ON THE CONNECTED COMPONENTS OF MODULI SPACES OF KISIN MODULES

NAOKI IMAI

ABSTRACT. We give a proof of a conjecture on the connected components of moduli spaces of Kisin module, which is valid also in the case $p = 2$.

INTRODUCTION

Let K be a p-adic field, and let $V_{\mathbb{F}}$ be a two-dimensional continuous representation of the absolute Galois group G_K over a finite field $\mathbb{F}$ of characteristic p. Take a ϕ-module $M_{\mathbb{F}}$ corresponding to the Galois representation $V_{\mathbb{F}}(-1)$. As in [Kis, Corollary 2.1.13], we can construct a moduli space $\mathscr{GR}_{V_{\mathbb{F}},0}$ of Kisin modules in $M_{\mathbb{F}}$, that is a projective scheme over $\mathbb{F}$. Let $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ be a closed subscheme of $\mathscr{GR}_{V_{\mathbb{F}},0}$ determined by the condition that p-adic Hodge type is $\mathbf{v} = 1$.

In the case $p > 2$, a Kisin module in $M_{\mathbb{F}}$ corresponds a finite flat models of $V_{\mathbb{F}}$, and $\mathscr{GR}_{V_{\mathbb{F}},0}$ is called a moduli space of finite flat models of $V_{\mathbb{F}}$. In this case, Kisin conjectured that the non-ordinary locus of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ is connected. (In fact, this is a special case of [Kis, Conjecture 2.4.16].) This conjecture was proved by Kisin in [Kis] if K is totally ramified over $\mathbb{Q}_p$, by Gee in [Gee] if $V_{\mathbb{F}}$ is the trivial representation, and by the author in [Ima] for general K and $V_{\mathbb{F}}$. In the proof in [Ima], we need the condition $p > 2$. In this paper, we prove the conjecture for all p. The main theorem is the following.

Theorem. *The non-ordinary locus of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ is geometrically connected.*

The outline of the proof is the same as the proof in [Ima], but we need some more sophisticated arguments to treat the case $p = 2$.

Acknowledgment. The author would like to thank Mark Kisin for suggesting him to consider the problem of the connectedness in the case $p = 2$. He is grateful to a referee for a careful reading of this paper and suggestions for improvements.

Notation. Throughout this paper, we use the following notation. Let p be a prime number, and k be a finite extension of $\mathbb{F}_p$ of cardinality $q = p^n$. The Witt ring of k is denoted by $W(k)$, and let $K_0 = W(k)[1/p]$. Let K be a totally ramified extension of K_0 of degree e, and $\mathcal{O}_K$ be the ring of integers of K. The absolute Galois group of K is denoted by G_K. Let $\mathbb{F}$ be a finite field of characteristic p. The formal power series ring of u over $\mathbb{F}$ is denoted by $\mathbb{F}[[u]]$, and its quotient field is denoted by $\mathbb{F}((u))$. Let v_u be the valuation of $\mathbb{F}((u))$ normalized by $v_u(u) = 1$. For a field F, the algebraic closure of F is denoted by $\overline{F}$ and the separable closure of F is denoted by F^{sep}.

1. Preliminaries

First of all, we recall some notation from [Kis], and the interested reader should consult [Kis] for more detailed definitions.

We put $\mathfrak{S} = W(k)[[u]]$. Let $\mathcal{O}_{\mathcal{E}}$ be the p-adic completion of $\mathfrak{S}[1/u]$. There is an action of ϕ on $\mathcal{O}_{\mathcal{E}}$ determined by Frobenius on $W(k)$ and $u \mapsto u^p$. We take and fix a uniformizer π of $\mathcal{O}_K$. We choose elements $\pi_m \in \overline{K}$ such that $\pi_0 = \pi$ and $\pi_{m+1}^p = \pi_m$ for $m \geq 0$, and put $K_\infty = \bigcup_{m\geq 0} K(\pi_m)$. Let $\Phi\mathrm{M}_{\mathcal{O}_{\mathcal{E}},\mathbb{F}}$ be the category of finite $\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F}$-modules M equipped with ϕ-semi-linear map $M \to M$ such that the induced $\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F}$-linear map $\phi^*(M) \to M$ is an isomorphism. Let $\mathrm{Rep}_{\mathbb{F}}(G_{K_\infty})$ be the category of finite-dimensional continuous representations of G_{K_∞} over $\mathbb{F}$. Then the functor

$$T : \Phi\mathrm{M}_{\mathcal{O}_{\mathcal{E}},\mathbb{F}} \to \mathrm{Rep}_{\mathbb{F}}(G_{K_\infty});\ M \mapsto \bigl(k((u))^{\mathrm{sep}} \otimes_{k((u))} M\bigr)^{\phi=1}$$

gives an equivalence of abelian categories as in [Kis, (1.1.12)]. Here ϕ acts on $k((u))^{\mathrm{sep}}$ by the p-th power map.

Let $V_{\mathbb{F}}$ be a continuous two-dimensional representation of G_K over $\mathbb{F}$. We take the ϕ-module $M_{\mathbb{F}} \in \Phi\mathrm{M}_{\mathcal{O}_{\mathcal{E}},\mathbb{F}}$ such that $T(M_{\mathbb{F}})$ is isomorphic to $V_{\mathbb{F}}(-1)|_{G_{K_\infty}}$. Here (-1) denotes the inverse of the Tate twist.

From now on, we assume $\mathbb{F}_{q^2} \subset \mathbb{F}$ and fix an embedding $k \hookrightarrow \mathbb{F}$. This assumption does not matter, because we may extend $\mathbb{F}$ to prove the main theorem. We consider the isomorphism

$$\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F} \cong k((u)) \otimes_{\mathbb{F}_p} \mathbb{F} \xrightarrow{\sim} \prod_{\sigma \in \mathrm{Gal}(k/\mathbb{F}_p)} \mathbb{F}((u))\ ;\ \Bigl(\sum a_i u^i\Bigr) \otimes b \mapsto \Bigl(\sum \sigma(a_i) b u^i\Bigr)_\sigma$$

and let $\epsilon_\sigma \in k((u)) \otimes_{\mathbb{F}_p} \mathbb{F}$ be the primitive idempotent corresponding to σ. Take $\sigma_1, \cdots, \sigma_n \in \mathrm{Gal}(k/\mathbb{F}_p)$ such that $\sigma_{i+1} = \sigma_i \circ \phi^{-1}$. Here we regard ϕ as the p-th power Frobenius, and use the convention that $\sigma_{n+i} = \sigma_i$. In the following, we often use such conventions. Then we have $\phi(\epsilon_{\sigma_i}) = \epsilon_{\sigma_{i+1}}$, and $\phi : M_{\mathbb{F}} \to M_{\mathbb{F}}$ determines $\phi : \epsilon_{\sigma_i} M_{\mathbb{F}} \to \epsilon_{\sigma_{i+1}} M_{\mathbb{F}}$.

For $(A_i)_{1\leq i\leq n} \in GL_2\bigl(\mathbb{F}((u))\bigr)^n$, we write

$$M_{\mathbb{F}} \sim (A_1, A_2, \ldots, A_n) = (A_i)_i$$

if there is a basis $\{e_1^i, e_2^i\}$ of $\epsilon_{\sigma_i} M_{\mathbb{F}}$ over $\mathbb{F}((u))$ such that $\phi\begin{pmatrix} e_1^i \\ e_2^i \end{pmatrix} = A_i \begin{pmatrix} e_1^{i+1} \\ e_2^{i+1} \end{pmatrix}$. We use the same notation for any sublattice $\mathfrak{M}_{\mathbb{F}} \subset M_{\mathbb{F}}$ similarly. Here and in the following, we consider only sublattices that are $\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathbb{F}$-modules.

Finally, for any sublattice $\mathfrak{M}_{\mathbb{F}} \subset M_{\mathbb{F}}$ with a chosen basis $\{e_1^i, e_2^i\}_{1\leq i\leq n}$ and $B = (B_i)_{1\leq i\leq n} \in GL_2\bigl(\mathbb{F}((u))\bigr)^n$, the module generated by the entries of $\left\langle B_i \begin{pmatrix} e_1^i \\ e_2^i \end{pmatrix} \right\rangle$ with the basis given by these entries is denoted by $B \cdot \mathfrak{M}_{\mathbb{F}}$. Note that $B \cdot \mathfrak{M}_{\mathbb{F}}$ depends on the choice of the basis of $\mathfrak{M}_{\mathbb{F}}$.

For each $\mathbb{Q}_p$-algebra embedding $\psi : K \to \overline{K}_0$, we put $v_\psi = 1$ and set $\mathbf{v} = (v_\psi)_\psi$. Then $\mathscr{GR}_{V_{\mathbb{F}},0}^{\mathbf{v}}$ is the moduli space of Kisin modules with p-adic Hodge type $\mathbf{v}$. The rational points of $\mathscr{GR}_{V_{\mathbb{F}},0}^{\mathbf{v}}$ are described as in the following.

Proposition 1.1. *If $\mathbb{F}'$ is a finite extension of $\mathbb{F}$, the elements of $\mathscr{GR}_{V_{\mathbb{F}},0}^{\mathbf{v}}(\mathbb{F}')$ naturally correspond to free $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}'$-submodules $\mathfrak{M}_{\mathbb{F}'} \subset M_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ of rank 2 that satisfy the following:*

(1) *$\mathfrak{M}_{\mathbb{F}'}$ is ϕ-stable.*

(2) *For some (so any) choice of $k[[u]]\otimes_{\mathbb{F}_p}\mathbb{F}'$-basis for $\mathfrak{M}_{\mathbb{F}'}$, and for each $\sigma\in\mathrm{Gal}(k/\mathbb{F}_p)$, the map*
$$\phi:\epsilon_\sigma\mathfrak{M}_{\mathbb{F}'}\to\epsilon_{\sigma\circ\phi^{-1}}\mathfrak{M}_{\mathbb{F}'}$$
has determinant αu^e for some $\alpha\in\mathbb{F}'[[u]]^\times$.

Proof. This is [Gee, Lemma 2.2]. □

2. Main theorem

To prove the main theorem, in fact we prove that the non-ordinary component of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ is rationally connected. We use the following two Lemmas to join two points by $\mathbb{P}^1$.

Lemma 2.1. *Suppose $x_1,x_2\in\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F})$ correspond to objects $\mathfrak{M}_{1,\mathbb{F}},\mathfrak{M}_{2,\mathbb{F}}$ of $(\mathrm{Mod}/\mathfrak{S})_{\mathbb{F}}$ respectively. We fix bases of $\mathfrak{M}_{1,\mathbb{F}},\mathfrak{M}_{2,\mathbb{F}}$ over $k[[u]]\otimes_{\mathbb{F}_p}\mathbb{F}$. We assume that there is a nilpotent element $N=(N_i)_{1\le i\le n}$ of $M_2\big(\mathbb{F}((u))\big)^n$ such that $\mathfrak{M}_{2,\mathbb{F}}=(1+N)\cdot\mathfrak{M}_{1,\mathbb{F}}$. Let $A=(A_i)_{1\le i\le n}$ be an element of $GL_2\big(\mathbb{F}((u))\big)^n$ such that $\mathfrak{M}_{1,\mathbb{F}}\sim A$. If $\phi(N_i)A_iN_{i+1}\in M_2\big(\mathbb{F}[[u]]\big)$ for all i, then there is a morphism $\mathbb{P}^1\to\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ sending 0 to x_1 and 1 to x_2.*

Proof. This is [Gee, Lemma 2.4]. □

Lemma 2.2. *Suppose $n\ge 2$. Let $\mathfrak{M}_{\mathbb{F}}$ be the object of $(\mathrm{Mod}/\mathfrak{S})_{\mathbb{F}}$ corresponding to a point $x\in\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F})$. Fix a basis of $\mathfrak{M}_{\mathbb{F}}$ over $k[[u]]\otimes_{\mathbb{F}_p}\mathbb{F}$. Consider $U^{(i)}=(U^{(i)}_j)_{1\le j\le n}\in GL_2\big(\mathbb{F}((u))\big)^n$ such that $U^{(i)}_i=\begin{pmatrix}u&0\\0&u^{-1}\end{pmatrix}$ and $U^{(i)}_j=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ for all $j\ne i$. If $U^{(i)}\cdot\mathfrak{M}_{\mathbb{F}}$ is ϕ-stable, it corresponds to a point $x'\in\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F})$, and there is a morphism $\mathbb{P}^1\to\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ sending 0 to x and 1 to x'. If $(U^{(i)})^{-1}\cdot\mathfrak{M}_{\mathbb{F}}$ is ϕ-stable, it corresponds to a point $x''\in\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F})$, and there is a morphism $\mathbb{P}^1\to\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ sending 0 to x and 1 to x''.*

Proof. This is [Ima, Lemma 2.3]. □

To prove the main theorem, it suffices to show the following theorem. The strategy of the proof is the same as in [Ima], and we focus on the changed points in the case $p=2$.

Theorem 2.3. *Let $\mathbb{F}'$ be a finite extension of $\mathbb{F}$. Suppose $x_1,x_2\in\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F}')$ correspond to objects $\mathfrak{M}_{1,\mathbb{F}'},\mathfrak{M}_{2,\mathbb{F}'}$ of $(\mathrm{Mod}/\mathfrak{S})_{\mathbb{F}'}$ respectively. If $\mathfrak{M}_{1,\mathbb{F}'}$ and $\mathfrak{M}_{2,\mathbb{F}'}$ are both non-ordinary, then x_1 and x_2 lie on the same connected component of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$.*

Proof. When $n=1$, this was proved in [Kis], and we did not use the condition $p>2$ in the proof. If $e<p-1$, then $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F}')$ is one point by [Ray, Theorem 3.3.3]. So we may assume $n\ge 2$ and $e\ge p-1$. Furthermore, replacing $V_{\mathbb{F}}$ by $V_{\mathbb{F}}\otimes_{\mathbb{F}}\mathbb{F}'$, we may assume $\mathbb{F}=\mathbb{F}'$.

In the case where $V_{\mathbb{F}}$ is reducible, the proof of [Ima, Theorem 2.4] goes on, even if $p=2$. So, by a base change, we may assume that $V_{\mathbb{F}}$ is absolutely irreducible. As in the proof of [Ima, Theorem 2.4], we can prove that, after extending the field $\mathbb{F}$, there exists a basis such that
$$M_{\mathbb{F}}\sim\left(\alpha_1\begin{pmatrix}0&u^{s_1}\\u^{t_1}&0\end{pmatrix},\alpha_2\begin{pmatrix}u^{s_2}&0\\0&u^{t_2}\end{pmatrix},\dots,\alpha_n\begin{pmatrix}u^{s_n}&0\\0&u^{t_n}\end{pmatrix}\right)$$

where $\alpha_i \in \mathbb{F}$, $0 \le s_i, t_i \le e$, $s_i + t_i = e$ and $|s_i - t_i| \le p+1$ for all i. Note that we have proved that we may assume $|s_i - t_i| \le p+1$ for all i in the last paragraph of [Ima, p. 1197]

Let $\mathfrak{M}_{\mathbb{F},0}$ be the $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$-module generated by the basis giving the above matrix expression. Then $\mathfrak{M}_{\mathbb{F},0}$ satisfies the condition in Proposition 1.1. We take the point x_0 of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F})$ corresponding to $\mathfrak{M}_{\mathbb{F},0}$. We are going to prove that x_0 and x_1 lie on the same connected component. We can prove that x_0 and x_2 lie on the same connected component by the same argument.

By the Iwasawa decomposition and the determinant conditions, we can take $B = (B_i)_{1 \le i \le n} \in GL_2\big(\mathbb{F}((u))\big)^n$ such that $\mathfrak{M}_{1,\mathbb{F}} = B \cdot \mathfrak{M}_{0,\mathbb{F}}$ and $B_i = \begin{pmatrix} u^{-a_i} & v_i \\ 0 & u^{a_i} \end{pmatrix}$ for $a_i \in \mathbb{Z}$ and $v_i \in \mathbb{F}((u))$. Then we put $r_i = v_u(v_i)$. Now we have

$$\phi(B_1)\begin{pmatrix} 0 & u^{s_1} \\ u^{t_1} & 0 \end{pmatrix} B_2^{-1} = \begin{pmatrix} \phi(v_1)u^{t_1+a_2} & u^{s_1-pa_1-a_2} - \phi(v_1)v_2u^{t_1} \\ u^{t_1+pa_1+a_2} & -v_2u^{t_1+pa_1} \end{pmatrix},$$
$$\phi(B_i)\begin{pmatrix} u^{s_i} & 0 \\ 0 & u^{t_i} \end{pmatrix} B_{i+1}^{-1} = \begin{pmatrix} u^{s_i-pa_i+a_{i+1}} & \phi(v_i)u^{t_i-a_{i+1}} - v_{i+1}u^{s_i-pa_i} \\ 0 & u^{t_i+pa_i-a_{i+1}} \end{pmatrix}$$

for $2 \le i \le n$. On the right-hand sides, every component of the matrices is integral because $\mathfrak{M}_{1,\mathbb{F}}$ is ϕ-stable.

First, we consider the case $t_1 + pa_1 + a_2 > e$. In this case,

$$(pr_1 + t_1 + a_2) + (r_2 + t_1 + pa_1) = e,\ s_1 - pa_1 - a_2 = pr_1 + r_2 + t_1 < 0$$

by the ϕ-stability and the determinant conditions of $\mathfrak{M}_{1,\mathbb{F}}$. We have $a_1 > r_1$, because $t_1 + pa_1 + a_2 > e \ge pr_1 + t_1 + a_2$. Similarly, we have $a_2 > r_2$, because $t_1 + pa_1 + a_2 > e \ge r_2 + t_1 + pa_1$.

We consider the following operations:

$$a_i \rightsquigarrow a_i - 1,\ v_i \rightsquigarrow uv_i,\ \text{if it preserves the } \phi\text{-stability of } B \cdot \mathfrak{M}_{0,\mathbb{F}}.$$

These operations replace x_1 by a point that lies on the same connected component as x_1 by Lemma 2.2. We prove that we can continue these operations until we get to the situation where $t_1 + pa_1 + a_2 \le e$. In other words, we reduce the problem to the case $t_1 + pa_1 + a_2 \le e$. If we can continue the operations endlessly, we get to the situation where $t_1 + pa_1 + a_2 \le e$, because the conditions $s_i - pa_i + a_{i+1} \ge 0$ for $2 \le i \le n$ exclude that both a_1 and a_2 remain bounded below. Suppose we cannot continue the operations. This is equivalent to the following condition:

$$s_n - pa_n + a_1 = 0 \text{ or } r_2 + t_1 + pa_1 \le p-1,$$
$$pr_1 + t_1 + a_2 = 0 \text{ or } t_2 + pa_2 - a_3 \le p-1,$$
$$s_{i-1} - pa_{i-1} + a_i = 0 \text{ or } t_i + pa_i - a_{i+1} \le p-1 \text{ for each } 3 \le i \le n.$$

If $e \ge p$, there are only the following two cases, because $(pr_1 + t_1 + a_2) + (r_2 + t_1 + pa_1) = e$ and $(s_i - pa_i + a_{i+1}) + (t_i + pa_i - a_{i+1}) = e$ for $2 \le i \le n$.

$$\text{Case 1}: pr_1 + t_1 + a_2 = 0,\ s_i - pa_i + a_{i+1} = 0 \text{ for } 2 \le i \le n.$$
$$\text{Case 2}: r_2 + t_1 + pa_1 \le p-1,\ t_i + pa_i - a_{i+1} \le p-1 \text{ for } 2 \le i \le n.$$

If $e = p-1$, clearly it is in Case 2.

In the Case 1, we have a contradiction as in the proof of [Ima, Theorem 2.4]. So we may assume that it is in the Case 2.

Then we can show that

$$r_i < a_i,\ pr_i + t_i - a_{i+1} = r_{i+1} + s_i - pa_i < 0 \text{ for } 2 \leq i \leq n$$

as in the proof of [Ima, Theorem 2.4]. Combining these equations with $s_1 - pa_1 - a_2 = pr_1 + r_2 + t_1$, we get

$$\begin{aligned}
-(p^n+1)r_1 &= (p^n+1)a_1 + (s_n - t_n) + p(s_{n-1} - t_{n-1}) + \\
&\qquad \cdots + p^{n-3}(s_3 - t_3) + p^{n-2}(s_2 - t_2) - p^{n-1}(s_1 - t_1), \\
-(p^n+1)r_2 &= (p^n+1)a_2 - (s_1 - t_1) - p(s_n - t_n) - \\
&\qquad \cdots - p^{n-3}(s_4 - t_4) - p^{n-2}(s_3 - t_3) - p^{n-1}(s_2 - t_2), \\
-(p^n+1)r_3 &= (p^n+1)a_3 + (s_2 - t_2) - p(s_1 - t_1) - \\
&\qquad \cdots - p^{n-3}(s_5 - t_5) - p^{n-2}(s_4 - t_4) - p^{n-1}(s_3 - t_3), \\
&\ \vdots \\
-(p^n+1)r_n &= (p^n+1)a_n + (s_{n-1} - t_{n-1}) + p(s_{n-2} - t_{n-2}) + \\
&\qquad \cdots + p^{n-3}(s_2 - t_2) - p^{n-2}(s_1 - t_1) - p^{n-1}(s_n - t_n).
\end{aligned}$$

As $|s_i - t_i| \leq p+1$ and

$$(p+1) + p(p+1) + \cdots + p^{n-1}(p+1) = \left(\frac{p^n - 1}{p-1}\right)(p+1) < 3(p^n+1),$$

we get $-a_i - 2 \leq r_i \leq -a_i + 2$. If $e = p$, as $|s_i - t_i| \leq p$ and

$$p + p^2 + \cdots + p^n = \left(\frac{p^n - 1}{p-1}\right)p < 2(p^n+1),$$

we get $-a_i - 1 \leq r_i \leq -a_i + 1$. If $e = p-1$, as $|s_i - t_i| \leq p-1$ and

$$(p-1) + p(p-1) + \cdots + p^{n-1}(p-1) = \left(\frac{p^n - 1}{p-1}\right)(p-1) < (p^n+1),$$

we get $-a_i = r_i$.

As $r_2 + t_1 + pa_1 \leq p-1$, we have

$$pa_1 \leq t_1 + pa_1 \leq p - 1 - r_2 \leq a_2 + p + 1.$$

For $2 \leq i \leq n$, as $t_i + pa_i - a_{i+1} \leq p-1$, we have

$$pa_i \leq t_i + pa_i \leq a_{i+1} + p - 1.$$

Take an index i_0 such that a_{i_0} is the greatest. If $2 \leq i_0 \leq n$, we get $a_{i_0} \leq 1$ by $pa_{i_0} \leq a_{i_0+1} + p - 1 \leq a_{i_0} + p - 1$. If $i_0 = 1$ and $a_1 \geq 3$, we get $a_2 \geq 3$, by $pa_1 \leq a_2 + p + 1$, and this contradicts the case where $2 \leq i_0 \leq n$. So, if $i_0 = 1$, we have $a_1 \leq 2$. Combining $-a_i - 2 \leq r_i$ and $r_i < a_i$, we get $a_i \geq 0$. Hence $0 \leq a_1 \leq 2$ and $0 \leq a_i \leq 1$ for $2 \leq i \leq n$.

First, we assume $a_2 = 0$. Now we have $-2 \leq r_2 \leq -1$. Comparing $t_1 + pa_1 + a_2 > e$ with $r_2 + t_1 + pa_1 \leq p-1$, we get $e \leq p - 2 - r_2$. If $r_2 = -2$, we get $e \leq p$. Then we have $-a_2 - 1 \leq r_2$, and this is a contradiction. If $r_2 = -1$, we get $e \leq p-1$. Then we have $-a_2 = r_2$, and this is a contradiction.

Next, we assume $a_2 = 1$. As $0 \leq t_i + pa_i - a_{i+1} \leq p-1$ for $2 \leq i \leq n$, we have $a_i = 1$ for all i and $t_i = 0$ for $2 \leq i \leq n$. As $r_2 + pa_1 + t_1 \leq p-1$, we have $r_2 \leq -1$.

As $pr_2+t_2-a_3=r_3+s_2-pa_2$, we have $r_3=pr_2+p-e-1\le -e-1$. If $e\ge p+1$, then $-a_3-2\le r_3$ and $r_3\le -e-1\le -4$. This is a contradiction. If $e=p$, then $-a_3-1\le r_3$ and $r_3\le -e-1\le -3$. This is a contradiction. If $e=p-1$, then $-a_3=r_3$ and $r_3\le -e-1\le -2$. This is a contradiction.

Thus we may assume $t_1+pa_1+a_2\le e$. We put $\mathfrak{M}_{3,\mathbb{F}}=\left(\begin{pmatrix}u^{-a_i} & 0\\ 0 & u^{a_i}\end{pmatrix}\right)_i\cdot\mathfrak{M}_{0,\mathbb{F}}$, then

$$\mathfrak{M}_{3,\mathbb{F}}\sim\left(\alpha_1\begin{pmatrix}0 & u^{s_1-pa_1-a_2}\\ u^{t_1+pa_1+a_2} & 0\end{pmatrix},\alpha_2\begin{pmatrix}u^{s_2-pa_2+a_3} & 0\\ 0 & u^{t_2+pa_2-a_3}\end{pmatrix},\right.$$
$$\left.\ldots,\alpha_n\begin{pmatrix}u^{s_n-pa_n+a_1} & 0\\ 0 & u^{t_n+pa_n-a_1}\end{pmatrix}\right)$$

and $\mathfrak{M}_{1,\mathbb{F}}=\left(\begin{pmatrix}1 & v_iu^{-a_i}\\ 0 & 1\end{pmatrix}\right)_i\cdot\mathfrak{M}_{3,\mathbb{F}}$. Note that $\mathfrak{M}_{3,\mathbb{F}}$ satisfies the conditions of Proposition 1.1, and let x_3 be the point of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ corresponding to $\mathfrak{M}_{3,\mathbb{F}}$. If we put $N_i=\begin{pmatrix}0 & v_iu^{-a_i}\\ 0 & 0\end{pmatrix}$, then

$$\phi(N_1)\begin{pmatrix}0 & u^{s_1-pa_1-a_2}\\ u^{t_1+pa_1+a_2} & 0\end{pmatrix}N_2=\begin{pmatrix}0 & \phi(v_1)v_2u^{t_1}\\ 0 & 0\end{pmatrix},$$
$$\phi(N_i)\begin{pmatrix}u^{s_i-pa_i+a_{i+1}} & 0\\ 0 & u^{t_i+pa_i-a_{i+1}}\end{pmatrix}N_{i+1}=0$$

for $2\le i\le n$. Here we have $v_u\bigl(\phi(v_1)v_2u^{t_1}\bigr)\ge 0$, because $s_1-pa_1-a_2\ge 0$ and $v_u\bigl(u^{s_1-pa_1-a_2}-\phi(v_1)v_2u^{t_1}\bigr)\ge 0$. Hence x_1 and x_3 lie on the same connected component by Lemma 2.1.

We are going to compare $\mathfrak{M}_{0,\mathbb{F}}$ and $\mathfrak{M}_{3,\mathbb{F}}$. First, we treat the case $e\ge p$. We consider the operations that decrease $|a_i|$ by 1 for an index i keeping the condition of ϕ-stability. By Lemma 2.2, these operations do not affect which of the connected components x_3 lies on. We prove that we can continue the operations until we have $a_i=0$ for all i, that is, x_0 and x_3 lie on the same connected component. Suppose that we cannot continue the operations and there is some nonzero a_i. The condition of ϕ-stability is equivalent to

$$C_1:0\le s_1-pa_1-a_2\le e,\ C_2:0\le s_2-pa_2+a_3\le e,$$
$$\ldots,\ C_n:0\le s_n-pa_n+a_1\le e.$$

Note that if $a_i\ne 0$ or $a_{i+1}\ne 0$, we can decrease $|a_i|$ or $|a_{i+1}|$ keeping C_i, because $e\ge p$.

We put

$$c_i=\sharp\bigl\{i\le j\le i+1\ \big|\ \text{we can decrease } |a_j| \text{ keeping } C_i\bigr\},$$

and claim that $\sharp\{j\mid a_j\ne 0\}=\sum_{i=1}^n c_i$. First, if $a_i\ne 0$, we have $c_{i-1}\ge 1$ and $c_i\ge 1$ from the above remark. So we have $\sharp\{j\mid a_j\ne 0\}\le\sum_{i=1}^n c_i$. Second, we count $a_i\ne 0$ in not both of C_{i-1} and C_i, because we cannot continue the operations. So we have $\sharp\{j\mid a_j\ne 0\}\ge\sum_{i=1}^n c_i$. Hence we have equality. From this equality, we have $a_i\ne 0$ and $c_i=1$ for all i. For $2\le i\le n$, we have $a_ia_{i+1}>0$ because $c_i=1$. So we have $a_1a_2>0$, but this contradicts $c_1=1$.

In the case $e = p-1$. We have $|pa_1+a_2| \leq p-1$ by C_1, and $|pa_i - a_{i+1}| \leq p-1$ by C_i for $2 \leq i \leq n$. Summing up these inequalities after multiplying some p-powers so that we can eliminate a_j for $j \neq i$, we get $|(p^n+1)a_i| \leq p^n - 1$. So we have $a_i = 0$ for all i.

Hence x_0 and x_3 lie on the same connected component. This completes the proof. □

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Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502 Japan

E-mail address: `naoki@kurims.kyoto-u.ac.jp`