On the exact WKB analysis of higher order simple-pole type operators

Dedicated to Professor Daisuke Fujiwara on his seventieth birthday

Takahiro Kawai
Research Institute for Mathematical Sciences
Kyoto University
Kyoto, 606-8502 Japan

Tatsuya Koike
Department of Mathematics
Graduate School of Science
Kobe University
Kobe, 657-8501 Japan

and

Yoshitsugu Takei
Research Institute for Mathematical Sciences
Kyoto University
Kyoto, 606-8502 Japan

The research of the authors has been supported in part by JSPS grants-in-aid No.20340028, No.21740098, No.21340029 and No.S-19104002.

Abstract

Higher order simple-pole type operators, that is, higher order linear ordinary differential operators with a large parameter η whose coefficients have simple poles at the origin, are discussed from the viewpoint of the exact WKB analysis. Making use of the technique of microdifferential operators, we clarify the singularity structure of the Borel transform of their WKB solutions.

1 Introduction

The purpose of this paper is to develop the exact WKB analysis for a class of higher order simple-pole type operators, which were introduced in our preceding paper [KKoT]. They are, in an intuitive description, higher order linear ordinary differential operators with a large parameter η whose coefficients may have simple poles at the origin of $\mathbb{C}$, and by their exact WKB analysis we mean the analytic study of the singularity structure of their Borel transformed WKB solutions. See [KT] and references cited therein for the general theory of the exact WKB analysis. The precise definition of the class of operators is given in Definition 2, and it is called the class (S) after Section 4, "Discussions and concluding remarks", of [KKoT]. The class (S) is larger than the class (S_0) of operators studied in [KKoT]; an important point is that the constant c which describes the Stokes phenomena in question (cf. [KKoT, (3.8) and (4.29)]) is not a genuine constant but rather an infinite series of η^{-1} for operators in (S). (Parenthetically we note that in Section 4 of this paper the object corresponding to c is designated by λ_0 when it is a genuine constant and by $\lambda(\eta)$ when it is an infinite series.) In order to overcome troubles caused by this fact we make use of microdifferential operators acting on the Borel transform of WKB solutions, in parallel with [AKT2] and [KKKoT]. See, e.g., [K^3] for the basic properties of microdifferential operators.

The plan of this paper is as follows. In Section 2 we introduce the class (S) of simple-pole type operators and establish the following decomposition theorem (Theorem 2.4):

For an m-th order ($m \geq 3$) operator $P(x, d/dx, \eta)$ in (S) we can find first order operators $Q^{(l)} = d/dx - \eta q^{(l)}(x, \eta)$ $(l = 1, 2, \cdots, m-2)$ and a second order operator R in (S) so that they satisfy

$$xP = Q^{(1)}Q^{(2)} \cdots Q^{(m-2)}xR \tag{1.1}$$

near the origin, where x stands for the multiplication operator and $q^{(l)}$ has the form

$$\sum_{k \geq 0} q_k^{(l)}(x)\eta^{-k}, \tag{1.2}$$

with holomorphic functions $\{q_k^{(l)}(x)\}$ satisfying the estimate (2.87). In accordance with the decomposition (1.1) we find in Theorem 2.5 that WKB solutions $\psi^{(l)}$ and $\psi_{\pm}$ of the equation $P\psi = 0$ so that $\psi^{(l)}$ has the form

$$\exp\left(-\int \Big(\sum_{k \geq 0} \tilde{q}_k^{(l)}(x)\eta^{-k+1}\Big)dx\right) \tag{1.3}$$

with

(1.4) $\quad \tilde{q}_0^{(l)} = q_0^{(l)}$ and $\tilde{q}_k^{(l)}$ being holomorphic near the origin

and that

$$R\psi_{\pm} = 0. \tag{1.5}$$

As $\tilde{q}_k^{(l)}$ is holomorphic near the origin, the WKB-theoretically interesting object is the operator R. Hence we concentrate our attention on analyzing the structure of second order simple-pole type operators. Then it is reasonable to analyze a Schrödinger operator $\tilde{L}$ obtained by eliminating the first order part from the operator R by the traditional guage transformation (3.4). We note that $\tilde{L}$ may acquire double poles through the elimination of the first order part. Thus our main task in Section 3 is to find an appropriate canonical form of the Schrödinger operator $\tilde{L}$. In this direction we obtain the following decisive result: the Borel transform $\tilde{L}_B$ of the operator $\tilde{L}$ is microlocally equivalent to the following operator

$$M_B = \frac{\partial^2}{\partial x^2} - \frac{1}{x}\frac{\partial^2}{\partial y^2} - \frac{\sum_{k\geq 0}\lambda_k(\partial/\partial y)^{-k}}{x^2} \quad (\lambda_k \in \mathbb{C})\,; \tag{1.6}$$

that is, Theorem 3.3 proves that there exist microlocally invertible microdifferential operators $\mathcal{X}$ and $\mathcal{Y}$ for which

$$h(x)\tilde{L}_B\mathcal{X} = \mathcal{Y}M_B \tag{1.7}$$

holds for some non-vanishing function $h(x)$ that appears in conjunction with a coordinate transformation that is needed to consider $\tilde{L}_B$ and M_B in the same coordinate system. Furthermore, microdifferential operators $\mathcal{X}$ and $\mathcal{Y}$ enjoy a beautiful and useful integral representation of the form (3.95). Concretely speaking, we find that the action of the operator $\mathcal{X}$ upon multi-valued analytic functions (such as Borel transformed WKB solutions) is expressed as an integral operator whose kernel function is a linear differential operator of infinite order (in the sense of [SKK]). Although M_B has a simple form, it is still difficult to analyze M_B as it stands. Hence in Section 4 we study its reduction to a further simplified operator

$$M_{0B} = \frac{\partial^2}{\partial x^2} - \frac{1}{x}\frac{\partial^2}{\partial y^2} - \frac{\lambda_0}{x^2}. \tag{1.8}$$

This reduction is again attained with the help of microdifferential operators. As the analytic structure of solutions of the equation $M_{0B}\varphi = 0$ is concretely studied in [Ko2], we combine the results in [Ko2] with the results in Sections 3 and 4 to obtain Theorem 5.1 that concretely describes the structure of Borel transformed WKB solutions for the equation $\tilde{L}\psi = 0$. We note that our reasoning has become a clear-cut one by the use of integral representation of the form (3.95), which enables us to find where the singularities of the integral are located. Thus we can bypass the hard and delicate computations which [Ko2] needed to analyze second order operators in class (S_0).

2 Decomposition theorem for simple-pole type operators

We begin our discussion by defining class (S) of simple-pole type operators. We also introduce an auxiliary class $(\tilde{S})$ of operators whose coefficients are holomorphic near the origin.

Definition 2.1. (i) Let P be an operator of the form

$$\frac{d^m}{dx^m} + \eta A_1(x,\eta)\frac{d^{m-1}}{dx^{m-1}} + \cdots + \eta^m A_m(x,\eta), \tag{2.1}$$

where η is a large parameter and

$$A_j(x,\eta) = \sum_{k\geq 0} A_{j,k}(x)\eta^{-k} \tag{2.2}$$

with $A_{j,k}$ being a meromorphic function on a neighborhood U of the origin in $\mathbb{C}$. Then we say that P is in class (S) if the following conditions (2.3)$\sim$ (2.6) are satisfied:

(2.3) $A_{1,0}$ is holomorphic on U,

(2.4) $xA_{1,k}$ $(k \geq 1)$ and $xA_{j,k}$ $(j = 2, 3, \cdots, m; k \geq 0)$ are holomorphic on U,

(2.5) for each compact set K in U there exists a constant C_K for which
$$\sup_{x\in K} |xA_{j,k}| \le C_K^{k+1} k!$$
holds for every k and $j = 1, 2, \cdots, m$,

(2.6) for $\alpha_j \underset{\text{def}}{=} \operatorname*{Res}_{x=0} A_{j,0}$ $(j = 2, 3, \cdots, m)$ we find
(2.6.a) $\alpha_2 \neq 0$, $\alpha_m \neq 0$,
and
(2.6.b) $f(\zeta) \underset{\text{def}}{=} \sum_{j=2}^{m} \alpha_j \zeta^{m-j} = 0$ has mutually different $(m-2)$ roots.

(ii) Class $(\tilde{S})$ consists of operators of the form xP for P in (S).

Remark 2.1. (i) It follows from the definition that an operator $\tilde{P}$ in $(\tilde{S})$ has the following form:

$$x\frac{d^m}{dx^m} + \eta \tilde{A}_1(x,\eta)\frac{d^{m-1}}{dx^{m-1}} + \cdots + \eta^m \tilde{A}_m(x,\eta), \tag{2.7}$$

where $\tilde{A}_j(x,\eta) = \sum_{k\ge 0} \tilde{A}_{j,k}(x)\eta^{-k}$ $(j = 1, 2, \cdots, m)$ satisfy

(2.8) $\tilde{A}_{j,k}$ $(1 \le j \le m; k \ge 0)$ are all holomorphic on U,

$$\tilde{A}_{1,0}(0) = 0, \tag{2.9}$$

$$\tilde{A}_{2,0}(0) \neq 0,\ \tilde{A}_{m,0}(0) \neq 0, \tag{2.10}$$

and

(2.11) $\tilde{f}(\zeta) \underset{\text{def}}{=} \sum_{j=2}^{m} \tilde{A}_{j,0}(0)\zeta^{m-j} = 0$ has mutually different $(m-2)$ roots.

(ii) Classes (S) and $(\tilde{S})$ were first introduced in [AKKoT2] as classes $(\tilde{K})$ and $(\tilde{\tilde{K}})$ respectively. The importance of class (S) was emphasized in Section 4 of [KKoT], from which this paper stems.

We first show the following decomposition theorem (Theorem 2.1) for operators in class $(\tilde{S})$. A corresponding result for operators in class (S) (Theorem 2.4 below) will follow from this theorem.

Theorem 2.1. *Let $\tilde{P}$ be an m-th $(m \geq 3)$ order operator in $(\tilde{S})$ that has the form (2.7). Then there exist an open neighborhood V of the origin, holomorphic functions $\tilde{q}_k(x)$ $(k = 0, 1, 2, \cdots)$ defined on V and an $(m-1)$-st order operator $\tilde{R}$ in $(\tilde{S})$ that is defined on V which satisfy the following relations (2.12) and (2.13) if we let $\tilde{q}(x, \eta)$ denote $\sum_{k\geq 0} \tilde{q}_k(x)\eta^{-k}$ and define a differential operator $\tilde{Q}$ by $d/dx - \eta\tilde{q}(x, \eta)$:*

$$\tilde{P} = \tilde{Q}\tilde{R}, \tag{2.12}$$

(2.13) *for each compact set K in V there exists a constant M_K for which the following relation holds for every k :*

$$\sup_{x\in K} |\tilde{q}_k(x)| \leq M_K^{k+1} k!$$

.

Proof. Let us write down the required operator $\tilde{R}$ as

$$x\frac{d^{m-1}}{dx^{m-1}} + \eta\tilde{a}_1(x, \eta)\frac{d^{m-2}}{dx^{m-2}} + \cdots + \eta^{m-1}\tilde{a}_{m-1}(x, \eta) \tag{2.14}$$

with

$$\tilde{a}_j(x, \eta) = \sum_{k\geq 0} \tilde{a}_{j,k}(x)\eta^{-k}, \tag{2.15}$$

and try to find $\{\tilde{a}_{j,k}\}$ together with $\{\tilde{q}_k\}$ so that (2.12) is satisfied. Then the comparison of coefficients of like orders of differentiation in

(2.12) entails the following relations:

(2.16)
$$\begin{cases} \tilde{A}_1 = \tilde{a}_1 - x\tilde{q} + \eta^{-1}, & (2.16.1) \\ \tilde{A}_2 = \tilde{a}_2 - \tilde{q}\tilde{a}_1 + \eta^{-1}\tilde{a}_1', & (2.16.2) \\ \tilde{A}_3 = \tilde{a}_3 - \tilde{q}\tilde{a}_2 + \eta^{-1}\tilde{a}_2', & (2.16.3) \\ \quad \vdots & \vdots \\ \tilde{A}_{m-1} = \tilde{a}_{m-1} - \tilde{q}\tilde{a}_{m-2} + \eta^{-1}\tilde{a}_{m-2}', & (2.16.m-1) \\ \tilde{A}_m \quad = \qquad - \tilde{q}\tilde{a}_{m-1} + \eta^{-1}\tilde{a}_{m-1}'. & (2.16.m) \end{cases}$$

Here and in what follows $\tilde{a}_1'$ etc. respectively stand for $d\tilde{a}_1/dx$ etc. In what follows we try to construct $\tilde{a}_{j,k}$ and $\tilde{q}_k$ by comparing the coefficients of like powers of η^{-1} in (2.16).

First the comparison of the top degree part, i.e., the degree 0 part of (2.16) results in the following relations.

$$(2.17) \qquad \begin{cases} \tilde{A}_{1,0} = \tilde{a}_{1,0} - x\tilde{q}_0, & (2.17.1) \\ \tilde{A}_{2,0} = \tilde{a}_{2,0} - \tilde{q}_0\tilde{a}_{1,0}, & (2.17.2) \\ \tilde{A}_{3,0} = \tilde{a}_{3,0} - \tilde{q}_0\tilde{a}_{2,0}, & (2.17.3) \\ \quad \vdots & \vdots \\ \tilde{A}_{m-1,0} = \tilde{a}_{m-1,0} - \tilde{q}_0\tilde{a}_{m-2,0}, & (2.17.m-1) \\ \tilde{A}_{m,0} \quad = \qquad - \tilde{q}_0\tilde{a}_{m-1,0}. & (2.17.m) \end{cases}$$

Solving the equations (2.17) for $a_{j,0}$ $(1 \leq j \leq m-1)$ and $\tilde{q}_0$, we find the following relations with the convention that $\tilde{A}_{0,0} = x$:

$$(2.18) \qquad \tilde{a}_{j,0} = \sum_{l=0}^{j} \tilde{A}_{l,0}\tilde{q}_0^{j-l} \quad (j = 1, 2, \cdots, m-1)$$

and

$$(2.19) \qquad \sum_{l=0}^{m} \tilde{A}_{l,0}\tilde{q}_0^{m-l} = 0.$$

To find the required holomorphic function $\tilde{q}_0(x)$, we introduce the following function:

$$\tilde{F}(x,\zeta) = \sum_{l=0}^{m} \tilde{A}_{l,0}(x)\zeta^{m-l}. \tag{2.20}$$

Then it follows from (2.9), (2.10) and (2.11) that the equation $\tilde{F}(0,\zeta) = 0$ has mutually different $(m-2)$ roots $\zeta = \zeta^{(p)}$ $(p = 1, 2, \cdots, m-2)$. Note that (2.10) guarantees

$$\zeta^{(p)} \neq 0 \quad (p = 1, 2, \cdots, m-2). \tag{2.21}$$

It is then clear that

$$\frac{\partial \tilde{F}}{\partial \zeta}(x,\zeta)\Big|_{(x,\zeta)=(0,\zeta^{(p)})} \neq 0 \tag{2.22}$$

for $p = 1, 2, \cdots, m-2$. Thus it follows from the implicit function theorem that there exist $(m-2)$ holomorphic functions $\{\zeta^{(p)}(x)\}$ which satisfy

$$\tilde{F}(x, \zeta^{(p)}(x)) = 0 \quad \text{and} \quad \zeta^{(p)}(0) = \zeta^{(p)}. \tag{2.23}$$

Let us choose one of them, say $\zeta^{(1)}(x)$, as $\tilde{q}_0(x)$, and let V denote its domain of definition. Then we can fix holomorphic functions $\tilde{a}_{j,0}$ $(j = 1, 2, \cdots, m-1)$ by (2.18). It is clear from (2.9), (2.10), (2.18) and (2.23) that we find

$$\tilde{a}_{1,0}(0) = 0, \tag{2.24}$$

$$\tilde{a}_{2,0}(0) \neq 0. \tag{2.25}$$

We also obtain by (2.10), (2.17.m) and (2.21) that

$$\tilde{a}_{m-1,0}(0) \neq 0. \tag{2.26}$$

To confirm that the equation

$$\sum_{j=2}^{m-1} \tilde{a}_{j,0}(0)\zeta^{m-1-j} = 0 \tag{2.27}$$

has $(m-3)$ (if $m \geq 4$) mutually distinct roots, we note that

(2.28) $$\tilde{F}(x,\zeta) = \big(\zeta - \tilde{q}_0(x)\big)\big(x\zeta^{m-1} + \tilde{a}_{1,0}(x)\zeta^{m-2} + \cdots + \tilde{a}_{m-1,0}(x)\big)$$

follows from (2.12). In fact, by replacing d/dx by ξ in (2.12) and comparing the part of (2.12) which has homogeneous degree m in (ξ, η), we immediately find (2.28) by setting $\xi = \zeta\eta$. If we set $x = 0$ in (2.28), the choice of $\tilde{q}_0(x)$ together with (2.20) and (2.24) guarantees that (2.27) has $(m-3)$ roots $\zeta = \zeta^{(p)}$ $(p = 2, 3, \cdots, m-2)$, which are mutually distinct. Thus the functions $\{\tilde{a}_{j,0}\}_{j=1,2,\cdots,m-1}$ meet the requirements (2.9), (2.10) and (2.11) that the top degree part of an operator in $(\tilde{S})$ should satisfy.

Next we construct $\tilde{q}_k$ $(k \geq 1)$ and $\{\tilde{a}_{j,k}\}_{1 \leq j \leq m-1}$ $(k \geq 1)$ in an inductive manner with respect to k so that they satisfy (2.16). Let us begin our discussion by explicitly writing down the degree $(-k)$ (in η) part of (2.16). Here and in what follows $\delta_{1,k}$ stands for Kronecker's delta.

(2.29)
$$\begin{cases}
\tilde{A}_{1,k} = \tilde{a}_{1,k} - x\tilde{q}_k + \delta_{1,k}, & (2.29.1)\\
\tilde{A}_{2,k} = \tilde{a}_{2,k} - \sum_{l=0}^{k} \tilde{q}_l \tilde{a}_{1,k-l} + \tilde{a}'_{1,k-1}, & (2.29.2)\\
\tilde{A}_{3,k} = \tilde{a}_{3,k} - \sum_{l=0}^{k} \tilde{q}_l \tilde{a}_{2,k-l} + \tilde{a}'_{2,k-1}, & (2.29.3)\\
\quad\vdots & \vdots\\
\tilde{A}_{m-1,k} = \tilde{a}_{m-1,k} - \sum_{l=0}^{k} \tilde{q}_l \tilde{a}_{m-2,k-l} + \tilde{a}'_{m-2,k-1}, & (2.29.m-1)\\
\tilde{A}_{m,k} \quad = \qquad - \sum_{l=0}^{k} \tilde{q}_l \tilde{a}_{m-1,k-l} + \tilde{a}'_{m-1,k-1}. & (2.29.m)
\end{cases}$$

For the sake of the clarity of description we rewrite (2.29) in a matrix form:

$$C\begin{pmatrix}\tilde{q}_k\\ \tilde{a}_{1,k}\\ \tilde{a}_{2,k}\\ \vdots\\ \tilde{a}_{m-1,k}\end{pmatrix}=\begin{pmatrix}\tilde{\mathfrak{a}}_{0,k}\\ \tilde{\mathfrak{a}}_{1,k}\\ \tilde{\mathfrak{a}}_{2,k}\\ \vdots\\ \tilde{\mathfrak{a}}_{m-1,k}\end{pmatrix},\tag{2.30}$$

where

$$C=\begin{pmatrix}x & -1 & & \text{\Large 0} & \\ \tilde{a}_{1,0} & \tilde{q}_0 & -1 & & \\ \tilde{a}_{2,0} & 0 & \tilde{q}_0 & -1 & \ddots & \\ \vdots & \vdots & & \ddots & -1\\ \tilde{a}_{m-1,0} & 0 & \text{\Large 0} & & \tilde{q}_0\end{pmatrix},\tag{2.31}$$

$$\tilde{\mathfrak{a}}_{0,k}=\delta_{1,k}-\tilde{A}_{1,k}\tag{2.32}$$

and

$$\tilde{\mathfrak{a}}_{j,k}=\tilde{a}'_{j,k-1}-\sum_{l=1}^{k-1}\tilde{q}_l\tilde{a}_{j,k-l}-\tilde{A}_{j+1,k}\quad(1\le j\le m-1).\tag{2.33}$$

First we note

(2.34)

$$\det C=x\begin{vmatrix}\tilde{q}_0 & -1 & & \text{\Large 0}\\ & \tilde{q}_0 & -1 & \ddots\\ & & \ddots & -1\\ \text{\Large 0} & & & \tilde{q}_0\end{vmatrix}+\begin{vmatrix}\tilde{a}_{1,0} & -1 & & \text{\Large 0}\\ \tilde{a}_{2,0} & \tilde{q}_0 & -1 & \ddots\\ \vdots & & \ddots & -1\\ \tilde{a}_{m-1,0} & \text{\Large 0} & & \tilde{q}_0\end{vmatrix}$$

$$=x\tilde{q}_0^{m-1}+\tilde{a}_{1,0}\tilde{q}_0^{m-2}+\begin{vmatrix}\tilde{a}_{2,0} & -1 & & \text{\Large 0}\\ \vdots & \tilde{q}_0 & -1 & \\ \vdots & & \ddots & -1\\ \tilde{a}_{m-1,0} & \text{\Large 0} & & \tilde{q}_0\end{vmatrix}$$

$$= \cdots = x\tilde{q}_0^{m-1} + \tilde{a}_{1,0}\tilde{q}_0^{m-2} + \cdots + \tilde{a}_{m-1,0}.$$

Hence it follows from (2.28) and the choice of $\tilde{q}_0$ that

$$\det C \neq 0 \quad \text{on} \quad V \tag{2.35}$$

if V is chosen further smaller if necessary. Therefore, if $\tilde{q}_k$ and $\{\tilde{a}_{j,k}\}_{1\leq j\leq m-1}$ have been found for $0 \leq k \leq k_0 - 1$, we can obtain $\tilde{q}_{k_0}(x)$ which is holomorphic on V by

$$\det \tilde{C}_{k_0} / \det C, \tag{2.36}$$

where

$$\tilde{C}_{k_0} = \begin{pmatrix} \tilde{\mathfrak{a}}_{0,k_0} & -1 & & 0 \\ \tilde{\mathfrak{a}}_{1,k_0} & \tilde{q}_0 & -1 & \ddots & \\ \vdots & & \ddots & -1 \\ \tilde{\mathfrak{a}}_{m-1,k_0} & 0 & & \tilde{q}_0 \end{pmatrix}. \tag{2.37}$$

Once $\tilde{q}_{k_0}$ is fixed, we can construct $\{\tilde{a}_{j,k_0}\}_{1\leq j\leq m-1}$ explicitly by solving (2.29) in an inductive manner with respect to j starting with

$$\tilde{a}_{1,k_0} = \tilde{A}_{1,k_0} + x\tilde{q}_{k_0} - \delta_{1,k_0}. \tag{2.38}$$

Thus we can find $\tilde{q}_k$ and $\{\tilde{a}_{j,k}\}_{1\leq j\leq m-1}$ so that (2.16) may be satisfied. Hence what remains to be done is the following estimation:

For each compact set K in V there exists a constant C_K for which

$$\sup_{x\in K} |\tilde{q}_k(x)| \leq C_K^{k+1} k! \tag{2.39}$$

and

$$\sup_{x\in K} |\tilde{a}_{j,k}(x)| \leq C_K^{k+1} k! \quad (1 \leq j \leq m-1) \tag{2.40}$$

hold for every k.

This estimation can be done in the same way as in [KKoT], but for the sake of completeness we briefly describe its core part.

In what follows we fix an arbitrary point x_0 in V, and we let $D(r)$ denote a closed disc centered at x_0 with radius r. Let r_0 be a positive number such that

$$D(r_0) \subset V. \tag{2.41}$$

It follows from (2.5) and the definition of class $(\tilde{S})$ that we have a constant M for which

$$\sup_{\substack{x \in D(r_1) \\ j=1,2,\cdots,m}} |\tilde{A}_{j,k}(x)| \leq k! M^{k+1} (r_0 - r_1)^{-k} \tag{2.42}$$

holds for any $r_1 < r_0$.

We now prove the existence of a constant C for which

$$\sup_{\substack{x \in D(r_1) \\ j=1,2,\cdots,m-1}} \{|\tilde{q}_k|, |\tilde{a}_{j,k}|\} \leq k! C^{k+1} (r_0 - r_1)^{-k} \tag{2.43.k}$$

holds for every k. As (2.43.0) is clear, we prove (2.43.k) by the induction on k. Since

$$\det \tilde{C}_{k_0} = \tilde{\mathfrak{a}}_{0,k_0} \tilde{q}_0^{m-1} + \tilde{\mathfrak{a}}_{1,k_0} \tilde{q}_0^{m-2} + \cdots + \tilde{\mathfrak{a}}_{m-1,k_0}, \tag{2.44}$$

(2.36) implies that it suffices to show

$$\sup_{\substack{x \in D(r_1) \\ j=1,2,\cdots,m-1}} |\tilde{\mathfrak{a}}_{j,k_0}| \leq k_0! C^{k_0+1} (r_0 - r_1)^{-k_0} \tag{2.45.k_0}$$

on the condition that (2.43.k) holds for $k = 0, 1, \cdots, k_0 - 1$. To prove (2.45.$k_0$) we follow the reasoning in [AKT1, Appendix, §A.1]. To dominate $|d\tilde{a}_{j,k_0-1}/dx|$ we use the following device: for each positive number r that is smaller than r_0 we use the induction hypothesis by defining

$$r_1 = r + \frac{r_0 - r}{k_0}. \tag{2.46}$$

Then we have

$$r_0 - r_1 = \Big(1 - \frac{1}{k_0}\Big)(r_0 - r). \tag{2.47}$$

Hence it follows from the induction hypothesis and Cauchy's formula that we have

$$\sup_{D(r)} \left| \frac{d\tilde{a}_{j,k_0-1}}{dk} \right| \le (k_0-1)! C^{k_0} (r_0-r_1)^{-k_0+1} \frac{k_0}{r_0-r} \tag{2.48}$$
$$\le k_0! C^{k_0} \Big(1-\frac{1}{k_0}\Big)^{-k_0+1} (r_0-r)^{-k_0}$$
$$\le k_0! C^{k_0} e (r_0-r)^{-k_0},$$

where $e = 2.718\cdots$. Since

$$\sum_{k=1}^{k_0-1} (k_0-k)!k! \le 4(k_0-1)! \tag{2.49}$$

holds for $k_0 \ge 2$, the definition (2.33) of $\tilde{\mathfrak{a}}_{j,k}$ implies

$$\sup_{\substack{D(r)\\ j=1,2,\cdots,m-1}} \left|\tilde{\mathfrak{a}}_{j,k_0}\right| \tag{2.50}$$
$$\le k_0! C^{k_0+1} \Big(eC^{-1} + 4k_0^{-1} + \big(\frac{M}{C}\big)^{k_0+1}\Big)(r_0-r)^{-k_0}.$$

Hence by choosing C sufficiently large we find

$$\sup_{\substack{D(r)\\ j=1,2,\cdots,m-1}} \left|\tilde{\mathfrak{a}}_{j,k_0}\right| \le k_0! C^{k_0+1} (r_0-r)^{-k_0}. \tag{2.51}$$

Since r is an arbitrary positive number that is smaller than r_0, this proves (2.45.k_0). Thus the induction proceeds. Therefore we find (2.43.k) for every k. Since x_0 is an arbitrary point in V, this implies the existence of a constant C_K for which (2.39) and (2.40) hold for every k. This completes the proof of Theorem 2.1.

Q.E.D.

Let us now consider the situation where the operator $\tilde{P}$ in $(\tilde{S})$ is of order $m \ge 4$. Then the repeated applications of Theorem 2.1 entail the existence of an operator $\tilde{R}$ in $(\tilde{S})$ and first order operators $\tilde{Q}^{(j)} = d/dx - \eta q^{(j)}(x,\eta)$ $(j = 1,2)$ so that they satisfy the following conditions:

$$\tilde{P} = \tilde{Q}^{(1)}\tilde{Q}^{(2)}\tilde{R}, \tag{2.52}$$

(2.53) $\tilde{q}^{(j)}(x,\eta) = \sum_{k\geq 0}\tilde{q}_k^{(j)}(x)\eta^{-k}$ $(j = 1,2)$, where $\{\tilde{q}_k^{(j)}(x)\}$ satisfy the same growth condition (2.13) that $\{\tilde{q}_k(x)\}$ satisfy.

We note that (2.11) together with (2.28) entail

$$\tilde{q}_0^{(1)}(0) \neq \tilde{q}_0^{(2)}(0). \tag{2.54}$$

Concerning the structure of $\tilde{Q}^{(1)}$ and $\tilde{Q}^{(2)}$ we find the following

Proposition 2.2. *Let $\tilde{Q}^{(1)}$ and $\tilde{Q}^{(2)}$ be as above. Then there uniquely exists another pair of operators $\hat{Q}^{(1)}$ and $\hat{Q}^{(2)}$ which satisfy the following*:

(2.55) $\hat{Q}^{(j)}$ *has the form* $d/dx - \eta\hat{q}^{(j)}(x,\eta)$ *with* $\hat{q}^{(j)}(x,\eta) = \sum_{k\geq 0}\hat{q}_k^{(j)}(x)\eta^{-k}$, *where* $\{\hat{q}_k^{(j)}\}$ *satisfy* (2.13),

$$\hat{q}_0^{(j)} = \tilde{q}_0^{(j)} \quad (j = 1,2), \tag{2.56}$$

$$\tilde{Q}^{(1)}\tilde{Q}^{(2)} = \hat{Q}^{(2)}\hat{Q}^{(1)}. \tag{2.57}$$

Proof. In order to attain (2.57), $\hat{q}^{(1)}$ and $\hat{q}^{(2)}$ should satisfy

$$\hat{q}^{(2)}(x,\eta) + \hat{q}^{(1)}(x,\eta) = \tilde{q}^{(1)}(x,\eta) + \tilde{q}^{(2)}(x,\eta) \tag{2.58}$$

$$\hat{q}^{(2)}\hat{q}^{(1)} - \eta^{-1}\frac{d\hat{q}^{(1)}}{dx} = \tilde{q}^{(1)}\tilde{q}^{(2)} - \eta^{-1}\frac{d\tilde{q}^{(2)}}{dx}. \tag{2.59}$$

From the logical viewpoint, solving these equations and solving (2.16) are different problems. But the procedure employed to solve (2.58) and

(2.59) is basically the same as that used in the proof of Theorem 2.1. Actually the problem is slightly easier this time, as the top degree parts are given by (2.56). Then the comparison of the coefficients of η^{-k} ($k \geq 1$) in (2.58) and (2.59) gives the following relations:

$$\hat{q}_k^{(2)} + \hat{q}_k^{(1)} = \tilde{q}_k^{(1)} + \tilde{q}_k^{(2)}, \tag{2.60}$$

$$\hat{q}_k^{(2)}\hat{q}_0^{(1)} + \sum_{l=1}^{k-1} \hat{q}_{k-l}^{(2)}\hat{q}_l^{(1)} + \hat{q}_0^{(2)}\hat{q}_k^{(1)} - \hat{q}_{k-1}^{(1)\prime} = \sum_{l=0}^{k} \tilde{q}_{k-l}^{(1)}\tilde{q}_l^{(2)} - \tilde{q}_{k-1}^{(2)\prime}. \tag{2.61}$$

Rewriting these in a matrix form, we find

$$C \begin{pmatrix} \hat{q}_k^{(2)} \\ \hat{q}_k^{(1)} \end{pmatrix} = \begin{pmatrix} \tilde{q}_k^{(1)} + \tilde{q}_k^{(2)} \\ \hat{q}_{k-1}^{(1)\prime} - \Bigl(\sum_{l=1}^{k-1} \hat{q}_{k-l}^{(2)}\hat{q}_l^{(1)}\Bigr) + \sum_{l=0}^{k} \tilde{q}_{k-l}^{(1)}\tilde{q}_l^{(2)} - \tilde{q}_{k-1}^{(2)\prime} \end{pmatrix} \tag{2.62}$$

where

$$C = \begin{pmatrix} 1 & 1 \\ \hat{q}_0^{(1)} & \hat{q}_0^{(2)} \end{pmatrix}. \tag{2.63}$$

It follows from (2.54) and (2.56) that

$$\det C\Bigr|_{x=0} \neq 0. \tag{2.64}$$

Therefore we can find $\hat{q}_k^{(2)}$ and $\hat{q}_k^{(1)}$ uniquely in an inductive manner with respect to k. By the same reasoning as in the proof of Theorem 2.1 we can confirm that they satisfy the growth order condition of the form (2.13).

Q.E.D.

Repeated applications of Theorem 2.1 show that an m-th ($m \geq 3$) order operator $\tilde{P}$ in $(\tilde{S})$ can be decomposed as follows:

$$\tilde{P} = \tilde{Q}^{(1)} \cdots \tilde{Q}^{(m-2)}\tilde{R}, \tag{2.65}$$

where

$$\tilde{Q}^{(l)} = \frac{d}{dx} - \eta \tilde{q}^{(l)}(x, \eta) \tag{2.66}$$

with

$$\tilde{q}^{(l)}(x, \eta) = \sum_{k \geq 0} \tilde{q}_k^{(l)}(x) \eta^{-k}. \tag{2.67}$$

To fix the situation let us suppose

$$\tilde{q}_0^{(j)}(x) = \zeta^{(j)}(x), \tag{2.68}$$

where $\{\zeta^{(j)}(x)\}_{j=1,2,\cdots,m-2}$ are solutions of (2.28). One implication of Proposition 2.2 is that, for a permutation σ of indices $\{1, 2, \cdots, m-2\}$, we can find another decomposition

$$\tilde{P} = \hat{Q}^{(\sigma(1))} \cdots \hat{Q}^{(\sigma(m-2))} \hat{R}, \tag{2.69}$$

where

$$\hat{Q}^{(\sigma(l))} = \frac{d}{d\eta} - \eta \hat{q}^{(\sigma(l))}(x, \eta) \tag{2.70}$$

with

$$\hat{q}_0^{(\sigma(l))}(x) = \zeta^{(\sigma(l))}(x) \tag{2.71}$$

and

$$\hat{R} = \tilde{R}. \tag{2.72}$$

The important point is that the second order operator $\tilde{R}$ is not altered although other factors $\tilde{Q}^{(j)}$ $(j = 1, 2, \cdots, m-2)$ are interchanged with necessary modifications of lower degree terms, namely $q_k^{(j)}$ with $k \geq 1$. This observation suggests that we should concentrate our attention on the structure of the equation $\tilde{R}\psi = 0$ from the viewpoint of WKB analysis near the origin. Actually we have the following

Theorem 2.3. *Let $\tilde{P}$ be an m-th $(m \geq 3)$ order operator in $(\tilde{S})$. Then, in parallel with the decomposition* (2.65), *we find the following m WKB solutions $\tilde{\psi}_\pm$ and $\hat{\psi}^{(j)}$ $(1 \leq j \leq m-2)$ of the equation $\tilde{P}\tilde{\psi} = 0$ near the origin so that they satisfy the following:*

$$\tilde{R}\tilde{\psi}_\pm = 0, \tag{2.73}$$

$$\hat{\psi}^{(j)} = \exp\Big(\eta \int_0^x \hat{q}^{(j)}(x,\eta)dx\Big), \tag{2.74}$$

where

$$\hat{q}^{(j)}(x,\eta) = \zeta^{(j)}(x) + \sum_{k\geq 1} \hat{q}_k^{(j)}(x)\eta^{-k} \tag{2.75}$$

with $\hat{q}_k^{(j)}(x)$ being holomorphic on a neighborhood V of the origin.

Proof. Since $\tilde{R}$ is a second order operator we can readily construct WKB solutions $\tilde{\psi}_\pm$ of (2.73). Then it follows from (2.65) that $P\tilde{\psi}_\pm = 0$. In order to confirm the existence of solutions of the form (2.74), we may repeat the reasoning in the proof of Theorem 2.1 to prove the decomposition $P = \hat{R}\hat{Q}$ with $\hat{Q}$ being $d/dx - \eta\hat{q}^{(j)}(x,\eta)$. Here we show another device used in [AKKoT2]. Let $\tilde{T}$ denote the adjoint operator $\tilde{P}^*$ of $\tilde{P}$. It follows from Remark 2.1 that $\tilde{P}^*$ belongs to $(\tilde{S})$, and hence we can apply Theorem 2.1 to $\tilde{P}^* = \tilde{T}$. Then we have

$$\tilde{T} = \tilde{Q}\tilde{R}, \tag{2.76}$$

$$\tilde{R} \text{ belongs to } (\tilde{S}), \tag{2.77}$$

$$\tilde{Q} = \frac{d}{dx} - \eta\tilde{q}^{(j)}(x,\eta), \tag{2.78}$$

where

$$\tilde{q}^{(j)}(x,\eta) = -\zeta^{(j)}(x) - \sum_{k\geq 1} \tilde{q}_k^{(j)}(x)\eta^{-k} \tag{2.79}$$

with $\tilde{q}_k^{(j)}(x)$ being holomorphic on a neighborhood V of the origin. Hence we find

(2.80) $$\tilde{P} = \tilde{T}^* = \tilde{R}^*\tilde{Q}^*,$$

while

(2.81) $$\tilde{Q}^* = -\frac{d}{dx} - \eta\tilde{q}^{(j)}(x,\eta).$$

Hence the equation $\tilde{P}\tilde{\psi} = 0$ is seen to have a WKB solution $\hat{\psi}^{(j)}(x,\eta)$ of the form of (2.74) if we choose $\hat{q}^{(j)} = -\tilde{q}^{(j)}$.

Q.E.D.

Remark 2.2. It follows from the assumptions (2.9), (2.10) and (2.11) together with Theorem 4.1 of [AKKoT1] that the logarithmic derivative of a WKB solution $\tilde{\psi}$ of $\tilde{P}\tilde{\psi} = 0$ is uniquely determined by its highest degree (in η) part. Hence a WKB solution $\tilde{\varphi}$ of $\tilde{P}\tilde{\psi} = 0$ such that the highest degree part of $d\log\tilde{\varphi}/dx$ coincides with $d\log\tilde{\psi}_+/dx$ (resp., $d\log\tilde{\psi}_-/dx$) should coincide with $\tilde{\psi}_+$ (resp., $\tilde{\psi}_-$), and we conclude $\tilde{R}\tilde{\varphi} = 0$. Therefore the WKB theoretic structure of the WKB solution $\tilde{\varphi}$ of the equation $\tilde{P}\tilde{\varphi} = 0$ can be clarified at least near the origin by the analysis of the second order equation $\tilde{R}\tilde{\psi} = 0$, which we will do in the subsequent sections.

We have so far discussed operators in $(\tilde{S})$. Results for operators in (S) are readily deduced from the corresponding results for operators in $(\tilde{S})$. For example, a simple algebraic argument deduces the following Theorem 2.4 from Theorem 2.1 and Proposition 2.2. We note that Theorem 2.4 below is basically the same as an announced result in [KKoT], i.e., Theorem 4.4 in [KKoT, Section 4].

Theorem 2.4. *Let P be an m-th $(m \geq 3)$ order operator in (S). Then there exist an open neighborhood V of the origin, meromorphic functions $q_k^{(l)}(x)$ $(l = 1, 2, \cdots, m-2;\ k \geq 0)$ defined on V and* *a second order operator R defined on V that belongs to (S) so that they may satisfy the following conditions* (2.84) $\sim$ (2.88), *if we let $q^{(l)}(x,\eta)$ and $Q^{(l)}$ $(l=1,2,\cdots,m-2)$ respectively denote*

$$\sum_{k\geq 0} q_k^{(l)}(x)\eta^{-k} \tag{2.82}$$

and

$$\frac{d}{dx}-\eta q^{(l)}(x,\eta): \tag{2.83}$$

(2.84) $q_k^{(l)}(x)$ $(k\neq 1)$ *is holomorphic on V,*

(2.85) $xq_1^{(l)}(x)$ *is holomorphic on V,*

$$\operatorname*{Res}_{x=0} q_1^{(l)}(x)=-1, \tag{2.86}$$

(2.87) *for each compact set K in V there exists a constant M_K for which*

$$\sup_{x\in K}|q_k^{(l)}(x)|\leq M_K^k k!$$

holds for each $l=1,2,\cdots,m-2$ and every $k\geq 2$,

$$P=Q^{(1)}Q^{(2)}\cdots Q^{(m-2)}R. \tag{2.88}$$

Furthermore the operator R is uniquely determined by P regardless of the choice of operators $\{Q^{(l)}\}_{1\leq l\leq m}$.

Proof. Let $\tilde{P}$ denote the operator xP. Then $\tilde{P}$ belongs to $(\tilde{S})$, and hence repeated applications of Theorem 2.1 enable us to find a second order operator $\tilde{R}$ in $(\tilde{S})$ and operators $\{\tilde{Q}^{(l)}\}_{1\leq l\leq m-2}$ described by (2.66) and (2.67) so that they satisfy

$$\tilde{P}=\tilde{Q}^{(1)}\tilde{Q}^{(2)}\cdots\tilde{Q}^{(m-2)}\tilde{R} \tag{2.89}$$

on an open neighborhood V of the origin. We note that operators $\{\tilde{Q}^{(l)}\}_{1\le l\le m-2}$ and $\tilde{R}$ are with holomorphic coefficients on V. On the other hand, the commutation relation between the differential operator d/dx and the multiplication operator x entails

$$\Big(\frac{d}{dx} - \eta\tilde{q}^{(l)}(x,\eta)\Big)x = x\Big(\frac{d}{dx} - \eta\tilde{q}^{(l)}(x,\eta) + x^{-1}\Big), \tag{2.90}$$

that is,

$$\tilde{Q}^{(l)}x = x(\tilde{Q}^{(l)} + x^{-1}). \tag{2.91}$$

Therefore, if we set

$$q^{(l)}(x,\eta) = \tilde{q}^{(l)}(x,\eta) - \eta^{-1}x^{-1} \tag{2.92}$$

and define $Q^{(l)}$ by $d/dx - \eta q^{(l)}$, we find the relation

$$\begin{aligned} xP &= \tilde{Q}^{(1)}\tilde{Q}^{(2)}\cdots\tilde{Q}^{(m-2)}xx^{-1}\tilde{R} \\ &= xQ^{(1)}Q^{(2)}\cdots Q^{(m-2)}x^{-1}\tilde{R}. \end{aligned} \tag{2.93}$$

Hence by setting $R = x^{-1}\tilde{R}$ we obtain

$$P = Q^{(1)}Q^{(2)}\cdots Q^{(m-2)}R \tag{2.94}$$

with R in (S). It is clear that $\{Q^{(l)}\}_{1\le l\le m-2}$ satisfy the required conditions including somewhat intriguing condition (2.86). Furthermore, as we can obtain (2.89) from (2.94) by reversing the above reasoning, Proposition 2.2 entails that the operator $R = x^{-1}\tilde{R}$ is uniquely fixed by P, regardless of the choice of $Q^{(l)}$'s. This completes the proof of the theorem.

Q.E.D.

In view of the way how to find $Q^{(l)}$'s and R from $\tilde{Q}^{(l)}$'s and $\tilde{R}$, we readily obtain the following Theorem 2.5 from Theorem 2.3:

Theorem 2.5. *Let P be an m-th $(m \geq 3)$ order simple-pole type operator in class (S). Then, in parallel with the decomposition* (2.88), *we find the following m WKB solutions $\psi_\pm$ and $\psi^{(j)}$ $(1 \leq j \leq m-2)$ of the equation $P\psi = 0$ near the origin so that they satisfy the following:*

$$R\psi_\pm = 0, \tag{2.95}$$

$$\psi^{(j)} = \exp\Big(\eta \int_0^x \hat{q}^{(j)}(x,\eta)dx\Big), \tag{2.96}$$

where

$$\hat{q}^{(j)}(x,\eta) = \zeta^{(j)}(x) + \sum_{k\geq 1} \hat{q}_k^{(j)}(x)\eta^{-k} \tag{2.97}$$

with $\hat{q}_k^{(j)}(x)$ being holomorphic on a neighborhood V of the origin.

Proof. Applying Theorem 2.3 to $\tilde{P} = xP$ we immediately find that it suffices to choose $\psi^{(j)} = \hat{\psi}^{(j)}$. Since $R = x^{-1}\tilde{R}$, it is also clear that we can take $\tilde{\psi}_\pm$ as $\psi_\pm$. Thus this theorem is an immediate consequence of Theorem 2.3.

Q.E.D.

3 A canonical form of a second order simple-pole type operator in (S)

In view of Theorem 2.5, our principal aim is now to clarify the WKB-theoretic structure of a second order simple-pole type operator R in class (S). To attain this aim let us first note the following well-known lemma.

Lemma 3.1. *Let R be a second order simple-pole type operator in (S) that has the following form:*

$$\frac{d^2}{dx^2} + \eta A_1(x,\eta)\frac{d}{dx} + \eta^2 A_2(x,\eta), \tag{3.1}$$

where

$$A_j(x,\eta) = \sum_{k\geq 0} A_{j,k}(x)\eta^{-k} \quad (j=1,2) \tag{3.2}$$

with $A_{1,0}$, $xA_{1,k}$ $(k \geq 1)$ and $xA_{2,k}$ $(k \geq 0)$ being holomorphic on a neighborhood U of the origin. Then the equation $R\psi = 0$ can be brought to the Schrödinger equation of the form

$$\frac{d^2\varphi}{dx^2} = \eta^2 Q(x,\eta)\varphi \tag{3.3}$$

through the gauge transformation

$$\varphi = \exp\Big(\frac{1}{2}\eta \int^x A_1(x,\eta)dx\Big)\psi, \tag{3.4}$$

where

$$Q = -A_2(x,\eta) + \frac{1}{4}A_1(x,\eta)^2 + \frac{1}{2}\eta^{-1}\frac{dA_1(x,\eta)}{dx}. \tag{3.5}$$

A straightforward computation validates Lemma 3.1. One important point in the above gauge transformation is that $\frac{1}{2}\eta\int^x A_{1,0}(x)dx$ is a holomorphic function. Hence it does not cause any problems in performing the Borel transformation of φ; this term only results in translating the Borel transform of ψ by $\frac{1}{2}\int^x A_{1,0}dx$ in the y (= the variable dual to η)-plane. For reference purposes we note the concrete form of the potential

$$Q = \sum_{k\geq 0} Q_k(x)\eta^{-k}. \tag{3.6}$$

$$Q_0 = -A_{2,0} + \frac{1}{4}A_{1,0}^2, \tag{3.7}$$

(3.8) $$Q_1 = -A_{2,1} + \frac{1}{2}A_{1,0}A_{1,1} + \frac{1}{2}\frac{dA_{1,0}}{dx},$$

(3.9) $$Q_k = -A_{2,k} + \frac{1}{4}\Big(\sum_{p+q=k} A_{1,p}A_{1,q}\Big) + \frac{1}{2}\frac{dA_{1,k-1}}{dx} \quad (k \geq 2).$$

Thus one immediately notices that, although Q_0 and Q_1 are with a simple pole at the origin, Q_k $(k \geq 2)$ is, in general, with a double pole. We note that, if the operator is in class (S_0) studied in [KKoT], Q_k $(k \geq 3)$ is with a simple pole at the origin. As we will see below, this means that we cannot employ the results of [Ko1] but that we should use the results in [Ko3] concerning the construction of the so-called "formal coordinate transformation" that brings the potential Q of the Schrödinger equation (3.3) to its canonical form given below. For the convenience of the reader we quote below the result of [Ko3] as Proposition 3.2. See [Ko3] for its proof. It is basically the same as the reasoning in Section 2 of [Ko1].

Proposition 3.2. ([Ko3, Proposition 2]) *Let $\tilde{L}$ denote the following Schrödinger operator*:

(3.10) $$\frac{d^2}{d\tilde{x}^2} - \eta^2\tilde{Q}(\tilde{x}, \eta),$$

where its potential

(3.11) $$\tilde{Q}(\tilde{x}, \eta) = \frac{\tilde{Q}_0(\tilde{x})}{\tilde{x}} + \eta^{-1}\frac{\tilde{Q}_1(\tilde{x})}{\tilde{x}} + \sum_{k\geq 2}\eta^{-k}\frac{\tilde{Q}_k(\tilde{x})}{\tilde{x}^2}$$

satisfies the following conditions:

(3.12) *each $\tilde{Q}_k(\tilde{x})$ is holomorphic on a neighborhood U of the origin,*

(3.13) $$\tilde{Q}_0(0) \neq 0,$$

(3.14) *for each compact set K in U there exists a constant C_K for which the following estimate holds for every k :*

$$\sup_{\tilde{x}\in K} |\tilde{Q}_k(\tilde{x})| \le C_K^{k+1} k!.$$

Then we can find a series $\lambda(\eta) = \sum_{j\ge 0} \lambda_j \eta^{-j}$, a neighborhood $\tilde{V}$ of $\{\tilde{x} = 0\}$ and a series $x(\tilde{x}, \eta)$ that is of the form

$$x_0(\tilde{x}) + \eta^{-1} x_1(\tilde{x}) + \eta^{-2} x_2(\tilde{x}) + \cdots, \tag{3.15}$$

where $\{x_j(\tilde{x})\}_{j\ge 0}$ are holomorphic on V, so that they satisfy the following conditions (3.16) $\sim$ (3.20):

(3.16) $x_j(0) = 0$ *for every* $j = 0, 1, 2, \cdots,$

$$\frac{dx_0}{d\tilde{x}}(0) \neq 0, \tag{3.17}$$

(3.18) *there exists a constant C for which the following estimate holds for every j :*

$$\sup_V |x_j(\tilde{x})| \le C^{j+1} j!$$

(3.19)

$$\tilde{Q}(\tilde{x}, \eta) = \Big(\frac{dx}{d\tilde{x}}\Big)^2 \left(\frac{1}{x(\tilde{x}, \eta)} + \eta^{-2} \frac{\lambda(\eta)}{x(\tilde{x}, \eta)^2} \right) - \frac{1}{2} \eta^{-2} \{x(\tilde{x}, \eta); \tilde{x}\},$$

$$\lambda_j = \tilde{Q}_{j+2}(0) \quad (j \ge 0). \tag{3.20}$$

Here and in what follows $\{x; \tilde{x}\}$ stands for the Schwarzian derivative, i.e.,

$$\{x; \tilde{x}\} = \frac{d^3x/d\tilde{x}^3}{dx/d\tilde{x}} - \frac{3}{2} \left(\frac{d^2x/d\tilde{x}^2}{dx/d\tilde{x}} \right)^2. \tag{3.21}$$

It is known (e.g., [KT, Chap.2]) that the relation (3.19) guarantees that the equation

$$\tilde{L}\tilde{\psi} = 0 \tag{3.22}$$

is WKB-theoretically equivalent to the equation

$$M\psi = 0, \tag{3.23}$$

where

$$M = \frac{d^2}{dx^2} - \eta^2\left(\frac{1}{x} + \eta^{-2}\frac{\lambda(\eta)}{x^2}\right), \tag{3.24}$$

in the sense that, for a WKB solution $\psi(x, \eta)$ of (3.23),

$$\tilde{\psi}(\tilde{x}, \eta) = \left(\frac{dx}{d\tilde{x}}\right)^{-1/2} \psi(x(\tilde{x}, \eta), \eta) \tag{3.25}$$

is a WKB solution of (3.22), and vice versa (with the use of the inverse formal coordinate transformation $\tilde{x} = \tilde{x}(x, \eta)$, whose existence is guaranteed by (3.17)). Since the Borel transform ψ_B of a WKB solution ψ of $M\psi = 0$ can be concretely analyzed with the help of microdifferential operators, as we will show in Section 4, we try to analyze the Borel transform $\tilde{\psi}_B$ of $\tilde{\psi}$ through (3.25). In the last century this was thought to be a hard task to carry out from the viewpoint of microlocal analysis for simple-pole type operators even when λ is a genuine constant (as opposed to an infinite series as in Proposition 3.2). (See, e.g., the introduction of [Ko2].) Fortunately, the recent study [AKT2] has made a breakthrough in this subject, and by employing the method developed in [AKT2] we can analyze the precise meaning of the equivalence of (3.22) and (3.23) through microlocal analysis applied to their Borel transforms. It should be worth emphasizing that one crucial point that enabled [AKT2] to formulate their results using particular integro-differential operators, which we also employ below, is the following elementary and widely-applicable observation: the inequality (3.26) below, which had been used in [AKT1] to analyze the structure of an equation near its simple turning point, can be improved as (3.27):

(3.26)
$$\sum_{\substack{j_1+j_2+\cdots+j_k=j\\ j_1,j_2,\cdots,j_k\geq 1}} j_1!j_2!\cdots j_k! \leq j!,$$

(3.27)
$$\sum_{\substack{j_1+j_2+\cdots+j_k=j\\ j_1,j_2,\cdots,j_k\geq 1}} j_1!j_2!\cdots j_k! \leq 4^{k-1}(j-k+1)!.$$

This fact indicates that the reasoning of [AKT2] has an ample application scope, and our reasoning below is one example that shows the wide applicability of the reasoning in [AKT2]. Although the proof of Theorems 3.3 and 3.5 is essentially the same as that of Theorems 1.6 and 1.7 of [KKKoT], we describe its core part below in view of its importance in the main theme of this paper — microlocal approach to the exact WKB analysis, WKB analysis based on the Borel transformation.

In order to deduce Theorems 3.3 and 3.5 below from Proposition 3.2, we first make some notational preparations. To begin with, we introduce the inverse function $g(x)$ of $x = x_0(\tilde{x})$, that is,

(3.28)
$$x = x_0(g(x)),\ \tilde{x} = g(x_0(\tilde{x})).$$

The existence of $g(x)$ near the origin is guaranteed by (3.17). Then, rewriting the Borel transform $\tilde{L}_B$ of the operator $\tilde{L}$ in (x, y)-coordinate, we find

(3.29)
$$\tilde{L}_B\Big|_{\tilde{x}=g(x)} = \Big(\frac{dg}{dx}\Big)^{-2}\left[\frac{\partial^2}{\partial x^2} - \left(\frac{d^2g/dx^2}{dg/dx}\right)\frac{\partial}{\partial x}\right] - \tilde{Q}\Big(g(x), \frac{\partial}{\partial y}\Big)\frac{\partial^2}{\partial y^2}$$

$$= \left(\frac{dg}{dx}\right)^{-2} \Bigg[\frac{\partial^2}{\partial x^2} - \left(\frac{d^2g/dx^2}{dg/dx}\right) \frac{\partial}{\partial x} - \frac{1}{x} \frac{\partial^2}{\partial y^2}$$

$$- \frac{(dg/dx)^2}{g(x)} \tilde{Q}_1(g(x)) \frac{\partial}{\partial y} - \frac{(dg/dx)^2}{g(x)^2} \left(\sum_{k \geq 2} \tilde{Q}_k(g(x)) \left(\frac{\partial}{\partial y}\right)^{2-k} \right) \Bigg].$$

Here we have used the relation

$$\frac{\tilde{Q}_0(\tilde{x})}{\tilde{x}} = \left(\frac{dx_0}{d\tilde{x}}\right)^2 \frac{1}{x_0(\tilde{x})}, \tag{3.30}$$

or

$$\frac{\tilde{Q}_0(g(x))}{g(x)} = \left(\frac{dg}{dx}\right)^{-2} \frac{1}{x}, \tag{3.31}$$

which is a consequence of the comparison of the top degree (in η) part of (3.19).

Let us now define microdifferential operators $\mathcal{L}$ and $\mathcal{M}$ respectively by

(3.32)

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} - \left(\frac{d^2g/dx^2}{dg/dx}\right) \frac{\partial}{\partial x} - \frac{1}{x} \frac{\partial^2}{\partial y^2} - \frac{(dg/dx)^2}{g(x)} \tilde{Q}_1(g(x)) \frac{\partial}{\partial y}$$

$$- \frac{(dg/dx)^2}{g(x)^2} \left(\sum_{k \geq 2} \tilde{Q}_k(g(x)) \left(\frac{\partial}{\partial y}\right)^{2-k} \right)$$

and

$$\mathcal{M} = \frac{\partial^2}{\partial x^2} - \frac{1}{x} \frac{\partial^2}{\partial y^2} - \sum_{k \geq 0} \frac{\lambda_k}{x^2} \left(\frac{\partial}{\partial y}\right)^{-k}. \tag{3.33}$$

We note that $\mathcal{M}$ is nothing but the Borel transform of the operator M. Then we have the following Theorem 3.3 that asserts that operators $\mathcal{L}$ and $\mathcal{M}$ are intertwined by microdifferential operators.

Theorem 3.3. *Let ω_0 be an open neighborhood of $x = 0$, and set*

$$\Omega_0 = \{(x, y; \xi, \eta) \in T^*\mathbb{C}^2_{(x,y)}; x \in \omega_0, \eta \neq 0\} \tag{3.34}$$

and

$$\Omega_0^* = \{(x, y; \xi, \eta) \in \Omega_0; x \neq 0\}. \tag{3.35}$$

Then there exist microdifferential operators $\mathcal{X}$ and $\mathcal{Y}$ defined on Ω_0 that satisfy

$$\mathcal{L}\mathcal{X} = \mathcal{Y}\mathcal{M} \tag{3.36}$$

on Ω_0^* and that are invertible on Ω_0.

Proof. To begin with we note that $\psi(x(\tilde{x}, \eta), \eta)$ which appears in the right-hand side of (3.25) can be formally written as

$$\sum_{n\geq 0} \frac{1}{n!}\Big(\sum_{k\geq 1} x_k(\tilde{x})\eta^{-k}\Big)^n \Big(\frac{\partial^n}{\partial x^n}\psi(x, \eta)\Big)\Big|_{x=x_0(\tilde{x})}. \tag{3.37}$$

Hence its Borel transform is expressed in (x, y)-coordinate as

$$\left(\sum_{n\geq 0} \frac{1}{n!}\Big(\sum_{k\geq 1} x_k(g(x))\Big(\frac{\partial}{\partial y}\Big)^{-k}\Big)^n \frac{\partial^n}{\partial x^n}\right)\psi_B(x, y). \tag{3.38}$$

If we employ the notation of the symbol calculus (cf. [A]), (3.38) is rewritten as

$$:\exp\left(\Big(\sum_{k\geq 1} x_k(g(x))\eta^{-k}\Big)\xi\right): \psi_B(x, y). \tag{3.39}$$

Here, and in what follows, we use the notation $:s(x, y, \xi, \eta):$ for a symbol s to designate the corresponding normal ordered product, that is, in the current situation a product in which all the multiplication operators by functions of x stand to the left of all the differential operators in x. We also use the notation $\sigma(\mathcal{X})$ to designate the symbol of a microdifferential operator $\mathcal{X}$. Having the expression (3.39) in mind, we try to find required operators $\mathcal{X}$ and $\mathcal{Y}$ in the following form:

$$\mathcal{X} = :C(x,\eta)\exp(r(x,\eta)\xi):, \tag{3.40}$$

$$\mathcal{Y} = :C^*(x,\eta)\exp(r(x,\eta)\xi):, \tag{3.41}$$

where $C(x,\eta)$, $C^*(x,\eta)$ and $r(x,\eta)$ are symbols of microdifferential operators respectively of order $0, 0$ and -1. Let $r_k(x)$ denote the coefficient of η^{-k} in r; that is,

$$r(x,\eta) = \sum_{k\geq 1} r_k(x)\eta^{-k}. \tag{3.42}$$

The computation below aims to relate the series $r(x,\eta)$, or rather

$$s(x,\eta) = x + r(x,\eta), \tag{3.43}$$

with $x(\tilde{x},\eta)$ given in Proposition 3.2 through the coordinate transformation $\tilde{x} = g(x)$. Since $\mathcal{X}$ is free from y and since

$$\frac{\partial^p}{\partial\xi^p}\sigma(\mathcal{L}) = 0 \quad \text{if } p \geq 3, \tag{3.44}$$

it follows from the symbol calculus that we find

$$\sigma(\mathcal{L}\mathcal{X}) = \sigma(\mathcal{L})\sigma(\mathcal{X}) + \sigma_\xi(\mathcal{L})\sigma_x(\mathcal{X}) + \frac{1}{2!}\sigma_{\xi\xi}(\mathcal{L})\sigma_{xx}(\mathcal{X}). \tag{3.45}$$

Here, and in what follows, we use the subscript x (resp., ξ) to designate the differentiation by x (resp., ξ), i.e., $r_x = dr/dx$, $r_{xx} = d^2r/dx^2$, etc. We also use the letter E as an abbreviation of

$$\exp(r(x,\eta)\xi). \tag{3.46}$$

Thus we obtain

(3.47)

$$
\begin{aligned}
&\sigma(\mathcal{L}\mathcal{X}) \\
&= \Big[\xi^2 - \frac{1}{x}\eta^2 - \frac{g_{xx}}{g_x}\xi - \frac{g_x^2}{g}\tilde{Q}_1(g(x))\eta \\
&\quad - \frac{g_x^2}{g^2}\Big(\sum_{k\geq 2}\tilde{Q}_k(g(x))\eta^{2-k}\Big)\Big]C(x,\eta)E \\
&\quad + \Big(2\xi - \frac{g_{xx}}{g_x}\Big)\Big(C_xE + r_x\xi CE\Big) \\
&\quad + \frac{1}{2!}(2)\big(C_{xx}E + 2C_xr_x\xi E + r_{xx}\xi CE + (r_x\xi)^2CE\big) \\
&= (1+r_x)^2\xi^2CE + \Big[\Big(2C_x - \frac{g_{xx}}{g_x}C\Big)(1+r_x) + r_{xx}C\Big]\xi E \\
&\quad + \Big[\Big(-\frac{1}{x}\eta^2 - \frac{g_x^2}{g}\tilde{Q}_1(g(x))\eta - \frac{g_x^2}{g^2}\Big(\sum_{k\geq 2}\tilde{Q}_k(g(x))\eta^{2-k}\Big)\Big)C \\
&\quad - \frac{g_{xx}}{g_x}C_x + C_{xx}\Big]E.
\end{aligned}
$$

In parallel with (3.47) we find

(3.48)

$$
\begin{aligned}
&\sigma(\mathcal{Y}\mathcal{M}) \\
&= (C^*E)\Big(\xi^2 - \frac{1}{x}\eta^2 - \sum_{k\geq 0}\frac{\lambda_k}{x^2}\eta^{-k}\Big) \\
&\quad + \sum_{n\geq 1}\frac{1}{n!}(r^nC^*E)\Big[(-1)^{n+1}n!x^{-(n+1)}\eta^2
\end{aligned}
$$

$$+\sum_{k\geq 0}(-1)^{n+1}(n+1)!x^{-(n+2)}\lambda_k\eta^{-k}\Big]$$

$$= C^*E\Big[\xi^2 - \sum_{n\geq 0}(-1)^n(r/x)^n x^{-1}\eta^2$$

$$- \Big(\sum_{k\geq 0}\lambda_k\eta^{-k}\Big)\Big(\sum_{n\geq 0}(-1)^n(n+1)(r/x)^n x^{-2}\Big)\Big]$$

$$= C^*E\Big[\xi^2 - \frac{\eta^2}{x(1+(r/x))} - \Big(\sum_{k\geq 0}\lambda_k\eta^{-k}\Big)\Big(\frac{1}{x^2(1+(r/x))^2}\Big)\Big]$$

$$= C^*E\Big[\xi^2 - \frac{\eta^2}{s} - \frac{(\sum_{k\geq 0}\lambda_k\eta^{-k})}{s^2}\Big].$$

Then the comparison of the coefficients of $\xi^l E$ ($l = 2, 1, 0$) in (3.47) and (3.48) entails the following relations:

$$(1 + r_x)^2 C = C^*, \tag{3.49}$$

$$\Big(2C_x - \frac{g_{xx}}{g_x}C\Big)(1 + r_x) + r_{xx}C = 0, \tag{3.50}$$

(3.51)

$$\Big(-\frac{1}{x}\eta^2 - \frac{g_x^2}{g}\tilde{Q}_1(g(x))\eta - \frac{g_x^2}{g^2}\Big(\sum_{k\geq 2}\tilde{Q}_k(g(x))\eta^{2-k}\Big)\Big)C - \frac{g_{xx}}{g_x}C_x + C_{xx}$$

$$= -C^*\Big(\frac{\eta^2}{s} + \frac{\sum_{k\geq 0}\lambda_k\eta^{-k}}{s^2}\Big).$$

Since we have

$$s_x = 1 + r_x, \quad s_{xx} = r_{xx}, \tag{3.52}$$

we can rewrite (3.50) as

$$\frac{C_x}{C} = \frac{1}{2}\Big(\frac{g_{xx}}{g_x} - \frac{s_{xx}}{s_x}\Big). \tag{3.53}$$

Hence we obtain

$$C = \gamma (g_x)^{1/2}(s_x)^{-1/2} \tag{3.54}$$

with some constant γ. In view of the way C and C^* appear in (3.49) and (3.50) , we may choose γ to be 1 without loss of generality. Then we find

$$C = (g_x)^{1/2}(s_x)^{-1/2}, \tag{3.55}$$

$$C^* = s_x^2 C = (s_x)^{3/2}(g_x)^{1/2}. \tag{3.56}$$

Substituting (3.56) into (3.51) , we obtain

$$\begin{aligned} &\frac{1}{x}\eta^2 + \frac{g_x^2}{g}\tilde{Q}_1(g(x))\eta + \frac{g_x^2}{g^2}\Big(\sum_{k\geq 2}\tilde{Q}_k(g(x))\eta^{2-k}\Big) \\ &= s_x^2\Big(\frac{\eta^2}{s} + \frac{\sum_{k\geq 0}\lambda_k\eta^{-k}}{s^2}\Big) - C^{-1}\Big(\frac{g_{xx}}{g_x}C_x - C_{xx}\Big). \end{aligned} \tag{3.57}$$

In order to relate (3.57) with (3.19), we first use (3.31) to rewrite η^2/x as

$$\frac{g_x^2}{g}\tilde{Q}_0(g(x))\eta^2. \tag{3.58}$$

Then (3.57) assumes the following form:

$$\begin{aligned} &\frac{1}{g}\tilde{Q}_0(g(x))\eta^2 + \frac{1}{g}\tilde{Q}_1(g(x))\eta + \frac{1}{g^2}\Big(\sum_{k\geq 2}\tilde{Q}_k(g(x))\eta^{2-k}\Big) \\ &= g_x^{-2}s_x^2\Big(\frac{\eta^2}{s} + \frac{\sum_{k\geq 0}\lambda_k\eta^{-k}}{s^2}\Big) - g_x^{-2}C^{-1}\Big(\frac{g_{xx}}{g_x}C_x - C_{xx}\Big). \end{aligned} \tag{3.59}$$

Our computation has so far been performed in (x, y)-coordinate, but the relation (3.19) is expressed with $\tilde{x}$-variable. Hence we substitute

$$x = x_0(\tilde{x}) \tag{3.60}$$

into (3.59), and we compare the resulting equation with (3.19). In order to facilitate the description of (3.59) in $\tilde{x}$-coordinate, we introduce $\tilde{s}(\tilde{x},\eta)$ and $\tilde{C}(\tilde{x},\eta)$ given by the following:

$$\tilde{s}(\tilde{x},\eta) = s(x_0(\tilde{x}),\eta), \tag{3.61}$$

$$\tilde{C}(\tilde{x},\eta) = C(x_0(\tilde{x}),\eta). \tag{3.62}$$

Then we immediately find

$$\begin{aligned}\frac{d\tilde{s}}{d\tilde{x}} &= \Big(\frac{ds}{dx}\Big|_{x=x_0(\tilde{x})}\Big)\frac{dx_0(\tilde{x})}{d\tilde{x}} \\ &= \Big(\frac{ds}{dx}\Big|_{x=x_0(\tilde{x})}\Big)\Big(\Big(\frac{dg}{dx}\Big)^{-1}\Big|_{x=x_0(\tilde{x})}\Big),\end{aligned} \tag{3.63}$$

that is,

$$\big(g_x^{-2}s_x^2\big)\Big|_{x=x_0(\tilde{x})} = \Big(\frac{d\tilde{s}(\tilde{x},\eta)}{d\tilde{x}}\Big)^2. \tag{3.64}$$

Thus the comparison of (3.19) and (3.59) expressed with $\tilde{x}$-variable tells us what remains to be done is to relate the Schwarzian derivative with

$$D(x,\eta) = g_x(x)^{-2}C(x,\eta)^{-1}\Big(\frac{g_{xx}(x)}{g_x(x)}C_x(x,\eta) - C_{xx}(x,\eta)\Big) \tag{3.65}$$

expressed with $\tilde{x}$-variable. By combining (3.55) and (3.63), we find

$$\tilde{C}(\tilde{x},\eta) = \Big(\frac{d\tilde{s}}{d\tilde{x}}\Big)^{-1/2}. \tag{3.66}$$

On the other hand, (3.54) also implies

$$C(x,\eta) = \tilde{C}(g(x),\eta). \tag{3.67}$$

Hence we have

$$C_x(x,\eta) = \Big(\frac{d\tilde{C}}{d\tilde{x}}\Big|_{\tilde{x}=g(x)}\Big)\Big(\frac{dg}{dx}\Big), \tag{3.68}$$

$$C_{xx}(x,\eta) = \Big(\frac{d^2\tilde{C}}{d\tilde{x}^2}\Big|_{\tilde{x}=g(x)}\Big)\Big(\frac{dg}{dx}\Big)^2 + \Big(\frac{d\tilde{C}}{d\tilde{x}}\Big|_{\tilde{x}=g(x)}\Big)\Big(\frac{d^2g}{dx^2}\Big). \tag{3.69}$$

Then it follows from the definition of $D(x,\eta)$ that

$$\begin{aligned} D(x,\eta) &= g_x(x)^{-2}C(x,\eta)^{-1}\Big[g_{xx}(x)\Big(\frac{d\tilde{C}}{d\tilde{x}}\Big|_{\tilde{x}=g(x)}\Big) \\ &\quad - g_x(x)^2\Big(\frac{d^2\tilde{C}}{d\tilde{x}}\Big|_{\tilde{x}=g(x)}\Big) - g_{xx}(x)\Big(\frac{d\tilde{C}}{d\tilde{x}}\Big|_{\tilde{x}=g(x)}\Big)\Big] \\ &= -C(x,\eta)^{-1}\Big(\frac{d^2\tilde{C}}{d\tilde{x}^2}\Big|_{\tilde{x}=g(x)}\Big). \end{aligned} \tag{3.70}$$

Since (3.66) entails

$$\begin{aligned} \frac{d^2\tilde{C}}{d\tilde{x}^2} &= \frac{3}{4}\,\tilde{s}_{\tilde{x}}^{-5/2}(\tilde{s}_{\tilde{x}\tilde{x}})^2 - \frac{1}{2}\,\tilde{s}_{\tilde{x}}^{-3/2}\,\tilde{s}_{\tilde{x}\tilde{x}\tilde{x}} \\ &= -\frac{1}{2}\,(\tilde{s}_{\tilde{x}})^{-1/2}\Big[(\tilde{s}_{\tilde{x}})^{-1}\,\tilde{s}_{\tilde{x}\tilde{x}\tilde{x}} - \frac{3}{2}\,(\tilde{s}_{\tilde{x}})^{-2}\,(\tilde{s}_{\tilde{x}\tilde{x}})^2\Big], \end{aligned} \tag{3.71}$$

and since

$$C(x_0(\tilde{x}),\eta) = \tilde{C}(\tilde{x},\eta) = (\tilde{s}_{\tilde{x}})^{-1/2}, \tag{3.72}$$

we conclude from (3.70) evaluated at $x = x_0(\tilde{x})$ that

$$D(x_0(\tilde{x}),\eta) = \frac{1}{2}\Big[(\tilde{s}_{\tilde{x}})^{-1}\tilde{s}_{\tilde{x}\tilde{x}\tilde{x}} - \frac{3}{2}(\tilde{s}_{\tilde{x}})^{-2}(\tilde{s}_{\tilde{x}\tilde{x}})^2\Big] = \frac{1}{2}\{\tilde{s};\tilde{x}\}. \tag{3.73}$$

Summing up, we find that (3.59) with x being set to be $x_0(\tilde{x})$ reads as follows:

$$\begin{aligned} &\frac{\tilde{Q}_0(\tilde{x})}{\tilde{x}}\eta^2 + \frac{\tilde{Q}_1(\tilde{x})}{\tilde{x}}\eta + \frac{\sum_{k\geq 2}\tilde{Q}_k(\tilde{x})\eta^{2-k}}{\tilde{x}^2} \\ &= \Big(\frac{d\tilde{s}}{d\tilde{x}}\Big)^2\Big(\frac{\eta^2}{\tilde{s}(\tilde{x},\eta)} + \frac{\sum_{k\geq 0}\lambda_k\eta^{-k}}{\tilde{s}(\tilde{x},\eta)^2}\Big) - \frac{1}{2}\{\tilde{s};\tilde{x}\}. \end{aligned} \tag{3.74}$$

Otherwise stated, the construction of the required operators $\mathcal{X}$ and $\mathcal{Y}$ is reduced to finding a series $\tilde{s}(\tilde{x},\eta)$ that satisfies (3.74). Comparing

(3.74) with (3.19), one immediately finds that $x(\tilde{x}, \eta)$ gives the required series. Note that (3.43) guarantees that the degree 0 (in η) part of $\tilde{s}(\tilde{x}, \eta)$, the non-free (i.e., a priori given) part in the series $\tilde{s}$, is $x_0(\tilde{x})$, which coincides with the degree 0 part of $x(\tilde{x}, \eta)$. Thus, if we define

$$r(x, \eta) = \sum_{j \geq 1} x_j(g(x))\eta^{-j}, \tag{3.75}$$

$$C(x, \eta) = (g_x)^{1/2}(s_x)^{-1/2}, \tag{3.76}$$

$$C^*(x, \eta) = (g_x)^{1/2}(s_x)^{3/2}, \tag{3.77}$$

the estimate (3.18) guarantees they are the symbols of microdifferential operators needed to define microdifferential operators $\mathcal{X}$ and $\mathcal{Y}$ which satisfy

$$\mathcal{L}\mathcal{X} = \mathcal{Y}\mathcal{M}. \tag{3.78}$$

Furthermore the top degree (in η) parts of C and C^* are both $(g_x)^{1/2}$, which is different from 0 at the origin. Hence both $\mathcal{X}$ and $\mathcal{Y}$ are invertible on Ω_0 if ω_0 is chosen sufficiently small. This completes the proof of Theorem 3.3.

Q.E.D.

The way of the above construction of the operator $\mathcal{X}$ indicates that $\mathcal{X}\psi_B(x, y)$ is the Borel transform of $(dx/d\tilde{x})^{-1/2}\psi(x(\tilde{x}, \eta), \eta)$ written down in (x, y)-coordinate. Since the structure of the Borel transformed WKB solutions of $M\psi = 0$ will be explicitly analyzed in the next section, we try to find some concrete form of the integral operator that realizes the action of the operator $\mathcal{X}$ upon multi-valued analytic functions such as Borel transformed WKB solutions. Since the operator $\mathcal{X}$ has the same form as that studied in [AKT2, Appendix C] (cf. [AKT2, (C.1) and (C.2)]) and since the basic estimate (C.10) on which the reasoning of [AKT2, Appendix C] relies is of the same form as (3.18), we obtain Theorem 3.5 below by exactly the same reasoning as in [AKT2].

Remark 3.1. Using this chance, we make a correction of misprints in [AKT2]: In (C.6) the exponent 1/2 in $(r_0^\dagger)^{1/2}$ should be $-1/2$. (Two places.)

In what follows we briefly describe how we can obtain Theorem 3.5. We refer the reader to [AKT2, Appendix C] for the details.

A crucial step in the proof is the following Proposition 3.4.

Proposition 3.4. ([AKT2, Proposition C.1]) *For a domain* U *in* $\mathbb{C}_x$ *let* Ω *denote*

$$\{(x, y; \xi, \eta) \in T^*(U \times \mathbb{C}_y); \eta \neq 0\}, \tag{3.79}$$

and let $P = P(x, \partial/\partial x, \partial/\partial y)$ *be a microdifferential operator of order* 0 *on* Ω *with the symbol*

$$\sigma(P) = \sum_{j \geq 0} \eta^{-j} P_j(x, \eta^{-1}\xi), \tag{3.80}$$

where $P_j(x, \zeta)$ *is entire for every* j *and satisfies the following condition:*

There exists a constant C_0 *such that for each compact set* K *in* $U \times \mathbb{C}_\zeta$ *we can find a constant* M_K *satisfying*

$$\sup_{(x,\zeta) \in K} |P_j(x, \eta)| \leq M_K j! C_0^j \tag{3.81}$$

for $j = 0, 1, 2, \cdots$. *Then the action of* P *upon a (multi-valued) analytic function* $\phi(x, y)$ *is represented in the following form:*

$$(P\phi)(x, y) = \int_{y_0}^{y} K(x, y - y', \partial/\partial x)\phi(x, y')dy', \tag{3.82}$$

where $K(x,y,\partial/\partial x)$ is a differential operator of infinite order (in the sense of [SKK]*) and that is defined on $\{(x,y); x\in U$ and $|y|<1/C_0\}$ and y_0 is a reference point that fixes the action of $(\partial/\partial y)^{-1}$ as an integral operator.*

See [AKT2] for the proof. Here we content ourselves with showing the explicit form (3.84) below of the kernel function K in terms of $\{P_j\}_{j\geq 0}$. For this purpose we expand P_j as

$$P_j(x,\zeta)=\sum_{k\geq 0} a_{jk}(x)\zeta^k. \tag{3.83}$$

Then we find

$$K(x,y,d/dx)=\sum_{k\geq 0}\Big(\sum_{j\geq 0} a_{jk}(x)\frac{y^{j+k-1}}{(j+k-1)!}\Big)\Big(\frac{d}{dx}\Big)^k. \tag{3.84}$$

We next show that the symbol of the operator $\mathcal{X}$ is actually of the form (3.80). If we introduce $\{h_j(x)\}_{j\geq 0}$ and $\{f_{l,j}(x)\}_{1\leq l\leq j}$ so that they satisfy

$$C=(g_x)^{1/2}(s_x)^{-1/2}=\sum_{j\geq 0}\eta^{-j}h_j(x), \tag{3.85}$$

$$\exp(r(x,\eta)\xi)=1+\sum_{1\leq l\leq j}\eta^{-j}\xi^l f_{l,j}(x), \tag{3.86}$$

it follows from the definition (3.40) of the operator $\mathcal{X}$ that its symbol is

$$\begin{aligned}(3.87)\qquad &\Big(\sum_{j\geq 0}\eta^{-j}h_j\Big)\Big(1+\sum_{1\leq l\leq j}\eta^{-j}\xi^l f_{l,j}\Big)\\ &=\Big(\sum_{j\geq 0}\eta^{-j}h_j\Big)\Big(1+\sum_{k\geq 0}\eta^{-k}\sum_{l\geq 1} f_{l,l+k}(\eta^{-1}\xi)^l\Big)\\ &=\sum_{m\geq 0}\eta^{-m}\Big[h_m+\sum_{l\geq 1}\Big(\sum_{\substack{j+k=m\\ j,k\geq 0}} h_j f_{l,l+k}\Big)(\eta^{-1}\xi)^l\Big].\end{aligned}$$

Remark 3.2. It immediately follows from (3.86) that

$$f_{l,j} = \frac{1}{l!} \sum_{\substack{j_1+j_2+\cdots+j_l=j \\ j_k \geq 1}} r_{j_1} r_{j_2} \cdots r_{j_l}. \tag{3.88}$$

Similarly, if we set

$$C^{-2} = g_x^{-1}(1 + r_x) = \sum_{j\geq 0} \eta^{-j} r_j^\dagger(x), \tag{3.89}$$

we find

(3.90)

$$\begin{cases} h_0 = (r_0^\dagger)^{-1/2} \\ h_j = (r_0^\dagger)^{-1/2} \displaystyle\sum_{1\leq l\leq j} \frac{(-1)^l \Gamma(l+1/2)}{l!\Gamma(1/2)} \sum_{\substack{j_1+j_2+\cdots+j_l=j \\ j_k\geq 1}} \frac{r_{j_1}^\dagger \cdots r_{j_l}^\dagger}{(r_0^\dagger)^l} \ (j \geq 1). \end{cases}$$

These expressions convince the reader of the importance of the improved estimate (3.27) in our reasoning. (See [AKT2, Remark C.2] for the details.)

It is now clear from (3.87) that we should choose

$$P_m(x,\zeta) = h_m + \sum_{l\geq 1} \Big(\sum_{\substack{j+k=m \\ j,k\geq 0}} h_j f_{l,l+k} \Big) \zeta^l \tag{3.91}$$

to find (3.80). The fact that $P_m(x,\zeta)$ is an entire function of ζ is confirmed by (3.18) and (3.27). We refer the reader to [AKT2, Appendix C] for the proof of this fact together with the estimate (3.81).

In order to employ Proposition 3.4 in the situation where the operand $\phi(x,y)$ is a Borel transformed WKB solution of the equation $M\psi = 0$, we prepare some notations needed for the concrete description. Let U and $S_\pm$ respectively denote the following sets:

$$U = \{(x,y) \in \mathbb{C}^2; |x|, |y| < \delta\} \tag{3.92}$$

and

$$S_{\pm} = \{(x, y) \in U; y = \pm 2\sqrt{x}\} \tag{3.93}$$

where δ is a sufficiently small positive number. We also define

$$U^* = U - \big(\{(x, y) \in U; x = 0\} \cup S_+ \cup S_-\big). \tag{3.94}$$

Then Proposition 3.4 reads as follows:

Theorem 3.5. *Let $\mathcal{X}$ be the microdifferential operator given by* (3.40) *together with* (3.75) *and* (3.76)*. Then its action upon a multi-valued analytic function $\varphi(x, y)$ defined on U^* is represented as an integro-differential operator of the form*

$$(\mathcal{X}\varphi)(x, y) = \int_{y_0}^{y} K(x, y - y', \partial/\partial x)\varphi(x, y')dy', \tag{3.95}$$

where $K(x, y, d/dx)$ is a differential operator of infinite order that is defined on $\{(x, y) \in \mathbb{C}^2; |x| < C$ and $|y| < C'$ for some positive constants C and $C'\}$, and $y = y_0$ is a reference point that fixes the action of $(\partial/\partial y)^{-1}$ as an integral operator. (*See* Figure 3.1 *below.*) *The operator $\mathcal{Y}$ given by* (3.41) *together with* (3.75) *and* (3.77) *also enjoys a similar expression.*

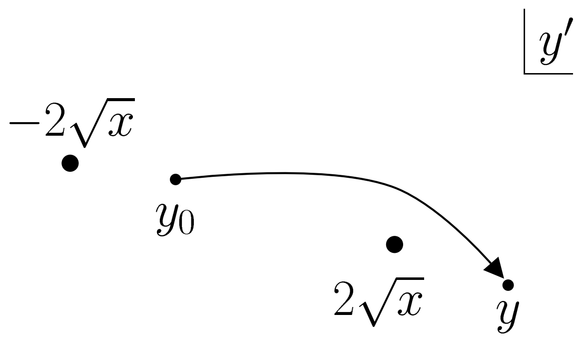

Figure 3.1.

Remark 3.3. We will show in the next section that the Borel transform $\psi_B(x, y)$ of a WKB solution ψ of the equation $M\psi = 0$ enjoys the analyticity properties required for the application of Theorem 3.5.

4 Structure of the Borel transforms of WKB solutions of the equation $M\psi = 0$

It follows from Theorems 3.3 and 3.5 that the study of Borel transformed WKB solutions for the simple-pole type Schrödinger equation $\tilde{L}\tilde{\psi} = 0$ is reduced to that for the equation $M\psi = 0$. The purpose of this section is to relate the Borel transformed WKB solution $\psi_B(x, y)$ for the equation $M\psi = 0$ and the Borel transformed WKB solution $\chi_B(x, y, \lambda_0)$ for the equation $M_0\chi = 0$, where M_0 designates the following Schrödinger operator:

$$\frac{d^2}{dx^2} - \eta^2\Big(\frac{1}{x} + \eta^{-2}\frac{\lambda_0}{x^2}\Big), \tag{4.1}$$

with λ_0 being a complex number. In what follows we let θ_0 denote the symbol $\sigma(\partial/\partial\lambda_0)$. Then, in parallel with Theorem 3.3 we find the following

Theorem 4.1. *Let U and U^* respectively denote*

$$\{(x, y, \lambda_0; \xi, \eta, \theta_0) \in T^*\mathbb{C}^3_{(x,y,\lambda_0)}; \eta \neq 0\} \tag{4.2}$$

and

$$\{(x, y, \lambda_0; \xi, \eta, \theta_0) \in U; x \neq 0\}, \tag{4.3}$$

and let $\mathcal{M}$ and $\mathcal{M}_0$ respectively denote the Borel transform of M and M_0. Then for

$$\rho(\eta) = \sum_{j\geq 1} \lambda_j \eta^{-j} \tag{4.4}$$

the microdifferential operator $\mathcal{R}$ given by

$$: \exp(\rho(\eta)\theta_0) : \tag{4.5}$$

is well-defined on U and it satisfies the following relation (4.6) *on* U^* :

$$\mathcal{M}\mathcal{R} = \mathcal{R}\mathcal{M}_0. \tag{4.6}$$

Proof. Since it follows from the definition that $\lambda_j = \tilde{Q}_{j-2}(0)$, we find a constant C by the assumption (3.14) so that the following relation may hold for every $j \geq 1$:

$$|\lambda_j| \leq C^j (j-1)!. \tag{4.7}$$

Hence $\rho(\eta)$ defines a symbol of a microdifferential operator of order at most -1. Thus $\mathcal{R}$ is a well-defined microdifferential operator on U. Further we find

$$\sigma(\mathcal{M}\mathcal{R}) = \sigma(\mathcal{M})\sigma(\mathcal{R}), \tag{4.8}$$

because $\sigma(\mathcal{R})$ is free from (x, y, λ_0). Since $\sigma(\mathcal{R})$ is free from ξ, $\sigma(\mathcal{M}_0)$ is free from y and $\partial^2\sigma(\mathcal{M}_0)/\partial\lambda_0^2$ vanishes identically, we obtain

$$\begin{aligned} \sigma(\mathcal{R}\mathcal{M}_0) &= \sigma(\mathcal{R})\sigma(\mathcal{M}_0) + \frac{\partial(\sigma(\mathcal{R}))}{\partial\theta_0}\frac{\partial(\sigma(\mathcal{M}_0))}{\partial\lambda_0} \\ &= \sigma(\mathcal{R})\sigma(\mathcal{M}_0) - \rho(\eta)\sigma(\mathcal{R})x^{-2} \\ &= \sigma(\mathcal{R})\sigma(\mathcal{M}). \end{aligned} \tag{4.9}$$

Therefore we find (4.6).

Q.E.D.

In view of the definition (4.5) of the operator $\mathcal{R}$, we find Proposition 3.4 is applicable to our case; that is, the action of $\mathcal{R}$ upon a multi-valued analytic function $\phi(x, y, \lambda_0)$ is represented in the following form:

$$(\mathcal{R}\phi)(x, y, \lambda_0) = \int_{y_0}^{y} R(y - y', \partial/\partial\lambda_0)\phi(x, y, \lambda_0)dy', \tag{4.10}$$

where $R(y, \partial/\partial\lambda_0)$ is a differential operator of infinite order and that is defined on $\{(x, y, \lambda_0) \in \mathbb{C}^3;\ |y| < C\}$ for some positive constant C. On the other hand, the Borel transform $\chi_{\pm,B}$ of WKB solutions $\chi_\pm$ of the equation $M_0\chi = 0$ is concretely written down in terms of the Gauss hypergeometric function in the following manner ([Ko1, p.49]):

(4.11)

$$\begin{aligned}
&\chi_{+,B}(x, y, \lambda_0) \\
&= \frac{1}{\sqrt{4\pi}}\, s^{-1/2} F\big(\alpha - \frac{1}{2}, \beta - \frac{1}{2}, \frac{1}{2}; s\big)\Big|_{s=\frac{y}{4\sqrt{x}}+\frac{1}{2}} \\
&= \frac{1}{\sqrt{4\pi}}\, s^{-1/2} \sum_{n\ge 0} \frac{\prod_{l=0}^{n-1}(\alpha + l - 1/2)(\beta + l - 1/2)}{\prod_{l=0}^{n-1}(l + 1/2)} \frac{s^n}{n!}\Big|_{s=\frac{y}{4\sqrt{x}}+\frac{1}{2}},
\end{aligned}$$

(4.12)
$$\begin{aligned}
&\chi_{-,B}(x, y, \lambda_0) \\
&= \frac{1}{\sqrt{-4\pi}}(1 - s)^{-1/2} F\big(\frac{3}{2} - \alpha, \frac{3}{2} - \beta, \frac{1}{2}; 1 - s\big)\Big|_{s=\frac{y}{4\sqrt{x}}+\frac{1}{2}} \\
&= \frac{1}{\sqrt{-4\pi}}(1 - s)^{-1/2} \\
&\quad \times \sum_{n\ge 0} \frac{\prod_{l=0}^{n-1}(l - \alpha + 3/2)(l - \beta + 3/2)}{\prod_{l=0}^{n-1}(l + 1/2)} \frac{(1 - s)^n}{n!}\Big|_{s=\frac{y}{4\sqrt{x}}+\frac{1}{2}},
\end{aligned}$$

where α and β are constants given by

(4.13) $$\alpha + \beta = 2, \quad \alpha\beta = -4\lambda_0.$$

Hence we find

(4.14) $$(\alpha + l - 1/2)(\beta + l - 1/2) = -4\lambda_0 + 2(l - 1/2) + (l - 1/2)^2$$

and

(4.15) $$(l - \alpha + 3/2)(l - \beta + 3/2) = -4\lambda_0 - 2(l + 3/2) + (l + 3/2)^2.$$

These relations guarantee that $\chi_{\pm,B}(x, y, \lambda_0)$ is non-singular outside $\{(x, y) \in \mathbb{C}^2; x = 0\} \cup \{(x, y) \in \mathbb{C}^2; y^2 = 4x\}$; $\chi_{\pm,B}(x, y, \lambda_0)$ depends holomorphically on λ_0 despite the fact that α and β become singular at $\lambda_0 = -1/4$. Thus we find $\mathcal{R}(\chi_{\pm,B})$, and hence $\psi_{\pm,B}$ also, to be non-singular on U^* given in (3.94) with δ_0 being sufficiently small as is required in Theorem 3.5. Furthermore, Gauss' connection formula

$$(4.16)\quad s^{-1/2}F\left(\alpha - \frac{1}{2}, \beta - \frac{1}{2}, \frac{1}{2}; s\right)$$
$$= 2\sqrt{1+4\lambda_0}\,\sin(\pi\sqrt{1+4\lambda_0})\,F\left(\alpha, \beta, \frac{3}{2}; 1-s\right)$$
$$+ \cos(\pi\sqrt{1+4\lambda_0})\,(1-s)^{-1/2}F\left(\frac{3}{2} - \alpha, \frac{3}{2} - \beta, \frac{1}{2}; 1-s\right)$$

entails the following Borel transformed version of the connection formulas for WKB solutions for $M_0\chi = 0$.

Proposition 4.2. ([Ko1, Proposition 3.1]) *The discontinuity* $\Delta_{S_+}\chi_{+,B}$ *(resp.,* $\Delta_{S_-}\chi_{-,B}$*) of* $\chi_{+,B}(x, y, \lambda_0)$ *(resp.,* $\chi_{-,B}(x, y, \lambda_0)$*) along the cut* $S_+ = \{(x, y) \in \mathbb{C}^2;\ \mathrm{Im}\, y = \mathrm{Im}(2\sqrt{x}),\ \mathrm{Re}\, y \geq \mathrm{Re}(2\sqrt{x})\}$ *(resp.,* $S_- = \{(x, y) \in \mathbb{C}^2;\ \mathrm{Im}\, y = \mathrm{Im}(-2\sqrt{x}),\ \mathrm{Re}\, y \geq \mathrm{Re}(-2\sqrt{x})\}$*) coincides with* $2i\cos(\pi\sqrt{1+4\lambda_0})\chi_{-,B}(x, y, \lambda_0)$ *(resp.,* $-2i\cos(\pi\sqrt{1+4\lambda_0})\chi_{+,B}(x, y, \lambda_0)$*).*

Let us concentrate our attention on one of them, i.e.,

$$(4.17)\qquad \Delta_{S_+}\chi_{+,B} = 2i\cos(\pi\sqrt{1+4\lambda_0})\chi_{-,B}.$$

We note that each term in (4.16) depends holomorphically on λ_0. Hence, by applying the integro-differential operator $\mathcal{R}$ to (4.11) and (4.12) , we observe that the singularities of $\mathcal{R}(\chi_{+,B})$ and $\mathcal{R}(\chi_{-,B})$ are confined to the set $\{(x, y) \in \mathbb{C}^2;\ y^2 = 4x\}$ near the origin, outside $\{x = 0\}$. Note that $\mathcal{R}(\chi_{+,B})$ (resp., $\mathcal{R}(\chi_{-,B})$) coincides with the Borel transformed WKB solution $\psi_{+,B}$ (resp., $\psi_{-,B}$) for the equation $M\psi = 0$ modulo holomorphic functions defined on a neighborhood of $(x, y, \lambda_0) = (0, 0, \lambda_0)$, which originate from the arbitrariness in the choice of the endpoint y_0 of the integral. Since $\chi_{-,B}$, and hence $\Delta_{S_+}\chi_{+,B}$ also, has the inverse of square-root type singularity along $\{y = 2\sqrt{x}\}$, we can "locally" normalize the action of the operator $\mathcal{R}$ by choosing

$$y_0 = 2\sqrt{x}. \tag{4.18}$$

Here by saying "locally" we mean that the class of the operands is restricted. Actually we have the following

Lemma 4.3. ([Ko2, Lemma 3.2]) *Let $\phi(x, \eta)$ be a Borel transformable (in the sense of* [KT] *) series of the following form:*

$$\exp(-s(x)\eta) \sum_{j \geq 0} \phi_j(x) \eta^{-j-1/2}. \tag{4.19}$$

Then we find the Borel transform $[\eta^{-n}\phi(x, \eta)]_B$ of $\eta^{-n}\phi(x, \eta)$ coincides with

$$\frac{1}{(n-1)!} \int_{s(x)}^{y} (y - y')^{n-1} \phi_B(x, y') dy' \tag{4.20}$$

for $n = 1, 2, \cdots$. Here ϕ_B denotes the Borel transform of $\phi(x, \eta)$.

The proof is a straightforward one, and we omit it here. The point is that, although $\partial_y^{-n}\phi_B$ is not canonically fixed in general, it can be well-defined consistently if the class of the target functions is restricted appropriately as above. In what follows we use the notation $\mathcal{R}_-$ when we want to emphasize the particular choice (4.18) of the endpoint y_0. (The reason of using $\mathcal{R}_-$, not $\mathcal{R}_+$, is that we want to stress that the starting formal series contains $\exp(-2\sqrt{x}\eta)$, not $\exp(2\sqrt{x}\eta)$.)

Now, the singular part of $\psi_{+,B}$ near $\{y = 2\sqrt{x}\}$ is the singular part of $\mathcal{R}_-(\chi_{+,B})$, which comes from the singular part of $\chi_{+,B}$. Therefore

(4.17) entails

$$\Delta_{S_+}(\psi_{+,B}) = 2i\mathcal{R}_-\big((\cos\pi\sqrt{1+4\lambda_0})\chi_{-,B}\big). \tag{4.21}$$

Summing up, we have the following

Theorem 4.4. *Using the notations introduced above, we find that the discontinuity $\Delta_{S_+}(\psi_{+,B})$ of the Borel transformed WKB solution ψ_+ of $M\psi = 0$ is expressed as*

$$2i\mathcal{R}_-\big((\cos\pi\sqrt{1+4\lambda_0})\chi_{-,B}(x,y,\lambda_0)\big), \tag{4.22}$$

where $\chi_{-,B}$ stands for the Borel transformed WKB solution χ_- of $M_0\chi = 0$.

Remark 4.1. Using the notations in Lemma 4.3, we can rewrite (4.22) as

$$\begin{aligned} &2i\Big[\Big(\sum_{n\geq 0}\frac{\rho(\eta)^n}{n!}\frac{\partial^n}{\partial\lambda_0^n}\Big)\Big(\big(\cos\pi\sqrt{1+4\lambda_0}\big)\chi_-(x,\eta,\lambda_0)\Big)\Big]_B \\ &= 2i\Big[\big(\cos\pi\sqrt{1+4(\lambda_0+\rho(\eta))}\big)\chi_-\big(x,\eta,\lambda_0+\rho(\eta)\big)\Big]_B \\ &= 2i\Big[\big(\cos\pi\sqrt{1+4\lambda(\eta)}\big)\psi_-\big(x,\eta,\lambda(\eta)\big)\Big]_B. \end{aligned} \tag{4.23}$$

This expression explains why (4.21) is called the Borel transformed version of the connection formula for the WKB solution ψ_+ of the equation $M\psi = 0$.

5 Structure of the Borel transforms of WKB solutions of the equation $\tilde{L}\tilde{\psi} = 0$

As in Section 3, we let $\mathcal{L}$ (resp., $\mathcal{M}$) denote the Borel transform multiplied by $(dg/dx)^2$ of a general simple-pole type Schrödinger operator $\tilde{L}$ written down in (x, y)-coordinate (resp., the Borel transform of the canonical simple-pole type Schrödinger operator M). (Note that the operators $\tilde{L}$ and M are respectively given by (3.10) and (3.24).) Then Theorem 3.3 asserts that operators $\mathcal{L}$ and $\mathcal{M}$ are microlocally intertwined (in the sense that they are mutually connected through the relation $\mathcal{L}\mathcal{X} = \mathcal{Y}\mathcal{M}$ with microlocally invertible operators $\mathcal{X}$ and $\mathcal{Y}$), and Theorem 3.5 shows that the intertwining operators $\mathcal{X}$ and $\mathcal{Y}$ enjoy beautiful integral representations. Further Theorem 4.1 shows that the operator $\mathcal{M}$ can be microlocally reduced to a simple operator $\mathcal{M}_0 = \dfrac{\partial^2}{\partial x^2} - \dfrac{1}{x}\dfrac{\partial^2}{\partial y^2} - \dfrac{\lambda_0}{x^2}$, which can be concretely analyzed. Summing up all these, we can concretely describe the analytic structure of the Borel transformed WKB solutions for the equation $\tilde{L}\tilde{\psi} = 0$, as is shown in Theorem 5.1 below. We basically use the same notations used in Sections 3 and 4, but we need the following additional notations to state Theorem 5.1 below: let V_0 be an open neighborhood of $(x, \lambda_0) = (0, \tilde{Q}_2(0))$ and set

$$W_0 = \{(x, y, \lambda_0; \xi, \eta, \theta_0) \in T^*_{(x,y,\lambda_0)}\mathbb{C}^3; (x, \lambda_0) \in V_0, \eta \neq 0\} \tag{5.1}$$

and

$$W_0^* = \{(x, y, \lambda_0; \xi, \eta, \theta_0) \in W_0; x \neq 0\}. \tag{5.2}$$

It is then clear that we may regard $\mathcal{X}$ and $\mathcal{Y}$ (resp., $\mathcal{L}$ and $\mathcal{M}$) as operators defined on W_0 (resp., W_0^*) which are free from $\partial/\partial\lambda_0$. To stay on the safer side, we note that $\tilde{\psi}_+(x, \eta)$ (resp., $\tilde{\psi}_-(x, \eta)$) designates the WKB solutions of $\tilde{L}\tilde{\psi} = 0$ that contains the factor $\exp\Big(\eta \int_0^{\tilde{x}} \sqrt{\tilde{Q}_0(\tilde{x})/\tilde{x}}\, d\tilde{x}\Big)$ (resp., $\exp\Big(-\eta \int_0^{\tilde{x}} \sqrt{\tilde{Q}_0(\tilde{x})/\tilde{x}}\, d\tilde{x}\Big)$.

Theorem 5.1. (i) *Using the notations of* Theorems 3.3 *and* 4.1, *we find the following*:

(5.3) $\mathcal{L}\mathcal{X}\mathcal{R} = \mathcal{Y}\mathcal{R}\mathcal{M}_0$ *holds on* W_0^*, (5.4) $\mathcal{XR}$ *and* $\mathcal{YR}$ *are well-defined and invertible on* W_0.

(ii) *There exists an open neighborhood* $\tilde{\omega}_0$ *of* $(\tilde{x}, y) = (0,0)$ *for which the following holds: the Borel transform* $\tilde{\psi}_{+,B}$ *(resp.,* $\tilde{\psi}_{-,B}$*) of* $\tilde{\psi}_+$ *(resp.,* $\tilde{\psi}_-$*) is analytic on*

$$\begin{aligned}(5.5)\qquad \tilde{\omega}_0 - \Big(&\{(\tilde{x}, y) \in \tilde{\omega}_0; \tilde{x} = 0\} \\ &\cup \{(\tilde{x}, y) \in \tilde{\omega}_0; y = \int_0^{\tilde{x}} \sqrt{\tilde{Q}_0(\tilde{x})/\tilde{x}}\, d\tilde{x}\} \\ &\cup \{(\tilde{x}, y) \in \tilde{\omega}_0; y = -\int_0^{\tilde{x}} \sqrt{\tilde{Q}_0(\tilde{x})/\tilde{x}}\, d\tilde{x}\}\Big).\end{aligned}$$

(iii) *Using the notations in* Proposition 4.2, *we find the following relations* (5.6) *and* (5.7)*, where* $\mathcal{X}_-$ *(resp.,* $\mathcal{X}_+$*) designates the integral operator given by* (3.95) *with* y_0 *being chosen to be* $2\sqrt{x}$ *(resp.,* $-2\sqrt{x}$*), and* $\mathcal{R}_+$ *denotes, in parallel with the case of* $\mathcal{R}_-$*, the operator* $\mathcal{R}$ *given by* (4.10) *with* y_0 *being chosen to be* $-2\sqrt{x}$ *:*

$$\begin{aligned}(5.6)\qquad &\Delta_{S_+}\tilde{\psi}_{+,B}(g(x), y) \\ &= 2i\mathcal{X}_-\mathcal{R}_-\big((\cos\pi\sqrt{1+4\lambda_0})\chi_{-,B}(x, y, \lambda_0)\big)\end{aligned}$$

holds for $x \neq 0$ *on a neighborhood of* $(x, y) = (0,0)$

$$\begin{aligned}(5.7)\qquad &\Delta_{S_-}\tilde{\psi}_{-,B}(g(x), y) \\ &= 2i\mathcal{X}_+\mathcal{R}_+\big((\cos\pi\sqrt{1+4\lambda_0})\chi_{+,B}(x, y, \lambda_0)\big)\end{aligned}$$

holds for $x \neq 0$ *on a neighborhood of* $(x, y) = (0,0)$.

References

[A] T. Aoki: Symbols and formal symbols of pseudodifferential operators, Advanced Studies in Pure Mathematics, **4**, Kinokuniya, 1984, pp.181–208.

[AKKoT1] T. Aoki, T.Kawai, T. Koike and Y. Takei: On the exact WKB analysis of operators admitting infinitely many phases, Adv. Math., **181** (2004), 165–189.

[AKKoT2] ______: A fresh glimpse into the Stokes geometry of Berk-Nevins-Roberts equation through a singular coordinate transformation, RIMS-Kôkyûroku, **1431**, 2005, pp.1–13.

[AKT1] T. Aoki, T. Kawai and T. Takei: The Bender-Wu analysis and the Voros theory, Special Functions, Springer-Verlag, 1991, pp.1–29.

[AKT2] ______: The Bender-Wu analysis and the Voros theory. II, Advanced Studies in Pure Mathematics, **54**, Math. Soc. Japan, 2009, pp.19–94.

[KKKoT] S. Kamimoto, T. Kawai, T. Koike and Y. Takei: On the WKB theoretic structure of a Schrödinger operator with a merging pair of a simple pole and a simple turning point, RIMS preprint (RIMS-**1678**), 2009. To appear in Kyoto Journal of Mathematics, **1** (2010).

[K^3] M. Kashiwara, T. Kawai and T. Kimura: Foundations of Algebraic Analysis, Princeton University Press, 1986. Japanese edition was published by Kinokuniya in 1980.

[KKoT] T. Kawai, T. Koike and Y. Takei: On the structure of higher order simple-pole type operators in exact WKB analysis, RIMS preprint (RIMS-**1658**), 2009. To appear in Funkcialaj Ekvacioj.

[KT] T. Kawai and Y. Takei: Algebraic Analysis of Singular Perturbation Theory, Amer. Math. Soc., 2005. Japanese edition was published by Iwanami in 1998.

[Ko1] T. Koike: On a regular singular point in the exact WKB analysis, Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear, Kyoto University Press, 2000, pp.39–54.

[Ko2] ———: On the exact WKB analysis of second order linear ordinary differential equations with simple poles, Publ. RIMS, Kyoto Univ., **36**(2000), 297–319.

[Ko3] ———: On a connection problem of simple pole type operators of second order in exact WKB analysis, RIMS-Kôkyûroku, **1433**, 2005, pp.9–26.

[SKK] M. Sato, T. Kawai and M. Kashiwara: Microfunctions and pseudo-differential equations, Lect. Notes in Math., **287**, Springer, 1973, pp.265–529.