

Title	Desingularization and Singularities of Some Moduli Scheme of Sheaves on a Surface
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Citation	Asian Journal of Mathematics (2009), 13(4): 465-472
Issue Date	2009-12
URL	http://hdl.handle.net/2433/145984
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Type	Journal Article
Textversion	author

DESINGULARIZATION AND SINGULARITIES OF SOME MODULI SCHEME OF SHEAVES ON A SURFACE

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ABSTRACT. Let X be a nonsingular projective surface over \mathbb{C} , and H_- and H_+ be ample line bundles on X in adjacent chamber of type (c_1, c_2) . Let $0 < a_- < a_+ < 1$ be adjacent minichambers, which are defined from H_- and H_+ , such that the moduli scheme $M(H_-)$ of rank-two a_- -stable sheaves with Chern classes (c_1, c_2) is non-singular. We shall construct a desingularization of $M(a_+)$ by using $M(a_-)$. As an application, we study whether singularities of $M(a_+)$ are terminal or not in some cases where X is ruled or elliptic.

1. INTRODUCTION

Let X be a projective non-singular surface over \mathbb{C} , H an ample line bundle on X . Denote by $M(H)$ the coarse moduli scheme of rank-two H -stable sheaves with fixed Chern class $(c_1, c_2) \in \text{NS}(X) \times \mathbb{Z}$. In this paper we think about singularities and desingularization of $M(H)$ from the view of wall-crossing problem of H and $M(H)$.

Let H_- and H_+ be ample line bundles on X separated by only one wall of type (c_1, c_2) . For a parameter $a \in (0, 1)$, one can define the a -stability of sheaves in such a way that a -stability of sheaves with fixed Chern class equals H_- -stability (resp. H_+ -stability) if a is sufficiently close to 0 (resp. 1), and there is a coarse moduli scheme $M(a)$ of rank-two a -stable sheaves with Chern classes (c_1, c_2) . Let a_- and $a_+ \in (0, 1)$ be parameters which are separated by only one miniwall. Assume $M_- = M(a_-)$ is non-singular. One can find such a_- when X is ruled or elliptic. We construct a desingularization $\tilde{\pi}_+ : \tilde{M} \rightarrow M_+$ of $M_+ = M(a_+)$ by using M_- and wall-crossing methods, and apply it to consider whether singularities of M_+ are terminal or not when X is ruled or elliptic.

Let $\overline{M}(H)$ denote the Gieseker-Maruyama compactification of $M(H)$. By [10], when X is minimal and its Kodaira dimension is positive, $\overline{M}(H)$ has the nef canonical divisor if $\dim \overline{M}(H)$ equals its expected dimension and if H is sufficiently close to K_X . Thus, to understand minimal models of a moduli scheme of stable sheaves, it can be meaningful to study singularities on $M(H)$. As a problem to be solved, it is desirable to extend results in this article to the case where M_- is not necessarily non-singular but its singularities are terminal (Remark 2.5).

Notation. For a k -scheme S , X_S is $X \times S$ and $\text{Coh}(X_S)$ is the set of coherent sheaves on X_S . For $s \in S$ and $E_S \in \text{Coh}(X_S)$, E_s means $E \otimes k(s)$. For E and $F \in \text{Coh}(X)$, $\text{ext}^i(E, F) := \dim \text{Ext}_X^i(E, F)$ and $\text{hom}(E, F) = \dim \text{Hom}_X(E, F)$. $\text{Ext}_X^i(E, E)^0$ indicates $\text{Ker}(\text{tr} : \text{Ext}_X^i(E, E) \rightarrow H^0(\mathcal{O}_X))$. For $\eta \in \text{NS}(X)$, we define $W^\eta \subset \text{Amp}(X)$ by $\{H \in \text{Amp}(X) \mid H \cdot \eta = 0\}$.

2. DESINGULARIZATION OF M_+ BY USING M_-

We begin with background materials. Let H_- and H_+ be ample divisors lying in neighboring chambers of type $(c_1, c_2) \in \text{NS}(X) \times \mathbb{Z}$, and H_0 an ample divisor in the wall W of type (c_1, c_2) which lies in the closure of chambers containing H_- and H_+ respectively. (Refer to [8] about the definition of wall and chamber.) Assume that $M = H_+ - H_-$ is effective. For a number $a \in [0, 1]$ one can define the a -stability of a torsion-free sheaf E using

$$P_a(E(n)) = \{(1-a)\chi(E(H_-)(nH_0)) + a\chi(E(H_+(nH_0)))\} / \text{rk}(E).$$

There is the coarse moduli scheme $\bar{M}(a)$ of rank-two a -semistable sheaves on X with Chern classes (c_1, c_2) . Denote by $M(a)$ its open subscheme of a -stable sheaves. When one replace H_\pm by NH_\pm if necessary, $M(0)$ (resp. $M(1)$) equals the moduli scheme of H_- -semistable (resp. H_+ -semistable) sheaves. There exist finite numbers $a_1 \dots a_l \in (0, 1)$ called minichambers such that $\bar{M}(a)$ and $M(a)$ changes only when a passes a miniwall. Refer to [2, Section 3] for details. Fix numbers a_- and a_+ separated by the only one miniwall, and indicate $\bar{M}_\pm = \bar{M}(a_\pm)$ and $M_\pm = M(a_\pm)$ for short. From [9, Section 2], the subset

$$\begin{aligned} \bar{M}_- \supset P_- &= \{[E] \mid E \text{ is not } a_+ \text{-semistable}\} \\ (\text{resp. } \bar{M}_+ \supset P_+ &= \{[E] \mid E \text{ is not } a_- \text{-semistable}\}) \end{aligned}$$

is contained in M_- (resp. M_+) and endowed with a natural closed subscheme structure of M_- (resp. M_+). Let η be a element of

$$A^+(W) = \{\eta \in \text{NS}(X) \mid \eta \text{ defines } W, 4c_2 - c_1^2 + \eta^2 \geq 0 \text{ and } \eta \cdot H_+ > 0\}.$$

After [2, Definition 4.2] we define

$$T_\eta = M(1, (c_1 + \eta)/2, n) \times M(1, (c_1 - \eta)/2, m),$$

where n and m are numbers defined by

$$n + m = c_2 - (c_1 - \eta^2)/4 \quad \text{and} \quad n - m = \eta \cdot (c_1 - K_X)/2 + (2a_0 - 1)\eta \cdot (H_+ - H_-),$$

and $M(1, (c_1 + \eta)/2)$ is the moduli scheme of rank-one torsion-free sheaves on X with Chern classes $((c_1 + \eta)/2, n)$. If F_{T_η} (resp. G_{T_η}) is the pull-back of a universal sheaf of $M(1, (c_1 + \eta)/2, n)$ (resp. $M(1, (c_1 - \eta)/2, m)$) to X_{T_η} , then we have an isomorphism

$$P_- \simeq \coprod_{\eta \in A^+(W)} \mathbb{P}_{T_\eta} \left(\text{Ext}_{X_{T_\eta/T_\eta}}^1(F_{T_\eta}, G_{T_\eta}(K_X)) \right) \quad (1)$$

from [9, Section 5].

Proposition 2.1 ([9] Proposition 4.9). *The blowing-up of M_- along P_- agrees with the blowing-up of M_+ along P_+ . So we have blowing-ups*

$$M_- \xleftarrow{\pi_-} B_{P_-}(M_-) = B_{P_+}(M_+) \xrightarrow{\pi_+} M_+.$$

By taking $4c_2 - c_1^2$ to be sufficiently large with respect to H_- and H_+ , we can assume from [6] and [7] that $M_\pm \supset \text{Sing}(M_\pm) := \{E \mid \text{ext}^2(E, E)^0 \neq 0\}$ satisfies $\text{codim}(M_\pm, \text{Sing}(M_\pm)) \geq 2$ and that $P_\pm \subset M_\pm$ is nowhere dense, and hence both M_- and M_+ are normal l.c.i. schemes and birationally equivalent. Suppose that

$A^+(W) = \{\eta\}$ for simplicity and denote $T_\eta = T$. From Hironaka's desingularization theorem, there is a sequence of blowing-ups

$$M_N \longrightarrow M_{N-1} \cdots \longrightarrow M_- \quad (2)$$

along non-singular centers $Z_i \subset M^i$ such that the ideal sheaf of \mathcal{O}_{M_N} generated by pull-back of the ideal sheaf of $P_- \subset M_-$ is invertible.

Claim 2.2. If we set

$$l_1 = \max\{\text{ext}^1(F_t, G_t(K_X)) \mid t \in T\},$$

then we can take the center Z_i in (2) so that the dimension of Z_i is not greater than $l_1 - 1 + \dim T$.

Proof. Since one can readily show $\text{ext}^2(F_t, G_t(K_X)) = \text{hom}(G_t, F_t) = 0$ for all $t \in T$, (1) implies that P_- is embedded in a \mathbb{P}^{l_1} -bundle over T . Thus for $s \in P_-$, the rank of $\Omega_{P_-} \otimes k(s)$ is not greater than $\dim T + l_1 - 1$. From the exact sequence

$$CN_{P_-/M_-} \longrightarrow \Omega_{M_-}|_{P_-} \longrightarrow \Omega_{P_-} \longrightarrow 0,$$

we can choose local coordinates $g_i \in \mathcal{O}_{M_-,s}$ so that g_i lies in $I_{P_-,s}$ for $i \leq \dim M_- - (\dim T + l_1 - 1)$. From [1, Thm. 1.10], one can choose the center Z_i in such a way that the ideal sheaf of Z_i contains the weak transform of I_{P_-} by $M_i \rightarrow M_-$, say I_i . If y is a local generator of the exceptional divisor of $M_1 \rightarrow M_-$, then g_i/y ($i \leq \dim M_- - (\dim T + l_1 - 1)$) are partial coordinating parameters of M_1 and belong to I_1 . Since I_{Z_1} contains I_1 , the claim holds for $i = 1$. For general i , one can verify the claim in the same way. \square

From Proposition 2.1, we obtain a morphism

$$M_N \longrightarrow B(M) := B_{P_-}(M_-) = B_{P_+}(M_+) \longrightarrow M_+$$

and a diagram

$$\begin{array}{ccc} & \tilde{M} := M_N & \\ \tilde{\pi}_- \swarrow & \downarrow \pi & \searrow \tilde{\pi}_+ \\ M_- & \longleftarrow B(M) \longrightarrow & M_+ \end{array} \quad (3)$$

Therefore we can regard \tilde{M} as a desingularization of M_+ .

Next let us calculate $K_{\tilde{M}} - \tilde{\pi}_+^* K_{M_+}$. If we denote by $D_i \subset \tilde{M}$ the pull-back of the exceptional divisor of $M^i \rightarrow M^{i-1}$, then

$$K_{\tilde{M}} - \tilde{\pi}_+^* K_{M_+} = \sum_i [\dim M_- - \dim Z_i - 1] D_i. \quad (4)$$

Next consider $\tilde{\pi}_-^*(K_{M_-}) - \tilde{\pi}_+^*(K_{M_+})$. By the proof of Proposition 2.1, which uses elementary transform, we have the following.

Proposition 2.3. *Denote the exceptional divisor $\pi_-^{-1}(P_-) = \pi_+^{-1}(P_+) \subset B(M)$ by D . Suppose we have a universal family $E_{M_-}^- \in \text{Coh}(X_{M_-})$ of M_- and a universal family $E_{M_+}^+ \in \text{Coh}(X_{M_+})$ of M_+ . If $p : D \rightarrow P_+ \rightarrow T$ is a natural map, then there*

are line bundles L_{\pm} on P_{\pm} and a line bundle L_0 on $B(M)$ such that we have exact sequences

$$0 \longrightarrow \pi_+^* E_{M_+}^+ \otimes L_0 \longrightarrow \pi_-^* E_{M_-}^- \xrightarrow{f} p^* G_T \otimes \pi_+^* L_+ \longrightarrow 0 \quad (5)$$

in $\text{Coh}(X_{B(M)})$ and

$$0 \longrightarrow \pi^* F_T \otimes \pi_-^* L_- \longrightarrow \pi_-^* (E_{M_-}^-)|_{X_D} \xrightarrow{f|_D} p^* G_T \otimes \pi_+^* L_+ \longrightarrow 0 \quad (6)$$

in $\text{Coh}(X_D)$.

The exact sequence (6) is the relative a_+ -Harder Narashimhan filtration of $E_{M_-}^-$. Here we remark that generally a universal family of M_- exists only étale-locally, but one can generalize this proposition to general case with straightforward labor. Suppose L_{\pm} and L_0 in this proposition are trivial for simplicity. From (5)

$$\begin{aligned} & \tilde{\pi}_-^* K_{M_-} - \tilde{\pi}_+^* K_{M_+} \\ &= \pi_-^* \det \mathbf{R}Hom_{X_{M_-}/M_-}(E_{M_-}^-, E_{M_-}^-) - \pi_+^* \det \mathbf{R}Hom_{X_{M_+}/M_+}(E_{M_+}^+, E_{M_+}^+) \\ &= \det \mathbf{R}Hom_{X_{B(M)}/B(M)}(\pi_-^* E_{M_-}^-, \pi_-^* E_{M_-}^-) \\ &\quad - \det \mathbf{R}Hom_{X_{B(M)}/B(M)}(\pi_+^* E_{M_+}^+, \pi_+^* E_{M_+}^+) \\ &= \det \mathbf{R}Hom_{X_{B(M)}/B(M)}(E_{B(M)}^-, E_{B(M)}^+) + \det \mathbf{R}Hom_{X_{B(M)}/B(M)}(E_{B(M)}^+, \pi^* G_T) \\ &\quad - \det \mathbf{R}Hom_{X_{B(M)}/B(M)}(E_{B(M)}^-, E_{B(M)}^+) \\ &\quad + \det \mathbf{R}Hom_{X_{B(M)}/B(M)}(\pi_+^* G_T, E_{B(M)}^+) \\ &= \det \mathbf{R}Hom_{X_{B(M)}/B(M)}(E_{B(M)}^-, G_D) + \det \mathbf{R}Hom_{X_{B(M)}/B(M)}(G_D, E_{B(M)}^+). \end{aligned}$$

If $i : D \hookrightarrow B(M)$ is inclusion, then by (6)

$$\begin{aligned} \det \mathbf{R}Hom_{X_{B(M)}/B(M)}(E_{B(M)}^-, G_D) &= \det i_* \mathbf{R}Hom_{X_D/D}(E_{B(M)}^-|_D, G_D) = \\ &= \det i_* \mathbf{R}Hom_{X_D/D}(F_D, G_D) + \det i_* \mathbf{R}Hom_{X_D/D}(G_D, G_D). \quad (7) \end{aligned}$$

Since $\det \mathcal{O}_D = D$, (7) equals $[\chi(F_t, G_t) + \chi(G_t, G_t)] D$ for any $t \in D$. By the Serre duality

$$\begin{aligned} & \det \mathbf{R}Hom_{X_{B(M)}/B(M)}(G_D, E_{B(M)}^+) \\ &= \det \mathbf{R}Hom_{B(M)}(\mathbf{R}Hom_{X_{B(M)}/B(M)}(E_{B(M)}^+, G_D(K_X)), \mathcal{O}_{B(M)}) \\ &= -\det \mathbf{R}Hom_{X_{B(M)}/B(M)}(E_{B(M)}^+, G_D(K_X)) \\ &= -\det i_* \mathbf{R}Hom_{X_D/D}(E_{B(M)}^+|_D, G_D(K_X)) \\ &= -[\chi(F_t, G_t(K_X)) + \chi(G_t, G_t(K_X))] D = -[\chi(G_t, F_t) + \chi(G_t, G_t)] D. \end{aligned}$$

Therefore

$$\pi_-^* K_{M_-} - \pi_+^* K_{M_+} = [\chi(F_t, G_t) - \chi(G_t, F_t)] D = 2(c_1(F_t) - c_1(G_t)) \cdot K_X. \quad (8)$$

Moreover, we put

$$\tilde{\pi}^* D = \sum_{i=0}^{N-1} \lambda_i D_i. \quad (9)$$

When $\dim M_- - (l_1 - 1 + \dim T) > 0$, all λ_i are 1. Indeed, the proof of Claim 2.2 says that some element $g \in I_{P_-}$ satisfies that if y is a local generator of the exceptional divisor of $M_1 \rightarrow M_-$, then g/y is a partial coordinating parameter of M_1 . Thus the pull-back of I_{P_-} by $M_1 \rightarrow M_-$ is divided by y , but cannot be divided by y^2 , which implies $\lambda_1 = 1$. One can show $\lambda_i = 1$ similarly. Consequently, from (4), (8) and (9), we have shown the following.

Proposition 2.4. *In the diagram (3) it holds that*

$$K_{\tilde{M}} - \tilde{\pi}_+^* K_{M_+} = \sum_{i=0}^{N-1} [\dim M_- - \dim Z_i - 1 + \lambda_i 2(c_1(F_t) - c_1(G_t)) \cdot K_X] D_i. \quad (10)$$

with $\lambda_i \geq 1$. If $\dim M_- > l_1 - 1 + \dim T$ then $\lambda_i = 1$ and

$$\begin{aligned} \dim M_- - \dim Z_i - 1 + 2\lambda_i(c_1(F_t) - c_1(G_t)) \cdot K_X &\geq \\ \dim M_- - (l_1 - 1 + \dim T) - 1 + 2(c_1(F_t) - c_1(G_t))K_X. \end{aligned}$$

One can use this proposition to verify whether singularities in M_+ is terminal or not.

Remark 2.5. It is desirable to extend this article to the case where M_- is not necessarily non-singular but its singularities are terminal. It is a problem that we can not use (4) since M_- is not non-singular.

3. EXAMPLES: RULED OR ELLIPTIC SURFACE

We shall give examples of M_{\pm} with M_- non-singular. If a surjective morphism $X \rightarrow C$ to a nonsingular curve C exists, then by [3, p.142] we have a (c_1, c_2) -suitable polarization, that is, an ample line bundle H such that H does not lie on any wall of type (c_1, c_2) , and for any wall $W = W^n$ of type (c_1, c_2) , we have $\eta \cdot f = 0$ or $\text{Sign}(f \cdot \eta) = \text{Sign}(H \cdot \eta)$. From [3, p.159, p.201], if X is a ruled surface or an elliptic surface, then any rank-two sheaf E of type (c_1, c_2) which is stable respect to (c_1, c_2) -suitable polarization is good, i.e. $\text{Ext}^2(E, E)^0 = 0$.

(A) First we suppose that X is a (minimal) ruled surface. When $c_1 \cdot f$ is odd $M(H)$ is empty for (c_1, c_2) -suitable polarization. Thus we assume $c_1 = 0$. If a rank-two sheaf E of type (c_1, c_2) is stable with respect to a polarization H such that $H \cdot K_X < 0$, then E is good and so $M(H)$ is nonsingular. Hence we assume that $W^{K_X} \cap \text{Amp}(X) \neq \emptyset$, so $2 \leq g = g(C)$ and $e(X) \leq 2g - 2$ from the description of $\text{Amp}(X)$ [4, Prop. V.2.21]. Since $\dim \text{NS}(X) = 2$, if we move polarization H from a (c_1, c_2) -suitable one, then $M(H)$ may begin to admit singularities when H passes the wall W^{K_X} . Let H_- and H_+ be ample line bundles separated by only one wall W^{K_X} . $M(H_-)$ is non-singular, and $E^+ \in \mathbb{P}_+$ has a non-trivial exact sequence

$$0 \longrightarrow G = L \otimes I_{Z_l} \longrightarrow E^+ \longrightarrow F = L^{-1} \otimes I_{Z_r} \longrightarrow 0 \quad (11)$$

with $-2L \sim mK_X$. About this filtration we have $\text{Ext}_-^2(E^+, E^+) = 0$ since $p_g(X) = 0$ (See [5, p. 49] for Ext_{\pm}), and

$$\text{ext}^2(E^+, E^+) = \text{ext}_+^2(E^+, E^+) = \text{ext}^2(L \otimes I_{Z_l}, L^{-1} \otimes I_{Z_r}) = \text{hom}(I_{Z_r}, \mathcal{O}(K_X + 2L) \otimes I_{Z_l}).$$

Since W^{K_X} defines a wall, $H^0(\mathcal{O}(K_X + 2L)) = 0$ unless $2L + K_X = 0$. Hence $\text{ext}^2(E^+, E^+)^0 \neq 0$ if and only if $-2L = K_X$ and $Z_l \subset Z_r$. As a result when one defines a -stability using H_\pm ,

$$\chi^a(E^+) - \chi^a(L \otimes I_{Z_l}) = Aa + B + l(Z_l)$$

for some constant A and B , and so the moduli scheme $M(a)$ of a -stable sheaves begins to admit singularities just when a passes a miniwall a_0 defined by

$$l(Z_l) = \begin{cases} c_2/2 - (g-1) & \text{if } c_2 \text{ is even} \\ (c_2 - 1)/2 - (g-1) & \text{if } c_2 \text{ is odd.} \end{cases}$$

Let a_- and a_+ be minichambers separated by only one miniwall a_0 . $M(a_+) = M_+$ has singularities along $P_+ \times_T T'$, where

$$T' = \{(L \otimes I_{Z_l}, L^{-1} \otimes I_{Z_r}) \mid -2L = K_X\}_{red} \subset M(1, K_X/2, l(Z_l)) \times M(1, -K_X/2, l(Z_r)).$$

(B) Suppose that X is an elliptic surface with a section σ and $c_1 = \sigma$. In contrast to ruled surfaces, $K_X^2 = 0$ and so $W^{K_X} \cap \text{Amp}(X)$ is always empty, though one can study some singularities appearing in $M(H)$ by Proposition 2.4. Let $\pi : X \rightarrow C$ be an elliptic fibration, $f \in \text{NS}(X)$ its fiber class, $d = -\deg R^1\pi_*(\mathcal{O}_X) - \sigma^2 \geq 0$. We have a natural map to a ruled surface $\kappa : X \rightarrow \mathbb{P}(\pi_*(\mathcal{O}(2\sigma))) = \mathbb{P}(\mathcal{E}_2)$. Since $\kappa_*(\sigma)$ is a section of $\mathbb{P}(\mathcal{E}_2)$, and since the pull-back of an ample line bundle by a finite map is ample, $L = af$ satisfies $W^{2L-c_1} \cap \text{Amp}(X) \neq \emptyset$ if $a > 0$ from the description of the ample cone of a ruled surface. Let c_1 be σ and $c_2 = (c_1 - L) \cdot L = a$. Then any sheaf E with non-trivial exact sequence

$$0 \longrightarrow F = L \longrightarrow E \longrightarrow G = L^{-1} \otimes c_1 \longrightarrow 0, \quad (12)$$

whose Chern class equals (c_1, c_2) , is stable with respect to a (c_1, c_2) -suitable ample line bundle. Indeed, $(2L - c_1) \cdot f < 0$ and so $\pi_*(\mathcal{O}(2L - c_1)) = 0$ and $R^1\pi_*(\mathcal{O}(2L - c_1))$ commutes with base change. Thus the exact sequence

$$0 \longrightarrow H^1(C, \pi_*(\mathcal{O}(2L - c_1))) \longrightarrow H^1(X, \mathcal{O}(2L - c_1)) \longrightarrow H^0(E, R^1\pi_*(\mathcal{O}(2L - c_1)))$$

shows that the restriction of the exact sequence (12) to a general fiber is non-trivial, and so a corollary of Artin's theorem for vector bundles on an elliptic curve [3, p. 89] and a basic property of a suitable polarization [3, p. 144] deduce that E is stable with respect to a suitable polarization. Thereby such E is good. Let H_- and H_+ be ample line bundles which lie in no wall of type (c_1, c_2) with $(2L - c_1) \cdot H_- < 0 < (2L - c_1) \cdot H_+$. One can define a -stability by them. Let a_0 be a miniwall such that $\chi^{a_0}(\mathcal{O}(L)) = \chi^{a_0}(\mathcal{O}(2L - c_1))$, $a_- < a_0 < a_+$ minichambers, and $M_\pm = M(a_\pm)$. Then some connected components of $P_- \subset M_-$ contains any sheaf E with non-trivial exact sequence (12), and some neighborhood of them in M_- is non-singular. It induces a desingularization of some open neighborhood of connected components \mathcal{K}_+ of P_+ consisting of sheaves E^+ with a non-trivial exact sequence

$$0 \longrightarrow L^{-1} \otimes c_1 \longrightarrow E^+ \longrightarrow L \longrightarrow 0$$

as in Section 2.

We have in case of (A) $\text{ext}^1(G, F) \leq 1$, and in case of (B) $\text{ext}^1(G, F) = h^0(c_1 - 2L + K_X) - \chi(c_1 - 2L) \leq 2c_2 + C(X)$ with some constant $C(X)$ independent of c_2 because $h^0(c_1 - 2L + K_X) = 0$ if $a = c_2$ is sufficiently large. Thus in both

cases one can show that, if c_2 is sufficiently large, then all singularities of M_+ along above-mentioned sheaves are terminal.

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