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<thead>
<tr>
<th>Title</th>
<th>On the connected components of moduli spaces of Kisin modules</th>
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<td>Kyoto University</td>
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ON THE CONNECTED COMPONENTS OF MODULI SPACES OF KISIN MODULES

NAOKI IMAI

Abstract. We give a proof of a conjecture on the connected components of moduli spaces of Kisin module, which is valid also in the case $p = 2$.

Introduction

Let $K$ be a $p$-adic field, and let $V_F$ be a two-dimensional continuous representation of the absolute Galois group $G_K$ over a finite field $\mathbb{F}$ of characteristic $p$. Take a $\phi$-module $M_F$ corresponding to the Galois representation $V_F$, that is a projective scheme over $\mathbb{F}$. Let $\mathcal{M}_{V_F,0}$ be a closed subscheme of $\mathcal{M}_{V_F,0}$ determined by the condition that $p$-adic Hodge type is $v = 1$.

In the case $p > 2$, a Kisin module in $M_F$ corresponds a finite flat models of $V_F$, and $\mathcal{M}_{V_F,0}$ is called a moduli space of finite flat models of $V_F$. In this case, Kisin conjectured that the non-ordinary locus of $\mathcal{M}_{V_F,0}$ is connected. (In fact, this is a special case of [Kis, Conjecture 2.4.16].) This conjecture was proved by Kisin in [Kis] if $K$ is totally ramified over $\mathbb{Q}_p$, by Gee in [Gee] if $V_F$ is the trivial representation, and by the author in [Ima] for general $K$ and $V_F$. In the proof in [Ima], we need the condition $p > 2$. In this paper, we prove the conjecture for all $p$. The main theorem is the following.

Theorem. The non-ordinary locus of $\mathcal{M}_{V_F,0}$ is geometrically connected.

The outline of the proof is the same as the proof in [Ima], but we need some more sophisticated arguments to treat the case $p = 2$.

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Notation. Throughout this paper, we use the following notation. Let $p$ be a prime number, and $k$ be a finite extension of $\mathbb{F}_p$ of cardinality $q = p^n$. The Witt ring of $k$ is denoted by $W(k)$, and let $K_0 = W(k)[1/p]$. Let $K$ be a totally ramified extension of $K_0$ of degree $e$, and $\mathcal{O}_K$ be the ring of integers of $K$. The absolute Galois group of $K$ is denoted by $G_K$. Let $\mathbb{F}$ be a finite field of characteristic $p$. The formal power series ring of $u$ over $\mathbb{F}$ is denoted by $\mathbb{F}[[u]]$, and its quotient field is denoted by $\mathbb{F}((u))$. Let $v_u$ be the valuation of $\mathbb{F}((u))$ normalized by $v_u(1) = 1$. For a field $F$, the algebraic closure of $F$ is denoted by $\overline{F}$ and the separable closure of $F$ is denoted by $F_{sep}$. 1
1. Preliminaries

First of all, we recall some notation from [Kis], and the interested reader should consult [Kis] for more detailed definitions.

We put \( \mathcal{G} = W(k)[[u]] \). Let \( \mathcal{O}_E \) be the \( p \)-adic completion of \( \mathcal{G}[1/u] \). There is an action of \( \phi \) on \( \mathcal{O}_E \) determined by Frobenius on \( W(k) \) and \( u \mapsto u^p \). We take and fix a uniformizer \( \pi \) of \( \mathcal{O}_K \). We choose elements \( \pi_m \in \bar{K} \) such that \( \pi_0 = \pi \) and \( \pi_{m+1}^p = \pi_m \) for \( m \geq 0 \), and put \( K_\infty = \bigcup_{m \geq 0} K(\pi_m) \). Let \( \Phi \mathcal{O}_{E,F} \) be the category of finite \( \mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{F} \)-modules \( M \) equipped with \( \phi \)-semi-linear map \( M \to M \) such that the induced \( \mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{F} \)-linear map \( \phi^*(M) \to M \) is an isomorphism. Let \( \text{Rep}_F(G_{K_\infty}) \) be the category of finite-dimensional continuous representations of \( G_{K_\infty} \) over \( \mathbb{F} \). Then the functor

\[
T : \Phi \mathcal{O}_{E,F} \to \text{Rep}_F(G_{K_\infty}) : M \mapsto (k((u))^\text{sep} \otimes_k k((u)) M)^{\phi = 1}
\]

gives an equivalence of abelian categories as in [Kis, (1.1.12)]. Here \( \phi \) acts on \( k((u))^{\text{sep}} \) by the \( p \)-th power map.

Let \( V_F \) be a continuous two-dimensional representation of \( G_K \) over \( \mathbb{F} \). We take the \( \phi \)-module \( M_F \in \Phi \mathcal{O}_{E,F} \) such that \( T(M_F) \) is isomorphic to \( V_F(-1)|_{G_{K_\infty}} \). Here \((-1)\) denotes the inverse of the Tate twist.

From now on, we assume \( \mathbb{F}_{\text{alg}} \subset \mathbb{F} \) and fix an embedding \( k \to \mathbb{F} \). This assumption does not matter, because we may extend \( \mathbb{F} \) to prove the main theorem. We consider the isomorphism

\[
\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{F} \cong k((u)) \otimes_{\mathbb{F}_p} \mathbb{F} \cong \prod_{\sigma \in \text{Gal}(k/\mathbb{F}_p)} \mathbb{F}(u) ; \left( \sum a_i u^i \right) \otimes b \mapsto \left( \sum \sigma(a_i)bu^i \right)_\sigma
\]

and let \( \epsilon_\sigma \in k((u)) \otimes_{\mathbb{F}_p} \mathbb{F} \) be the primitive idempotent corresponding to \( \sigma \). Take \( \sigma_1, \cdots, \sigma_n \in \text{Gal}(k/\mathbb{F}_p) \) such that \( \sigma_{i+1} = \sigma_i \circ \phi^{-1} \). Here we regard \( \phi \) as the \( p \)-th power Frobenius, and use the convention that \( \sigma_{n+1} = \sigma_n \). In the following, we often use such conventions. Then we have \( \phi(\epsilon_{\sigma_i}) = \epsilon_{\sigma_{i+1}} \), and \( \phi : M_F \to M_F \) determines \( \phi : \epsilon_\sigma, M_F \to \epsilon_{\sigma_{i+1}}, M_F \).

For \( (A_i)_{1 \leq i \leq n} \in GL_2(\mathbb{F}(u))^n \), we write

\[
M_F \sim (A_1, A_2, \ldots, A_n) = (A_i)
\]

if there is a basis \( \{e_1, e_2\} \) of \( \epsilon_\sigma, M_F \) over \( \mathbb{F}(u) \) such that \( \phi \left( \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \right) = A_i \left( \begin{pmatrix} e_1^{i+1} \\ e_2^{i+1} \end{pmatrix} \right) \).

We use the same notation for any sublattice \( \mathcal{M}_F \subset M_F \) similarly. Here and in the following, we consider only sublattices that are \( \mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{F} \)-modules.

Finally, for any sublattice \( \mathcal{M}_F \subset M_F \) with a chosen basis \( \{e_1, e_2\}_{1 \leq i \leq n} \) and \( B = (B_i)_{1 \leq i \leq n} \in GL_2(\mathbb{F}(u))^n \), the module generated by the entries of \( \left( B_i \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \right) \) with the basis given by these entries is denoted by \( B \cdot \mathcal{M}_F \). Note that \( B \cdot \mathcal{M}_F \) depends on the choice of the basis of \( \mathcal{M}_F \).

For each \( \mathbb{Q}_p \)-algebra embedding \( \psi : K \to \overline{K} \), we put \( v_\psi = 1 \) and set \( v = (v_\psi)_\psi \). Then \( \mathcal{M}_{v,0} \) is the moduli space of Kisin modules with \( p \)-adic Hodge type \( v \). The rational points of \( \mathcal{M}_{v,0} \) are described as in the following.

**Proposition 1.1.** If \( \mathbb{F}' \) is a finite extension of \( \mathbb{F} \), the elements of \( \mathcal{M}_{v,0}(\mathbb{F}') \) naturally correspond to free \( k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}' \)-submodules \( \mathcal{M}_{\mathbb{F}'} \subset M_F \otimes_{\mathbb{F}_p} \mathbb{F}' \) of rank 2 that satisfy the following:

1. \( \mathcal{M}_{\mathbb{F}'} \) is \( \phi \)-stable.
(2) For some (so any) choice of \(k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}'\)-basis for \(M_{\mathbb{F}'}\), and for each \(\sigma \in \text{Gal}(k/\mathbb{F}_p)\), the map

\[
\phi : \varepsilon_{\mathbb{F}'} \to \varepsilon_{\mathbb{F}'_n}
\]

has determinant \(a\alpha^e\) for some \(\alpha \in \mathbb{F}'[[u]]^\times\).

**Proof.** This is [Gee, Lemma 2.2]. \(\square\)

### 2. MAIN THEOREM

To prove the main theorem, in fact we prove that the non-ordinary component of \(\mathcal{X}_{V_0,0}\) is rationally connected. We use the following two Lemmas to join two points by \(\mathbb{P}^1\).

**Lemma 2.1.** Suppose \(x_1, x_2 \in \mathcal{X}_{V_0,0}(\mathbb{F})\) correspond to objects \(M_{1,2,2}\) of \((\text{Mod}/\mathfrak{S})_{\mathbb{F}}\) respectively. We fix bases of \(M_{1,2,2}\) over \(k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}\). We assume that there is a nilpotent element \(N = (N_i)_{1 \leq i \leq n}\) of \(M_2(\mathbb{F}[[u]])^n\) such that \(M_{1,2} = (1 + N) \cdot M_{1,2}\). Let \(A = (A_i)_{1 \leq i \leq n}\) be an element of \(GL_2(\mathbb{F}[[u]])^n\) such that \(M_{1,2} \sim A\). If \(\phi(N_i)A_iN_{i+1} \in M_2(\mathbb{F}[[u]])\) for all \(i\), then there is a morphism \(\mathbb{P}^1 \to \mathcal{X}_{V_0,0}\) sending \(0\) to \(x_1\) and \(1\) to \(x_2\).

**Proof.** This is [Gee, Lemma 2.4]. \(\square\)

**Lemma 2.2.** Suppose \(n \geq 2\). Let \(M_{\mathbb{F}}\) be the object of \((\text{Mod}/\mathfrak{S})_{\mathbb{F}}\) corresponding to a point \(x \in \mathcal{X}_{V_0,0}(\mathbb{F})\). Fix a basis of \(M_{\mathbb{F}}\) over \(k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}\). Consider \(U^{(i)} = (U^{(i)}_{j})_{1 \leq j \leq n}\) \(\in GL_2(\mathbb{F}[[u]])^{n}\) such that \(U^{(i)}_{j} = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}\) and \(U^{(i)}_{j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) for all \(j \neq i\). If \(U^{(i)} : M_{\mathbb{F}}\) is \(\phi\)-stable, it corresponds to a point \(x' \in \mathcal{X}_{V_0,0}(\mathbb{F})\), and there is a morphism \(\mathbb{P}^1 \to \mathcal{X}_{V_0,0}\) sending \(0\) to \(x\) and \(1\) to \(x'\). If \((U^{(i)})^{-1} : M_{\mathbb{F}}\) is \(\phi\)-stable, it corresponds to a point \(x'' \in \mathcal{X}_{V_0,0}(\mathbb{F})\), and there is a morphism \(\mathbb{P}^1 \to \mathcal{X}_{V_0,0}\) sending \(0\) to \(x\) and \(1\) to \(x''\).

**Proof.** This is [Ima, Lemma 2.3]. \(\square\)

To prove the main theorem, it suffices to show the following theorem. The strategy of the proof is the same as in [Ima], and we focus on the changed points in the case \(p = 2\).

**Theorem 2.3.** Let \(\mathbb{F}'\) be a finite extension of \(\mathbb{F}\). Suppose \(x_1, x_2 \in \mathcal{X}_{V_0,0}(\mathbb{F}')\) correspond to objects \(M_{1,2,2}\) of \((\text{Mod}/\mathfrak{S})_{\mathbb{F}'}\) respectively. If \(M_{1,2,2}\) are both non-ordinary, then \(x_1\) and \(x_2\) lie on the same connected component of \(\mathcal{X}_{V_0,0}\).

**Proof.** When \(n = 1\), this was proved in [Kis], and we did not use the condition \(p > 2\) in the proof. If \(e < p - 1\), then \(\mathcal{X}_{V_0,0}(\mathbb{F}')\) is one point by [Ray, Theorem 3.3.3]. So we may assume \(n \geq 2\) and \(e \geq p - 1\). Furthermore, replacing \(V_{\mathbb{F}}\) by \(V_{\mathbb{F}'} \otimes_{\mathbb{F}_p} \mathbb{F}'\), we may assume \(V_{\mathbb{F}'} = \mathbb{F}'\).

In the case where \(V_{\mathbb{F}}\) is reducible, the proof of [Ima, Theorem 2.4] goes on, even if \(p = 2\). So, by a base change, we may assume that \(V_{\mathbb{F}'}\) is absolutely irreducible. As in the proof of [Ima, Theorem 2.4], we can prove that, after extending the field \(\mathbb{F}\), there exists a basis such that

\[
M_{\mathbb{F}} \sim \begin{pmatrix}
\alpha_1 \begin{pmatrix} 0 & u^{s_1} \\ u^{t_1} & 0 \end{pmatrix} & \alpha_2 \begin{pmatrix} u^{s_2} & 0 \\ 0 & u^{s_2} \end{pmatrix} & \cdots & \alpha_n \begin{pmatrix} u^{s_n} & 0 \\ 0 & u^{s_n} \end{pmatrix}
\end{pmatrix}
\]
where \( \alpha_i \in \mathbb{F} \), \( 0 \leq s_i, t_i \leq e \), \( s_i + t_i = e \) and \( |s_i - t_i| \leq p + 1 \) for all \( i \). Note that we have proved that we may assume \( |s_i - t_i| \leq p + 1 \) for all \( i \) in the last paragraph of [Ima, p. 1197]

Let \( \mathcal{M}_{\mathcal{F},0} \) be the \( k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F} \)-module generated by the basis giving the above matrix expression. Then \( \mathcal{M}_{\mathcal{F},0} \) satisfies the condition in Proposition 1.1. We take the point \( x_0 \) of \( \mathcal{C}_{\mathcal{F},0}(\mathbb{F}) \) corresponding to \( \mathcal{M}_{\mathcal{F},0} \). We are going to prove that \( x_0 \) and \( x_1 \) lie on the same connected component. We can prove that \( x_0 \) and \( x_2 \) lie on the same connected component by the same argument.

By the Iwasawa decomposition and the determinant conditions, we can take \( B = (B_i)_{1 \leq i \leq n} \in GL_2(\mathbb{F}(u)) \) such that \( \mathcal{M}_{1,\mathcal{F}} = B \cdot \mathcal{M}_{0,\mathcal{F}} \) and \( B_i = \begin{pmatrix} u & v_i \\ 0 & u_i \end{pmatrix} \) for \( a_i \in \mathbb{Z} \) and \( v_i \in \mathbb{F}(u) \). Then we put \( r_i = v_u(v_i) \). Now we have

\[
\phi(B_1) \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix} B_2^{-1} = \begin{pmatrix} \phi(v_1)u^{*1+a_2} & u^{*1-pa_1-a_2} - \phi(v_1)v_2u^{*1} \\ -v_2u^{*1+pa_1} & -v_2u^{*1+pa_1} \end{pmatrix},
\]

\[
\phi(B_2) \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} B_3^{-1} = \begin{pmatrix} \phi(v_3)u^{*1-pa_1+1-a_1} - v_3u^{*1+1-a_1} & 0 \\ u^{*1+pa_1-a_1+1} & u^{*1+pa_1-a_1+1} \end{pmatrix}
\]

for \( 2 \leq i \leq n \). On the right-hand sides, every component of the matrices is integral because \( \mathcal{M}_{1,\mathcal{F}} \) is \( \phi \)-stable.

First, we consider the case \( t_1 + pa_1 + a_2 > e \). In this case,

\[
(pr_1 + t_1 + a_2) + (r_2 + t_1 + pa_1) = e, \quad s_1 - pa_1 - a_2 = pr_1 + r_2 + t_1 < 0
\]

by the \( \phi \)-stability and the determinant conditions of \( \mathcal{M}_{1,\mathcal{F}} \). We have \( a_1 > r_1 \), because \( t_1 + pa_1 + a_2 > e \geq pr_1 + t_1 + a_2 \). Similarly, we have \( a_2 > r_2 \), because \( t_1 + pa_1 + a_2 > e \geq r_2 + t_1 + pa_1 \).

We consider the following operations:

\[
a_i \sim a_i - 1, \quad v_i \sim uv_i, \quad \text{if it preserves the } \phi \text{-stability of } B \cdot \mathcal{M}_{0,\mathcal{F}}.
\]

These operations replace \( x_1 \) by a point that lies on the same connected component as \( x_1 \) by Lemma 2.2. We prove that we can continue these operations until we get to the situation where \( t_1 + pa_1 + a_2 \leq e \). In other words, we reduce the problem to the case \( t_1 + pa_1 + a_2 \leq e \). If we can continue the operations endlessly, we get to the situation where \( t_1 + pa_1 + a_2 \leq e \), because the conditions \( s_i - pa_i + a_{i+1} \geq 0 \) for \( 2 \leq i \leq n \) exclude that both \( a_1 \) and \( a_2 \) remain bounded below. Suppose we cannot continue the operations. This is equivalent to the following condition:

\[
s_n - pa_n + a_1 = 0 \quad \text{or} \quad r_2 + t_1 + pa_1 \leq p - 1, \quad pr_1 + t_1 + a_2 = 0 \quad \text{or} \quad t_2 + pa_2 - a_3 \leq p - 1, \quad s_{i-1} - pa_{i-1} + a_i = 0 \quad \text{or} \quad t_i + pa_i - a_{i+1} \leq p - 1 \quad \text{for each } 3 \leq i \leq n.
\]

If \( e \geq p \), there are only the following two cases, because \( (pr_1 + t_1 + a_2) + (r_2 + t_1 + pa_1) = e \) and \( (s_i - pa_i + a_{i+1}) + (t_i + pa_i - a_{i+1}) = e \) for \( 2 \leq i \leq n \).

Case 1: \( pr_1 + t_1 + a_2 = 0, \quad s_i - pa_i + a_{i+1} = 0 \) for \( 2 \leq i \leq n \).

Case 2: \( r_2 + t_1 + pa_1 \leq p - 1, \quad t_i + pa_i - a_{i+1} \leq p - 1 \) for \( 2 \leq i \leq n \).

If \( e = p - 1 \), clearly it is in Case 2.

In the Case 1, we have a contradiction as in the proof of [Ima, Theorem 2.4]. So we may assume that it is in the Case 2.
Then we can show that
\[ r_i < a_i, \ p r_i + t_i - a_{i+1} = r_{i+1} + s_i - p a_i < 0 \text{ for } 2 \leq i \leq n \]
as in the proof of [Ima, Theorem 2.4]. Combining these equations with \( s_1 - p a_1 - a_2 = p r_1 + r_2 + t_1 \), we get
\[
- (p^n + 1)r_1 = (p^n + 1)a_1 + (s_n - t_n) + p(s_n-1 - t_{n-1}) + \cdots + p^{n-3}(s_3 - t_3) + p^{n-2}(s_2 - t_2) - p^{n-1}(s_1 - t_1),
\]
\[
- (p^n + 1)r_2 = (p^n + 1)a_2 - (s_1 - t_1) - p(s_n - t_n) - \cdots - p^{n-3}(s_4 - t_4) - p^{n-2}(s_3 - t_3) - p^{n-1}(s_2 - t_2),
\]
\[
- (p^n + 1)r_3 = (p^n + 1)a_3 + (s_2 - t_2) - p(s_1 - t_1) - \cdots - p^{n-3}(s_5 - t_5) - p^{n-2}(s_4 - t_4) - p^{n-1}(s_3 - t_3),
\]
\[
\vdots
\]
\[
- (p^n + 1)r_n = (p^n + 1)a_n + (s_{n-1} - t_{n-1}) + p(s_{n-2} - t_{n-2}) + \cdots + p^{n-3}(s_2 - t_2) - p^{n-2}(s_1 - t_1) - p^{n-1}(s_0 - t_0).
\]
As \( |s_i - t_i| \leq p + 1 \) and
\[
(p + 1) + p(p + 1) + \cdots + p^{n-1}(p + 1) = \left( \frac{p^n - 1}{p - 1} \right)(p + 1) < 3(p^n + 1),
\]
we get \(-a_i - 2 \leq r_i \leq -a_i + 2\). If \( e = p \), as \( |s_i - t_i| \leq p \) and
\[
p + p^2 + \cdots + p^n = \left( \frac{p^n - 1}{p - 1} \right)p < 2(p^n + 1),
\]
we get \(-a_i - 1 \leq r_i \leq -a_i + 1\). If \( e = p - 1 \), as \( |s_i - t_i| \leq p - 1 \) and
\[
(p - 1) + p(p - 1) + \cdots + p^{n-1}(p - 1) = \left( \frac{p^n - 1}{p - 1} \right)(p - 1) < (p^n + 1),
\]
we get \(-a_i = r_i\).
As \( r_2 + t_1 + pa_1 \leq p - 1 \), we have
\[
pa_1 \leq t_1 + pa_1 \leq p - 1 - r_2 \leq a_2 + p + 1.
\]
For \( 2 \leq i \leq n \), as \( t_i + pa_i - a_{i+1} \leq p - 1 \), we have
\[
pa_i \leq t_i + pa_i \leq a_{i+1} + p - 1.
\]
Take an index \( i_0 \) such that \( a_{i_0} \) is the greatest. If \( 2 \leq i_0 \leq n \), we get \( a_{i_0} \leq 1 \) by \( pa_{i_0} \leq a_{i_0+1} + p - 1 \leq a_{i_0} + p - 1 \). If \( i_0 = 1 \) and \( a_1 \geq 3 \), we get \( a_2 \geq 3 \), by \( pa_1 \leq a_2 + p + 1 \), and this contradicts the case where \( 2 \leq i_0 \leq n \). So, if \( i_0 = 1 \), we have \( a_1 \leq 2 \). Combining \(-a_i - 2 \leq r_i \) and \( r_i < a_i \), we get \( a_i \geq 0 \). Hence \( 0 \leq a_i \leq 2 \) and \( 0 \leq a_i \leq 1 \) for \( 2 \leq i \leq n \).

First, we assume \( a_2 = 0 \). Now we have \(-2 \leq r_2 \leq -1 \). Comparing \( t_1 + pa_1 + a_2 \geq c \) with \( r_2 + t_1 + pa_1 \leq p - 1 \), we get \( c \leq p - 2 - r_2 \). If \( r_2 = -2 \), we get \( c \leq p \). Then we have \(-a_2 - 1 \leq r_2 \), and this is a contradiction. If \( r_2 = -1 \), we get \( c \leq p - 1 \). Then we have \(-a_2 = r_2 \), and this is a contradiction.

Next, we assume \( a_2 = 1 \). As \( 0 \leq t_i + pa_i - a_{i+1} \leq p - 1 \) for \( 2 \leq i \leq n \), we have \( a_i = 1 \) for all \( i \) and \( t_i = 0 \) for \( 2 \leq i \leq n \). As \( r_2 + pa_1 + t_1 \leq p - 1 \), we have \( r_2 \leq -1 \).
As \( pr_2 + t_2 - a_3 = r_3 + s_2 - pa_2 \), we have \( r_1 = pr_2 + p - e - 1 \leq -e - 1 \). If \( e \geq p + 1 \), then \(-a_3 - 2 \leq r_3 \) and \( r_3 \leq -e - 1 \leq -4 \). This is a contradiction. If \( e = p \), then \(-a_3 - 1 \leq r_3 \) and \( r_3 \leq -e - 1 \leq -3 \). This is a contradiction. If \( e = p - 1 \), then \(-a_3 = r_3 \) and \( r_3 \leq -e - 1 \leq -2 \). This is a contradiction.

Thus we may assume \( t_1 + pa_1 + a_2 \leq e \). We put \( \mathcal{M}_{3,F} = \begin{pmatrix} u^{-a_1} & 0 \\ 0 & u^{a_1} \end{pmatrix}_i \cdot \mathcal{M}_{0,F} \), then

\[
\mathcal{M}_{3,F} \sim \begin{pmatrix} 0 \\ u_{t_1 + pa_1 + a_2} \end{pmatrix} \cdot \begin{pmatrix} u^{-a_1} & u^{a_1} \\ 0 & 0 \end{pmatrix}, \quad \mathcal{M}_{0,F} = \begin{pmatrix} 0 \\ u_{t_1 + pa_2 + a_3} \end{pmatrix}
\]

and \( \mathcal{M}_{1,F} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \mathcal{M}_{3,F} \). Note that \( \mathcal{M}_{3,F} \) satisfies the conditions of Proposition 1.1, and let \( x_3 \) be the point of \( \mathcal{R}^{\mathbb{N}}_{V_0} \) corresponding to \( \mathcal{M}_{3,F} \). If we put \( N_i = \begin{pmatrix} 0 \\ v_i u^{-a_1} \end{pmatrix} \), then

\[
\phi(N_1) \begin{pmatrix} 0 \\ u_{t_1 + pa_1 + a_2} \end{pmatrix} N_2 = \begin{pmatrix} 0 \\ \phi(v_1) v_2 u_{t_1} \end{pmatrix}, \\
\phi(N_1) \begin{pmatrix} u^{a_1} \\ 0 \end{pmatrix} N_{i+1} = 0
\]

for \( 2 \leq i \leq n \). Here we have \( v_i (\phi(v_1) v_2 u_{t_1}) \geq 0 \), because \( s_1 - pa_1 - a_2 \geq 0 \) and \( v_n (u^{a_1} - a_2 - \phi(v_1) v_2 u_{t_1}) \geq 0 \). Hence \( x_1 \) and \( x_3 \) lie on the same connected component by Lemma 2.1.

We are going to compare \( \mathcal{M}_{0,F} \) and \( \mathcal{M}_{3,F} \). First, we treat the case \( e \geq p \). We consider the operations that decrease \( |a_i| \) by 1 for an index \( i \) keeping the condition of \( \phi \)-stability. By Lemma 2.2, these operations do not affect which of the connected components \( x_3 \) lies on. We prove that we can continue the operations until we have \( a_i = 0 \) for all \( i \), that is, \( x_0 \) and \( x_3 \) lie on the same connected component. Suppose that we cannot continue the operations and there is some nonzero \( a_i \). The condition of \( \phi \)-stability is equivalent to

\[
C_1 : 0 \leq s_1 - pa_1 - a_2 \leq e, \quad C_2 : 0 \leq s_2 - pa_2 + a_3 \leq e, \\
\ldots, \quad C_n : 0 \leq s_n - pa_n + a_1 \leq e.
\]

Note that if \( a_i \neq 0 \) or \( a_{i+1} \neq 0 \), we can decrease \( |a_i| \) or \( |a_{i+1}| \) keeping \( C_i \), because \( e \geq p \).

We put

\[
c_i = \sharp \{ i \leq j \leq i + 1 \mid \text{we can decrease } |a_j| \text{ keeping } C_i \},
\]

and claim that \( \sharp \{ j \mid a_j \neq 0 \} = \sum_{i=1}^{n} c_i \). First, if \( a_i \neq 0 \), we have \( c_{i-1} \geq 1 \) and \( c_i \geq 1 \) from the above remark. So we have \( \sharp \{ j \mid a_j \neq 0 \} \leq \sum_{i=1}^{n} c_i \). Second, we count \( a_i \neq 0 \) in not both of \( C_{i-1} \) and \( C_i \), because we cannot continue the operations. So we have \( \sharp \{ j \mid a_j \neq 0 \} \geq \sum_{i=1}^{n} c_i \). Hence we have equality. From this equality, we have \( a_i \neq 0 \) and \( c_i = 1 \) for all \( i \). For \( 2 \leq i \leq n \), we have \( a_i a_{i+1} > 0 \) because \( c_i = 1 \). So we have \( a_1 a_2 > 0 \), but this contradicts \( c_1 = 1 \).
In the case $e = p - 1$. We have $|pa_1 + a_2| \leq p - 1$ by $C_1$, and $|pa_i - a_{i+1}| \leq p - 1$ by $C_i$ for $2 \leq i \leq n$. Summing up these inequalities after multiplying some $p$-powers so that we can eliminate $a_j$ for $j \neq i$, we get $|(p^n + 1)a_i| \leq p^n - 1$. So we have $a_i = 0$ for all $i$.

Hence $x_0$ and $x_3$ lie on the same connected component. This completes the proof.

\[ \square \]

**References**


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