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"Making the Case for a Low Intertemporal Elasticity of Substitution"

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Making the Case for a Low Intertemporal Elasticity of Substitution*

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Abstract

We provide two ways to reconcile small values of the intertemporal elasticity of substitution (IES) that range between 0.35 and 0.5 with empirical evidence that the IES is large. This is done using a model in which all agents have identical preferences and the same access to asset markets. We also conduct an encompassing test. That test indicates that specifications of the model with small values of the IES are more plausible than specifications with a large IES.

Keywords: Uncertainty; Intertemporal elasticity of substitution; Risk aversion; Business Cycles; Growth.

JEL Classification numbers: E21, E32, O41.

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1 Introduction

One of the most important margins in intertemporal decision making is the response of expected consumption growth to a change in the expected interest rate.

Empirical evidence from macroeconomic data about the magnitude of the intertemporal elasticity of substitution (IES) is mixed.

Simple and intuitive arguments for a large IES are provided by two sets of facts that we will refer to as the levels facts and difference facts. The levels facts are closely related to the risk-free rate puzzle discussed by Weil (1989). He points out that it is difficult to reconcile empirical observations about the average *level* of risk free returns and the average *level* of consumption growth with the intertemporal Euler equation when the maintained level of risk aversion is large and the discount factor is less than one. The observation that low risk aversion is required to account for the average level of risk free returns can also be stated in terms of the IES. Under the assumption of time additive preferences the IES is the inverse of the risk aversion coefficient and only high values of the IES can be reconciled with data on the average level of consumption growth and the real risk-free interest rate. Guvenen (2006), for instance, derives a lower bound on the IES of about 0.7 using data on consumption growth and real returns using U.S data.

A second set of observations that has been used to argue that the IES is large concerns cross-country *differences* in consumption growth and real returns. Lucas (1990) observes that a small IES implies that a permanent one percentage difference in the growth rate of consumption is associated with larger proportionate variations in the real return on capital. He goes on to argue that it is hard to reconcile a value of the IES as low as 1/2 with the small differences in measured real returns on capital across different countries. We will subsequently refer to this observation as the difference facts.

Perhaps the most influential empirical evidence in favor of a low value of the IES is provided by Hall (1988). He estimates the IES using U.S. data on aggregate consumption and interest rates. The resulting estimates of the IES are close to zero and sometimes even negative. Hall (1988) concludes that the value of the IES is probably 0.2 or lower. Low estimated values of the IES have been documented in more recent research as well. Campbell (2003) finds evidence of a low IES in a variety of countries and Yogo (2004) estimates the IES to be 0.2 for the U.S. using an estimation strategy that controls for weak instruments.[1]

This paper makes two contributions. First, we provide two distinct ways to reconcile a low value of the true IES with the empirical evidence that it is large. Our resolutions arise in a model in which all agents have identical preferences and the same access to asset markets. These results do not necessarily imply that the IES is low. The model can also account for the same macroeconomic evidence with a high IES. This motivates our second contribution

[1] Yogo's (2004) estimates of the IES are 0.5 or less in the ten other countries he considers.

which is to provide empirical evidence that the true value of the IES is low. We make this case by confronting respectively the maintained hypothesis of a high value of the IES and the maintained hypothesis of a low IES with the regression-based evidence that the IES is low. An encompassing test indicates that the maintained hypothesis of a low IES (0.35 or lower) is consistent with the regression based evidence but that the maintained hypothesis of a large value of the IES of e.g. 1.5 is inconsistent with this evidence.

Agents in our model have Epstein and Zin (1989) (see also Weil (1990)) preferences and growth is endogenous. Epstein-Zin preferences are convenient because they allow us to make a meaningful distinction between intertemporal substitution and risk aversion. One parameter determines the IES or the desired response of expected consumption growth to an increase in the expected interest rate. A second parameter governs risk aversion: the change in sure consumption that renders a household indifferent to a small wealth bet. These preferences also nest, as a special case, time additive preferences over consumption of the form: $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$. Our specification of endogenous AK growth in conjunction with an i.i.d. shock structure makes it possible to get nearly closed form solutions.[2]

We consider two versions of the model one with complete markets and the other with incomplete markets for human capital.

Our model with complete markets has the property that the IES must be large in order to account for the levels facts if the preference discount factor is restricted to be less than one. This restriction, though is not necessary for existence of an equilibrium in our model (see also Abel (1999) and Kocherlakota (1990)). In our model equilibrium requires instead that the effective preference discount factor be less than one.[3] Moreover, for reasons described by Reis (2009) data on consumption and real returns does not identify the preference discount factor but instead identifies the *effective* preference discount factor which depends on both the IES and the preference discount factor. Under the assumption of complete markets our model is consistent with data on average consumption growth and real returns with either a large IES in conjunction with a low value of the preference discount factor or alternatively a very low value of the IES with a preference discount factor that is greater than one.

We go on to show that the complete markets model is also consistent with international evidence on differences in the return on capital described by Lucas and documented here when the value of the IES as low as about 0.5.

These findings are robust in the sense that they hold for a very wide range of settings of the risk aversion parameter. The model produces the same results when agents are risk neutral or alternatively highly risk averse with risk aversion coefficients of 100.

The assumption of i.i.d. shocks is convenient because it allows us to solve the model

[2]Epaulard and Pommeret (2003) use this same strategy to solve a complete markets AK model with Epstein-Zin preferences.

[3]We formally define the effective preference discount factor below. In words though it determines where expected present value utility is summable in a growing economy.

exactly and provide a simple characterization of the equilibrium. However, the i.i.d. assumption renders the model inconsistent with the observed persistence in output at business cycle frequencies. We next relax this assumption and use perturbation methods to solve the model with persistent shocks. The results are virtually identical to the i.i.d case. Allowing for persistent shocks does improve the model's implications for the business cycle. Simulated data from the model exhibit a similar degree of persistence to what we see in U.S. data. Interestingly, the success of the model in accounting for the business cycle is robust to the value of the IES. Our model reproduces volatility and persistence properties of the U.S. data equally well with an IES of 0.5 or 1.5.

We next show that regression based estimates of the IES can be used to discriminate between the maintained hypothesis of a large IES and the maintained hypothesis of a low IES. Campbell (2003) and Yogo (2004) find that the estimated coefficient in a two stage least squares (2SLS) regression of consumption growth and the interest rate is low regardless of which variable is treated as the dependent variable. Yogo (2004) argues that this is a problem due to weak instruments. We these facts to conduct an encompassing test. When we posit low values of the IES, simulated data from our model successfully reproduces these empirical results for either regression. However, when we posit a large value of the IES instead regressions using data generated from the model estimate the IES to be close to its true (large) value. This result arises even when the sample size is small. Under the maintained hypothesis of a large IES the response of consumption to interest rate movements is so large that there is no weak instrument problem. 2SLS estimates of the IES are large regardless of which variable appears on the left hand side of the regression.

We also consider a market structure where households face uninsurable idiosyncratic shocks to their human capital. Human wealth is harder to collateralize than financial wealth and the return on human wealth is typically estimated to be higher than the return on financial wealth. The IES measures the response of expected consumption growth to a change in the expected return on *total wealth* of an individual. Under this form of market incompleteness, a distinction arises between the return on physical capital, human capital and total wealth. This distinction has some important implications. As the variance of the idiosyncratic shock is increased from zero, the return on total wealth also rises and this acts to lower the effective discount factor. A higher return on total wealth then makes it possible to reconcile a low IES with both the levels facts and the difference facts while at the same time maintaining a value of the preference discount factor that is less than one.

The incomplete markets model imposes stronger restrictions on the range of values of both the IES and the RRA coefficients and this evidence suggests that the magnitude of the IES is much less than one. Values of the IES above 0.35 imply that the return on human capital is too low and requires households to be nearly risk neutral if one is to reproduce the measured variability in labor income. And a value of the IES much below 0.2 implies

that the return on human capital is implausibly large. The incomplete market specification produces the best empirical fit with an IES of between 0.25 and 0.3 and a RRA coefficient of about 2.

Our work is most closely related to research by Guvenen (2006) who provides an alternative reconciliation of a low IES with the aggregate evidence of a high IES. He considers an incomplete markets model and posits two types of individuals: a small group with a high IES can participate in asset markets, and a second much larger group with a low IES are not allowed to purchase equity. The high IES individuals determine the return on assets but only constitute a small fraction of total consumption.

Our resolutions differ from that of Guvenen (2006) in that all households in our economy have identical preferences and the same access to financial markets. The business cycle properties of our complete markets model with persistent shocks are also similar to those reported in Guvenen (2006). However, the models we consider here do not have very rich implications for the cross-sectional distribution of consumption or wealth. The assumption of unit root idiosyncratic shocks implies that both cross-sectional distributions fan out over time.

2 The model

We consider an endogenous growth model with aggregate and idiosyncratic risk. The specific model of risk and growth follows Krebs (2003). However, we generalize his preference structure and consider the recursive utility function proposed by Epstein and Zin (1989) and Weil (1989).

2.1 The Model

Consider an economy with a single final good, y_t, that is produced by perfectly competitive firms with a Cobb-Douglas production technology using physical and human capital as inputs.

$$y_t = A_t k_{t-1}^{\alpha} h_{t-1}^{1-\alpha} \tag{1}$$

where k_{t-1} and h_{t-1} denote the aggregate stock of physical and human capital at the beginning of period t, respectively, and A_t is an exogenous productivity shock in period t.

The final good, y_t, can either be consumed or invested in either type of capital:

$$y_t = c_t + i_{k,t} + i_{h,t} \tag{2}$$

where $i_{k,t}$ and $i_{h,t}$ denote, respectively, investment in physical and human capital in period t.

The aggregate stocks of physical and human capital evolve according to

$$k_t = i_{k,t} + (1 - \delta_{k,t})k_{t-1} \tag{3}$$

$$h_t = i_{h,t} + (1 - \delta_{h,t})h_{t-1} \tag{4}$$

where $\delta_{k,t}$ and $\delta_{h,t}$ are exogenous, stochastic rates of depreciation of physical and human capital. We will use $k_{-1}, h_{-1} > 0$ to denote the initial stocks of physical and human capital.

The economy is inhabited by a large number of households with identical preferences. The period t utility function, u_t is given by

$$u_t = \left\{ c_{i,t}^{1-\frac{1}{\psi}} + \beta \left(E_t \left[u_{t+1}^{1-\gamma} \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\psi}}} \tag{5}$$

where $c_{i,t}$ is the amount of the good consumed in period t by individual i. Here, ψ measures the intertemporal elasticity of substitution and γ measures the degree of relative risk aversion. The constant relative risk aversion utility function is obtained as a special case when $\psi = 1/\gamma$. The flow budget constraint for individual i is:

$$c_{i,t} + k_{i,t} + h_{i,t} = R_{k,t}k_{i,t-1} + R_{h,i,t}h_{i,t-1} \tag{6}$$

and the returns on each type of asset are given by:

$$R_{k,t} = 1 + r_{k,t} - \delta_{k,t}$$

$$R_{h,i,t} = 1 + r_{h,t} - \delta_{h,t} + \eta_{i,t}$$

An important distinction between the two types of capital is that human capital is subject to an idiosyncratic uninsurable shock, $\eta_{i,t}$. An individual's situation in period t is summarized by a realization of the aggregate state $S_t \in S$, and a realization of the individual specific state $s_{it} \in s$. Shocks to productivity and depreciation are assumed to depend only on the aggregate state: $A_t = A(S_t)$, $\delta_{k,t} = \delta_k(S_t)$, $\delta_{h,t} = \delta_h(S_t)$. The idiosyncratic shock to the return on human capital depends on the individual specific state: $\eta_{i,t} = \eta(s_{i,t})$.

In order to retain analytical tractability, we assume for now that S_t is identically and independently distributed (i.i.d.) over time with probability distribution $\pi(S_t)$, $S_t \in S$ and that s_{it} is independently distributed across individuals and over time with probability distribution $\pi(s_{i,t})$, $s_{it} \in s$.

2.2 Competitive equilibrium

We describe the equilibrium for this economy in three steps. First, we characterize the solution to the household's problem under the assumption that returns to each asset are i.i.d. Second, we derive restrictions on factor inputs from firm optimization. Third, we combine the restrictions of household and firm optimization and characterize the competitive equilibrium.

Note that the household's budget constraint can be expressed as:

$$c_{i,t} + k_{i,t} + h_{i,t} = a_{i,t}$$
$$a_{i,t+1} = R_{k,t+1}k_{i,t} + R_{h,i,t+1}h_{i,t}$$

Next let $\omega_{k,t}$ denote the share of physical capital in total capital:

$$\omega_{k,t} \equiv \frac{k_t}{k_t + h_t}.$$

Then we can rewrite the household budget constraint as:

$$a_{i,t+1} = (a_{i,t} - c_{i,t}) \left[R_{k,t+1}\omega_{k,i,t} + R_{h,i,t+1}(1 - \omega_{k,i,t})\right]. \tag{7}$$

Under the assumption that the returns on each type of capital are i.i.d. the individual's state can be summarized by $a_{i,t}$ and the optimization problem for an individual has the following recursive representation:

$$v(a_i) = \max_{c_i, a_i', \omega_{k,i}} \left\{ c_i^{1-\frac{1}{\psi}} + \beta \left(E\left[v(a_i')^{1-\gamma}\right]\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\psi}}} \tag{8}$$
$$\text{s.t.} \quad a_i' = (a_i - c_i)\left\{R_k'\omega_{k,i} + R_{h,i}'(1 - \omega_{k,i})\right\}$$

with $a_i', c_i \geq 0$ and $\omega_{k,i} \in (0, 1)$. In the above expression E is the expectation operator.

A characterization of the solution to the household's problem is given in the following lemma.

Lemma 1. *Suppose that returns are i.i.d. and*

$$\beta^\psi \rho^{\psi-1} < 1 \tag{9}$$

where

$$\rho \equiv \max_{w_{k,i}} \left(E\left[\left\{R_k' w_{k,i} + R_{h,i}'(1 - w_{k,i})\right\}^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}} \tag{10}$$

then the solution to the household's problem (8) has the following properties:

1. *The optimal portfolio weight for each individual is constant over time and satisfies $\omega_{k,i,t} = \omega_k$ for all i and t, where*

$$\omega_k \equiv \arg\max_{w_k} \left(E\left[\left\{R_k' w_k + R_{h,i}'(1 - w_k)\right\}^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}. \tag{11}$$

2. *The optimal consumption, $c_{i,t}$, is linear in wealth and given by:*

$$c_{i,t} = \omega_c a_{i,t} \tag{12}$$

where

$$\omega_c \equiv 1 - \beta^\psi \rho^{\psi-1} \tag{13}$$

3. The value function, $v(a_i)$ is also linear in wealth and given by:

$$v(a_i) = \bar{v} a_i \tag{14}$$

where

$$\bar{v} \equiv \omega_c^{\frac{1}{1-\psi}} \tag{15}$$

Proof. We start by conjecturing that the value function is linear in assets $v(a_i) = \bar{v} a_i$ and then verify that this conjecture is correct. Using the conjectured form of the value function and the household budget constraint we have:

$$\left(E\left[v(a_i')^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}} = \bar{v}(a_i - c_i)\left(E\left[\left\{R_k'\omega_k + R_{h,i}'(1-\omega_k)\right\}^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}} \tag{16}$$

Then observe that ω_k will not depend on i, since η_i' is identically distributed across agents nor will it depend on R_k' or $R_{h,i}'$ since asset returns are also i.i.d. by assumption. It follows that the optimal portfolio weight is identical across individuals and given by (11). [4]

Next substitute (10) and (16) into (8) to get:

$$\bar{v} a_i = \max_{c_i} \left\{ c_i^{1-\frac{1}{\psi}} + \beta \bar{v}^{1-\frac{1}{\psi}} (a_i - c_i)^{1-\frac{1}{\psi}} \rho^{1-\frac{1}{\psi}} \right\}^{\frac{1}{1-\frac{1}{\psi}}} \tag{17}$$

Taking derivatives with respect to c_i yields:

$$c_i^{-\frac{1}{\psi}} = \beta \bar{v}^{1-\frac{1}{\psi}} \rho^{1-\frac{1}{\psi}} (a_i - c_i)^{-\frac{1}{\psi}} \tag{18}$$

or

$$c_i = \left(1 + \beta^{\psi} \rho^{\psi-1} \bar{v}^{\psi-1}\right)^{-1} a_i \tag{19}$$

Using equations (17) and (19) we can now express c_i and $\bar{v}$ in terms of a_i. The resulting expressions are given in equations (12) and (15). □

From Lemma 1 it can be seen that the inequality in equation (9) implies that $\omega_c \in (0, 1)$. We will refer to $\beta^{\psi}\rho^{\psi-1}$ as the effective discount factor. Restricting the effective discount factor to be less than one insures that the expected present value of utility is bounded. We wish to emphasize that this inequality can be satisfied when $\beta > 1$.[5] Below we will report some parameterizations of the model that have the property that the effective discount factor is less than one but the preference discount factor exceeds one.

[4] Although we have not explicitly included a risk free asset here, the fact that the portfolio weight is identical across all individuals implies the portfolio weight on the risk-free asset will be zero in equilibrium. see Constantinides and Duffie (1992) and Heathcote, Storesletten and Violante (2008) for a similar result.

[5] See Kocherlakota (1990) for further discussion of this point.

Note next that firm optimization implies that inputs are chosen to satisfy:

$$r_{k,t} = \alpha A_t k_{t-1}^{\alpha-1} h_{t-1}^{1-\alpha} \tag{20}$$
$$r_{h,t} = (1-\alpha) A_t k_{t-1}^{\alpha} h_{t-1}^{-\alpha}.$$

Define the return on wealth for individual i as:

$$R'_{a,i} \equiv R'_k \omega_k + R'_{h,i}(1-\omega_k) \tag{21}$$

Then the first order condition for the portfolio allocation problem is given by:

$$E_t \left[R_a'^{-\gamma} \left(R'_{k,} - R'_{h,i} \right) \right] = 0. \tag{22}$$

Next use the definition of ω_k to express the real returns on capital and human capital as:

$$R'_k = \alpha A' \omega_k^{\alpha-1} (1-\omega_k)^{1-\alpha} + 1 - \delta'_k \tag{23}$$

$$R'_{h,i} = (1-\alpha) A' \omega_k^{\alpha} (1-\omega_k)^{-\alpha} + 1 - \delta'_h + \eta'_i \tag{24}$$

Now if (23) and (24) are substituted into equation (22) this optimality condition can be expressed in terms of the single choice variable ω_k. Using the optimal choice of ω_k, ρ can now be expressed as:

$$\rho = \left(E_t \left[R_{a,t+1}^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}} \tag{25}$$

Given solutions to the individual's problem and firm's problem we can now characterize the competitive equilibrium allocations.

Proposition 1. *Let $R_{a,t+1}$, ω_k, and ρ be given by equations (21), (22) , and (25) and assume that the effective discount factor satisfies (9). Then the competitive equilibrium allocations are:*

$$c_{i,t} = \omega_c a_{i,t} \tag{26}$$
$$k_{i,t} = (1-\omega_c)\omega_k a_{i,t} \tag{27}$$
$$h_{i,t} = (1-\omega_c)(1-\omega_k) a_{i,t} \tag{28}$$
$$u_{i,t} = \bar{v} a_{i,t} \tag{29}$$
$$a_{i,t+1} = (1-\omega_c) R_{a,i,t+1} a_{i,t} \tag{30}$$

Using these allocations it is straightforward to derive implications for other moments of interest. Express $R_{a,i,t+1}$ as

$$R_{a,i,t+1} = \bar{R}_{a,t+1} + (1-\omega_k)\eta_{i,t+1} \tag{31}$$

where $\bar{R}_{a,t+1}$ is the cross-sectional mean of $R_{a,i,t+1}$:

$$\bar{R}_{a,t+1} \equiv A_{t+1}\omega_k^{\alpha}(1-\omega_k)^{1-\alpha} + 1 - \delta_{h,t+1} + \omega_k(\delta_{h,t+1} - \delta_{k,t+1}). \tag{32}$$

Then we have

$$E_t[R_{a,i,t+1}] = E_t[\bar{R}_{a,t+1}] \tag{33}$$

Individual wealth follows

$$a_{i,t+1} = R_{a,i,t+1}(1-\omega_c)a_{i,t} \tag{34}$$

and thus the growth rate of individual i's wealth satisfies

$$E_t\left[\frac{a_{i,t+1}}{a_{i,t}}\right] = (1-\omega_c)E_t[R_{a,i,t+1}]. \tag{35}$$

Define a_{t+1} to be aggregate per capita wealth. Then

$$a_{t+1} = (1-\omega_c)\bar{R}_{a,t+1}a_t \tag{36}$$

and it follows that the growth rate of aggregate wealth satisfies:

$$E_t\left[\frac{a_{t+1}}{a_t}\right] = (1-\omega_c)E_t[\bar{R}_{a,t+1}] = E_t\left[\frac{a_{i,t+1}}{a_{i,t}}\right] \tag{37}$$

We can derive implications for the expected growth rate of aggregate per capita consumption in a similar way.

The growth rate of individual i's consumption satisfies:

$$\frac{c_{i,t+1}}{c_{i,t}} = \frac{a_{i,t+1}}{a_{i,t}} = (1-\omega_c)R_{a,i,t+1} \tag{38}$$

And the growth rate of aggregate per capita consumption satisfies:

$$\frac{c_{t+1}}{c_t} = (1-\omega_c)\bar{R}_{a,t+1} \tag{39}$$

Taken together we get

$$E_t\left[\frac{c_{t+1}}{c_t}\right] = E_t\left[\frac{c_{i,t+1}}{c_{i,t}}\right] \tag{40}$$

or that the expected growth rate of aggregate consumption and individual consumption are the same. Using these results we can also derive expressions for the variance of individual and aggregate consumption. They are given by:

$$\operatorname{var}\left[\frac{c_{i,t+1}}{c_{i,t}}\right] = (1-\omega_c)^2 \operatorname{var}[R_{a,i,t+1}] \tag{41}$$

$$\operatorname{var}\left[\frac{c_{t+1}}{c_t}\right] = (1-\omega_c)^2 \operatorname{var}[\bar{R}_{a,t+1}] \tag{42}$$

The expected growth rate of aggregate per capita output can be derived in the following way. Recall that

$$k_t = (1-\omega_c)\omega_k a_t,$$
$$h_t = (1-\omega_c)(1-\omega_k)a_t$$

It follows that aggregate per capita output is given by:

$$y_t = A_t \omega_k^{\alpha}(1-\omega_k)^{1-\alpha}(1-\omega_c)a_{t-1} \tag{43}$$

and that the growth rate of aggregate output is

$$\frac{y_{t+1}}{y_t} = \frac{A_{t+1}}{A_t}(1-\omega_c)\bar{R}_{a,t+1} \tag{44}$$

Although the risk free asset is not held in equilibrium it is straightforward to derive an expression for it by resolving the asset allocation problem and allowing for a third asset. This yields the following first order condition:

$$E_t\left[(R'_{a,i})^{-\gamma}(R'_{a,i} - R_f)\right] = 0. \tag{45}$$

Solving for R_f yields:

$$R_f = \frac{E_t\left[(R'_{a,i})^{1-\gamma}\right]}{E_t\left[(R'_{a,i})^{-\gamma}\right]} \tag{46}$$

3 Empirical Results

In this section we describe how we assign parameters to the model, explain how we reconcile a low value of the IES in the model with some of the empirical evidence that it is large and provide some new empirical evidence that the value of the IES is low.

3.1 Model Parameterization

We will report simulation results for two market structures. The complete market structure assumes that households can insure their idiosyncratic shocks to human capital. The allocations and prices for this market structure are found by setting $\eta_{i,t} = 0$ for all i and all t. The remainder of the stochastic specification for the complete markets case is as follows.

The aggregate state has two outcomes $S_t \in \{1, 2\}$. We assume that $\Pr(S_t = 1) = \Pr(S_t = 2) = \frac{1}{2}$. For convenience aggregate shocks to each type of depreciation are assumed to be perfectly correlated so that $\delta_{k,t} = \delta_{h,t} = \delta_t$, for all t. Under these assumptions human and physical capital have identical risk properties, earn the same real return and thus the model reduces to an AK setup.

The productivity and depreciation shocks are parameterized in the following way:

$$A(S_t) = \begin{cases} E(A) - \sigma(A), & \text{if } S_t = 1 \\ E(A) + \sigma(A), & \text{if } S_t = 2 \end{cases}$$

$$\delta(S_t) = \begin{cases} E(\delta) - \sigma(\delta), & \text{if } S_t = 1 \\ E(\delta) + \sigma(\delta), & \text{if } S_t = 2 \end{cases}$$

The incomplete market structure maintains the same structure for the aggregate shocks. The idiosyncratic shock $\eta_{i,t}$ though now has two states $s_{i,t} = \{1, 2\}$ with $\Pr(s_{i,t} = 1) = \Pr(s_{i,t} = 2) = \frac{1}{2}$.

We want to illustrate that a low value of the IES in the model is consistent with empirical evidence used to assert that it is large. We choose a calibration strategy that allows us to reproduce some basic facts from the data as we vary the IES and the relative risk aversion parameters. We set the capital share parameter $\alpha = 0.3$ and set the average depreciation rate $E(\delta) = 0.08$. The remaining parameters $\{E(A), \sigma(A), \sigma(\delta), \beta\}$ are set to match the following targets: an investment share of output of 0.25; output growth of 2 percent per annum; a standard deviation of output growth of 0.0237; and a standard deviation of consumption growth of 0.0168.[6]

Our calibration strategy has the property that the *effective* discount factor is held fixed as ψ and γ are varied. It also implies that the value of β changes as we alter γ and ψ. We view this as an attractive way to proceed because it makes it possible to disentangle the effects of a change in ψ on discounting from its effects on intertemporal substitution. If we were to fix β instead then a change in ψ would affect both the effective discount rate and also the IES.

The incomplete markets model is calibrated to reproduce the same facts as the complete markets model and the standard deviation of the idiosyncratic shock to human capital is set to 0.18.[7]

3.2 The facts to be explained

We consider two types of facts that have led macroeconomists to conclude that the IES is large. Campbell (2003) and Guvenen (2006) among others have pointed out that it is hard to reconcile a low value of the IES with data on the average *level* of per capita consumption growth and the average *level* of real interest rates. We refer to these observations as the levels facts.

Lucas (1990) argues that it is difficult to reconcile a low IES with international evidence on cross-country *differences* in average output growth rates and cross-country differences in the return on capital. We refer to these observations as the difference facts.

[6] The choices for the standard deviation of output and consumption growth are taken from Krebs (2003).
[7] This value is also taken from Krebs (2003).

We now consider each type of evidence in turn.

3.2.1 Levels Facts

One can derive a lower bound on the size of the IES using data on the average *level* of the growth rate of consumption and the average *level* of a safe short-tem bond by appealing to the intertemporal first order condition of an individual.

Consider the Euler equation for the risk-free return in our economy:

$$1 = E_t \left[\beta^\theta \left(\frac{c_{i,t+1}}{c_{i,t}} \right)^{-\frac{\theta}{\psi}} R_{a,i,t+1}^{\theta-1} R_{f,t+1} \right]$$

where $\theta = \frac{1-\gamma}{1-1/\psi}$ and $R_{f,t+1}$ is the risk-free rate on a discount bond that pays off in period $t+1$.[8] Next factor out the risk-free rate to get:

$$1 = R_{f,t+1} \beta^\theta E_t \left[\left(\frac{c_{i,t+1}}{c_{i,t}} \right)^{-\frac{\theta}{\psi}} R_{a,i,t+1}^{\theta-1} \right] \tag{47}$$

Now take a second order approximation of (47) and solve for $R_{f,t+1}$:[9]

$$\begin{aligned} \ln(R_{f,t+1}) = & -\ln(\beta) + \frac{1}{\psi} E_t \left[\ln \left(\frac{c_{i,t+1}}{c_{i,t}} \right) \right] \\ & - \frac{\theta}{2\psi^2} \operatorname{var}_t \left[\ln \frac{c_{i,t+1}}{c_{i,t}} \right] - \frac{1-\theta}{2} \operatorname{var}_t \left[\ln(R_{a,i,t+1}) \right] \end{aligned} \tag{48}$$

Finally, assume perfect foresight: drop expectations; assume all agents are identical; and set $R_{a,t+1} = R_{f,t+1}$ to get:

$$\ln(R_{f,t+1}) = -\ln(\beta) + \frac{1}{\psi} \ln \left(\frac{c_{t+1}}{c_t} \right) \tag{49}$$

This equation has been used by Weil (1989) and Campbell (2003) to document the risk-free rate puzzle. It has also been used by Guvenen (2006) to derive a lower bound on the IES. To illustrate how this is done, following Guvenen (2006), suppose that consumption grows at the rate of 2 percent per annum, that the risk-free interest rate is 3 percent and impose an upper bound of one on β. If $\beta = 1$ equation (49) yields a value of $\psi = 0.66$. Note that this is a lower bound on the value of the IES in the sense that if $\beta < 1$ instead the resulting value of ψ is larger.

We wish to point out though that the choice of an upper bound of one for β is arbitrary. Under the assumption of perfect foresight the restriction on β that is required for existence of an equilibrium in our model is $\beta < R_f^{1/\psi - 1}$. What this means is that if we posit a value of $\psi < 1$ there will be an interval of values of $\beta > 1$ that are consistent with equilibrium.

[8] This condition also holds if the i.i.d. assumption is relaxed.

[9] See e.g. Campbell (2003) for this derivation.

3.2.2 Difference Facts

A second simple and persuasive argument for why the IES is close to one is provided by Lucas (1990) when justifying his calibration of the relative risk aversion coefficient for the following utility function: $u(c) = \frac{c_t^{1-\sigma}}{1-\sigma}$:

> If two countries have consumption growth rates differing by one percentage point, their interest rates must differ by σ percentage points (assuming similar time discount factors). A value of σ as high as 4 would thus produce cross-country interest differentials much higher than anything we observe, and from this viewpoint even $\sigma = 2$ seems high.

Using our notation $\gamma = \sigma = 1/\psi$ and Lucas' observation implies that a value of the IES of even 1/2 is low. In order to add more empirical content to his observation we report international differences in rates of return on capital in Table 1. The data are based on capital output ratios reported in Prescott (2002). To infer a return on capital we assume a capital share of 0.3 for all countries and a depreciation rate of 0.08. These are the same values of these two parameters that we use in calibrating our model. We are interested in comparing steady-states so we limit attention to advanced economies. The results reported in Table 1 show that the maximum difference in returns on capital for these countries is 2.5 percentage points.

The second part of Lucas' reasoning pertains to output variability. Table 1 also reports average growth rates of output for the same countries. This data is taken from Maddison's webpage (http://www.ggdc.net/maddison) and is the sample average over a sample period that extends from 1980 to 2006. The main thing to note about this data is that the range of output growth differences for these countries is 1.2 percent which is very close to the figure of one percent posited by Lucas.

3.3 Reconciling a low IES with evidence that the IES is high

We first report results for the complete markets version of our model and then turn to discuss how the answer changes when households face uninsured idiosyncratic risk.

3.3.1 Complete Markets

Table 2 reports the return on capital and the risk free rate implied by the complete markets specification. How well does this specification do in accounting for the level facts? The growth rate of consumption is 2 percent for each configuration of preference parameters which is the same value used when we discussed the levels facts above. The range of values for the risk free return reported in Table 2 are also close to the value of 3 percent we posited above. The risk free return in Table 2 ranges from a high of 4 percent when $\gamma = 2$ to a low

of 2.4 percent when $\gamma = 100$. Following the steps described in Section 3.2.1, if we substitute the average value of consumption growth and the average value of the risk free return from the simulations into equation (49) and set $\beta = 1$, the resulting lower bound on the IES ranges from 0.51 when $\gamma = 2$ to 0.83 when $\gamma = 100$.

Interestingly, these imputed lower bounds are sometimes inconsistent with the true value of the IES used when simulating the model. Consider for instance, the case where $\gamma = 2$. For that choice of the relative risk aversion parameter the true value of ψ of 1.5 is well above the lower bound of 0.51. However, the true values of ψ of 0.35 and 0.1 lie below this imputed lower bound.

Why is this derivation failing? The reason why the perfect foresight lower bound fails is due to the maintained assumption when computing it that the maximum size of the preference discount factor is one. Our calibration strategy yields values of the discount factor that are larger than one. Consider row 5 of Table 2 which reports the calibrated value of the preference discount factor. In virtually all cases β exceeds one when ψ falls below the perfect foresight lower bound.[10]

One might wonder whether it is reasonable to allow β to exceed one. In the context of the complete market specification there is a clear answer to this question. This specification can only reproduce investment's share of output and the average growth rate of consumption if the effective discount factor is held fixed. There are, also some other good reasons for calibrating the effective discount factor and allowing the preference discount factor to adjust.[11] First, as we pointed out above existence of equilibrium requires the effective preference discount factor to be less than one. There is no such restriction on β. It can either be less than one or exceed one. Second, for Epstein-Zin preferences a change in ψ has two effects. It changes the IES and also changes the way individuals discount future utility. By allowing β to adjust we can control for this second effect and isolate the effects of a change in ψ on the IES. Third, in our complete market specification there is a direct way to measure the effective discount factor but no direct way to measure β. Under perfect foresight the effective discount factor in our model can be expressed as:

$$-\ln(\beta_{eff}) = \ln(R_f) - \ln(g_c) \tag{50}$$

where $\beta_{eff} \equiv \beta^{\psi}\rho^{\psi-1}$ and consumption growth, g_c, is constant. It follows that the effective discount factor is an exact function of consumption growth and the risk free rate under the assumption of perfect foresight. Table 2 illustrates that perfect foresight is a reasonable benchmark for moderate values of γ. Consider the final row of Table 2. This row reports

[10]The sole exception occurs when $\gamma = 100$. For that parameterization the true IES is below the perfect foresight lower bound and $\beta < 1$. A value of risk aversion of this magnitude might appear to be very large. There is a sense though in which it is too small. The equity premium produced by the model with $\gamma = 100$ is only 1.6 percent. In U.S. data the average value of the equity premium is above 6 percent.

[11]The arguments we are about to provide are also made in Reis (2009).

the value of β that emerges if one uses (49) to solve for β using the value of ψ and R_f in the corresponding column along with a consumption growth rate of 2 percent. When risk aversion is two or four the value of β in the final row of Table 2 is close to its true value. The perfect foresight assumption breaks down though when risk aversion is large and ψ is small. For instance, when $\psi = 0.1$ the value of β implied by (49) is 1.19 while its true value is 1.070.

A final noteworthy feature of the complete market specification is that ψ has no effect on the size of the risk free return. The reason for this is that under complete markets $\delta_k = \delta_h = \delta$ and the return on total assets in (21) becomes:

$$R'_a = A'\alpha^{\alpha}(1-\alpha)^{1-\alpha} + (1-\delta')$$

It then follows from (46) that the value of the risk free return is independent of ψ.

Next we consider facts related to Lucas' observations on international return differentials. To construct return differentials associated with a one percentage permanent change in consumption growth we alter the average level of productivity, A, to induce a 1 percent change in the growth rate of consumption. Using this new stochastic steady-state we then calculate the expected return on physical capital.

Table 3 reports results that pertain to the difference facts. We report the simulated return differentials for capital and the risk-free return associated with a one percentage permanent change in consumption growth for alternative configurations of the preference parameters. The most striking feature of Table 3 is that ψ, the IES, is the key preference parameter that matters for the return on assets across steady-states. For values of ψ that are less than one, the risk aversion coefficient γ has a tiny effect on asset returns. For instance, when $\psi = 0.25$ the change in the return on physical capital across steady-states is 4.2 percent when γ is either 2 or 4 and 4.3 percent when $\gamma = 100$. The value of γ has a somewhat larger but still small effect on the returns when $\psi = 1.5$.

The evidence we provided above suggested that the difference in the return on capital associated with a permanent change in consumption growth of one percent is about 2.5 percentage points or less. From Table 3 we can see that a value of ψ of 0.35 yields a 3 percentage change in the return on capital. For purposes of comparison if $\psi = 0.5$ instead the change in the return in capital falls to 2 percent. Thus, the smallest value of ψ that can be rendered consistent with the difference facts lies somewhere between 0.35 and 0.5.

Up to this point we have assumed that the shocks are i.i.d. This assumption is very convenient because it is easy to solve and analyze the model. Moreover, the solution is exact. The only equilibrium object that cannot be solve for analytically is ω_k. It is computed numerically using the exact nonlinear equilibrium condition (10). However, the assumption of i.i.d. shocks limits the applicability of this model to business cycle analysis. It is also possible that allowing for persistent shocks might upset some of our findings. To investigate

these possibilities we now relax the assumption of i.i.d shocks and allow the shock process for the production function A_t to follow an A.R. 1 instead. We use a second order perturbation method to solve the model. We also relax the assumption that the support of the shocks is discrete and allow them to have continuous support. The model is calibrated in the same way as before.[12]

We set the persistence parameter on technology to 0.8 and recalibrate the model to match the same targets as before. Recall that our calibration strategy insures that the model can reproduce both the volatility of consumption growth and the volatility of output growth. The relative volatility of investment to output for this specification is 1.5. We repeated the same experiment reported in Table 3 to see whether allowing for persistence affects our previous conclusions. With persistent shocks to technology the percentage difference in the real return on capital associated with a reduction in the growth rate of consumption of 1 percent is 2.94 percent as compared to 2.96 percent for the case of i.i.d. shocks (see the column of Table 3 with $\gamma = 4$ and $\psi = 0.35$). The corresponding change in the risk-free return is now 2.93 percent as compared to 2.96 percent in Table 3. The results are very similar when we set $\psi = 1.5$ and $\gamma = 4$ instead. Overall, introducing persistence has a negligible effect on the conclusions we drew assuming i.i.d. shocks.

Allowing for persistent shocks to A significantly enhances the business cycle properties for our model. To illustrate this point we Hodrick-Prescott filtered log-level simulated data for consumption, output and investment using a smoothing parameter of 100. When $\psi = 0.35$ and $\gamma = 4$ the first order serial correlation of consumption, output and investment are respectively, 0.49, 0.45 and 0.4. For purposes of comparison Guvenen (2006) reports HP filtered serial correlations statistics for U.S. data. They are 0.57 for consumption, 0.52 for output and 0.36 for investment. If we set $\psi = 1.5$ and $\gamma = 4$ instead, then the model implies that the first order serial correlations of consumption, output and investment are virtually unchanged: 0.5 for consumption, 0.45 for output and 0.4 for investment. Overall, the model with persistent shocks to A captures the general pattern and the magnitude of persistence in these variables.

Another noteworthy property of the model is that the size of the IES is irrelevant for the volatility and persistence of output, consumption and investment. The business cycle performance of our model along these dimensions does not change as the value of the IES is varied from 0.35 to 1.5.

[12] As a check on this solution method we first solve the model for the case of i.i.d. shocks and generate Monte Carlo draws of length 2500. This produces small differences as compared to our previous method which is nearly analytic. For instance, when the IES parameter, ψ, is 0.35 and the relative risk aversion parameter, γ, is set to 4, the return on capital is now 4.08 percent as compared to a value of 4.00 percent reported in Table 2. The value of the risk-free return is now 3.77 percent as compared to the value of 3.89 percent reported in Table 2 and the difference in the calibrated value of the preference discount factor across the two solution methods is 0.0006.

The evidence we have presented so far opens the door to the possibility that the IES is low but does not rule out the possibility of a large IES. On the one hand, the we can account for the level and difference facts with a value of the IES that is less than 1/2 in conjunction with a value of $\beta > 1$. On the other hand, the same facts can be accounted for with a value of the IES that is greater than one and a value of $\beta < 1$. We now turn to describe some evidence that can be used to discriminate among these two possibilities.

Campbell (2003) and Yogo (2004) estimate the IES using 2SLS regressions of consumption growth and risk free interest rates of the following form:

$$\ln\left(\frac{c_{t+1}}{c_t}\right) = \mu + \psi \ln(R_{f,t+1}) + \epsilon_{t+1}. \tag{51}$$

where ϵ_{t+1} is an innovation that is orthogonal to information in period t. [13]

They find that when consumption growth is the dependent variable the estimate of the IES is low. However, when the interest rate appears on the left hand side instead, the implied value of the IES is large. They also find that it is not unusual for the estimated coefficient for the IES to be negative. A third result that emerges from their work is that the standard errors are smaller when consumption growth is the dependent variable. Yogo (2004) argues that these findings reflect a weak instruments problem and that this problem is particularly severe when the interest rate is the dependent variable.

We use these results to conduct an encompassing test of the maintained hypothesis of a low value of the IES versus the maintained hypothesis of a large value of the IES. Results from this test are reported in Table 4. The top row of this Table reports the maintained value of ψ used to simulate data. We simulate data using the specification with serially correlated shocks to A. The regressions are estimated by 2SLS. In the first stage the explanatory variable is regressed on the first lag of consumption growth and the interest rate. Then in the second stage the predicted value from the first regression is used as the explanatory variable.

Observe that when the true value of ψ is small the resulting Monte Carlo results are consistent with the previous findings of Hall (1988), Campbell (2003) and Yogo (2004). The value of the second stage regression coefficient is small irrespective of the choice of the dependent variable. This result occurs for sample sizes of 60, 120 and 240. It is also interesting that for the smaller sample sizes that value of the IES is negative as is found when using actual data. Finally, observe that the standard errors are smaller when consumption growth is the dependent variable in the second stage regression. As noted by Yogo (2004) this specification has less severe weak instrument problems. This follows from the fact that consumption growth is nearly iid.[14]

The maintained hypothesis of a large IES, however, fails to reproduce the empirical

[13] This equation follows from (48).

[14] See Yogo (2004) for details.

evidence of small and even negative regression coefficients. There is enough information to identify the value of the IES in even the shortest sample period and the resulting 2SLS estimates are greater than one in either regression.

These results make intuitive sense. When the true IES is low consumption is not very responsive to movements in expected interest rates and it is not surprising that one would need a long sample of data to precisely estimate its value. However, when the true IES is large consumption is very responsive and one can readily identify the IES using even a relatively short data sample.

On the basis of this evidence we conclude that on net the maintained hypothesis of a low IES is more plausible than the alternative of a high IES. The low IES maintained hypothesis encompasses the empirical evidence in favor of a high IES (the levels and difference facts). It is also consistent with the empirical estimates reported in Hall (1998), Campbell (2003) and Yogo (2004). The high IES maintained hypothesis can also account for the empirical evidence of a high IES. However, it fails to encompass regression evidence that suggests that IES is low.

3.3.2 Incomplete Markets

Now we turn to discuss results for the incomplete markets economy. Allowing for uninsured risk in human capital allows us to reconcile low values of the IES with the levels and difference facts using parameterizations in which the preference discount factor is less than 1. The incomplete markets model also turns out to have some additional restrictions that prove helpful in identifying the value of the IES.

Table 5 reports results that pertain to the levels facts. Recall that our calibration implies that the growth rate of consumption is two percent. Observe next that the risk-free rate in Table 5 has about the same magnitudes as Table 2. It ranges from 3.8 percent to 3.9 percent. Specifications with $\psi = 0.15$ are just as successful in reproducing these levels facts as $\psi = 1.5$.

Suppose that we use (49) to derive a lower bound on ψ by setting $R_f = 1.039$, $g_c = 1.02$ and $\beta = 1$. The resulting lower bound on ψ is 0.52. Interestingly, the results reported in the final four columns of Table 5 all violate this lower bound. Under complete markets this occurred only when $\beta > 1$. Here, however, the value of β is less than one. Why is the lower bound being violated with specifications in which $\beta < 1$?

In the complete markets specification we found that the perfect foresight linear Euler equation did a good job of recovering the true value of β when γ was 2 or 4. The second from final row of Table 5 shows that the value of β implied by equation (49) is very far from its true value when markets are incomplete. The bias is large for all configurations of the preference parameters.

The source of the bias is due to the distinction between equations (48) and (49). There

are two distinctions between these two equations. The first distinction follows from the fact that the log of expected consumption growth and the expectation of log consumption growth are not the same. When markets are complete idiosyncratic risk is insured and the variance of individual consumption growth is small. This results in a small Jensen's inequality term. With incomplete markets the variance of individual consumption is much larger and the Jensen's inequality term cannot be ignored. A second distinction is that (48) has two variance terms. These two terms disappear under the assumption of perfect foresight. In our incomplete markets specification the two variance terms are also very large. To provide the reader with a sense of the magnitude of these factors consider, for example, the parameterization with $\psi = 0.15$ and $\gamma = 4$. For this parameterization the difference between $1/\psi E_t \ln(c_{i,t+1}/c_{i,t})$ and $1/\psi \ln(E_t c_{i,t+1}/c_{i,t})$ is -0.051 and the size of the two variance terms is -0.2. These differences between (48) and (49) are almost precisely big enough to account for the fact that $\beta < 1$ with incomplete markets. To see why this is so consider the final two rows of Table 5 which report the value of β implied by equations (49) and (48) respectively. In computing β we use the true value of ψ and various other statistics from each simulation. From these estimates we can see that there is a large bias associated with estimates of β based on (49). However, (48) works extremely well.

Table 5 has some other noteworthy features. One of them is that the value of the IES does not matter for asset returns. In our incomplete markets specification human capital is subject to uninsurable idiosyncratic risk. As the risk aversion coefficient γ is increased individuals will lower their allocation to human capital and in equilibrium the return on human capital will rise. Changing ψ, in contrast, continues to have no effect on the portfolio allocation decision (see (11)).

Comparing Table 5 with Table 2, reveals that the value of β in Table 5 is always lower than its corresponding value with complete markets. The pattern of movements is also different. In Table 2 the preference discount factor increases as ψ is lowered. However, with incomplete markets the value of β falls as ψ is lowered. To see why this is the case note that the definition of the effective discount factor can be expressed as:

$$\ln \beta = \frac{1}{\psi}(\ln \beta_{eff} + \ln \rho) - \ln \rho \tag{52}$$

Under our calibration strategy changing ψ has no effect on either β_{eff} or ρ. In the complete markets specification $(\ln \beta_{eff} + \ln \rho)$ is positive and β increases when ψ is reduced. This effect makes it difficult to entertain low values of ψ while at the same time keeping $\beta < 1$. With incomplete markets, however, the sign of $(\ln \beta_{eff} + \ln \rho)$ is negative and a lower ψ also lowers β.

A final distinction between the complete markets results in Table 2 and the incomplete markets results in Table 5 is the effect of changes in γ on β. In the specification with complete markets β *falls* if γ is increased when $\psi < 1$ whereas β *increases* if γ is increased

when $\psi > 1$. Under the assumption of incomplete markets an increase in γ lowers β in Table 5 both when ψ is less than one and also when ψ is greater than one. We can offer some intuition for this property of the incomplete markets model. Consider first the case where $\psi < 1$. Equation (39) and $(1 - \omega_c) = \beta^{\psi}\rho^{\psi-1}$ yield

$$\ln(\beta^{\psi}\rho^{\psi-1}) = \ln g_c - \ln(E\bar{R}_{a,t+1}) \tag{53}$$

or

$$\psi \ln(\beta) = -(1-\psi)(\ln(E\bar{R}_{a,t+1}) - \ln \rho) - \psi \ln(E\bar{R}_{a,t+1}) + \ln g_c \tag{54}$$

Note that an increase in γ also increases, $(\ln(E\bar{R}_{a,t+1}) - \ln \rho)$, which is a form of risk premium for holding portfolio a. The expected cross-sectional return on assets, $E\bar{R}_{a,t+1}$, increases because households demand a higher return when they are more risk averse. Since ρ is a concave function of $R_{a,i,t}$ the difference also increases. When $\psi > 1$, the answer will depend on the magnitude of ρ. In Table 5 when $\psi = 1.5$ increasing γ from 2 to 4 results in an increase in ρ and thus β falls.

Table 6 reports the comparative steady-state results for the incomplete markets economy that pertain to the difference facts. Recall that the nature of the experiment is to consider a change in the level of technology A that produces a 1 percent change in consumption growth and to then determine the corresponding change in the return on physical capital. The incomplete markets model is more successful in reproducing the difference facts than the complete markets specification in the sense that it is possible to reconcile a lower value of ψ with empirical evidence that cross-country differences in the return on capital are small. For instance, when $\gamma = 4$ and $\psi = 0.35$ the difference in the return on capital associated with a one percentage change in consumption is 1.82 percent. This value is lower than the value of 2.9 percent that was produced by the complete markets model. It is also less than the value of 2.5 percent that we reported in Table 1 and even falls below Lucas' (1990) figure of 2 percent. If one uses a value of 2.5 percent from Table 1 as a cutoff point we can entertain values of ψ that are less than 0.2.

This reconciliation of a low IES with the difference facts is not entirely satisfactory. In the incomplete markets model a high value of γ is needed to reconcile low values of ψ with the facts on cross country returns on capital. High values of γ also have implications for average returns and in particular the return on human capital. For instance, using the same value of $\psi = 0.35$ when $\gamma = 4$ the average return on human capital is 26.71 percent (see Table 5). When $\gamma = 2$ it falls to 14.35 percent.

The fact that the return on human capital is independent of the setting of ψ suggests a strategy for jointly identifying both *psi* and γ. Table 7 reports alternative combinations of γ and ψ that are consistent with a 3 percent change in the return on physical capital.

To get some idea of the plausible range of values for the return on human capital, Christiansen, Jensen and Nielsen (2007) report returns for alternative professions. The

results we report in the final column of Table 7 (25 percent) correspond to the returns of physicians reported by Christiansen, Jensen and Nielsen (2007). They report returns on human capital for lawyers to be a bit lower at about 14.5 percent. Palacios-Huerta (2003) reports returns on human capital by highest degree attained, race and gender. Returns for white males with a college degree and one to five years of experience are 14.2 percent. Returns for white males with no college and one to five years of experience are 5.9 percent. Krebs (2003) argues that a return of about 12 percent is reasonable using the same model as ours with time additive preferences.

Taken together these estimates suggest that results reported in the middle two columns of Table 6 are the most relevant. These columns yield an IES of between 0.3 and 0.25 and a risk aversion coefficient of around 2. Looking across all four columns one sees that the moments we have chosen impose tight restrictions on the values of ψ and γ. On the one hand, values of ψ much above 0.35 are ruled out by the restriction that γ be non-negative. On the other hand, values of ψ lower than 0.15 imply implausibly large values of the return on human capital.

The parameterizations that emerge from our analysis are quite close to the parameterization used by Gomes and Michaelides (2007) in a model that accounts for a variety of asset pricing anamolies with two types of agents, life-cycle effects, limited participation and incomplete markets. They choose the relative risk aversion coefficients to be 1.1 and 5 and set the IES to respectively 0.1 and 0.4 for the two types of agents.

It is computationally more burdensome activity to solve and simulate the incomplete market version of our model with persistent shocks. However, our complete market results suggest that the results we have documented here with i.i.d. shocks are likely to be robust to this assumption.

4 Concluding Remarks

This paper has developed two ways to reconcile a low IES with previous evidence that has been used to argue that the IES is close to one. When markets are complete our reconciliation requires a value of the preference discount factor that is larger than one. When markets are incomplete the preference discount factor for our specifications with a low IES is always less than one.

We also provided some new empirical evidence that suggests a low value of the IES is more plausible than a large value. Our specifications that posit low values of the IES encompass empirical observations that suggest the IES is large. However, our specifications with a large IES fail to encompass regression evidence that the IES is low.

One distinct advantage of our framework is that it is easy to compute the exact equilibrium allocations and prices. There is only one nonlinear equation to be solved. The rest of

the equilibrium can be calculated in closed form. However, there are also costs. We require that shocks be i.i.d both across individuals and over time. For the complete market model we relaxed this assumption and computed numerical solutions. Those solutions indicated that the results are robust to the maintained assumption that shocks are i.i.d.

A remaining question is whether one can also reconcile a low value of the IES with the equity premium puzzle. The results for the incomplete market model suggest that if the return on equity has an idiosyncratic uninsured component that this is possible. The excess returns on human capital in Table 7 are as large as 20 percent. The question then is why would equity returns have an idiosyncratic component? Kiyotaki and Moore (2005) provide an answer to this question. They consider a setting where agents experience idiosyncratic liquidity shocks. These liquidity shocks introduce an idiosyncratic component into the return on equity. In our current work we are considering extensions to our model along these lines.

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Table 1

International estimates of the real return on capital*

Country	Annual real return on capital (capital share 0.3)	Annual Average Growth Rate of Output per capita
Australia	4.5%	2.0%
Denmark	3.1%	1.8%
Finland	3.1%	2.4%
France	5.6%	1.5%
Germany	3.1%	1.3%
Italy	3.5%	1.7%
Japan	4.0%	2.0%
Norway	3.5%	2.5%
United Kingdom	3.5%	2.1%
United States	5.0%	1.9%

*Capital output ratios are taken from Prescott (2002). We assume Cobb-Douglas technology with a capital share of 0.3 in all countries and a common depreciation rate of 0.08 percent per year. The data on per capita output growth rates are from Maddison's webpage (http://www.ggdc.net/maddison) and are average growth rates for the sample period 1980-2006.

Table 2
Complete Markets Model

Implications for returns and prefeence discount rates

Intertemporal Elasticity of Substitution (ψ)	1.5	1.5	1.5	0.35	0.35	0.35	0.1	0.1	0.1
Relative risk aversion coefficient (γ)	2	4	100	2	4	100	2	4	100
Expected real return on capital	4.00%	4.00%	4.00%	4.00%	4.00%	4.00%	4.00%	4.00%	4.00%
Real risk-free return	3.94%	3.89%	2.41%	3.94%	3.89%	2.41%	3.94%	3.89%	2.41%
Preference discount factor (β)	0.97	0.97	0.98	1.02	1.02	1.00	1.17	1.16	1.070
Effective preference discount factor	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98
Value of β implied by perfect foresight	0.97	0.98	0.99	1.02	1.02	1.03	1.17	1.17	1.19

Table 3
Complete markets model

Effect of a 1 percent reduction in consumption growth on the real return on capital

Intertemporal Elasticity of Substitution (ψ)	1.5	1.5	1.5	0.35	0.35	0.35	0.25	0.25	0.25
Relative risk aversion coefficient (γ)	2	4	100	2	4	100	2	4	100
Percentage difference in real return on capital	0.68%	0.68%	0.98%	2.96%	2.96%	3.04%	4.2%	4.2%	4.3%
Percentage difference in risk-free return	0.68%	0.68%	0.98%	2.96%	2.96%	3.04%	4.2%	4.2%	4.4%

Table 4

Complete Markets Model

Two stage least squares estimates of the slope coefficients in the regressions $\Delta c_{t+1} = \alpha + \psi r_{f,t+1} + \varepsilon_{t+1}$ and $r_{f,t+1} = \tau + \psi^{-1}\Delta c_{t+1} + u_{t+1}$ using synthetic data generated from the complete markets model.*

$\psi = 0.1$

	$\hat{\psi}$			$\hat{\psi}^{-1}$		
Sample Size	MC Mean	Mean MC SE	SD MC Estimates	MC Mean	Mean MC SE	SD MC Estimates
60	-0.34	1.21	1.2	0.06	1.27	0.92
120	-0.19	0.78	0.77	0.33	3.57	1.93
240	0.01	0.51	0.54	0.32	4.79	2.76

$\psi = 0.25$

	$\hat{\psi}$			$\hat{\psi}^{-1}$		
Sample Size	MC Mean	Mean MC SE	SD MC Estimates	MC Mean	Mean MC SE	SD MC Estimates
60	-0.16	1.2	1.2	-0.04	3.12	1.51
120	-0.02	0.76	0.76	0.2	1.4	1.26
240	0.18	0.51	0.54	0.31	3.81	2.48

$\psi = 0.35$

	$\hat{\psi}$			$\hat{\psi}^{-1}$		
Sample Size	MC Mean	Mean MC SE	SD MC Estimates	MC Mean	Mean MC SE	SD MC Estimates
60	-0.08	1.2	1.32	-0.05	5.21	2.25
120	0.19	0.76	0.76	0.36	2.49	1.54
240	0.29	0.5	0.49	0.89	5.67	2.8

$\psi = 1.5$

	$\hat{\psi}$			$\hat{\psi}^{-1}$		
Sample Size	MC Mean	Mean MC SE	SD MC Estimates	MC Mean	Mean MC SE	SD MC Estimates
60	1.43	1.2	1.17	0.52	1.25	0.94
120	1.44	0.75	0.71	0.58	0.38	0.36
240	1.56	0.49	0.5	0.62	0.24	0.24

*The model parameterization is $\gamma = 2$, $\rho_a = 0.8$, $\rho_d = 0.005$ and $(\beta, \psi) = \{(1.168, 0.1), (1.040, 0.25), (1.017, 0.35), (0.974, 1.5)\}$. The number of Monte Carlo repetitions is 100 and the instruments are $\{\Delta c_t, r_{f,t}\}$. We report the mean estimate of the slope coefficient, the mean standard error (SE) and the standard deviation (SD) of the estimate across Monte Carlo draws.

Table 5

Incomplete markets model

Implications for returns and preference discount rates

Intertemporal elasticity of substitution (ψ)	1.5	1.5	0.35	0.35	0.25	0.15
Relative risk aversion coefficient (γ)	2	4	2	4	4	4
Real return on physical capital	3.99%	3.95%	3.99%	3.95%	3.95%	3.95%
Real return on human capital	14.35%	26.71%	14.35%	26.71%	26.71%	26.71%
Real return on total assets	9.75%	14.09%	9.75%	14.09%	14.09%	14.09%
Real risk-free return	3.93%	3.83%	3.93%	3.83%	3.83%	3.83%
Preference discount factor	0.93	0.90	0.92	0.85	0.82	0.75
Effective preference discount factor	0.93	0.89	0.93	0.89	0.89	0.89
Value of β implied by perfect foresight	0.97	0.98	1.02	1.02	1.04	1.10
Value of β implied by second order approximation	0.93	0.90	0.91	0.84	0.81	0.74

Table 6

Incomplete markets model

Effect of a 1 percent reduction in consumption growth on real returns

Intertemporal elasticity of substitution (ψ)	1.5	1.5	0.35	0.35	0.25	0.15
Relative risk aversion coefficient (γ)	2	4	2	4	4	4
Percentage difference in real return on physical capital	0.64%	0.56%	2.45%	1.82%	2.24%	2.59%
Percentage difference in real return on wealth	0.73%	0.78%	2.83%	2.57%	3.19%	3.72%
Percentage difference in risk free rate	0.64%	0.56%	2.46%	1.82%	2.24%	2.59%

Table 7
Incomplete Markets Model

Risk Aversion and the real return on human capital

Intertemporal elasticity of substitution (ψ)	0.350	0.300	0.250	0.15
Relative risk aversion coefficient (γ)	0.5	1.7	2.4	3.7
Return on physical capital	4.0%	4.0%	4.0%	4.0%
Return on human capital	6.1%	12.4%	16.7%	25.0%
Return on total assets	5.4%	8.8%	10.7%	13.6%
Effective discount factor	0.97	0.94	0.92	0.90