Variational Inequality Formulation of a Class of Multi-Leader-Follower Games [*§]

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Abstract: The multi-leader-follower game can be looked on as a generalization of the Nash equilibrium problem and the Stackelberg game, which contains several leaders and a number of followers. Recently, the multi-leader-follower game has been drawing more and more attention, for example, in electricity power markets. However, when we formulate a general multi-leader-follower game as a single-level game, it will give rise to a lot of problems, such as the lack of convexity and the failure of constraint qualifications. In this paper, to get rid of these difficulties, we focus on a class of multi-leader-follower games that satisfy some particular, but still reasonable assumptions, and show that these games can be formulated as ordinary Nash equilibrium problems, and then as variational inequalities. We establish some results on the existence and uniqueness of a leader-follower Nash equilibrium. We also present illustrative numerical examples from an electricity power market model.

keywords: Multi-leader-follower game · Nash equilibrium problem · variational inequality · deregulated electricity market model

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1 Introduction

Since introduced by Nash in [1, 2], the Nash equilibrium problem (NEP) has received a lot of academic attention from more and more researchers. It also has been playing an essential

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role in many application areas of economics, engineering and science; see [3, 4].

The Stackelberg game, also called the single-leader-follower game, arises from the oligopolistic competition. In a Stackelberg game, there is a distinctive player, called the leader, who optimizes the upper-level problem and a number of remaining players, called the followers, who optimize the lower-level problems jointly. In particular, the leader can anticipate the response of the followers, and then use this ability to select his optimal strategy. At the same time, each follower selects her optimal strategy according to the strategies of the leader and the followers.

When dealing with more complex practical problems, such as a deregulated electricity market, we have to consider the competition among several firms and a different type of agents. The corresponding problem is called the multi-leader-follower game, which has several leaders and followers. Each leader can also anticipate the response of the followers, and uses this ability to select his strategy to compete with the other leaders. At the same time, each follower selects his optimal strategy according to the strategies of all leaders as well as the other followers. We are particularly interested in the situation, where no player can improve his status by changing his strategy unilaterally, which we call a leader-follower Nash equilibrium, or simply a L/F Nash equilibrium.

The multi-leader-follower game has recently been studied by some researchers and used to model several problems in applications. Pang and Fukushima [5] introduced a class of remedial models for the multi-leader-follower game, that can be formulated as a generalized Nash equilibrium problem (GNEP) with convexified strategy sets; they also proposed some oligopolistic competition models in electricity power markets, that led to multi-leader-follower games. Based on the strong stationarity conditions of each leader in a multi-leader-follower game, Leyffer and Munson [6] derived a family of nonlinear complementarity problem (NCP), nonlinear programming problem, and mathematical program with equilibrium constraints (MPEC) formulations of the multi-leader-follower game. They also reformulated the game as a square nonlinear complementarity problem by imposing an additional restriction. Outrata [7] converted a kind of multi-leader-follower games to the equilibrium problems with equilibrium constraints (EPECs) and presented some Karush-Kuhn-Tucher (KKT) type of necessary conditions for equilibria. Sherali [8] considered a special multi-leader-follower game where each leader anticipates the response explicitly by the aggregate follower reaction curve. He also showed the existence and uniqueness of the equilibrium of the game called generalized-Stackelberg-Nash Cournot equilibrium, and then proposed a numerical approach to find a generalized-Stackelberg-Nash Cournot equilibrium. Unlike the above deterministic multi-leader-follower games, DeMiguel and Xu [9] considered a stochastic multi-leader-follower game. They introduced a new concept called stochastic multiple-leader Stackelberg-Nash-Cournot equilibrium and showed the existence and uniqueness results under some assumptions. They proposed also a numerical approach to seek the Stackelberg-Nash-Cournot equilibrium with a sample average approximation method. Even with these efforts, we still have to face a lot of problems when we deal with the multi-leader-follower game, because of the inherent difficulties such as the lack of convexity and the failure of constraint qualifications.

In this paper, we consider a simplified multi-leader-follower game with one follower, which still has wide applications; see [5, 10, 11]. Under some particular assumptions on the cost functions of both leaders and follower, as well as the constraints in the follower's problem, we show that the game can be reduced to a NEP, which may further be reformulated as a variational inequality (VI). Moreover, under suitable assumptions, we establish the existence and uniqueness of a L/F Nash equilibrium. We also consider an optimization reformulation of the VI and give some conditions that guarantee a stationarity point of the optimization problem to be a L/F Nash equilibrium. Finally, we present illustrative numerical examples of an electricity power market that consists of two or three firms as the leaders and the independent system operator (ISO) as the follower.

The organization of the paper is as follows. In the next section, we collect some basic definitions and present some preliminary results that will be used later. In Section 3, we introduce the particular multi-leader-follower game considered in the paper, and formulate it as a NEP. In Section 4, we reformulate it as a VI and show some conditions that ensure the existence and uniqueness of a L/F Nash equilibrium. In Sections 5 and 6, we give an application in an electricity power market, and show some illustrative numerical examples from this model. Finally, in Section 7, we conclude the paper.

2 Preliminaries

We start this section with some notations. A function $G : \mathbb{R}^n \to \mathbb{R}^t$ is called a C^k function iff it is k times continuously differentiable. The gradient $\nabla f(x)$ of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is regarded as a column vector. Furthermore, we denote the $n \times m$ (transposed) Jacobian matrix of a differentiable function $F : \mathbb{R}^n \to \mathbb{R}^m$ at a given point x by $\nabla F(x)$. For a real-valued function $f(x, y)$ with the variable $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, the partial gradients with respect to x and y are denoted by $\nabla_x f(x, y) \in \mathbb{R}^n$ and $\nabla_y f(x, y) \in \mathbb{R}^m$, respectively. The partial Hessian matrix obtained by first differentiating with respect to y and then with respect to x is written as the matrix $\nabla^2_{yx} f(x, y) \in \mathbb{R}^{m\times n}$. A vector is regarded as a column vector.

However, if a vector x is composed of several subvectors, $x^1, \cdots, x^n$, it is denoted, for the sake of notation, as $(x^1, \cdots, x^n)$ instead of $((x^1)^T, \cdots, (x^n)^T)^T$, where T denotes transposition.

2.1 Multi-Leader-Follower Games with One Follower

In this subsection, we describe a multi-leader-follower game with one follower and define the corresponding L/F Nash equilibrium. First, we introduce the NEP and the Nash equilibrium.

In a NEP, there are N players labelled by integers $\nu = 1, \cdots, N$. Player ν's strategy is denoted by the vector $x^\nu \in \mathbb{R}^{n_\nu}$ and his cost function $\theta_\nu(x)$ depends on the strategies of all players, which are collectively denoted by the vector $x \in \mathbb{R}^n$ consisting of the subvectors $x^\nu \in \mathbb{R}^{n_\nu}$, $\nu = 1, \cdots, N$, and $n = n_1 + \cdots + n_N$. Player ν's strategy set $X^\nu \subseteq \mathbb{R}^{n_\nu}$ is independent of the strategies of the other players, which are denoted collectively by $x^{-\nu} = (x^1, \cdots, x^{\nu-1}, x^{\nu+1}, \cdots, x^N) \in \mathbb{R}^{n-n_\nu}$. For every fixed, but arbitrary, player ν, $x^{-\nu} \in X^{-\nu} := \prod_{\nu'=1,\nu'\neq\nu}^N X^{\nu'}$, which consists of all players' strategies except player ν, player ν solves the following optimization problem with his own variable x^ν:

$$\underset{x^\nu}{\text{minimize}} \quad \theta_\nu(x^\nu, x^{-\nu}) \quad \text{s.t.} \quad x^\nu \in X^\nu, \tag{1}$$

where we denote $\theta_\nu(x) = \theta_\nu(x^\nu, x^{-\nu})$ to emphasize the role of x^ν in this problem. A strategy tuple $x^* \equiv (x^{*,\nu})_{\nu=1}^N \in X \equiv \prod_{\nu=1}^N X^\nu$ is called a *Nash Equilibrium* (*NE*) iff for all $\nu = 1, \cdots, N$,

$$\theta_\nu(x^{*,\nu}, x^{*,-\nu}) \leq \theta_\nu(x^\nu, x^{*,-\nu}), \quad \forall x^\nu \in X^\nu.$$

A typical multi-leader-follower game with N leaders and one follower can be described as follows. Let $X^\nu \subseteq \mathbb{R}^{n_\nu}$ denote the strategy set of leader $\nu, \nu = 1, \cdots, N$. We assume that the strategy set of each leader X^ν be independent of the strategies of other rival leaders. We also denote each leader's objective function by $\theta_\nu(x^\nu, x^{-\nu}, y), \nu = 1, \cdots, N$, which is dependent on his own strategy x^ν and all the other rival leader's strategies denoted by $x^{-\nu} \in X^{-\nu} \equiv \prod_{\nu'=1,\nu'\neq\nu}^N X^{\nu'}$ and also on the follower's strategy denoted by y.

Let $\gamma(x, y)$ and $K(x)$ denote, respectively, the follower's objective function and strategy set, which depend on the leaders' strategy tuple x. For each given strategy tuple x of the leaders, the follower chooses his strategy by solving the following optimization problem with variable y:

$$\underset{y}{\text{minimize}} \quad \gamma(x, y) \quad \text{s.t.} \quad y \in K(x).$$

We denote the set of optimal solutions to this problem by $Y(x)$. Now, we define the concept of Nash equilibrium for the above multi-leader-follower game with N leaders and one follower.

Definition 2.1. *A strategy tuple $x^* = (x^{*,1}, \cdots, x^{*,N}) \in X$ of the leaders' strategies is called a L/F Nash equilibrium, where L/F means leader-follower, iff there exist N strategies of the follower denoted by a tuple $(y^{*,1}, \cdots, y^{*,N})$ such that $y^{*,\nu} \in Y(x^*), \nu = 1, \cdots, N$, and $(x^{*,\nu}, y^{*,\nu})$ is an optimal solution of leader ν's problem, $\nu = 1, \cdots, N$, which is to seek (x^ν, y^ν) to*

$$\underset{x^\nu, y^\nu}{\text{minimize}} \quad \theta_\nu(x^\nu, x^{*,-\nu}, y^\nu) \quad \text{s.t.} \quad x^\nu \in X^\nu, y^\nu \in Y(x^\nu, x^{*,-\nu}).$$

Here, for each $\nu = 1, \cdots, N$, $Y(x^\nu, x^{*,-\nu})$ is the set of the follower's optimal responses anticipated by leader ν, when leader ν chooses strategy x^ν and the other leaders' strategies $x^{-\nu}$ are fixed at $x^{*,-\nu}$. The above definition of a L/F Nash equilibrium is based on the assumption that leader ν optimizes his objective value by anticipating the follower's response y^ν from the response set $Y(x^\nu, x^{*,-\nu})$ optimistically. This definition for the multi-leader-follower game extends that of an optimistic Stackelberg game considerd by Lignola and Morgan [12]. Note that the follower will actually choose his optimal strategy $y^* \in Y(x^*)$ responding to all leaders' strategies x^*, which may be different from the optimistic anticipations y^ν by the leaders. However, such complication is completely avoided, when the follower's optimal response $Y(x)$ is a singleton for any x, which is the case for the game we consider in Section 3 of this paper.

2.2 Variational Inequality

The variational inequality VI(S, F) is to find a vector $x^* \in S$ such that

$$F(x^*)^T(x - x^*) \geq 0 \quad \text{for all } x \in S,$$

where $S \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a given function. Here we assume the function F to be continuously differentiable. Applications of a variational inequality can be found in various areas, such as transportation systems, mechanics, and economics; see [13, 14].

Suppose that the feasible set S can be represented as

$$S := \{x \in \mathbb{R}^n : h(x) = 0, g(x) \leq 0\},$$

where $h : \mathbb{R}^n \to \mathbb{R}^p$ is an affine function and $g : \mathbb{R}^n \to \mathbb{R}^m$ is a continuously differentiable convex function. Then under a suitable constraint qualification, a solution x of VI(S, F), along with some Lagrange multipliers $\mu \in \mathbb{R}^p$ and $\lambda \in \mathbb{R}^m$, satisfies the following KKT conditions:

$$\begin{aligned} F(x) + \nabla h(x)\mu + \nabla g(x)\lambda = 0, \\ h(x) = 0, \\ g(x) \leq 0, \lambda \geq 0, \lambda^T g(x) = 0. \end{aligned} \tag{2}$$

Conversely, if a triple $(x, \mu, \lambda) \in \mathbb{R}^{n+p+m}$ satisfies (2), then x solves VI(S, F). We call a triple $(x, \mu, \lambda) \in \mathbb{R}^{n+p+m}$ a KKT triple iff it satisfies the above KKT conditions. Moreover, the corresponding x-vector is called a KKT point. For simplicity, when there is no possibility of confusion, we shall often refer to a KKT triple simply as a KKT point.

The following definition will be useful in this paper.

Definition 2.2. *Let $X \subseteq \mathbb{R}^n$ be a convex set. The mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be*

(a) *monotone on X iff*

$$[F(x) - F(y)]^T(x - y) \geq 0 \quad \forall x, y \in X;$$

(b) *strictly monotone on X iff*

$$[F(x) - F(y)]^T(x - y) > 0 \quad \forall x, y \in X, x \neq y;$$

(c) *strongly monotone on X with modulus $\alpha > 0$ iff*

$$[F(x) - F(y)]^T(x - y) \geq \alpha\|x - y\|^2 \quad \forall x, y \in X.$$

Clearly, strong monotonicity implies strict monotonicity, and strict monotonicity implies monotonicity. The following proposition shows the well known relations between these monotonicity properties of F and the positive semidefiniteness or positive definiteness of the Jacobian matrix $\nabla F(x)$ [15].

Proposition 2.1. *Let mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable and $X \subseteq \mathbb{R}^n$ be a convex set. Then*

(a) *F is monotone on X if and only if the Jacobian matrix $\nabla F(x)$ is positive semidefinite for all $x \in X$;*

(b) *F is strictly monotone on X if the Jacobian matrix $\nabla F(x)$ is positive definite for all $x \in X$;*

(c) *F is strongly monotone on X if and only if the Jacobian matrix $\nabla F(x)$ is uniformly positive definite on X, which is equivalent to saying that the minimum eigenvalues of the symmetric matrices $\nabla F(x) + \nabla F(x)^T$ are bounded away from zero on X.*

As to the existence and uniqueness of a solution in the variational inequality, a number of results are known. One of the most fundamental results relies on the compactness of the set X. Other existence results can be obtained by imposing another condition, such as coerciveness of the function F, instead of the compactness of X. On the other hand, under some monotonicity assumptions on F, the following two results on the uniqueness of a solution can be presented [16].

Proposition 2.2. *If F is strictly monotone on S, and the VI(S, F) has at least one solution, then the solution is unique.*

Proposition 2.3. *If F is strongly monotone on S, then there exists a unique solution to the VI(S, F).*

The following proposition shows a basic relation between the NEP and the VI [17].

Proposition 2.4. *Consider the NEP where each player ν solves the optimization problem* (1). *If each X^ν is a nonempty, closed and convex subset of $\mathbb{R}^{n_\nu}$ and, for each fixed $x^{-\nu}$, the function $\theta_\nu(x^\nu, x^{-\nu})$ is convex and continuously differentiable in x^ν, then a strategy tuple $x = (x^1, \cdots, x^N)$ is a Nash equilibrium if and only if x solves the VI(X, F), where*

$$X \equiv \prod_{\nu=1}^{N} X^\nu \quad \mathit{and} \quad F(x) \equiv (\nabla_{x^\nu}\theta_\nu(x))_{\nu=1}^{N}.$$

3 Multi-Leader-Follower Games with Special Structure

In this section, we concentrate on a multi-leader-follower game with the following special structure:

Leader ν's Problem $(\nu = 1, \cdots, N)$.

$$\begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \theta_\nu(x^\nu, x^{-\nu}, y) := \omega_\nu(x^\nu, x^{-\nu}) + \varphi_\nu(x^\nu, y) \\ \text{subject to} & g^\nu(x^\nu) \leq 0,\ h^\nu(x^\nu) = 0. \end{array}$$

Follower's Problem.

$$\begin{array}{ll} \underset{y}{\text{minimize}} & \gamma(x, y) := \psi(y) - \sum_{\nu=1}^{N} \varphi_\nu(x^\nu, y) \\ \text{subject to} & y \in \mathcal{Y}. \end{array}$$

Here, all functions are assumed to be C^2 functions. In particular, each function $\omega_\nu : \mathbb{R}^n \to \mathbb{R}$ is convex with respect to the variable x^ν. Functions $g^\nu : \mathbb{R}^{n_\nu} \to \mathbb{R}^{s_\nu}$ are all convex and

$h^\nu : \mathbb{R}^{n_\nu} \to \mathbb{R}^{t_\nu}$ are all affine. Functions ψ, φ_ν and set $\mathcal{Y}$ are assumed to have the following explicit representations:

$$\begin{aligned}\varphi_\nu(x^\nu, y) &\equiv (x^\nu)^T D_\nu y,\ \nu = 1, \cdots, N,\\ \psi(y) &\equiv \frac{1}{2} y^T B y + c^T y,\\ \mathcal{Y} &\equiv \{y \in R^m | Ay + a = 0\}.\end{aligned}$$

Here, $D_\nu \in \mathbb{R}^{n_\nu \times m}, \nu = 1, \cdots, N, c \in \mathbb{R}^m$, and matrix $B \in \mathbb{R}^{m\times m}$ is assumed to be symmetric and positive definite. Moreover, $A \in \mathbb{R}^{p\times m}$, $a \in \mathbb{R}^p$, and matrix A has full row rank. Note that the strategy set of the follower's problem $\mathcal{Y}$ is an affine subset of $\mathbb{R}^m$.

Then the above multi-leader-follower game can be written as follows:

Leader ν's Problem ($\nu = 1, \cdots, N$).

$$\begin{aligned}\underset{x^\nu}{\text{minimize}} \quad & \omega_\nu(x^\nu, x^{-\nu}) + (x^\nu)^T D_\nu y\\ \text{subject to} \quad & g^\nu(x^\nu) \leq 0,\ h^\nu(x^\nu) = 0.\end{aligned}$$

Follower's Problem.

$$\begin{aligned}\underset{y}{\text{minimize}} \quad & \frac{1}{2} y^T B y + c^T y - \sum_{\nu=1}^{N} (x^\nu)^T D_\nu y\\ \text{subject to} \quad & Ay + a = 0.\end{aligned}$$

In this game, the objective functions of N leaders and the follower contain some related terms. Specifically, the second term of the objective function appears in the follower's objective function in the negated form. Therefore, the game partly contains a kind of zero-sum structure between each leader and the follower. An application of such special multi-leader-follower games will be presented with some illustrative numerical examples later.

In the remainder of the paper, for simplicity, we will mainly consider the following game with two leaders, labelled I and II. The results presented below can be extended to the case of more than two leaders in a straightforward manner.

Leader I's Problem.

$$\begin{aligned}\underset{x^{\mathrm{I}}}{\text{minimize}} \quad & \omega_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + (x^{\mathrm{I}})^T D_{\mathrm{I}} y\\ \text{subject to} \quad & g^{\mathrm{I}}(x^{\mathrm{I}}) \leq 0,\ h^{\mathrm{I}}(x^{\mathrm{I}}) = 0.\end{aligned}$$

Leader II's Problem.

$$\begin{aligned}\underset{x^{\mathrm{II}}}{\text{minimize}} \quad & \omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + (x^{\mathrm{II}})^T D_{\mathrm{II}} y\\ \text{subject to} \quad & g^{\mathrm{II}}(x^{\mathrm{II}}) \leq 0,\ h^{\mathrm{II}}(x^{\mathrm{II}}) = 0.\end{aligned}$$

Follower's Problem.

$$\begin{aligned}
\underset{y}{\text{minimize}} \quad & \frac{1}{2} y^T B y + c^T y - (x^{\mathrm{I}})^T D_{\mathrm{I}} y - (x^{\mathrm{II}})^T D_{\mathrm{II}} y \\
\text{subject to} \quad & Ay + a = 0.
\end{aligned}$$

Since the follower's problem is a strictly convex quadratic programming problem with equality constraints, it is equivalent to finding a pair $(y, \lambda) \in \mathbb{R}^{m \times p}$ satisfying the following KKT system of linear equations:

$$\begin{aligned}
By + c - (D_{\mathrm{I}})^T x^{\mathrm{I}} - (D_{\mathrm{II}})^T x^{\mathrm{II}} + A^T \lambda = 0, \\
Ay + a = 0.
\end{aligned} \tag{3}$$

Note that, under the given assumptions, a KKT pair (y, λ) exists uniquely for each $(x^{\mathrm{I}}, x^{\mathrm{II}})$ and is denoted by $(y(x^{\mathrm{I}}, x^{\mathrm{II}}), \lambda(x^{\mathrm{I}}, x^{\mathrm{II}}))$. By direct calculations, we have

$$\begin{aligned}
y(x^{\mathrm{I}}, x^{\mathrm{II}}) = & -B^{-1}c - B^{-1}A^T(AB^{-1}A^T)^{-1}(a - AB^{-1}c) \\
& + [B^{-1}(D_{\mathrm{I}})^T - B^{-1}A^T(AB^{-1}A^T)^{-1}AB^{-1}(D_{\mathrm{I}})^T]x^{\mathrm{I}} \\
& + [B^{-1}(D_{\mathrm{II}})^T - B^{-1}A^T(AB^{-1}A^T)^{-1}AB^{-1}(D_{\mathrm{II}})^T]x^{\mathrm{II}}, \\
\lambda(x^{\mathrm{I}}, x^{\mathrm{II}}) = & (AB^{-1}A^T)^{-1}(a - AB^{-1}c) + (AB^{-1}A^T)^{-1}AB^{-1}(D_{\mathrm{I}})^T x^{\mathrm{I}} \\
& + (AB^{-1}A^T)^{-1}AB^{-1}(D_{\mathrm{II}})^T x^{\mathrm{II}}.
\end{aligned}$$

Substituting $y(x^{\mathrm{I}}, x^{\mathrm{II}})$ for y in the leaders' problems, the leaders' objective functions can be rewritten as

$$\begin{aligned}
\omega_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + (x^{\mathrm{I}})^T D_{\mathrm{I}} y = & \, \omega_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + (x^{\mathrm{I}})^T D_{\mathrm{I}} r \\
& + (x^{\mathrm{I}})^T D_{\mathrm{I}} G x^{\mathrm{I}} + (x^{\mathrm{I}})^T D_{\mathrm{I}} H x^{\mathrm{II}},
\end{aligned} \tag{4}$$

$$\begin{aligned}
\omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + (x^{\mathrm{II}})^T D_{\mathrm{II}} y = & \, \omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + (x^{\mathrm{II}})^T D_{\mathrm{II}} r \\
& + (x^{\mathrm{II}})^T D_{\mathrm{II}} G x^{\mathrm{I}} + (x^{\mathrm{II}})^T D_{\mathrm{II}} H x^{\mathrm{II}},
\end{aligned} \tag{5}$$

where $G \in \mathbb{R}^{m \times n_{\mathrm{I}}}$, $H \in \mathbb{R}^{m \times n_{\mathrm{II}}}$, and $r \in \mathbb{R}^m$ are given by

$$\begin{aligned}
G &= B^{-1}(D_{\mathrm{I}})^T - B^{-1}A^T(AB^{-1}A^T)^{-1}AB^{-1}(D_{\mathrm{I}})^T, \\
H &= B^{-1}(D_{\mathrm{II}})^T - B^{-1}A^T(AB^{-1}A^T)^{-1}AB^{-1}(D_{\mathrm{II}})^T, \\
r &= -B^{-1}c - B^{-1}A^T(AB^{-1}A^T)^{-1}(a - AB^{-1}c).
\end{aligned}$$

Let the functions defined by (4) and (5) be denoted as $\Theta_{\mathrm{I}} : \mathbb{R}^{n_{\mathrm{I}}+n_{\mathrm{II}}} \to \mathbb{R}$ and $\Theta_{\mathrm{II}} : \mathbb{R}^{n_{\mathrm{I}}+n_{\mathrm{II}}} \to \mathbb{R}$, respectively. Then we can formulate the above multi-leader-follower game as the following NEP, which we call the NEP$(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$, where $X^\nu = \{x^\nu : g^\nu(x^\nu) \le 0, h^\nu(x^\nu) = 0\}, \nu = \mathrm{I}, \mathrm{II}$:

Leader I's Problem.

$$\begin{aligned} \underset{x^{\mathrm{I}}}{\text{minimize}} \quad & \Theta_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \\ \text{subject to} \quad & g^{\mathrm{I}}(x^{\mathrm{I}}) \leq 0,\ h^{\mathrm{I}}(x^{\mathrm{I}}) = 0. \end{aligned}$$

Leader II's Problem.

$$\begin{aligned} \underset{x^{\mathrm{II}}}{\text{minimize}} \quad & \Theta_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \\ \text{subject to} \quad & g^{\mathrm{II}}(x^{\mathrm{II}}) \leq 0,\ h^{\mathrm{II}}(x^{\mathrm{II}}) = 0. \end{aligned}$$

Remark 3.1. Instead of solving the KKT system (3) explicitly, we may leave it as additional constraints in each leader's problem. This results in a GNEP, which has attracted much attention recently [5, 18].

4 Existence and Uniqueness of L/F Nash Equilibrium

In this section, we further reformulate the NEP$(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$ derived in the previous section as a variational inequality, and discuss the existence and uniqueness of a L/F Nash equilibrium. Throughout this section, we assume the sets X^{I} and X^{II} are nonempty.

In the NEP$(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$, the Hessian matrices of the leaders' objective functions are calculated as

$$\begin{aligned} \nabla^2_{x^{\mathrm{I}}}\Theta_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) &= \nabla^2_{x^{\mathrm{I}}}\omega_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + 2(D_{\mathrm{I}}B^{-\frac{1}{2}})P(D_{\mathrm{I}}B^{-\frac{1}{2}})^T, \\ \nabla^2_{x^{\mathrm{II}}}\Theta_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) &= \nabla^2_{x^{\mathrm{II}}}\omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + 2(D_{\mathrm{II}}B^{-\frac{1}{2}})P(D_{\mathrm{II}}B^{-\frac{1}{2}})^T, \end{aligned}$$

where matrix $P \in \mathbb{R}^{m\times m}$ is given by

$$P = I - B^{-\frac{1}{2}}A^T(AB^{-1}A^T)^{-1}AB^{-\frac{1}{2}}.$$

Since P is a projection matrix, it is symmetric, idempotent, and positive semidefinite. By the given assumption, functions $\omega_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}})$ and $\omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}})$ are both convex with respect to x^{I} and x^{II}, respectively. Therefore the leaders' objective functions in the NEP$(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$ are both convex with respect to x^{I} and x^{II}, respectively.

By Proposition 2.4, we can reformulate the NEP$(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$ as the following variational inequality denoted by the VI$(X, \widehat{F})$: Find a vector $x^* \in X := X^{\mathrm{I}} \times X^{\mathrm{II}}$ such that

$$\widehat{F}(x^*)^T(x - x^*) \geq 0 \quad \text{for all } x \in X,$$

where $x = (x^{\mathrm{I}}, x^{\mathrm{II}}) \in \mathbb{R}^{n_{\mathrm{I}}+n_{\mathrm{II}}}$, $X^\nu = \{x^\nu : g^\nu(x^\nu) \le 0, h^\nu(x^\nu) = 0\}, \nu = \mathrm{I}, \mathrm{II}$ and the function $\widehat{F} : \mathbb{R}^{n_{\mathrm{I}}+n_{\mathrm{II}}} \to \mathbb{R}^{n_{\mathrm{I}}+n_{\mathrm{II}}}$ is defined by

$$\widehat{F}(x) \equiv \begin{pmatrix} \nabla_{x^{\mathrm{I}}}\Theta_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \\ \nabla_{x^{\mathrm{II}}}\Theta_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \end{pmatrix} = \begin{pmatrix} \nabla_{x^{\mathrm{I}}}\omega_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + D_{\mathrm{I}}r + 2D_{\mathrm{I}}Gx^{\mathrm{I}} + D_{\mathrm{I}}Hx^{\mathrm{II}} \\ \nabla_{x^{\mathrm{II}}}\omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + D_{\mathrm{II}}r + D_{\mathrm{II}}Gx^{\mathrm{I}} + 2D_{\mathrm{II}}Hx^{\mathrm{II}} \end{pmatrix}. \tag{6}$$

Here we notice that matrices $D_{\mathrm{I}}G$ and $D_{\mathrm{II}}H$ are symmetric from the definitions of G and H, respectively.

The VI Lagrangian function is written as

$$\widehat{L}(x, \mu, \lambda) = \widehat{F}(x) + \begin{pmatrix} \sum_{i=1}^{s_{\mathrm{I}}} \lambda_i^{\mathrm{I}} \nabla g_i^{\mathrm{I}}(x^{\mathrm{I}}) \\ \sum_{i=1}^{s_{\mathrm{II}}} \lambda_i^{\mathrm{II}} \nabla g_i^{\mathrm{II}}(x^{\mathrm{II}}) \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^{t_{\mathrm{I}}} \mu_j^{\mathrm{I}} \nabla h_j^{\mathrm{I}}(x^{\mathrm{I}}) \\ \sum_{j=1}^{t_{\mathrm{II}}} \mu_j^{\mathrm{II}} \nabla h_j^{\mathrm{II}}(x^{\mathrm{II}}) \end{pmatrix},$$

where $\lambda = (\lambda^{\mathrm{I}}, \lambda^{\mathrm{II}}) \in \mathbb{R}^{s_{\mathrm{I}}+s_{\mathrm{II}}}$ and $\mu = (\mu^{\mathrm{I}}, \mu^{\mathrm{II}}) \in \mathbb{R}^{t_{\mathrm{I}}+t_{\mathrm{II}}}$. The Jacobin matrix $\nabla_x \widehat{L}(x, \mu, \lambda)$ of the VI Lagrangian function with respect to x is written as

$$\begin{aligned} \nabla_x \widehat{L}(x, \mu, \lambda) = &\begin{pmatrix} \nabla^2_{x^{\mathrm{II}}}\omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) & \nabla^2_{x^{\mathrm{II}}x^{\mathrm{I}}}\omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \\ \nabla^2_{x^{\mathrm{I}}x^{\mathrm{II}}}\omega_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) & \nabla^2_{x^{\mathrm{II}}}\omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \end{pmatrix} \\ &+ \begin{pmatrix} 2D_{\mathrm{I}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{I}})^T & D_{\mathrm{I}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{II}})^T \\ D_{\mathrm{II}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{I}})^T & 2D_{\mathrm{II}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{II}})^T \end{pmatrix} \\ &+ \begin{pmatrix} \sum_{i=1}^{s_{\mathrm{I}}} \lambda_i^{\mathrm{I}} \nabla^2 g_i^{\mathrm{I}}(x^{\mathrm{I}}) & 0 \\ 0 & \sum_{i=1}^{s_{\mathrm{II}}} \lambda_i^{\mathrm{II}} \nabla^2 g_i^{\mathrm{II}}(x^{\mathrm{II}}) \end{pmatrix}. \end{aligned}$$

Lemma 4.1. *The matrix*

$$\begin{pmatrix} 2D_{\mathrm{I}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{I}})^T & D_{\mathrm{I}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{II}})^T \\ D_{\mathrm{II}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{I}})^T & 2D_{\mathrm{II}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{II}})^T \end{pmatrix}$$

is positive semidefinite.

Proof. Set $Q = B^{-\frac{1}{2}}PB^{-\frac{1}{2}}$, which is symmetric and positive semidefinite. Then we can write

$$\begin{aligned} &\begin{pmatrix} 2D_{\mathrm{I}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{I}})^T & D_{\mathrm{I}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{II}})^T \\ D_{\mathrm{II}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{I}})^T & 2D_{\mathrm{II}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{II}})^T \end{pmatrix} \\ = &\begin{pmatrix} 2D_{\mathrm{I}}Q(D_{\mathrm{I}})^T & D_{\mathrm{I}}Q(D_{\mathrm{II}})^T \\ D_{\mathrm{II}}Q(D_{\mathrm{I}})^T & 2D_{\mathrm{II}}Q(D_{\mathrm{II}})^T \end{pmatrix}. \end{aligned}$$

Consider the following products of matrices:

$$
\begin{aligned}
&\begin{pmatrix} D_{\mathrm{I}} & \frac{1}{2}D_{\mathrm{I}} \\ \frac{1}{2}D_{\mathrm{II}} & D_{\mathrm{II}} \end{pmatrix} \begin{pmatrix} 2Q & 0 \\ 0 & 2Q \end{pmatrix} \begin{pmatrix} D_{\mathrm{I}} & \frac{1}{2}D_{\mathrm{I}} \\ \frac{1}{2}D_{\mathrm{II}} & D_{\mathrm{II}} \end{pmatrix}^T \\
&= \begin{pmatrix} \frac{5}{2}D_{\mathrm{I}}Q(D_{\mathrm{I}})^T & 2D_{\mathrm{I}}Q(D_{\mathrm{II}})^T \\ 2D_{\mathrm{II}}Q(D_{\mathrm{I}})^T & \frac{5}{2}D_{\mathrm{II}}Q(D_{\mathrm{II}})^T \end{pmatrix} \\
&= 2\begin{pmatrix} 2D_{\mathrm{I}}Q(D_{\mathrm{I}})^T & D_{\mathrm{I}}Q(D_{\mathrm{II}})^T \\ D_{\mathrm{II}}Q(D_{\mathrm{I}})^T & 2D_{\mathrm{II}}Q(D_{\mathrm{II}})^T \end{pmatrix} - \begin{pmatrix} \frac{3}{2}D_{\mathrm{I}}Q(D_{\mathrm{I}})^T & 0 \\ 0 & \frac{3}{2}D_{\mathrm{II}}Q(D_{\mathrm{II}})^T \end{pmatrix}.
\end{aligned}
$$

Since the matrices $\begin{pmatrix} 2Q & 0 \\ 0 & 2Q \end{pmatrix}$ and $\begin{pmatrix} \frac{3}{2}D_{\mathrm{I}}Q(D_{\mathrm{I}})^T & 0 \\ 0 & \frac{3}{2}D_{\mathrm{II}}Q(D_{\mathrm{II}})^T \end{pmatrix}$ are both positive semidefinite, the matrix

$$
\begin{pmatrix} 2D_{\mathrm{I}}Q(D_{\mathrm{I}})^T & D_{\mathrm{I}}Q(D_{\mathrm{II}})^T \\ D_{\mathrm{II}}Q(D_{\mathrm{I}})^T & 2D_{\mathrm{II}}Q(D_{\mathrm{II}})^T \end{pmatrix}
$$

is also positive semidefinite. □

Theorem 4.1. *If the function*

$$
F_0(x) = F_0(x^{\mathrm{I}}, x^{\mathrm{II}}) \equiv \begin{pmatrix} \nabla_{x^{\mathrm{I}}}\omega_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \\ \nabla_{x^{\mathrm{II}}}\omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \end{pmatrix} \tag{7}
$$

is strictly monotone, and the NEP$(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$ *has at least one solution, then the solution is unique.*

Proof. By Lemma 4.1, the matrix

$$
\begin{pmatrix} 2D_{\mathrm{I}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{I}})^T & D_{\mathrm{I}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{II}})^T \\ D_{\mathrm{II}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{I}})^T & 2D_{\mathrm{II}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{II}})^T \end{pmatrix}
$$

is positive semidefinite. It then follows from the definitions of matrices G, H and P that the function

$$
\widetilde{F}(x) \equiv \begin{pmatrix} D_{\mathrm{I}}r + 2D_{\mathrm{I}}Gx^{\mathrm{I}} + D_{\mathrm{I}}Hx^{\mathrm{II}} \\ D_{\mathrm{II}}r + D_{\mathrm{II}}Gx^{\mathrm{I}} + 2D_{\mathrm{II}}Hx^{\mathrm{II}} \end{pmatrix}
$$

is monotone. By the given assumption, the function $\widehat{F} = F_0 + \widetilde{F}$ is therefore strictly monotone. Moreover, by Proposition 2.2, if the VI$(X, \widehat{F})$ has at least one solution, then the solution is unique. In view of (6), this in turn implies that the NEP$(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$ has a unique solution by Proposition 2.4. □

Theorem 4.2. *If the function F_0 defined by* (7) *is strongly monotone, then the NEP*$(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$ *has a unique solution.*

Proof. In a similar manner to the proof of Theorem 4.1, we can deduce that function $\widehat{F} = F_0 + \widetilde{F}$ is strongly monotone. Then, by Proposition 2.3, the VI$(X, \widehat{F})$ has a unique solution. Hence, by Proposition 2.4, the NEP$(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$ has a unique solution. $\square$

Remark 4.1. The NEP$(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$ has a nonempty, compact solution set when function $\widehat{F}$ is coercive or X is bounded.

5 The Multi-Leader-Follower Game in Deregulated Electricity Market

Privatization and restructuring of the edregulated electricity markets have taken place in many countries, although an excessive free market also has a possibility to bring about some trouble. Under this situation, a lot of researchers have paid much attention to the noncooperative competition problems in this area, see [5, 10, 19]. In this section, we present a simple model of competitive bidding under some macroeconomic regulation. We will show that it can be formulated as the multi-leader-follower game that we have considered in the previous sections.

In this model, there are several firms and one market maker, called the ISO, who employs a market cleaning mechanism to collect the electricity from firms by paying the bid costs, determine the price of electricity and sell it to consumers. We omit the problem of consumers, which means any quantity of electricity power can be consumed. The structure of the model can be described as follows. Again, for simplicity, we assume there are only two firms I and II. The two firms are competing for market power in an electricity network with M nodes. We assume that firms I and II produce the electricity with fixed quantities a^{I} and a^{II}, respectively, and send it to all nodes. A firm receives its profit by dispatching electricity with the bid parameters to the ISO at each node. Let $\rho^\nu = (\rho_1^\nu, \cdots, \rho_M^\nu)^T$, $\nu = \mathrm{I}, \mathrm{II}$, denote firm ν's bid parameter vectors, where the components ρ_i^ν are the bid parameters to nodes $i = 1, \cdots, M$. The vector $q = (q^{\mathrm{I}}, q^{\mathrm{II}}) \in \mathbb{R}^{2M}$ with $q^\nu = (q_1^\nu, \cdots, q_M^\nu)^T, \nu = \mathrm{I}, \mathrm{II}$, denotes the quantities supplied by the firms, or more specifically, q_i^{I} and q_i^{II} denote the quantities of electricity supplied by firm I and firm II at nodes $i = 1, \cdots, M$, respectively. Each firm ν will submit a bid function $b_\nu(q, \rho^\nu)$ to the ISO. The function b_ν represents how much revenue firm ν will receive by selling the electricity power. At the same time, each firm also needs to consider the transaction cost $\omega_\nu(\rho^\nu) = \frac{1}{2}(\rho^\nu)^T diag(\zeta_1^\nu, \cdots, \zeta_M^\nu)\rho^\nu$, where $\zeta_i^\nu, i = 1, \cdots, M$, are given positive constants.

Each firm ν tries to determine its bid parameter vector ρ^ν by minimizing the difference between its transaction cost and revenue, i.e., by solving the following optimization problem with variables ρ^ν:

$$\begin{aligned} \underset{\rho^\nu}{\text{minimize}} \quad & \frac{1}{2}(\rho^\nu)^T diag(\zeta_1^\nu, \cdots, \zeta_M^\nu)\rho^\nu - b_\nu(q, \rho^\nu) \\ \text{subject to} \quad & 0 \le \rho_i^\nu \le \xi_i^\nu,\ i = 1, \cdots, M, \end{aligned}$$

where ξ_i^ν, $i = 1, \cdots, M$, are positive empirical upper bounds set by the firm $\nu, \nu = \mathrm{I}, \mathrm{II}$. We further assume that the firms' bid functions are given by

$$b_\nu(q, \rho^\nu) \equiv \rho_1^\nu q_1^\nu + \cdots + \rho_M^\nu q_M^\nu = -(\rho^\nu)^T D_\nu q,\ \nu = \mathrm{I}, \mathrm{II},$$

where $D_\nu \in \mathbb{R}^{M \times 2M}$ are bid matrices of firm ν, which are defined by $D_{\mathrm{I}} = (-I, \mathbf{0})$ and $D_{\mathrm{II}} = (\mathbf{0}, -I)$, where I is the identity matrix and $\mathbf{0}$ is the zero matrix. We will write the constraints of the above problem as

$$g^\nu(\rho^\nu) \le 0,$$

where

$$g^\nu(\rho^\nu) = \begin{pmatrix} -\rho_1^\nu, & \cdots, & -\rho_M^\nu, & \rho_1^\nu - \xi_1^\nu, & \cdots, & \rho_M^\nu - \xi_M^\nu \end{pmatrix}^T.$$

Remark 5.1. Pang and Fukushima [5] also considered a multi-leader-follower game with an application in a deregulated electricity market model where the firms are required to bid on their revenue functions. Here we extend their model by considering the transaction cost in each leader's objective function.

Moreover, we assume that some economic interventionism works in this electricity model in order to maintain some equilibrium between the quantities of electricity at each node from two firms, which is represented by some quadratic terms denoted by $\frac{1}{2}\varepsilon_i(\frac{q_i^{\mathrm{I}}}{a^{\mathrm{I}}} - \frac{q_i^{\mathrm{II}}}{a^{\mathrm{II}}})^2$, $i = 1, \cdots, M$, where $a^\nu, \nu = \mathrm{I}, \mathrm{II}$, are the quantities of electricity produced by firm I, II, respectively, and each economic interventionism parameter ε_i is positive and small. We assume $a^\nu, \nu = \mathrm{I}, \mathrm{II}$, are fixed. Under the economic interventionism, the ISO tries to let each firm ν supply the electricity in such a way that the quantity tends to be proportional to his total amount a^ν at each node. At the same time, the ISO employs a market mechanism to determine a set of nodal prices and electricity quantities from each firm at each node in order to maximize its profit (the revenue minus the bid costs), or minimize the negative profit. We further assume that, at each node, the affine demand curves determine the prices p_i as a function of the total quantity of electricity from firms I and II as follows.

$$p_i(q_i^{\mathrm{I}}, q_i^{\mathrm{II}}) := \alpha_i - \beta_i(q_i^{\mathrm{I}} + q_i^{\mathrm{II}}),\ i = 1, \cdots, M,$$

where α_i and β_i are given positive constants.

Then the ISO minimizes its negative profit by solving the following optimization problem with variables $q = (q^{\mathrm{I}}, q^{\mathrm{II}})$:

$$\begin{aligned} \underset{q}{\text{minimize}} \quad & \sum_{i=1}^{M} \Big[\frac{\beta_i}{2}(q_i^{\mathrm{I}} + q_i^{\mathrm{II}})^2 - \alpha_i(q_i^{\mathrm{I}} + q_i^{\mathrm{II}})\Big] + \frac{1}{2}\sum_{i=1}^{M} \varepsilon_i \big(\frac{q_i^{\mathrm{I}}}{a^{\mathrm{I}}} - \frac{q_i^{\mathrm{II}}}{a^{\mathrm{II}}}\big)^2 + b_{\mathrm{I}}(q, \rho^{\mathrm{I}}) + b_{\mathrm{II}}(q, \rho^{\mathrm{II}}) \\ \text{subject to} \quad & \sum_{i=1}^{M} q_i^{\mathrm{I}} - a^{\mathrm{I}} = 0, \\ & \sum_{i=1}^{M} q_i^{\mathrm{II}} - a^{\mathrm{II}} = 0. \end{aligned}$$

Note that the first two terms of the ISO's objective function is rewritten as

$$\sum_{i=1}^{M} \Big[\frac{\beta_i}{2}(q_i^{\mathrm{I}} + q_i^{\mathrm{II}})^2 - \alpha_i(q_i^{\mathrm{I}} + q_i^{\mathrm{II}})\Big] + \frac{1}{2}\sum_{i=1}^{M} \varepsilon_i \big(\frac{q_i^{\mathrm{I}}}{a^{\mathrm{I}}} - \frac{q_i^{\mathrm{II}}}{a^{\mathrm{II}}}\big)^2 = \frac{1}{2} q^T B q + c^T q,$$

where

$$B = \begin{pmatrix} \beta_1 + \frac{2\varepsilon_1}{(a^{\mathrm{I}})^2} & \cdots & 0 & \beta_1 - \frac{\varepsilon_1}{a^{\mathrm{I}} a^{\mathrm{II}}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \beta_M + \frac{2\varepsilon_M}{(a^{\mathrm{I}})^2} & 0 & \cdots & \beta_M - \frac{\varepsilon_M}{a^{\mathrm{I}} a^{\mathrm{II}}} \\ \beta_1 - \frac{\varepsilon_1}{a^{\mathrm{I}} a^{\mathrm{II}}} & \cdots & 0 & \beta_1 + \frac{2\varepsilon_1}{(a^{\mathrm{II}})^2} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \beta_M - \frac{\varepsilon_M}{a^{\mathrm{I}} a^{\mathrm{II}}} & 0 & \cdots & \beta_M + \frac{2\varepsilon_M}{(a^{\mathrm{II}})^2} \end{pmatrix}, \ c = \begin{pmatrix} -\alpha_1 \\ \vdots \\ -\alpha_M \\ -\alpha_1 \\ \vdots \\ -\alpha_M \end{pmatrix}.$$

Notice that matrix B is positive definite. Moreover, the constraints of the ISO can be rewritten as

$$Aq + a = 0,$$

where

$$A = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{2 \times 2M}, \ a = \begin{pmatrix} -a^{\mathrm{I}} \\ -a^{\mathrm{II}} \end{pmatrix}.$$

Therefore, the electricity market model under consideration can be formulated as the following multi-leader-follower game:

Firm (Leader) I's Problem.

$$\begin{aligned} \underset{\rho^{\mathrm{I}}}{\text{minimize}} \quad & \frac{1}{2}(\rho^{\mathrm{I}})^T diag(\zeta_1^{\mathrm{I}}, \cdots, \zeta_M^{\mathrm{I}})\rho^{\mathrm{I}} + (\rho^{\mathrm{I}})^T D_{\mathrm{I}} q \\ \text{subject to} \quad & g^{\mathrm{I}}(\rho^{\mathrm{I}}) \leq 0. \end{aligned}$$

Firm (Leader) II's Problem.

$$\begin{aligned} \underset{\rho^{\mathrm{II}}}{\text{minimize}} \quad & \frac{1}{2}(\rho^{\mathrm{II}})^T diag(\zeta_1^{\mathrm{II}}, \cdots, \zeta_M^{\mathrm{II}})\rho^{\mathrm{II}} + (\rho^{\mathrm{II}})^T D_{\mathrm{II}} q \\ \text{subject to} \quad & g^{\mathrm{II}}(\rho^{\mathrm{II}}) \leq 0. \end{aligned}$$

ISO (Follower)'s Problem.

$$\begin{aligned} \underset{q}{\text{minimize}} \quad & \frac{1}{2} q^T B q + c^T q - (\rho^{\mathrm{I}})^T D_{\mathrm{I}} q - (\rho^{\mathrm{II}})^T D_{\mathrm{II}} q \\ \text{subject to} \quad & Aq + a = 0. \end{aligned}$$

In light of the analysis in the previous sections, we can further reformulate the multi-leader-follower game as the following VI: Find a vector $\rho^* = (\rho^{*,\mathrm{I}}, \rho^{*,\mathrm{II}}) \in X = X^{\mathrm{I}} \times X^{\mathrm{II}}$ such that

$$\widehat{F}(\rho^*)^T(\rho - \rho^*) \geq 0 \quad \text{for all } \rho \in X, \tag{8}$$

where

$$\begin{aligned} & \rho = (\rho^{\mathrm{I}}, \rho^{\mathrm{II}}) = (\rho_1^{\mathrm{I}}, \cdots, \rho_M^{\mathrm{I}}, \rho_1^{\mathrm{II}}, \cdots, \rho_M^{\mathrm{II}})^T, \\ & X^\nu = \{\rho^\nu : g^\nu(\rho^\nu) \leq 0\},\ \nu = \mathrm{I}, \mathrm{II}, \\ & g^\nu(\rho^\nu) = (-\rho_1^\nu, \cdots, -\rho_M^\nu, \rho_1^\nu - \xi_1^\nu, \cdots, \rho_M^\nu - \xi_M^\nu)^T,\ \nu = \mathrm{I}, \mathrm{II}, \\ & \widehat{F}(\rho) = \begin{pmatrix} diag(\zeta_1^{\mathrm{I}}, \cdots, \zeta_M^{\mathrm{I}})\rho^{\mathrm{I}} + D_{\mathrm{I}} r + 2D_{\mathrm{I}} G \rho^{\mathrm{I}} + D_{\mathrm{I}} H \rho^{\mathrm{II}} \\ diag(\zeta_1^{\mathrm{II}}, \cdots, \zeta_M^{\mathrm{II}})\rho^{\mathrm{II}} + D_{\mathrm{II}} r + D_{\mathrm{II}} G \rho^{\mathrm{I}} + 2D_{\mathrm{II}} H \rho^{\mathrm{II}} \end{pmatrix}. \end{aligned}$$

Remark 5.2. The assumption that the follower's problem does not contain inequality constraints would narrow down the range of applications of the model. In fact, the approach presented in this paper cannot be extended directly to the inequality constrained case. Nevertheless we may still try to make some further efforts to deal with inequality constraints based on the current approach. Suppose that the follower solves the optimization problem

$$\begin{aligned} \underset{q}{\text{minimize}} \quad & \frac{1}{2} q^T B y + c^T q - (\rho^{\mathrm{I}})^T D_{\mathrm{I}} q - (\rho^{\mathrm{II}})^T D_{\mathrm{II}} q \\ \text{subject to} \quad & Aq + a = 0,\ q \geq 0. \end{aligned}$$

A possible idea to deal with the inequality constraint $q \geq 0$ is the following: First, we just ignore it, apply the above-mentioned VI formulation, and find a L/F Nash equilibrium. Next, we check the components of q in the equilibrium. If they are all nonnegative, then we accept the current equilibrium as a solution of the problem. Otherwise, we set the negative components of q to be 0, or in other words, we discard those components. Then we try to find a L/F Nash equilibrium of the reduced problem by ignoring the inequality constraint $q \geq 0$ again. Repeating this heuristic procedure, we will eventually obtain an approximate L/F Nash equilibrium of the original game.

6 Numerical Experiments

In this section, we show numerical results for the electricity market model described in the previous section.

Note that X in (8) can be represented as $X = \{\rho \in \mathbb{R}^{2M} | 0 \leq \rho_i^\nu \leq \xi_i^\nu, i = 1, \cdots, M, \nu = \mathrm{I}, \mathrm{II}\}$. Thus the VI is a box-constrained variational inequality (BVI), denoted by $\mathrm{BVI}(X, \widehat{F}(\rho))$. To solve the BVI, Kanzow and Fukushima [20] present a nonsmooth Newton-type method applied to the nonlinear equation involving the natural residual of the BVI. The algorithm uses the D-gap function to ensure global convergence of the Newton-type method. We use the following parameter setting in the implementation of Algorithm 3.2 in [20]:

$$\alpha = 0.9, \quad \beta = 1.2, \ \ \delta = 0.6, \ \omega = 10^{-6},$$
$$\sigma = 10^{-4}, \ p = 2.1, \ \ \eta = 0.8, \ \tau = 10^{-6}.$$

First we solve a model with two firms I, II, who dispatch the electricity to two nodes $i = 1, 2$. We set the problem data as follows:

$$\alpha_1 = 1.5, \alpha_2 = 1.8; \beta_1 = 0.6, \beta_2 = 0.7; a^{\mathrm{I}} = 1.2, a^{\mathrm{II}} = 1.8;$$
$$\xi_1^{\mathrm{I}} = 1, \xi_2^{\mathrm{I}} = 1, \xi_1^{\mathrm{II}} = 1, \xi_2^{\mathrm{II}} = 1; \zeta_1^{\mathrm{I}} = 1.2; \zeta_2^{\mathrm{I}} = 1, \zeta_1^{\mathrm{II}} = 1.3, \zeta_2^{\mathrm{II}} = 1.5.$$

We also set the problem data $\varepsilon = (\varepsilon_1, \varepsilon_2)$ as listed in Table 1, where the corresponding computational results $\rho^* = (\rho_1^{*,\mathrm{I}}, \rho_2^{*,\mathrm{I}}, \rho_1^{*,\mathrm{II}}, \rho_2^{*,\mathrm{II}})^T$ and $q^* = (q_1^{*,\mathrm{I}}, q_2^{*,\mathrm{I}}, q_1^{*,\mathrm{II}}, q_2^{*,\mathrm{II}})^T$ along with the objective values of the firms and the ISO are shown.

We may observe from the table that the economic interventionism terms play an important role in the distribution of electricity quantities at each node. As the economic interventionism parameters $\varepsilon_i, i = 1, 2$, become larger, the ratio of electricity quantities supplied by firm I and firm II gets closer to the ratio of the amount of electricity $a^{\mathrm{I}} : a^{\mathrm{II}} = 1 : 1.5$. For example, when the ε changes from $(0.001, 0.001)$ to $(0.05, 0.05)$, the ratio at two nodes changes from $q_1^{*,\mathrm{I}} : q_1^{*,\mathrm{II}} = 1 : 1.5973$ and $q_2^{*,\mathrm{I}} : q_2^{*,\mathrm{II}} = 1 : 1.4223$ to $1 : 1.5938$ and $1 : 1.4251$, respectively. Also in this procedure, both firms' optimal profits are nondecreasing, but the ISO's profit is decreasing.

We also solved a larger electricity market model with three firms and five nodes, where the following problem data are used:

Table 1. The L/F Nash Equilibrium with Two Firms and Two Nodes

ε	(0.001, 0.001)	(0.005, 0.005)	(0.01, 0.01)	(0.05, 0.05)
ρ^*	(0.4139, 0.4137, 0.6429, 0.6428)	(0.4142, 0.4135, 0.6430, 0.6427)	(0.4146, 0.4132, 0.6432, 0.6426)	(0.4173, 0.4113, 0.6440, 0.6419)
valL1	-0.2483	-0.2483	-0.2483	-0.2485
valL2	-0.5786	-0.5786	-0.5786	-0.5786
q^*	(0.5331, 0.6669, 0.8515, 0.9485)	(0.5330, 0.6670, 0.8512, 0.9488)	(0.5330, 0.6670, 0.8510, 0.9490)	(0.5327, 0.6673, 0.8490, 0.9510)
valF	-1.8425	-1.8424	-1.8424	-1.8422
Iter	2	2	2	2

Note1. valL1, valL2 and valF denote the optimal values of firm I, firm II and the ISO, respectively, and Iter denotes the number of iterations.

Table 2. The L/F Nash Equilibrium with Three Firms and Five Nodes

ε	(0.001, 0.001, 0.001, 0.001, 0.001)	(0.005, 0.005, 0.005, 0.005, 0.005)	(0.01, 0.01, 0.01, 0.01, 0.01)	(0.05, 0.05, 0.05, 0.05, 0.05)
ρ^*	(0.1799, 0.1789, 0.1788, 0.1791, 0.1791; 0.3296, 0.3285, 0.3285, 0.3287, 0.3288; 0.2440, 0.2430, 0.2429, 0.2432, 0.2432)	(0.1831, 0.1780, 0.1776, 0.1790, 0.1790; 0.3327, 0.3277, 0.3273, 0.3285, 0.3287; 0.2469, 0.2421, 0.2418, 0.2432, 0.2431)	(0.1869, 0.177, 0.1762, 0.1789, 0.1788; 0.3364, 0.3267, 0.3259, 0.3283, 0.3286; 0.2505, 0.2411, 0.2404, 0.2431, 0.2430)	(0.2125, 0.1700, 0.1667, 0.1781, 0.1778; 0.3620, 0.3197, 0.3165, 0.3266, 0.3282; 0.2744, 0.2339, 0.2309, 0.2428, 0.2421)
$valL1$	-0.1078	-0.1091	-0.1107	-0.1202
$valL2$	-0.3952	-0.3978	-0.401	-0.4201
$valL3$	-0.2224	-0.2244	-0.2268	-0.2414
q^*	(0.5334, 0.1327, 0.1511, 0.1837, 0.1990; 1.0716, 0.2572, 0.2886, 0.4082, 0.3744; 0.8488, 0.1936, 0.2133, 0.2533, 0.2911)	(0.5317, 0.1334, 0.1516, 0.1841, 0.1993; 1.0668, 0.2590, 0.2905, 0.4087, 0.3750; 0.8450, 0.1949, 0.2146, 0.2539, 0.2916)	(0.5296, 0.1341, 0.1521, 0.1845, 0.1996; 1.0610, 0.2612, 0.2927, 0.4094, 0.3757; 0.8405, 0.1965, 0.2162, 0.2547, 0.2921)	(0.5154, 0.1394, 0.1559, 0.1875, 0.2018; 1.0217, 0.2756, 0.3079, 0.4141, 0.3808; 0.8098, 0.2072, 0.2267, 0.2602, 0.2960)
$valF$	-6.4479	-6.4419	-6.4346	-6.3871
Iter	4	23	3	14

Note 2. valL3 denotes the optimal value of firm III.

$\alpha_1 = 1.2, \alpha_2 = 1.5, \alpha_3 = 1.6, \alpha_4 = 1.8, \alpha_5 = 1.9; a^{\mathrm{I}} = 1.2, a^{\mathrm{II}} = 2.4, a^{\mathrm{III}} = 1.8;$

$\beta_1 = 0.2, \beta_2 = 0.5, \beta_3 = 0.6, \beta_4 = 0.7, \beta_5 = 0.8; \xi_1^{\mathrm{I}} = 1, \xi_2^{\mathrm{I}} = 3, \xi_3^{\mathrm{I}} = 5, \xi_4^{\mathrm{I}} = 2, \xi_5^{\mathrm{I}} = 6;$

$\xi_1^{\mathrm{II}} = 1, \xi_2^{\mathrm{II}} = 1, \xi_3^{\mathrm{II}} = 4, \xi_4^{\mathrm{II}} = 1.5, \xi_5^{\mathrm{II}} = 5; \xi_1^{\mathrm{III}} = 1, \xi_2^{\mathrm{III}} = 1.7, \xi_3^{\mathrm{III}} = 1, \xi_4^{\mathrm{III}} = 1, \xi_5^{\mathrm{III}} = 2;$

$\zeta_1^{\mathrm{I}} = 1.1, \zeta_2^{\mathrm{I}} = 1.4, \zeta_3^{\mathrm{I}} = 1.7, \zeta_4^{\mathrm{I}} = 1.2, \zeta_5^{\mathrm{I}} = 1.3; \zeta_1^{\mathrm{II}} = 1.2, \zeta_2^{\mathrm{II}} = 1.5, \zeta_3^{\mathrm{II}} = 1.8, \zeta_4^{\mathrm{II}} = 1.5,$

$\zeta_5^{\mathrm{II}} = 1.3; \zeta_1^{\mathrm{III}} = 1.3, \zeta_2^{\mathrm{III}} = 1.6, \zeta_3^{\mathrm{III}} = 1.9, \zeta_4^{\mathrm{III}} = 1.2, \zeta_5^{\mathrm{III}} = 1.4.$

We also set $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5)$ as listed in Table 2, where the corresponding numerical results are shown. We may also observe similar properties to those for the pervious model from the table. When the economic interventionism parameters $\varepsilon_i, i = 1, \cdots, 5$, become larger, the ratio of electricity quantities supplied by firms I , II and III gets closer to the ratio of the amount of electricity $a^{\mathrm{I}} : a^{\mathrm{II}} : a^{\mathrm{III}} = 1 : 2 : 1.5$, and the optimal profit of each firm increases, while that of the ISO decreases.

7 Conclusions

Considering the difficulties of general multi-leader-follower games, this paper has focused on a multi-leader-follower game with some particular structure, which contains one follower whose optimization problem is a strictly convex quadratic program with equality constraints. Then the multi-leader-follower game is formulated as a NEP, which can further be formulated as a VI. We have shown the existence and uniqueness of a L/F Nash equilibrium. We have also presented an application in electricity power markets with some numerical results.

References

1. Nash, J.F.: Equilibrium points in *n*-person games. Proc. Natl. Acad. Sci. U.S.A. **36**, 48-49 (1950)
2. Nash, J.F.: Non-cooperative games. Ann. Math. **54**, 286-295 (1951)
3. Bai, X., Shahidehpour, S.M., Ramesh, V.C., Yu, E.: Transmission analysis by Nash game method. IEEE Trans. Power Syst. **12**, 1046-1052 (1997)
4. Song, H., Liu, C.C., Lawarre, J., Bellevue, W.A.: Nash equilibrium bidding strategies in a bilateral electricity market. IEEE Trans. Power Syst. **17**, 73-79 (2002)
5. Pang, J.S., Fukushima, M.: Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games. Comput. Manage. Sci., **2**, 21-56 (2005)

6. Leyffer, S., Munson, T.: Solving multi-leader-common-follower games. Optim. Method. Softw. **25**, 601-623 (2010)

7. Outrata, J.V.: A note on a class of equilibrium problems with equilibrium constraints. Kybernetika. **40**, 585-594 (2004)

8. Sherali, H.D.: A multiple leader Stackelberg model and analysis. Oper. Res. **32**, 390-404 (1984)

9. DeMiguel, V., Xu, H.: A stochastic multiple-leader Stackelberg model: Analysis, computation, and application. Oper. Res. **57**, 1220-1235 (2009)

10. Hobbs, B.F., Metzler, C., Pang, J.S.: Strategic gaming analysis for electric power networks: An MPEC approach. IEEE Trans. Power Syst. **15**, 638-645 (2000)

11. Hu, X., Ralph, D.: Using EPECs to model bilevel games in restructured electricity markets with locational prices. Oper. Res. **55**, 809-827 (2007)

12. Lignola, M.B., Morgan, J.: Stability of regularized bilevel programming problems. J. Optim. Theory Appl. **93**, 575-596 (1997)

13. Ferris, M.C., Pang, J.S.: Engineering and economic applications of complementarity problems. SIAM Rev. **39**, 669-713 (1997)

14. Nagurney, A.: Network Economics: A Variational Inequality Approach. Kluwer Academic Publishers, Boston (1993)

15. Ortega, J.M., Rheinboldt, W.C.: Iterative Solution of Nonlinear Equations in Several Variables. Academic Press, New York (1970)

16. Harker, P.T., Pang, J.S.: Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications. Math. Program. **48**, 161-200 (1990)

17. Facchinei, F., Pang, J.S.: Finite-Dimensional Variational Inequalities and Complementarity Problems. Volume I. Springer, New York (2003)

18. Facchinei, F., Kanzow, C.: Generalized Nash equilibrium problems. Ann. Oper. Res. **175**, 177-211 (2010)

19. Cardell, J.B., Hitt, C.C., Hogan, W.W.: Market power and strategic interaction in electricity networks. Resour. Energy Econ. **19**, 109-137 (1997)

20. Kanzow, C., Fukushima, M.: Solving box constrained variational inequalities by using the natural residual with D-gap function globalization. Oper. Res. Lett. **23**, 45-51 (1998)