangle$ converges strongly to $\langle dr_w, df_\infty \rangle$ at every $w_\infty \in W$. □

REMARK 4.6. We shall introduce the following important method to get a uniformly L^2-Hessian estimates by using cut-off functions with good properties constructed by Cheeger-Colding: Let (M, m) be a pointed n-dimensional complete Riemannian manifold with $\mathrm{Ric}_M \geq -(n-1)$, R a positive number and f a C^2-function on $B_R(m)$. Assume that there exists $L \geq 1$ such that

$$|\nabla f|_{L^\infty(B_R(m))} + \int_{B_R(m)} |\Delta f|^2 d\underline{\mathrm{vol}} \leq L.$$

Then, we have

$$\int_{B_r(m)} |\mathrm{Hess}_f|^2 d\underline{\mathrm{vol}} < C(n, r, R, L)$$

for every $0 < r < R$. The proof is as follows. By the standard smoothing argument, without loss of generality, we can assume that f is a smooth function. By [2, Theorem 8.16], there exists a smooth function ϕ on M such that $0 \leq \phi \leq 1$, $\phi|_{B_r(m)} = 1$, $\mathrm{supp}\,\phi \subset B_R(m)$, $|\nabla \phi| \leq C(n, r, R)$ and $|\Delta \phi| \leq C(n, r, R)$. By Bochner's formula, we have

$$-\frac{1}{2}\Delta|\nabla(\phi f)|^2 \geq |\mathrm{Hess}_{\phi f}|^2 - \langle \nabla \Delta(\phi f), \nabla(\phi f)\rangle - (n-1)|\nabla(\phi f)|^2.$$

Thus, we have

$$\begin{aligned}
\int_{B_r(m)} |\mathrm{Hess}_f|^2 d\underline{\mathrm{vol}} &\leq \int_{B_R(m)} |\mathrm{Hess}_{\phi f}|^2 d\underline{\mathrm{vol}} \\
&\leq \int_{B_R(m)} (\Delta(\phi f))^2 \, d\underline{\mathrm{vol}} + C(n, R, L) \\
&\leq \int_{B_R(m)} \left((f\Delta\phi)^2 + (\phi \Delta f)^2 + |\langle \nabla f, \nabla \phi\rangle|^2\right) d\underline{\mathrm{vol}} + C(n, R, L) \\
&\leq C(n, r, R, L).
\end{aligned}$$

This observation performs a crucial role to study limit functions of harmonic functions in subsection 4.4.

The following proposition follows from Corollary 3.37 directly.

PROPOSITION 4.7. *Let $\{(M_i, m_i)\}_i$ be a sequence of pointed n-dimensional complete Riemannian manifolds with $\mathrm{Ric}_{M_i} \geq -(n-1)$, (Y, y, υ) a Ricci limit space of $\{(M_i, m_i, \underline{\mathrm{vol}})\}_i$, w^1_∞, w^2_∞ points in Y, and w^1_i, w^2_i points in M_i for every i, satisfying that $w^j_i \to w^j_\infty$ for every j. Then $\langle dr_{w^1_i}, dr_{w^2_i}\rangle$ converges strongly to $\langle dr_{w^1_\infty}, dr_{w^2_\infty}\rangle$ at every $z \in Y \setminus (C_{w^1_\infty} \cup C_{w^2_\infty} \cup \{w^1_\infty, w^2_\infty\})$.*

4.2 Pointwise weak convergence of L^∞-functions

Our aims in this subsection are to give the following notion and its fundamental properties.

DEFINITION 4.8 (Pointwise weak convergence of L^∞-functions). Let R be a positive number, w_∞ a point in $B_R(x_\infty)$ and $\{f_i\}_{1\le i\le\infty}$ a sequence of L^∞-functions f_i on $B_R(x_i)$ with $\sup_i |f_i|_{L^\infty(B_R(x_i))} < \infty$. We say that f_i *converges weakly to* f_∞ *at* w_∞ if for every $\epsilon > 0$, there exists $r > 0$ such that

$$\limsup_{i\to\infty}\left|\frac{1}{\upsilon_i(B_t(w_i))}\int_{B_t(w_i)} f_i d\upsilon_i - \frac{1}{\upsilon_\infty(B_t(w_\infty))}\int_{B_t(w_\infty)} f_\infty d\upsilon_\infty\right| \le \epsilon$$

for every $0 < t < r$ and every $w_i \to w_\infty$.

It is clear that if f_i converges strongly to f_∞ at w_∞, then f_i converges weakly to f_∞ at w_∞. We skip the proof of the next proposition because it is not difficult to check it.

PROPOSITION 4.9 (Linearlity of weak convergence). *Let R be a positive number, w_∞ a point in $B_R(x_\infty)$ and a_i, b_i, c_i, d_i L^∞-functions on $B_R(x_i)$ for every $1 \le i \le \infty$ with $\sup_i(|a_i| + |b_i| + |c_i| + |d_i|)_{L^\infty(B_R(x_i))} < \infty$. Assume that a_i, b_i converge strongly to a_∞, b_∞ at w_∞, respectively, and that c_i, d_i converge weakly to c_∞, d_∞ at w_∞, respectively. Then $a_i c_i + b_i d_i$ converges weakly to $a_\infty c_\infty + b_\infty d_\infty$ at w_∞.*

PROPOSITION 4.10. *Let $\{A_i\}_{1\le i\le\infty}$ be a sequnece of Borel subsets A_i of $B_R(x_i)$ and w_∞ a point in $\mathrm{Leb}A_\infty$. Assume that 1_{A_i} converges weakly to 1_{A_∞} at w_∞. Then 1_{A_i} converges strongly to 1_{A_∞} at w_∞.*

PROOF. Fix $\epsilon > 0$. Let $\{w_i\}_i$ be a sequence of points w_i in Z_i satisfying $w_i \to w_\infty$. There exists $r > 0$ such that $\upsilon_\infty(B_t(w_\infty) \cap A_\infty)/\upsilon_\infty(B_t(w_\infty)) \ge 1 - \epsilon$ and

$$\limsup_{i\to\infty}\left|\frac{1}{\upsilon_i(B_t(w_i))}\int_{B_t(w_i)} 1_{A_i} d\upsilon_i - \frac{1}{\upsilon_\infty(B_t(w_\infty))}\int_{B_t(w_\infty)} 1_{A_\infty} d\upsilon_\infty\right| < \epsilon$$

for every $0 < t < r$. Fix $0 < t < r$. Then we have

$$\begin{aligned}
&\frac{1}{\upsilon_i(B_t(w_i))}\int_{B_t(w_i)}\left|1_{A_i} - \frac{1}{\upsilon_\infty(B_t(w_\infty))}\int_{B_t(w_\infty)} 1_{A_\infty} d\upsilon_\infty\right| d\upsilon_i \\
&\le \frac{1}{\upsilon_i(B_t(w_i))}\int_{B_t(w_i)}\left|1_{A_i} - \frac{1}{\upsilon_i(B_t(w_i))}\int_{B_t(w_i)} 1_{A_i} d\upsilon_i\right| d\upsilon_i + \epsilon \\
&= \frac{1}{\upsilon_i(B_t(w_i))}\int_{A_i}\frac{\upsilon_i(B_t(w_i)\setminus A_i)}{\upsilon_i(B_t(w_i))} d\upsilon_i + \frac{1}{\upsilon_i(B_t(w_i))}\int_{B_t(w_i)\setminus A_i}\frac{\upsilon_i(A_i)}{\upsilon_i(B_t(w_i))} d\upsilon_i + \epsilon \\
&\le 2\frac{\upsilon_i(B_t(w_i)\setminus A_i)}{\upsilon_i(B_t(w_i))} + \epsilon < 2\frac{\upsilon_\infty(B_t(w_\infty)\setminus A_\infty)}{\upsilon_\infty(B_t(w_\infty))} + 2\epsilon < 5\epsilon.
\end{aligned}$$

for every sufficiently large i. Similarly, we have

$$\frac{1}{\upsilon_\infty(B_t(w_\infty))}\int_{B_t(w_\infty)}\left|1_{A_\infty}-\frac{1}{\upsilon_i(B_t(w_i))}\int_{B_t(w_i)}1_{A_i}d\upsilon_i\right|d\upsilon_\infty<5\epsilon$$

for every sufficiently large i. Thus, we have the assertion. $\square$

The next proposition follows from an argument similar to the proof of Proposition 2.5:

Proposition 4.11. *Let R be a positive number, $\{K_i\}_{1\le i\le\infty}$ a sequence of Borel subsets K_i of $\overline{B}_R(x_i)$, and $\{f_i\}_{1\le i\le\infty}$ of nonnegative valued L^∞-functions f_i on $\overline{B}_R(x_i)$ with* $\sup_i|f_i|_{L^\infty(B_R(x_i))}<\infty$. *Assume that K_∞ is compact,* $\limsup^{GH}_{i\to\infty}K_i\subset K_\infty$ *and that f_i converges weakly to f_∞ at a.e. $w\in K_\infty$. Then we have*

$$\limsup_{i\to\infty}\int_{K_i}f_id\upsilon_i\le\int_{K_\infty}f_\infty d\upsilon_\infty.$$

We shall give a fundamental result about this weak convergence:

Proposition 4.12. *Let R be a positive number, $\{A_i\}_{1\le i\le\infty}$ a sequence of Borel subsets A_i of $\overline{B}_R(x_i)$, and $\{f_i\}_{1\le i\le\infty}$ of L^∞-functions f_i on $\overline{B}_R(x_i)$ with* $\sup_i|f_i|_{L^\infty(B_R(x_i))}<\infty$. *Assume that 1_{A_i} converges weakly to 1_{A_∞} at a.e. $w\in B_R(x_\infty)$ and that f_i converges weakly to f_∞ at a.e. $w\in A_\infty$. Then, we have*

$$\lim_{i\to\infty}\int_{A_i}f_id\upsilon_i=\int_{A_\infty}f_\infty d\upsilon_\infty.$$

Proof. It follows from (the proof of) Proposition 4.9 and 4.10 that $f_i1_{A_i}$ converges weakly to $f_\infty 1_{A_\infty}$ at a.e. $w_\infty\in B_R(x_\infty)$. Thus, without loss of generality, we can assume that $A_i=B_R(x_i)$ for every $1\le i\le\infty$. Fix $\epsilon>0$. Let $L\ge1$ with $\sup_i|f_i|_{L^\infty(B_R(x_i))}+\upsilon_\infty(B_R(x_\infty))<L$. There exists a Borel subset $\hat{K}_\infty$ of $B_R(x_\infty)$ such that $\upsilon(B_R(x_\infty)\setminus\hat{K}_\infty)=0$ and that for every $w_\infty\in\hat{K}_\infty$, there exists $t_{w_\infty}>0$ such that $\overline{B}_{10t_{w_\infty}}(w_\infty)\subset B_R(x)$ and

$$\limsup_{i\to\infty}\left|\frac{1}{\upsilon_i(B_s(w_i))}\int_{B_s(w_i)}f_id\upsilon_i-\frac{1}{\upsilon_\infty(B_s(w_\infty))}\int_{B_s(w_\infty)}f_\infty d\upsilon_\infty\right|<\epsilon$$

for every $0<s<t_{w_\infty}$ and every $w_i\to w_\infty$. By Lemma 2.4, there exists a pairwise disjoint collection $\{\overline{B}_{r_i}(x_i)\}_i$ such that $x_i\in\hat{K}_\infty$, $r_i<<t_{x_i}$, and $\hat{K}_\infty\setminus\bigcup_{i=1}^N\overline{B}_{r_i}(x_i)\subset\bigcup_{i=N+1}^\infty\overline{B}_{5r_i}(x_i)$ for every N. Fix N satisfying $\sum_{i=N+1}^\infty\upsilon_\infty(B_{r_i}(x_i))<\epsilon$. Then, we have $\sum_{i=N+1}^\infty\upsilon_\infty(B_{5r_i}(x_i))<2^{5\kappa(1)}\epsilon$. For every i,j, let $x_i(j)$ be a point in Z_j satisfying $x_i(j)\to$

x_i. Then we have

$$\begin{aligned}\int_{B_R(x_\infty)} f_\infty d\upsilon_\infty &= \sum_{i=1}^N \int_{B_{r_i}(x_i)} f_\infty d\upsilon_\infty \pm \Psi(\epsilon; \kappa(1), L) \\ &= \sum_{i=1}^N \int_{B_{r_i}(x_i(j))} f_j d\upsilon_j \pm \Psi(\epsilon; \kappa(1), L) \\ &= \int_{B_R(x_j)} f_j d\upsilon_j \pm \left(\int_{B_R(x_j)\setminus \bigcup_{i=1}^N \overline{B}_{r_i}(x_i(j))} |f_j| d\upsilon_j + \Psi(\epsilon; \kappa(1), L)\right).\end{aligned}$$

for every sufficiently large j. On the other hand, by Proposition 2.3 and Proposition 2.5, we have

$$\begin{aligned}\limsup_{j\to\infty} \int_{B_R(x_j)\setminus \bigcup_{i=1}^N \overline{B}_{r_i}(x_i(j))} |f_j| d\upsilon_j &\le L \limsup_{j\to\infty} \upsilon_j \left(\overline{B}_R(x_j) \setminus \bigcup_{i=1}^N B_{r_i}(x_i(j))\right) \\ &\le L\upsilon_\infty \left(\hat{K}_\infty \setminus \bigcup_{i=1}^N B_{r_i}(x_i)\right) \\ &\le \Psi(\epsilon; \kappa(1), L).\end{aligned}$$

Therefore, we have the assertion. □

Next corollary follows from Proposition 4.12 directly.

Corollary 4.13. *Let R be a positive number, N a positive integer, $\{r_j\}_{1\le j\le N}$ a collection of positive numbers, $\{z_j\}_{1\le j\le N}$ of points in Y, and $\{f_i\}_{1\le i\le\infty}$ a sequence of L^∞-functions f_i on $B_R(x_i)$ with $\sup_i |f_i|_{L^\infty(B_R(x_i))} < \infty$. Assume that f_i converges weakly to f_∞ at a.e. $w \in B_R(x_\infty) \setminus \bigcup_{i=1}^N B_{r_i}(z_i)$. Then, we have*

$$\lim_{j\to\infty} \int_{B_R(x_j)\setminus \bigcup_{i=1}^N B_{r_i}(z_i(j))} f_j d\upsilon_j = \int_{B_R(x_\infty)\setminus \bigcup_{i=1}^N B_{r_i}(z_i)} f_\infty d\upsilon_\infty$$

for every $z_i(j) \to z_i$.

4.3 Convergence of the differentials of Lipschitz functions

A purpose of this subsection is to give the definition of a convergence: $df_i \to df_\infty$. See Definition 1.1 or Definition 4.15. Throughout this subsection, we fix the following situation:

1. Let $\{(M_i, m_i)\}_{1\le i<\infty}$ be a sequence of pointed n-dimensional complete Riemannian manifolds with $\mathrm{Ric}_{M_i} \ge -(n-1)$.

2. Let (Y, y, υ) be a Ricci limit space of $\{(M_i, m_i, \underline{\mathrm{vol}})\}_i$.

3. Let R be a positive number, $\{x_i\}_{1\le i<\infty}$ a sequence of points x_i in M_i, and x_∞ a point in Y satisfying $x_i \to x_\infty$.

4. Let $\{f_i\}_{1\le i\le\infty}$ be a sequence of Lipschitz functions f_i on $B_R(x_i)$ with $\sup_i(\mathbf{Lip}f_i + |f_i|_{L^\infty(B_R(x_i))}) < \infty$.

In this setting, we recall that f_i *converges to* f_∞ at $w_\infty \in B_R(x_\infty)$ if $f_i(w_i) \to f_\infty(w_\infty)$ holds for every $w_i \to w_\infty$. See section 1.2. We denote it by $f_i \to f_\infty$ at w_∞. We remark that it is easy to check that the following conditions are equivalent:

1. f_i converges strongly to f_∞ at w_∞.

2. $f_i \to f_\infty$ at w_∞.

3. f_i converges weakly to f_∞ at w_∞.

We shall consider a convergence of the L^2-energy of Lipschitz functions.

Definition 4.14 (Pointwise upper semicontinuity of L^2-energy). We say that *L^2-energy of $\{f_i\}_i$ are upper semicontinuous at* $w_\infty \in B_R(x_\infty)$ if for every $\epsilon > 0$, there exists $r > 0$ such that

$$\limsup_{i\to\infty} \frac{1}{\underline{\mathrm{vol}}\, B_t(w_i)} \int_{B_t(w_i)} (\mathrm{Lip} f_i)^2 d\underline{\mathrm{vol}} \le \frac{1}{\upsilon(B_t(w_\infty))} \int_{B_t(w_\infty)} (\mathrm{Lip} f_\infty)^2 d\upsilon + \epsilon$$

for every $0 < t < r$ and every $w_i \to w_\infty$.

By the definition, if $(\mathrm{Lip} f_i)^2$ converges weakly to $(\mathrm{Lip} f_\infty)^2$ at w_∞, then L^2-energy of $\{f_i\}_i$ are upper semicontinuous at w_∞. We shall give the definition of a convergence of the differentials of Lipschitz functions:

Definition 4.15 (Convergence of the differentials of Lipschitz functions). We say that *df_i converges to df_∞ at* $w_\infty \in B_R(x_\infty)$ if the following properties hold:

1. $\langle dr_{z_i}, df_i\rangle$ converges weakly to $\langle dr_{z_\infty}, df_\infty\rangle$ at w_∞ for every $z_i \to z_\infty$

2. L^2-energy of $\{f_i\}_i$ are upper semicontinuous at w_∞.

Then we denote it by $df_i \to df_\infty$ at w_∞. Moreover, for a subset A of $B_R(x_\infty)$, if $f_i \to f_\infty$ and $df_i \to df_\infty$ at every $a \in A$, then we denote it by $(f_i, df_i) \to (f_\infty, df_\infty)$ on A.

Proposition 4.16. *Let w_i be a point in M_i for every $i < \infty$, and w_∞ a point in Y with $w_i \to w_\infty$. Then we have $(r_{w_i}, dr_{w_i}) \to (r_{w_\infty}, dr_{w_\infty})$ on Y.*

Proof. It follows from Proposition 4.7 and Proposition 4.12 directly. □

The following theorem is the main result in this subsection:

THEOREM 4.17. *Let* $\{g_i\}_{1\le i\le\infty}$ *be a sequence of Lipschitz functions* g_i *on* $B_R(x_i)$*, and* A *a Borel subset of* $B_R(x_\infty)$*. Assume that* $\sup_i(\mathbf{Lip}g_i + |g_i|_{L^\infty(B_R(x_i))}) < \infty$*,* $df_i \to df_\infty$ *and* $dg_i \to dg_\infty$ *on* A*. Then,* $\langle df_i, dg_i\rangle$ *converges strongly to* $\langle df_\infty, dg_\infty\rangle$ *at a.e.* $w_\infty \in A$*.*

PROOF. By Theorem 3.16 and Lemma 3.48, there exist collections of Borel subset $\{A_j\}_j$ of A, of positive integers $\{k_j\}_j$ with $1 \le k_j \le n$, and of points $\{x_l^j\}_{j,1\le l\le k_j}$ in Y such that the following properties hold:

1. $\upsilon\left(A \setminus \bigcup_{j=1}^\infty A_j\right) = 0$ and $A_j \subset Y \setminus \bigcup_{l=1}^{k_j}(C_{x_l^j} \cup \{x_l^j\})$ for every j.
2. For every $w \in A_j$, there exists $a_1^j, \ldots, a_{k_j}^j, b_1^j, \ldots, b_{k_j}^j \in \mathbf{R}$ such that

$$\lim_{r\to 0}\frac{1}{\upsilon(B_r(w_\infty))}\int_{B_r(w_\infty)}\left|df_\infty - d\left(\sum_{l=1}^{k_j}a_l^j r_{x_l^j}\right)\right|^2 + \left|dg_\infty - d\left(\sum_{l=1}^{k_j}b_l^j r_{x_l^j}\right)\right|^2 d\upsilon = 0.$$

Fix j and $w_\infty \in A_j$. Let $a_1^j, \ldots, a_{k_j}^j, b_1^j, \ldots, b_{k_j}^j \in \mathbf{R}$ as above, and $L \ge 1$ with $\sup_i(\mathbf{Lip}f_i + \mathbf{Lip}g_i) + \sum_{l=1}^{k_j}((a_l^j)^2 + (b_l^j)^2) \le L$. Take $\tau > 0$ with $w \in \bigcup_{l=1}^{k_j}(D_{x_l^j}^\tau \setminus B_\tau(x_l^j))$. Let $x_l^j(i) \to x_l^j$ and $w_i \to w_\infty$. Fix $\epsilon > 0$. Then, there exists $r > 0$ such that

$$\frac{1}{\upsilon(B_t(w_\infty))}\int_{B_t(w_\infty)}\left|df_\infty - d\left(\sum_{l=1}^{k_j}a_l^j r_{x_l^j}\right)\right|^2 + \left|dg_\infty - d\left(\sum_{l=1}^{k_j}b_l^j r_{x_l^j}\right)\right|^2 d\upsilon \le \epsilon,$$

$$\limsup_{i\to\infty}\frac{1}{\underline{\mathrm{vol}}\,B_t(w_i)}\int_{B_t(w_i)}(\mathrm{Lip}f_i)^2 d\underline{\mathrm{vol}} \le \frac{1}{\upsilon(B_t(w_\infty))}\int_{B_t(w_\infty)}(\mathrm{Lip}f_\infty)^2 d\upsilon + \epsilon,$$

$$\limsup_{i\to\infty}\frac{1}{\underline{\mathrm{vol}}\,B_t(w_i)}\int_{B_t(w_i)}(\mathrm{Lip}g_i)^2 d\underline{\mathrm{vol}} \le \frac{1}{\upsilon(B_t(w_\infty))}\int_{B_t(w_\infty)}(\mathrm{Lip}g_\infty)^2 d\upsilon + \epsilon,$$

$$\limsup_{i\to\infty}\left|\frac{1}{\underline{\mathrm{vol}}\,B_t(w_i)}\int_{B_t(w_i)}\langle df_i, dr_{x_l^j(i)}\rangle d\underline{\mathrm{vol}} - \frac{1}{\upsilon(B_t(w_\infty))}\int_{B_t(w_\infty)}\langle df_\infty, dr_{x_l^j}\rangle d\upsilon\right| < \epsilon$$

and

$$\limsup_{i\to\infty}\left|\frac{1}{\underline{\mathrm{vol}}\,B_t(w_i)}\int_{B_t(w_i)}\langle dg_i, dr_{x_l^j(i)}\rangle d\underline{\mathrm{vol}} - \frac{1}{\upsilon(B_t(w_\infty))}\int_{B_t(w_\infty)}\langle dg_\infty, dr_{x_l^j}\rangle d\upsilon\right| < \epsilon$$

for every l and every $0 < t < r$. Fix $0 < t << \min\{r, \epsilon, \tau\}$. Then, by Corollary 3.37, we have

$$\frac{1}{\upsilon(B_t(w_\infty))}\int_{B_t(w_\infty)}\left|\langle df_\infty, dg_\infty\rangle - \frac{1}{\upsilon(B_t(w_\infty))}\int_{B_t(w_\infty)}\left\langle d\left(\sum_{l=1}^{k_j}a_l^j r_{x_l^j}\right), d\left(\sum_{l=1}^{k_j}b_l^j r_{x_l^j}\right)\right\rangle d\upsilon\right| d\upsilon$$
$$\le \Psi(\epsilon; n, L)$$

and

$$
\begin{aligned}
&\frac{1}{\upsilon(B_t(w_\infty))}\int_{B_t(w_\infty)}\left|\langle df_\infty, dg_\infty\rangle - \frac{1}{\upsilon(B_t(w))}\int_{B_t(w)}\langle df_\infty, dg_\infty\rangle d\upsilon\right| d\upsilon \\
&= \frac{1}{\upsilon(B_t(w_\infty))}\int_{B_t(w_\infty)}\Bigg|\left\langle d\left(\sum_{l=1}^{k_j} a_l^j r_{x_l^j}\right), d\left(\sum_{l=1}^{k_j} b_l^j r_{x_l^j}\right)\right\rangle \\
&\quad - \frac{1}{\upsilon(B_t(w_\infty))}\int_{B_t(w_\infty)}\left\langle d\left(\sum_{l=1}^{k_j} a_l^j r_{x_l^j}\right), d\left(\sum_{l=1}^{k_j} b_l^j r_{x_l^j}\right)\right\rangle d\upsilon\Bigg| d\upsilon \pm \Psi(\epsilon; n, L) \\
&= \Psi(\epsilon; n, L).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\frac{1}{\underline{\mathrm{vol}}\, B_t(w_i)}\int_{B_t(w_i)}\left|df_i - d\left(\sum_{l=1}^{k_j} a_l^j r_{x_l^j(i)}\right)\right|^2 d\underline{\mathrm{vol}} \\
&= \frac{1}{\underline{\mathrm{vol}}\, B_t(w_i)}\int_{B_t(w_i)}|df_i|^2 d\underline{\mathrm{vol}} - \sum_{l=1}^{k_j}\frac{a_l^j}{\underline{\mathrm{vol}}\, B_t(w_i)}\int_{B_t(w_i)}\langle df_i, dr_{x_l^j(i)}\rangle d\underline{\mathrm{vol}} \\
&+ \sum_{l,\hat{l}}\frac{a_l^j a_{\hat{l}}^j}{\underline{\mathrm{vol}}\, B_t(w_i)}\int_{B_t(w_i)}\langle dr_{x_l^j(i)}, dr_{x_{\hat{l}}^j(i)}\rangle d\underline{\mathrm{vol}} \\
&\le \frac{1}{\upsilon(B_t(w_\infty))}\int_{B_t(w_\infty)}|df_\infty|^2 d\upsilon - \sum_{l=1}^{k}\frac{a_l^j}{\upsilon(B_t(w_\infty))}\int_{B_t(w_\infty)}\langle df_\infty, dr_{x_l^j}\rangle d\upsilon \\
&+ \sum_{l,\hat{l}}\frac{a_l^j a_{\hat{l}}^j}{\upsilon(B_t(w_\infty))}\int_{B_t(w_\infty)}\langle dr_{x_l^j}, dr_{x_{\hat{l}}^j}\rangle d\upsilon + \Psi(\epsilon; n, L) \\
&= \frac{1}{\upsilon(B_t(w_\infty))}\int_{B_t(w_\infty)}\left|df_\infty - d\left(\sum_{l=1}^{k_j} a_l^j r_{x_l^j}\right)\right|^2 d\upsilon + \Psi(\epsilon; n, L) \le \Psi(\epsilon; n, L).
\end{aligned}
$$

for every sufficiently large i. Similarly, we have

$$
\frac{1}{\underline{\mathrm{vol}}\, B_t(w_i)}\int_{B_t(w_i)}\left|dg_i - d\left(\sum_{l=1}^{k_j} b_l^j r_{x_l^j(i)}\right)\right|^2 d\underline{\mathrm{vol}} \le \Psi(\epsilon; n, L)
$$

for every sufficiently large i. Especially, we have

$$
\begin{aligned}
&\frac{1}{\underline{\mathrm{vol}}\, B_t(w_i)}\int_{B_t(w_i)}\left|\langle df_i, dg_i\rangle - \frac{1}{\underline{\mathrm{vol}}\, B_t(w_i)}\int_{B_t(w_i)}\left\langle d\left(\sum_{l=1}^{k_j} a_l^j r_{x_l^j(i)}\right), d\left(\sum_{l=1}^{k_j} b_l^j r_{x_l^j(i)}\right)\right\rangle d\underline{\mathrm{vol}}\right| d\underline{\mathrm{vol}} \\
&\le \Psi(\epsilon; n, L).
\end{aligned}
$$

Therefore, by Corollary 3.37, we have the assertion. □

Corollary 4.18. *Let Ω be a non-empty open subset of $B_R(x_\infty)$. Assume that $df_i \to df_\infty$ at a.e. $w \in \Omega$. Then $df_i \to df_\infty$ on Ω.*

Proof. The assertion follows from Proposition 4.12 and Theorem 4.17. $\square$

Corollary 4.19. *Let $\{g_i\}_{1\le i\le\infty}$ be a sequence of Lipschitz functions g_i on $B_R(x_i)$ with $\sup_i(\mathbf{Lip} g_i + |g_i|_{L^\infty(B_R(x_i))}) < \infty$, and A a Borel subset of $B_R(x_\infty)$. Assume that $(f_i, df_i) \to (f_\infty, df_\infty)$ and $(g_i, dg_i) \to (g_\infty, dg_\infty)$ on A. Then, $(f_i + g_i, d(f_i + g_i)) \to (f_\infty + g_\infty, d(f_\infty + g_\infty))$ at a.e. $w_\infty \in A$, and $(f_i g_i, d(f_i g_i)) \to (f_\infty g_\infty, d(f_\infty g_\infty))$ at a.e. $w_\infty \in A$.*

Proof. By Theorem 4.17, there exists a Borel subset $\hat{A}$ of A such that $\upsilon(A \setminus \hat{A}) = 0$ and that $|df_i|^2, \langle df_i, dg_i\rangle$ and $|dg_i|^2$ converge strongly to $|df_\infty|^2, \langle df_\infty, dg_\infty\rangle$ and $|dg_\infty|^2$ on $\hat{A}$, respectively. Since $|d(f_i g_i)|^2 = f_i^2|dg_i|^2 + 2f_i g_i\langle df_i, dg_i\rangle + g_i|df_i|^2$, by Proposition 4.3, $|d(f_i g_i)|^2$ converges strongly to $f_\infty^2|dg_\infty|^2 + 2f_\infty g_\infty\langle df_\infty, dg_\infty\rangle + g_\infty^2|df_\infty|^2 = |d(f_\infty g_\infty)|^2$ on $\hat{A}$. On the other hand, since $d(f_i g_i) = g_i df_i + f_i dg_i$, by Proposition 4.9, $\langle dr_{z_i}, d(f_i g_i)\rangle$ converges weakly to $g_\infty\langle dr_{z_\infty}, df_\infty\rangle + f_\infty\langle dr_{z_\infty}, dg_\infty\rangle = \langle dr_{z_\infty}, d(f_\infty g_\infty)\rangle$ on $\hat{A}$ for every $z_i \to z_\infty$. Therefore we have $(f_i g_i, d(f_i g_i)) \to (f_\infty g_\infty, d(f_\infty g_\infty))$ on $\hat{A}$. Similarly, we have $(f_i + g_i, d(f_i + g_i)) \to (f_\infty + g_\infty, d(f_\infty + g_\infty))$ on $\hat{A}$. $\square$

Corollary 4.20. *Let k be a positive integer, $\{A_i\}_{1\le i\le\infty}$ a sequence of Borel subsets A_i of $\overline{B}_R(x_i)$, $\{f_i^l, g_i^l\}_{1\le i\le\infty, 1\le l\le k}$ a collection of Lipschitz functions f_i^l, g_i^l on $B_R(x_i)$ with $\sup_i(\mathbf{Lip}\, f_i^l + \mathbf{Lip}\, g_i^l) < \infty$, and $\{F_i\}_{1\le i\le\infty}$ a sequence of continuous functions on $\mathbf{R}^k$. Assume that the following properties hold:*

1. *F_i converges to F_∞ with respect to the compact uniformly topology.*
2. *1_{A_i} converges weakly to 1_{A_∞} at a.e. $w_\infty \in B_R(x_\infty)$.*
3. *$df_i^l \to df_\infty^l$ and $dg_i^l \to dg_\infty^l$ at a.e. $w_\infty \in A_\infty$ for every $1 \le l \le k$.*

Then we have

$$\lim_{i\to\infty}\int_{A_i} F_i(\langle df_i^1, dg_i^1\rangle, \ldots, \langle df_i^k, dg_i^k\rangle) d\underline{\mathrm{vol}} = \int_{A_\infty} F_\infty(\langle df_\infty^1, dg_\infty^1\rangle, \ldots, \langle df_\infty^k, dg_\infty^k\rangle) d\upsilon.$$

Proof. The assertion follows from Proposition 4.3, Proposition 4.10 and Theorem 4.17. $\square$

We shall end this subsection by giving several remarks:

Remark 4.21. By several arguments in section 3, and the proof of Theorem 4.17, we can also show the following: Assume that the following properties hold:

1. L^2-energy of $\{f_i\}_i$ are upper semicontinuous at every $\alpha \in B_R(x_\infty)$,

2. There exist a dense subset A of $B_R(x_\infty)$ and a Borel subset $\hat{A}$ of $B_R(x_\infty)$ such that $\upsilon(B_R(x_\infty) \setminus \hat{A}) = 0$ and that $\langle dr_{w_i}, df_i\rangle$ converges weakly to $\langle dr_{w_\infty}, df_\infty\rangle$ at every $\alpha \in \hat{A}$ for every $w_\infty \in A$ and every $w_i \to w_\infty$.

Then, $df_i \to df_\infty$ on $B_R(x_\infty)$.

REMARK 4.22. Let $\{(Y_i, y_i, \upsilon_i)\}_{1\le i\le\infty}$ be a sequence of Ricci limit spaces and $\{f_i\}_{1\le i\le\infty}$ a sequence of Lipschitz functions f_i on $B_R(y_i)$. Then, similarly, we can also define a notion of convergence: $df_i \to df_\infty$ and give several properties as above.

REMARK 4.23. Let (Y, y, υ) be a Ricci limit space and $\{f_i\}_{1\le i\le\infty}$ a sequence of Lipschitz functions on $B_R(y)$ with $\sup_i \mathbf{Lip} f_i < \infty$. Then, $df_i \to df_\infty$ on $B_R(y)$ (in the sense of Definition 4.15 with respect to the convergence $(Y, y, \upsilon) \stackrel{(\mathrm{id}_Y, R_i, \epsilon_i)}{\to} (Y, y, \upsilon)$) if and only if $|\mathrm{Lip}(f_i - f_\infty)|_{L^2(B_R(y))} \to 0$. We shall check it below. By Corollary 4.20, it suffices to check 'if' part. Assume that $|\mathrm{Lip}(f_i - f_\infty)|_{L^2(B_R(y))} \to 0$. Then, it is clear that L^2-energy of $\{f_i\}_i$ are upper semicontinuous at every $w \in B_R(y)$. On the other hand, by Proposition 4.16, we have $\lim_{i\to\infty} |\mathrm{Lip}(r_{x_i} - r_{x_\infty})|_{L^2(B_R(y))} = 0$ for every $x_i \to x_\infty \in Y$. Especially, $\langle dr_{x_i}, df_i\rangle$ converges weakly to $\langle dr_{x_\infty}, df_\infty\rangle$ at every $w \in B_R(y)$. Thus, $df_i \to df_\infty$ on $B_R(y)$.

4.4 An approximation theorem

Throughout this subsection, we shall use the following notation (same to one used in previous subsection): Let $\{(M_i, m_i)\}_i$ be a sequence of pointed n-dimensional complete Riemannian manifolds with $\mathrm{Ric}_{M_i} \ge -(n-1)$, (Y, y, υ) a Ricci limit space of $\{(M_i, m_i, \underline{\mathrm{vol}})\}_i$, x_i a point in M_i for every $i < \infty$, x_∞ a point in Y satisfying $(M_i, m_i, x_i, \underline{\mathrm{vol}}) \stackrel{(\phi_i, R_i, \epsilon_i)}{\to} (Y, y, x_\infty, \upsilon)$. A purpose in this subsection is to give the following approximation theorem. Roughly speaking, it means that for a given Lipschitz function f_∞ on $B_R(x_\infty)$, there exists a sequence of Lipschitz functions f_i on $B_R(x_i)$ approximating the given function with respect to the topology: "$(f_i, df_i) \to (f_\infty, df_\infty)$".

THEOREM 4.24 (Approximation theorem). *Let L, R be positive numbers, f_∞ an L-Lipschitz function on $\overline{B}_R(x_\infty)$, A_∞ a compact subset of $\overline{B}_R(x_\infty)$, $\{A_i\}_{1\le i<\infty}$ a sequence of Borel subsets A_i of $\overline{B}_R(x_i)$, and $\{f_i\}_{1\le i<\infty}$ a sequence of L-Lipschitz functions f_i on A_i. Assume that $\limsup^{GH}_{i\to\infty} A_i \subset A_\infty$ and that $f_\infty|_{A_\infty}$ is an extension of $\{f_i\}_i$ asymptotically. Then, for every $\epsilon > 0$, there exist an open subset Ω_ϵ of $B_R(x_\infty) \setminus A_\infty$, and a sequence $\{f_i^\epsilon\}_{1\le i\le\infty}$ of $C(n, L)$-Lipschitz functions f_i^ϵ on $B_R(x_i)$ such that $(f_i^\epsilon, df_i^\epsilon) \to (f_\infty^\epsilon, df_\infty^\epsilon)$ on Ω_ϵ, $f_i^\epsilon|_{A_i} = f_i|_{A_i}$ for every $1 \le i \le \infty$, and*

$$\frac{\upsilon(B_R(x_\infty) \setminus (\Omega_\epsilon \cup A_\infty))}{\upsilon(B_R(x_\infty))} + |f_\infty - f_\infty^\epsilon|_{L^\infty(B_R(x_\infty))} + |\mathrm{Lip}(f_\infty^\epsilon - f_\infty)|_{L^2(B_R(x_\infty))} < \epsilon.$$

PROOF. Fix sufficiently small $\epsilon > 0$ and $\xi > 0$ (we will decide ξ later). By Lemma 3.13 and (the proof of) Theorem 3.16, there exist collections of pairwise disjoint Borel subsets $\{E_j\}_j$ of $B_R(x_\infty)$, of positive numbers $\{\tau_j\}_j$, of positive integers $\{k_j\}_j$ with $1 \le k_j \le n$, and of points $\{x_l^j\}_{j,1\le l\le k_j}$ in Y such that the following properties hold:

1. $\upsilon_\infty\left(B_R(x_\infty)\setminus \bigcup_j E_j\right) = 0$ and $E_j \subset \bigcap_{l=1}^{k_j}\left(\mathcal{D}^{\tau_j}_{x_l^j}\setminus B_{\tau_j}(x_l^j)\right)$ for every j.
2. For every $w \in E_j$, we have
$$\langle dr_{x_l^j}, dr_{x_{\hat{l}}^j}\rangle(w) = \lim_{r\to 0}\frac{1}{\upsilon(B_r(w))}\int_{B_r(w)}\langle dr_{x_l^j}, dr_{x_{\hat{l}}^j}\rangle d\upsilon = \delta_{l,\hat{l}} \pm \epsilon.$$
3. For every $w \in E_j$, there exists $r_w > 0$ such that $r_w << \tau_j$, $\overline{B}_{10r_w}(w) \subset B_R(x_\infty)$ and
$$\frac{1}{\upsilon(B_t(w))}\int_{B_t(w)}\left|df_\infty - d\left(\sum_{l=1}^{k_j} a_l^j(w) r_{x_l^j}\right)\right|^2 d\upsilon < \epsilon$$
for every $0 < t < r_w$.

Put $X = \bigcup_{j=1}^\infty (E_j \setminus \overline{B}_{5\xi}(A_\infty))$. By Proposition 2.4, there exists a pairwise disjoint collection $\{\overline{B}_{r_i}(z_i)\}_i \subset B_R(x_\infty)$ such that $z_i \in X$, $r_i << \min\{r_{z_i}, \epsilon, \xi\}$ and $X \setminus \bigcup_{i=1}^N \overline{B}_{r_i}(z_i) \subset \bigcup_{i=N+1}^\infty \overline{B}_{5r_i}(z_i)$ for every N. For every i, let $l(i)$ with $z_i \in E_{l(i)}$. Without loss of generality, we can assume that $l(i) = i$. Fix N satisfying $\sum_{i=N+1}^\infty \upsilon(B_{r_i}(z_i)) < \epsilon$. Let $z_i(j) \to z_i$ and $x_m^l(j) \to x_m^l$. Define functions F_i^j on $B_{r_i}(z_i(j))$, and F_i on $B_{r_i}(z_i)$ by
$$F_i^j = \sum_{m=1}^{k_i} a_m^i r_{x_m^i(j)} + C_i,\ \ F_i = \sum_{m=1}^{k_i} a_m^i r_{x_m^i} + C_i,$$
where C_i is the constant defined by satisfying $F_i(z_i) = f_\infty(z_i)$, and $a_m^i = a_m^i(z_i)$.

CLAIM 4.25. *We have* $\mathbf{Lip}F_i^j + \mathbf{Lip}F_i \le C(n, L)$ *for every* i, j.

The proof is as follows. Since
$$\begin{aligned}|df_\infty(z_i)|^2 &= \sum_{s,t} a_s^i a_t^i \langle dr_{x_s^i}, dr_{x_t^i}\rangle(z_i)\\ &= \sum_{s,t} a_s^i a_t^i(\delta_{s,t} \pm \epsilon)\\ &= (1\pm\epsilon)\sum_{s=1}^{k_i}(a_s^i)^2 \pm \Psi(\epsilon; n)\sum_{s=1}^{k_i}(a_s^i)^2 = (1\pm\Psi(\epsilon;n))\sum_{s=1}^{k_i}(a_s^i)^2\end{aligned}$$
and $|df_\infty|(z_i) \le L$, we have $\sum_{m=1}^{k_i}(a_m^i)^2 \le L^2 + \Psi(\epsilon; n, L)$. Therefore we have Claim 4.25.

We remark that $\{\overline{B}_{r_i}(z_i(j))\}_{1\le i\le N}$ is a pairwise disjoint collection for every sufficiently large j. Define functions F_j on $\bigcup_{m=1}^N \overline{B}_{(1-\xi)r_i}(z_i(j))$, and F_∞ on $\bigcup_{m=1}^N \overline{B}_{(1-\xi)r_i}(z_i)$ by $F_j|_{B_{(1-\xi)r_i}(z_i(j))} = F_j^i|_{B_{(1-\xi)r_i}(z_i(j))}, F_\infty|_{B_{(1-\xi)r_i}(z_i)} = F_j|_{B_{(1-\xi)r_i}(z_i)}$ for every sufficiently large j.

CLAIM 4.26. *We have* $\mathbf{Lip}F_j + \mathbf{Lip}F_\infty \le C(n,L) + \xi^{-1}\Psi(\epsilon; n, L)$ *for every sufficiently large* j.

The proof is as follows. By Claim 4.25, we have $\mathbf{Lip}(F_j|_{\overline{B}_{(1-\xi)r_i}(z_i(j))}) + \mathbf{Lip}(F_\infty|_{\overline{B}_{(1-\xi)r_i}(z_i)}) \le C(n,L)$ for every i, j. Let j_0 satisfying that $\epsilon_j << \min\{\xi r_1, \ldots, \xi r_N\}$ for every $j \ge j_0$. Fix $j \ge j_0$, $1 \le l < m \le N$, $w_l(j) \in \overline{B}_{(1-\xi)r_l}(z_l(j))$ and $w_m(j) \in \overline{B}_{(1-\xi)r_m}(z_m(j))$. Since $\overline{B}_{r_l}(z_l(j)) \cap \overline{B}_{r_m}(z_m(j)) = \emptyset$, there exists $\alpha(j) \in \partial B_{r_l}(z_l)$ such that $\overline{w_l(j), \alpha(j)} + \overline{\alpha(j), w_m(j)} = \overline{w_l(j), w_m(j)}$. Thus we have $\overline{w_l(j), w_m(j)} \ge \overline{w_l(j), \alpha(j)} \ge \xi r_l$. Similarly, we have $\overline{w_l(j), w_m(j)} \ge \xi r_m$. Thus, we have $\overline{w_l(j), w_m(j)} \ge \xi(r_l + r_m)/2$. On the other hand, since

$$\frac{1}{\upsilon(B_{10r_l}(z_l))}\int_{B_{10r_l}(z_l)} \left|\mathrm{Lip}\left(f_\infty - \sum_{s=1}^{k_l} a_s^l r_{x_s^l}\right)\right|^2 d\upsilon < \epsilon,$$

by the segment inequality on limit spaces [6, Theorem 2.6], there exist points $\hat{z}_l, \phi_j(\hat{w_l(j)})$ in $B_{r_l}(z_l)$ and a minimal geodesic γ from $\hat{z}_l$ to $\phi_j(\hat{w_l(j)})$ such that $\overline{z_l, \hat{z}_l} + \overline{\phi_j(w_l(j)), \phi_j(\hat{w_l(j)})} < \Psi(\epsilon; n) r_l$ and

$$\int_0^{\overline{\hat{z}_l, \phi_j(\hat{w_l(j)})}} \mathrm{Lip}\left(f_\infty - \sum_{s=1}^{k_l} a_s^l r_{x_s^l}\right)(\gamma(t))dt < \Psi(\epsilon; n) r_l.$$

Therefore we have

$$\begin{aligned}&\left| f_\infty(\hat{z}_l) - \sum_{s=1}^{k_l} a_s^l r_{x_s^l}(\hat{z}_l) - \left(f_\infty(\phi_j(\hat{z_l(j)})) - \sum_{s=1}^{k_l} a_s^l r_{x_s^l}(\phi_j(\hat{z_l(j)}))\right)\right| \\ &\le \int_0^{\overline{\hat{z}_l, \phi_j(\hat{w_l(j)})}} \mathrm{Lip}\left(f_\infty - \sum_{s=1}^{k_l} a_s^l r_{x_s^l}\right)(\gamma(t))dt < \Psi(\epsilon; n) r_l.\end{aligned}$$

Thus we have

$$\left| f_\infty(z_l) - \sum_{s=1}^{k_l} a_s^l r_{x_s^l}(z_l) - \left(f_\infty(\phi_j(z_l(j))) - \sum_{s=1}^{k_l} a_s^l r_{x_s^l}(\phi_j(z_l(j)))\right)\right| \le \Psi(\epsilon; n, L) r_l.$$

Especially, we have $|F_j(w_l(j)) - f_\infty \circ \phi_j(w_l(j))| \le \Psi(\epsilon; n, L) r_l$. Similarly, we have $|F_j(w_m(j)) - f_\infty \circ \phi_j(w_m(j))| \le \Psi(\epsilon; n, L) r_m$ and $|F_\infty - f_\infty| \le \Psi(\epsilon; n, L) r_l$ on $\overline{B}_{(1-\xi)r_l}(z_l)$. Therefore we have

$$\begin{aligned}|F_j(w_l(j)) - F_j(w_m(j))| &\le |f_\infty \circ \phi_j(w_l(j)) - f_\infty \circ \phi_j(w_l(j))| + \Psi(\epsilon; n, L)(r_l + r_m) \\ &\le L\overline{\phi_j(w_l(j)), \phi_j(w_m(j))} + \Psi(\epsilon; n, L)(r_l + r_m) \\ &\le L(\overline{w_l(j), w_m(j)} + \epsilon_j) + \Psi(\epsilon; n, L)(r_l + r_m) \\ &\le L\overline{w_l(j), w_m(j)} + \Psi(\epsilon; n, L)(r_l + r_m) \le (L + \xi^{-1}\Psi(\epsilon; n, L))\overline{w_l(j), w_m(j)}.\end{aligned}$$

Thus, by Claim 4.25, we have $\mathbf{Lip}F_j \le C(n,L) + \xi^{-1}\Psi(\epsilon; n, L)$. Similarly, we have $\mathbf{Lip}F_\infty \le C(n,L) + \xi^{-1}\Psi(\epsilon; n, L)$. Therefore we have Claim 4.26.

Claim 4.27. *We have* $\bigcup_{i=1}^N \overline{B}_{(1-\xi)r_i}(z_i(j)) \subset M_i \setminus B_{2\xi}(A_i)$ *and* $\bigcup_{i=1}^N \overline{B}_{(1-\xi)r_i}(z_i) \subset Y \setminus B_{2\xi}(A_\infty)$ *for every sufficiently large* j.

The proof is as follows. It is easy to check that $\bigcup_{i=1}^N \overline{B}_{r_i}(z_i) \subset Y \setminus B_{2\xi}(A_\infty)$. On the other hand, by the assumption, there exists i_0 such that $\phi_i(A_i) \subset B_\xi(A_\infty)$ and $\epsilon_i << \min_{1\le j\le N}\{\xi r_j\}$ for every $i \ge i_0$. Thus, since $\phi_i\left(\bigcup_{i=1}^N \overline{B}_{(1-\xi)r_i}(z_i(j))\right) \subset \bigcup_{i=1}^N \overline{B}_{r_i}(z_i) \subset Y \setminus B_{4\xi}(A_\infty)$ for every $i \ge i_0$, we have Claim 4.27.

Claim 4.28. *We have*

$$\lim_{i\to\infty} \sup_{A_i} |f_i - f_\infty \circ \phi_i| = 0.$$

The proof is done by a contradiction. Assume that the assertion is false. Then, there exist $\tau > 0$, a subsequence $\{n(i)\}_i$ of $\mathbf{N}$, and $\alpha_{n(i)} \in A_{n(i)}$ such that $|f_{n(i)}(\alpha_{n(i)}) - f_\infty \circ \phi_{n(i)}(\alpha_{n(i)})| > \tau$. Without loss of generality, we can assume that there exists $\alpha_\infty \in Y$ such that $\phi_{n(i)}(\alpha_{n(i)}) \to \alpha_\infty$. Thus, $\liminf_{i\to\infty} |f_{n(i)}(\alpha_{n(i)}) - f_\infty(\alpha_\infty)| \ge \tau$. On the other hand, we have $\alpha_\infty \in \overline{A_\infty} = A_\infty$. Since $f_\infty|_{A_\infty}$ is an extension of $\{f_i\}_i$ asymptotically, this is a cotradiction. Therefore we have Claim 4.28.

Put $W_j = \bigcup_{m=1}^N B_{(1-\xi)r_i}(z_i(j))$ and $W_\infty = \bigcup_{m=1}^N B_{(1-\xi)r_i}(z_i)$. By Claim 4.27, we can define Lipschitz functions G_j on $W_j \cup A_j$, and G_∞ on $W_\infty \cup A_\infty$ by $G_j|_{W_j} = F_j|_{W_j}$, $G_j|_{A_j} = f_j$, $G_\infty|_{W_\infty} = F_\infty|_{W_\infty}$ and $G_\infty|_{A_\infty} = f_\infty|_{A_\infty}$ for every sufficiently large j.

Claim 4.29. *We have* $\mathbf{Lip}\, G_j + \mathbf{Lip}\, G_\infty \le C(n,L) + \xi^{-1}\Psi(\epsilon; n, L)$ *for every sufficiently large* j.

The proof is as follows. Put $\xi_j = \sup_{A_j} |f_j - f_\infty \circ \phi_j|$. Then by the proof of Claim 4.26, there exists j_0 such that

$$\begin{aligned}
|G_j(\alpha_j) - G_j(\beta_j)| &= |F_j(\alpha_j) - f_j(\beta_j)| \\
&\le |F_\infty \circ \phi_j(\alpha_j) - f_\infty \circ \phi_j(\beta_j)| + \Psi(\epsilon; n, L) r_i + \xi_j \\
&\le |f_\infty \circ \phi_j(\alpha_j) - f_\infty \circ \phi_j(\beta_j)| + \Psi(\epsilon; n, L) r_i + \xi_j \\
&\le L\overline{\phi_j(\alpha_j), \phi_j(\beta_j)} + \Psi(\epsilon; n, L) r_i \\
&\le L(\overline{\alpha_j, \beta_j} + \epsilon_j) + \Psi(\epsilon; n, L)\xi \le (L + \Psi(\epsilon; n, L))\overline{\alpha_j, \beta_j}.
\end{aligned}$$

for every $j \ge j_0$, every $\alpha_j \in \overline{B}_{(1-\xi)r_i}(z_i(j))$ and every $\beta_j \in A_j$. Therefore, by Claim 4.26, we have $\mathbf{Lip} G_j \le C(n,L) + \xi^{-1}\Psi(\epsilon; n, L)$ for every sufficiently large j. Similarly, we have $\mathbf{Lip} G_\infty \le C(n,L) + \xi^{-1}\Psi(\epsilon; n, L)$. Thus, we have Claim 4.29.

For $\Psi = \Psi(\epsilon; n, L)$ as in Claim 4.29, put $\xi = \sqrt{\Psi}$. Let f_j^ϵ be a Lipschitz function on M_j and f_∞^ϵ a Lipschitz function on Y satisfying that $\mathbf{Lip} f_j^\epsilon = \mathbf{Lip} G_j$, $\mathbf{Lip} f_\infty^\epsilon = \mathbf{Lip} G_\infty$, $f_j^\epsilon|_{W_j \cup A_j} = F_j|_{W_j \cup A_j}$ and $f_\infty^\epsilon|_{W_\infty \cup A_\infty} = F_\infty|_{W_\infty \cup A_\infty}$. Put $\Omega_\epsilon = W_\infty$. Then, by Proposition 4.16 and Corollary 4.19, we have $(f_i^\epsilon, df_i^\epsilon) \to (f_\infty^\epsilon, df_\infty^\epsilon)$ on Ω_ϵ. On the other hand, we have $\upsilon(B_R(x_\infty)\setminus(\Omega_\epsilon\cup A_\infty))\le \upsilon(X\setminus\Omega_\epsilon)+\upsilon(\overline{B}_{5\xi}(A_\infty)\setminus A_\infty)\le\sum_{i=N+1}^{\infty}\upsilon(B_{5r_i}(z_i))+\upsilon(\overline{B}_{5\xi}(A_\infty)\setminus A_\infty)+\Psi(\epsilon;n,L)\le C(n)\epsilon+\upsilon(\overline{B}_{5\xi}(A_\infty)\setminus A_\infty)+\Psi(\epsilon;n,L)$ and

$$\begin{aligned}\int_{B_R(x_\infty)}|df_\infty-df_\infty^\epsilon|^2d\upsilon &\le \int_X|df_\infty-df_\infty^\epsilon|^2d\upsilon+\int_{\overline{B}_{5\xi}(A_\infty)}|df_\infty-df_\infty^\epsilon|^2d\upsilon\\ &\le\sum_{i=1}^N\int_{B_{(1-\xi)r_i}(z_i)}|df_\infty-df_\infty^\epsilon|^2d\upsilon\\ &+5L^2\upsilon(B_{5\xi}(A_\infty)\setminus A_\infty)+\int_{A_\infty}|df_\infty^\epsilon-df_\infty|^2d\upsilon+\Psi(\epsilon;n,L)\\ &\le\sum_{i=1}^N\epsilon\upsilon(B_{(1-\xi)r_i}(z_i))+5L^2\upsilon(B_{5\xi}(A_\infty)\setminus A_\infty)+\Psi(\epsilon;n,L)\\ &\le\epsilon\upsilon(B_R(x_\infty))+5L^2\upsilon(B_{5\xi}(A_\infty)\setminus A_\infty)+\Psi(\epsilon;n,L).\end{aligned}$$

We remark that since A_∞ is compact, we have $\lim_{r\to 0}\upsilon(B_r(A_\infty)\setminus A_\infty)=0$. Put $\tau(r)=\upsilon(B_r(A_\infty)\setminus A_\infty)$. On the other hand, by the proof of Claim 4.26, we have $|f_\infty^\epsilon-f_\infty|<\Psi(\epsilon;n,L)$ on $\Omega_\epsilon\cup A_\infty$. For every $w\in B_R(x_\infty)$, there exists $\hat{w}\in\Omega_\epsilon\cup A_\infty$ such that $\overline{w,\hat{w}}<\Psi(\epsilon,\tau(5\xi);n,L,\upsilon(B_R(x_\infty)))$. Therefore, we have $|f_\infty^\epsilon(w)-f_\infty(w)|\le|f_\infty^\epsilon(\hat{w})-f_\infty(\hat{w})|+\Psi(\epsilon,\tau(5\xi);n,L,\upsilon(B_R(x_\infty)))\le\Psi(\epsilon,\tau(5\xi);n,L,\upsilon(B_R(x_\infty)))$. Thus, we have $|f_\infty^\epsilon-f_\infty|<\Psi(\epsilon,\tau(5\xi);n,L,\upsilon(B_R(x_\infty)))$ on $B_R(x_\infty)$. Since it is not difficult to check that $|\mathrm{Lip}(f_\infty^\epsilon-f_\infty)|_{L^2(B_R(x_\infty))}\le\Psi(\epsilon;n,L,R,\upsilon(B_R(x_\infty)))$, we have the assertion. $\square$

By using Theorem 4.24, we shall give a sufficient condition to satisfy pointwise upper semicontinuity of L^2-energy:

PROPOSITION 4.30. *Let R be a positive number, f_i a C^2-function on $B_R(x_i)$ for every $i<\infty$, and f_∞ a Lipschitz function on $\overline{B}_R(x_\infty)$. Assume that*

$$\sup_i\left(\mathbf{Lip}f_i+\int_{B_R(x_i)}|\Delta f_i|d\underline{\mathrm{vol}}\right)<\infty$$

and $f_i\to f_\infty$ on $B_R(x_\infty)$. Then, we have

$$\limsup_{i\to\infty}\int_{B_R(x_i)}(\mathrm{Lip}f_i)^2d\underline{\mathrm{vol}}\le\int_{B_R(x_\infty)}(\mathrm{Lip}f_\infty)^2d\upsilon.$$

Especially, L^2-energy of $\{f_i\}_i$ are upper semicontinuous at every $w\in B_R(x_\infty)$.

PROOF. Let $g_i=\Delta f_i$. First, we shall remark the following:

CLAIM 4.31. *We have*

$$\int_{B_R(x_i)}|d(f_i+k)|^2d\underline{\mathrm{vol}}-2\int_{B_R(x_i)}g_i(f_i+k)d\underline{\mathrm{vol}}\ge\int_{B_R(x_i)}|df_i|^2d\underline{\mathrm{vol}}-2\int_{B_R(x_i)}g_if_id\underline{\mathrm{vol}}$$

for every Lipschitz function k on $B_R(x_i)$, which has compact support.

Claim 4.31 follows from the equality:

$$\int_{B_R(x_i)} |d(f_i+k)|^2 d\underline{\mathrm{vol}} - 2\int_{B_R(x_i)} g_i(f_i+k)d\underline{\mathrm{vol}} = \int_{B_R(x_i)} |df_i|^2 d\underline{\mathrm{vol}} - 2\int_{B_R(x_i)} g_i f_i d\underline{\mathrm{vol}} + \int_{B_R(x_i)} |dk|^2 d\underline{\mathrm{vol}}.$$

Fx $\epsilon > 0$. Let $L \ge 1$ with

$$\sup_i \left(\mathbf{Lip} f_i + |f_i|_{L^\infty(B_R(x_i))} + \int_{B_R(x_i)} |g_i| d\underline{\mathrm{vol}} \right) < L.$$

Since $\limsup^{GH}_{i\to\infty} A_{R-\epsilon,R}(x_i) \subset A_{R-\epsilon,R}(x_\infty)$, by Theorem 4.24, there exist a sequence $\{f_i^\epsilon\}_{1\le i\le\infty}$ of $C(n,L)$-Lipschitz functions f_i^ϵ on $B_R(x_i)$, and an open set $\Omega_\epsilon \subset B_R(x_\infty) \setminus A_{R-\epsilon,R}(x_\infty)$ such that $f_i^\epsilon|_{A_{R-\epsilon,R}(x_i)} = f_i|_{A_{R-\epsilon,R}(x_i)}$ for every $1 \le i \le \infty$, $(f_i^\epsilon, df_i^\epsilon) \to (f_\infty^\epsilon, df_\infty^\epsilon)$ on Ω_ϵ, and

$$\frac{\upsilon\left(B_R(x_\infty)\setminus(\Omega_\epsilon \cup A_{R-\epsilon,R}(x_\infty))\right)}{\upsilon(B_R(x_\infty))} + |f_\infty - f_\infty^\epsilon|_{L^\infty(B_R(x_\infty))} + |\mathrm{Lip}(f_\infty^\epsilon - f_\infty)|_{L^2(B_R(x_\infty))} < \epsilon.$$

By Claim 4.31, we have

$$\int_{B_R(x_i)} |df_i^\epsilon|^2 d\underline{\mathrm{vol}} - 2\int_{B_R(x_i)} g_i f_i^\epsilon d\underline{\mathrm{vol}} \ge \int_{B_R(x_i)} |df_i|^2 d\underline{\mathrm{vol}} - 2\int_{B_R(x_i)} g_i f_i d\underline{\mathrm{vol}}.$$

By Proposition 2.4, without loss of generality, we can assume that there exists a pairwise disjoint finite collection $\{\overline{B}_{r_i}(z_i)\}_{1\le i\le N}$ such that $\Omega_\epsilon = \bigcup_{i=1}^N B_{r_i}(z_i)$. Let $z_i(j) \to z_i$. Put $\Omega_\epsilon(j) = \bigcup_{i=1}^N B_{r_i}(z_i(j))$. Since $\underline{\mathrm{vol}}(\Omega_\epsilon(j) \cup A_{R-\epsilon,R}(x_j))/\underline{\mathrm{vol}}\, B_R(x_j) \ge 1-\epsilon$ for every sufficiently large j, by Proposition 4.12, we have

$$\left| \int_{B_R(x_j)} |df_j^\epsilon|^2 d\underline{\mathrm{vol}} - \int_{B_R(x_\infty)} |df_\infty|^2 d\upsilon \right| < \Psi(\epsilon; n, L, R).$$

On the other hand, since $\sup_{B_R(x_j)} |f_j^\epsilon - f_j| \le C(n,R,L)\sup_{\Omega_\epsilon(j)} |f_j^\epsilon - f_j|$ and $\limsup_{j\to\infty}\sup_{\Omega_\epsilon(j)} |f_j^\epsilon - f_j| \le \sup_{\Omega_\epsilon} |f_\infty^\epsilon - f_\infty|$, we have

$$\left| \int_{B_R(x_j)} g_j f_j^\epsilon d\underline{\mathrm{vol}} - \int_{B_R(x_j)} g_j f_j d\underline{\mathrm{vol}} \right| \le \sup_{B_R(x_j)} |f_j^\epsilon - f_j| \int_{B_R(x_j)} |g_j| d\underline{\mathrm{vol}} \le \Psi(\epsilon; n, R, L)$$

for every sufficiently large j. Therefore, we have

$$\limsup_{i\to\infty} \int_{B_R(x_i)} |df_i|^2 d\underline{\mathrm{vol}} \le \int_{B_R(x_\infty)} |df_\infty|^2 d\upsilon + \Psi(\epsilon; n, L, R).$$

By letting $\epsilon \to 0$, we have the assertion. □

Next corollary follows from Remark 4.6 and Proposition 4.30 directly.

Corollary 4.32. *Let R be a positive number, f_i a C^2-function on $B_R(x_i)$ for every $i < \infty$, and f_∞ a Lipschitz function on $B_R(x_\infty)$. Assume that*

$$\sup_i \left(\mathbf{Lip} f_i + \int_{B_R(x_i)} |\Delta f_i|^2 d\underline{\mathrm{vol}} \right) < \infty$$

and $f_i \to f_\infty$ on $B_R(x_\infty)$. Then, we have $(f_i, df_i) \to (f_\infty, df_\infty)$ on $B_R(x_\infty)$.

Next we shall consider a convergence of the equations $\Delta f_i = g_i$ with respect to the measured Gromov-Hausdorff convergence:

Corollary 4.33. *Let R be a positive number, f_i a C^2-function on $B_R(x_i)$ for every $i < \infty$, and f_∞ a Lipschitz function on $B_R(x_\infty)$ with $\sup_i(\mathbf{Lip} f_i + |\Delta f_i|_{L^\infty(B_R(x_i))}) < \infty$. Assume that $f_i \to f_\infty$ on $B_R(x_\infty)$ and that there exists a L^∞-function g_∞ on $B_R(x_\infty)$ such that Δf_i converges weakly to g_∞ at a.e. $w \in B_R(x_\infty)$. Then, we have*

$$\int_{B_R(x_\infty)} \langle df_\infty, dk_\infty \rangle d\upsilon = \int_{B_R(x_\infty)} k_\infty g_\infty d\upsilon$$

for every Lipschitz function k_∞ on $B_R(x_\infty)$, which has compact support.

Proof. By Corollary 4.32, we have $(f_i, df_i) \to (f_\infty, df_\infty)$ on $B_R(x_\infty)$. Let $L \geq 1$ with $\sup_i(\mathbf{Lip} f_i + |f_i|_{L^\infty(B_R(x_i))} + |\Delta f_i|_{L^\infty(B_R(x_i))}) < L$. Put $r = \sup_{w \in \mathrm{supp}\, k_\infty} \overline{x_\infty, w}$ and $g_i = \Delta f_i$. Then, we have $r < R$. Fix $\epsilon > 0$ with $\epsilon < R - r$. By Theorem 4.24, there exist a sequence $\{k_i^\epsilon\}_{1 \leq i \leq \infty}$ of $C(n, L)$-Lipschitz functions k_i^ϵ on $B_R(x_i)$, and an open set $\Omega_\epsilon \subset B_R(x_\infty) \setminus A_{R-\epsilon, R}(x_\infty)$ such that $k_i^\epsilon|_{A_{R-\epsilon,R}(x_i)} = 0$ for every $1 \leq i \leq \infty$, $(k_i^\epsilon, dk_i^\epsilon) \to (k_\infty^\epsilon, dk_\infty^\epsilon)$ on Ω_ϵ and

$$\frac{\upsilon\left(B_R(x_\infty) \setminus (\Omega_\epsilon \cup A_{R-\epsilon,R}(x_\infty))\right)}{\upsilon(B_R(x_\infty))} + |k_\infty - k_\infty^\epsilon|_{L^\infty(B_R(x_\infty))} + |\mathrm{Lip}(k_\infty^\epsilon - k_\infty)|_{L^2(B_R(x_\infty))} < \epsilon.$$

By Proposition 4.9, $k_i^\epsilon g_i$ converges weakly to $k_\infty^\epsilon g_\infty$ at a.e. $w \in \Omega_\epsilon$. By an argument similar to the proof of Proposition 4.30, and Proposition 4.12, we have

$$\left| \int_{B_R(x_i)} \langle df_i, dk_i^\epsilon \rangle d\underline{\mathrm{vol}} - \int_{B_R(x_\infty)} \langle df_\infty, dk_\infty^\epsilon \rangle d\upsilon \right| + \left| \int_{B_R(x_i)} g_i k_i^\epsilon d\underline{\mathrm{vol}} - \int_{B_R(x_\infty)} g_\infty k_\infty^\epsilon d\upsilon \right| < \Psi(\epsilon; n, L, R)$$

for every sufficiently large i. Since

$$\int_{B_R(x_i)} \langle df_i, dk_i^\epsilon \rangle d\underline{\mathrm{vol}} = \int_{B_R(x_i)} g_i k_i^\epsilon d\underline{\mathrm{vol}},$$

we have

$$\int_{B_R(x_\infty)} \langle df_\infty, dk_\infty \rangle d\upsilon = \int_{B_R(x_\infty)} g_\infty k_\infty d\upsilon \pm \Psi(\epsilon; n, L, R).$$

By letting $\epsilon \to 0$, we have the assertion. □

We shall recall the notion of *(2-) harmonic* for Lipschitz functions on Ricci limit spaces. For a Lipschitz function f on $B_R(x_\infty)$, we say that f *is harmonic on* $B_R(x_\infty)$ if

$$\int_{B_R(x_\infty)} |df|^2 dv \le \int_{B_R(x_\infty)} |d(f+k)|^2 dv$$

for every Lipschitz function k on on $B_R(x_\infty)$ which has compact support. We remark that the notion of harmonic function for $H_{1,2}$-functions is well-defined. See section 7 in [2]. The following corollary follows from Corollary 4.32 and 4.33 directly. See also [11].

COROLLARY 4.34. *Let R be a positive number, f_i a harmonic function on $B_R(x_i)$ for every $i < \infty$, and f_∞ a Lipschitz function on $B_R(x_\infty)$ with $\sup_i \mathbf{Lip} f_i < \infty$. Assume that $f_i \to f_\infty$ on $B_R(x_\infty)$. Then, we have $(f_i, df_i) \to (f_\infty, df_\infty)$ on $B_R(x_\infty)$. Moreover, we have*

$$\int_{B_R(x_\infty)} \langle df_\infty, dk_\infty \rangle dv = 0$$

for every Lipschitz function k_∞ on $B_R(x_\infty)$, which has compact support. Especially f_∞ is harmonic on $B_R(x_\infty)$.

5 Appendix: A proof of Claim 3.24

In this appendix, we shall give a proof of Claim 3.24. Define functions π_1, f_r^A on $\mathbf{R}^k$ by $\pi_1((x_1, \ldots, x_k)) = x_1$, $f_r^A(x) = H^{k-1}\left(\overline{B}_r(x) \cap A \cap \pi_1^{-1}(\pi_1(x))\right) 1_A(x)$. We remark that by the definition of $sl_1 - \mathrm{Leb}A$,

$$sl_1 - \mathrm{Leb}A = \left\{ a = (a_1, \ldots, a_k) \in A; \liminf_{r \to 0} \frac{H^{k-1}(\overline{B}_r(a) \cap A \cap \pi_1^{-1}(\pi_1(a)))}{\omega_{k-1} r^{k-1}} = 1 \right\}.$$

First, assume that A is compact.

CLAIM 5.1. *The function f_r^A is an upper semi-continuous function on $\mathbf{R}^k$. Especially, f_r^A is a H^k-measurable function.*

PROOF. Let $\{x_i\}_{1 \le i \le \infty}$ be a sequence of points in $\mathbf{R}^k$ with $x_i \to x_\infty$. It suffices to check that $\limsup_{i \to \infty} f_r^A(x_i) \le f_r^A(x_\infty)$ under the assumption: $x_j \in A$ for every j. Fix $\delta > 0$. Let $\{n(i)\}_{i \in \mathbf{N}}$ be a subsequence of $\mathbf{N}$ satisfying $\lim_{j \to \infty} H^{k-1}\left(\overline{B}_r(x_{n(j)}) \cap A \cap \pi_1^{-1}(\pi_1(x_{n(j)}))\right) = \limsup_{i \to \infty} H^{k-1}\left(\overline{B}_r(x_i) \cap A \cap \pi_1^{-1}(\pi_1(x_i))\right)$. On the other hand, since $\{\overline{B}_r(x_{n(j)}) \cap A \cap \pi_1^{-1}(\pi_1(x_{n(j)}))\}_j$ is precompact with respect to the Hausdroff distance on $\mathbf{R}^k$, without loss of generality, we can assume that there exists a compact subset K_∞ of $\mathbf{R}^k$ such that $\overline{B}_r(x_{n(j)}) \cap A \cap \pi_1^{-1}(\pi_1(x_{n(j)}))$ converges to K_∞ with respect to the Hausdorff distance on $\mathbf{R}^k$. Then, it is easy to check $K_\infty \subset \overline{B}_r(x_\infty) \cap A \cap \pi_1^{-1}(\pi_1(x_\infty))$. There exists a finite collection $\{B_{r_i}(y_i)\}_{1 \le i \le N}$ such that $r_i << \delta$, $\overline{B}_r(x_\infty) \cap A \cap \pi_1^{-1}(\pi_1(x_\infty)) \subset \bigcup_{i=1}^N B_{r_i}(y_i)$ and

$\left| H^{k-1}(\overline{B}_r(x_\infty) \cap A \cap \pi_1^{-1}(\pi_1(x_\infty))) - \sum_{i=1}^N \omega_{k-1} r_i^{k-1} \right| < \delta$. Since $\overline{B}_r(x_\infty) \cap A \cap \pi_1^{-1}(\pi_1(x_\infty))$ is compact, there exists $\tau_0 > 0$ such that $B_{\tau_0}(\overline{B}_r(x_\infty) \cap A \cap \pi_1^{-1}(\pi_1(x_\infty))) \subset \bigcup_{i=1}^N B_{r_i}(y_i)$. Since $\overline{B}_r(x_{n(j)}) \cap A \cap \pi_1^{-1}(\pi_1(x_{n(j)})) \subset B_{\tau_0}(K_\infty)$ for every sufficiently large j, we have $\overline{B}_r(x_{n(j)}) \cap A \cap \pi_1^{-1}(\pi_1(x_{n(j)})) \subset \bigcup_{i=1}^N B_{r_i}(y_i)$. Thus, we have $H^{k-1}\left(\overline{B}_r(x_{n(j)}) \cap A \cap \pi_1^{-1}(\pi_1(x_{n(j)}))\right) \leq \sum_{i=1}^N H^{k-1}\left(\overline{B}_r(y_i) \cap \pi_1^{-1}(\pi_1(x_{n(j)}))\right) \leq \sum_{i=1}^N \omega_{k-1} r^{k-1} \leq H^{k-1}(\overline{B}_r(x_\infty) \cap A \cap \pi_1^{-1}(\pi_1(x_\infty))) + \delta$ for every sufficiently large j. Therefore, we have Claim 5.1. □

By Claim 5.1, we have the statement 1 in Claim 3.24. The statement 2 follows from Lebesgue differentiation theorem on Euclidean spaces. Finally, by Fubini's theorem, we have

$$H^k(A \setminus sl_1 - \mathrm{Leb}A) = \int_{\mathbf{R}} H^{k-1}\left(A \cap (\{t\} \times \mathbf{R}^{k-1}) \setminus sl_1 - \mathrm{Leb}A\right) dt = 0.$$

Thus, we have the statement 3. Therefore, we have Claim 3.24 if A is compact.

We shall give a proof of Claim 3.24 in general case. Fix $R > 0$. There exists a sequence of compact subsets $\{K_i\}_i$ of $B_R(0_k) \cap A$ such that $H^k(B_R(0_k) \cap A \setminus K_i) \to 0$. Then, we have $sl_1 - \mathrm{Leb}K_i \subset sl_1 - \mathrm{Leb}(B_R(0_k) \cap A)$. Thus, we have $H^k(B_R(0_k) \cap A \setminus sl_1 - \mathrm{Leb}(B_R(0_k) \cap A)) \leq H^k(B_R(0_k) \cap A \setminus sl_1 - \mathrm{Leb}K_i) \leq H^k(B_R(0_k) \cap A \setminus K_i) + H^k(K_i \setminus sl_1 - \mathrm{Leb}K_i) \overset{i\to\infty}{\to} 0$ as an outer measure. Thus, $sl_1 - \mathrm{Leb}(B_R(0) \cap A)$ is a H^k-measurable set. Since $sl_1 - \mathrm{Leb}A = \bigcup_{N \in \mathbf{N}} (sl_1 - \mathrm{Leb}(A \cap B_N(0)))$, we have the statement 1 in Claim 3.24. By Lebesgue differentiation theorem and Fubini's theorem, we have the statements 2 and 3. Thus, we have Claim 3.24.

References

[1] J. Cheeger, Differentiability of Lipschitz functions on metric measure spaces, Geom. Funct. Anal. 9 (1999), 428-517.

[2] J. Cheeger, Degeneration of Riemannian metrics under Ricci curvature bounds, Lezioni Fermiane, Scuola Normale Superiore, Pisa, 2001.

[3] J. Cheeger and T. H. Colding, Lower bounds on Ricci curvature and the almost rigidity of warped products, Ann. of Math. 144 (1996), 189-237.

[4] J. Cheeger and T. H. Colding, On the structure of spaces with Ricci curvature bounded below, I, J. Differential Geom. 45 (1997), 406-480.

[5] J. Cheeger and T. H. Colding, On the structure of spaces with Ricci curvature bounded below, II, J. Differential Geom. 54 (2000), 13-35.

[6] J. Cheeger and T. H. Colding, On the structure of spaces with Ricci curvature bounded below, III, J. Differential Geom. 54 (2000), 37-74.

[7] S. Y. Cheng and S. T. Yau, Differential equations on Riemannian manifolds and their geometric applications, Comm. Pure Appl. Math. 28 (1975), 333-354.

[8] T. H. Colding, Ricci curvature and volume convergence, Ann. of Math. 145 (1997), 477-501.

[9] T. H. Colding and W. P. Minicozzi II, Harmonic functions on manifolds, Ann. of Math. 146 (1997), 725-747.

[10] T. H. Colding and W. P. Minicozzi II, Liouville theorems for harmonic sections and applications, Comm. Pure Appl. Math. 51 (1998), 113-138.

[11] Y. Ding, Heat kernels and Green's functions on limit spaces, Comm. Anal. Geom. 10 (2002), 475-514.

[12] H. Federer, Geometric measure theory, Springer, Berlin-New York, 1969.

[13] K. Fukaya, Collapsing of Riemannian manifolds and eigenvalues of the laplace operator, Invent. Math. 87 (1987), 517-547.

[14] K. Fukaya, Hausdorff convergence of Riemannian manifolds and its applications, Recent topics in differential and analytic geometry, 143-238, Adv. Stud. Pure Math. 18-I, Academic Press, Boston, MA, (1990).

[15] M. Fukushima, Dirichlet forms and Markoff processes, North Holland (Amsterdam) 1980.

[16] D. Gilbarg and N. Trudinger, Elliptic partial differential equations of second order, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin (2001).

[17] M. Gromov, Metric Structures for Riemannian and Non-Riemannian Spaces, Birkhauser Boston Inc, Boston, MA, 1999, Based on the 1981 French original [MR 85e:53051], With appendices by M. Katz, P. Pansu, and S. Semmes, Translated from the French by Sean Michael Bates.

[18] P. Hajlasz, Sobolev spaces on an arbitrary metric space, J. Potential Anal. 5 (1995), 403-415.

[19] P. Hajlasz and P. Koskela, Sobolev meets Poincare, C. R. Acad. Sci. Paris 320 (1995), 1211-1215.

[20] J. Heinonen, T. Kilpeläinen and O. Martio, Nonlinear potential theory for degenerate elliptic equations, Clarendon Press (Oxford, Tokyo, New York) 1993.

[21] S. Honda, Ricci curvature and almost spherical multi-suspension, Tohoku Math. J. 61 (2009), 499-522.

[22] S. Honda, Bishop-Gromov type inequality on Ricci limit spaces, J. Math. Soc. Japan, to appear.

[23] S. Honda, On Low Dimensional Ricci limit spaces, submitted.

[24] S. Honda, Harmonic functions on asymptotic cones with Euclidean volume growth, preprint.

[25] S. Honda, (in preparation).

[26] A. Kasue, Convergence of Riemannian manifolds and Laplace operators, I, Ann. Inst. Fourier 52 (2002), 1219-1257.

[27] A. Kasue, Convergence of Riemannian manifolds and Laplace operators, II, Potential Anal. 24 (2006), 137-194.

[28] A. Kasue and H. Kumura, Spectral convergence of Riemannian manifolds, Tohoku Math. J. 4 (1994), 147-179.

[29] A. Kasue and H. Kumura, Spectral convergence of Riemannian manifolds II, Tohoku Math. J. 2 (1996), 71-120.

[30] K. Kuwae and T. Shioya, Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry, Comm. Anal. Geom. 11 (2003), 599-673.

[31] P. Li, The theory of harmonic functions and its relation to geometry, Proc. Sympos. Pure Math., Amer. Math. Soc. Vol. 54, Part 1, (Ed. R. Green and S. T. Yau), 1993.

[32] P. Li, Harmonic functions on complete Riemannian manifolds, "Handbook of Gemetric Analysis, Vol. I", Advanced Lectures in Mathematics, 7 (2008), Higher Education Press, 195-227.

[33] P. Li and L-F. Tam, Green's functions, harmonic functions, and volume comparison, J. Differential Geom. 41 (1995), 277-318.

[34] J. Lott, Optimal transport and Ricci curvature for metric-measure spaces, in Surveys in Differential Geometry, vol. XI, Metric and Comparison Geometry, eds. J. Cheeger and K. Grove, International Press, Somerville, MA, p. 229-257 (2007).

[35] C. B. Morrey Jr., Multiple integrals in the calculus of variations, Springer, New York, 1966.

[36] R. Schoen and S. T. Yau, Lectures on Differential Geometry, International Press, 1995.

[37] N. Shanmugalingam, Harmonic functions on metric spaces, Illinois J. Math. 45 (2001), 1021-1050.

[38] L. M. Simon, Lectures on Geometric Measure Theory, Proc. of the Center for Mathematical Analysis 3, Australian National University, 1983.

[39] S. T. Yau, Harmonic functions on complete Riemannian manifolds, Comm. Pure and Appl. Math. 28 (1975), 201-228.

[40] S. T. Yau, Some function-theoretic properties of complete Riemannian manifold and their applications to geometry, Indiana Univ. Math. J. 25 (1976), 659-670.

Research Institute for Mathematical
Sciences,
Kyoto University,
Kyoto 606-8502
Japan

E-mail address: honda@kurims.kyoto-u.ac.jp