ON THE CONNECTED COMPONENTS OF MODULI SPACES
OF KISIN MODULES

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Abstract. We give a proof of a conjecture on the connected components of
moduli spaces of Kisin module, which is valid also in the case \( p = 2 \).

Introduction

Let \( K \) be a \( p \)-adic field, and let \( V_F \) be a two-dimensional continuous representation
of the absolute Galois group \( G_K \) over a finite field \( \mathbb{F} \) of characteristic \( p \). Take
a \( \phi \)-module \( M_\phi \) corresponding to the Galois representation \( V_F \). As in [Kis, Corollary 2.1.13],
we can construct a moduli space \( \mathcal{M}_{V_F,0}^\times \) of Kisin modules in \( M_\phi \),
that is a projective scheme over \( \mathbb{F} \). Let \( \mathcal{M}_{V_F,0} \) be a closed subscheme of \( \mathcal{M}_{V_F,0}^\times \)
determined by the condition that \( p \)-adic Hodge type is \( v = 1 \).

In the case \( p > 2 \), a Kisin module in \( M_\phi \) corresponds a finite flat models of
\( V_F \), and \( \mathcal{M}_{V_F,0} \) is called a moduli space of finite flat models of \( V_F \). In this case,
Kisin conjectured that the non-ordinary locus of \( \mathcal{M}_{V_F,0}^\times \) is connected. (In fact,
this is a special case of [Kis, Conjecture 2.4.16].) This conjecture was proved by
Kisin in [Kis] if \( K \) is totally ramified over \( \mathbb{Q}_p \), by Gee in [Gee] if \( V_F \) is the trivial
representation, and by the author in [Ima] for general \( K \) and \( V_F \). In the proof in
[Ima], we need the condition \( p > 2 \). In this paper, we prove the conjecture for all
\( p \). The main theorem is the following.

Theorem. The non-ordinary locus of \( \mathcal{M}_{V_F,0}^\times \) is geometrically connected.

The outline of the proof is the same as the proof in [Ima], but we need some
more sophisticated arguments to treat the case \( p = 2 \).

Acknowledgment. The author would like to thank Mark Kisin for suggesting him
to consider the problem of the connectedness in the case \( p = 2 \). He is grateful to a
referee for a careful reading of this paper and suggestions for improvements.

Notation. Throughout this paper, we use the following notation. Let \( p \) be a prime
number, and \( k \) be a finite extension of \( \mathbb{F}_p \) of cardinality \( q = p^n \). The Witt ring
of \( k \) is denoted by \( W(k) \), and let \( K_0 = W(k)[1/p] \). Let \( K \) be a totally ramified
extension of \( K_0 \) of degree \( e \), and \( \mathcal{O}_K \) be the ring of integers of \( K \). The absolute
Galois group of \( K \) is denoted by \( G_K \). Let \( \mathbb{F} \) be a finite field of characteristic \( p \).
The formal power series ring of \( u \) over \( \mathbb{F} \) is denoted by \( \mathbb{F}[[u]] \), and its quotient field is
denoted by \( \mathbb{F}((u)) \). Let \( v_u \) be the valuation of \( \mathbb{F}((u)) \) normalized by \( v_u(u) = 1 \). For
a field \( F \), the algebraic closure of \( F \) is denoted by \( \overline{F} \) and the separable closure of
\( F \) is denoted by \( F^\mathrm{sep} \).
1. Preliminaries

First of all, we recall some notation from [Kis], and the interested reader should consult [Kis] for more detailed definitions.

We put $\mathcal{S} = W(k)[[u]]$. Let $\mathcal{O}_\mathcal{S}$ be the $p$-adic completion of $\mathcal{S}[1/u]$. There is an action of $\phi$ on $\mathcal{O}_\mathcal{S}$ determined by Frobenius on $W(k)$ and $u \mapsto u^p$. We take and fix a uniformizer $\pi$ of $\mathcal{O}_K$. We choose elements $\pi_m \in \mathcal{K}$ such that $\pi_0 = \pi$ and $\pi_{m+1}^p = \pi_m$ for $m \geq 0$, and put $K_\infty = \bigcup_{m \geq 0} K(\pi_m)$. Let $\text{Phi}\mathcal{O}_\mathcal{S}, \mathcal{F}$ be the category of finite $\mathcal{O}_\mathcal{S} \otimes_{\mathcal{Z}_p} \mathcal{F}$-modules $M$ equipped with $\phi$-semi-linear map $M \to M$ such that the induced $\mathcal{O}_\mathcal{S} \otimes_{\mathcal{Z}_p} \mathcal{F}$-linear map $\phi^* (M) \to M$ is an isomorphism. Let $\text{Rep}_\mathcal{F}(G_{K_\infty})$ be the category of finite-dimensional continuous representations of $G_{K_\infty}$ over $\mathcal{F}$.

Then the functor

$$T : \text{Phi}\mathcal{O}_\mathcal{S}, \mathcal{F} \to \text{Rep}_\mathcal{F}(G_{K_\infty}) : M \mapsto (k((u))^{\text{sep}} \otimes_{k((u))} M)^{\phi=1}$$

gives an equivalence of abelian categories as in [Kis, (1.1.12)]. Here $\phi$ acts on $k((u))$ by the $p$-th power map.

Let $V_\mathcal{F}$ be a continuous two-dimensional representation of $G_K$ over $\mathcal{F}$. We take the $\phi$-module $M_\mathcal{F} \in \text{Phi}\mathcal{O}_\mathcal{S}, \mathcal{F}$ such that $T(M_\mathcal{F})$ is isomorphic to $V_\mathcal{F}(-1)|_{G_{K_\infty}}$. Here $(-1)$ denotes the inverse of the Tate twist.

From now on, we assume $\mathcal{O}_\mathcal{S} \subseteq \mathcal{F}$ and fix an embedding $k \to \mathcal{F}$. This assumption does not matter, because we may extend $\mathcal{F}$ to prove the main theorem. We consider the isomorphism

$$\mathcal{O}_\mathcal{S} \otimes_{\mathcal{Z}_p} \mathcal{F} \cong k((u)) \otimes_{\mathcal{F}_p} \mathcal{F} \cong \prod_{\sigma \in \text{Gal}(k/\mathcal{F}_p)} \mathcal{F}((u)) : \left( \sum a_i u^i \right) \otimes b \mapsto \left( \sum \sigma(a_i) b u^i \right)_{\sigma}$$

and let $\epsilon_\sigma \in k((u)) \otimes_{\mathcal{F}_p} \mathcal{F}$ be the primitive idempotent corresponding to $\sigma$. Take $\sigma_1, \ldots, \sigma_n \in \text{Gal}(k/\mathcal{F}_p)$ such that $\sigma_{i+1} = \sigma_i \circ \phi^{-1}$. Here we regard $\phi$ as the $p$-th power Frobenius, and use the convention that $\sigma_{n+1} = \sigma_1$. In the following, we often use such conventions. Then we have $\phi(\epsilon_\sigma) = \epsilon_{\sigma_{i+1}}$, and $\phi : M_\mathcal{F} \to M_\mathcal{F}$ determines $\phi : \epsilon_\sigma M_\mathcal{F} \to \epsilon_{\sigma_{i+1}} M_\mathcal{F}$.

For $(A_i)_{1 \leq i \leq n} \in GL_2(\mathcal{F}((u)))^n$, we write

$$M_\mathcal{F} \sim (A_1, A_2, \ldots, A_n) = (A_i)$$

if there is a basis $\{e_1^i, e_2^i\}$ of $\epsilon_\sigma M_\mathcal{F}$ over $\mathcal{F}((u))$ such that $\phi \left( \begin{pmatrix} e_1^i \\ e_2^i \end{pmatrix} \right) = A_i \left( \begin{pmatrix} e_1^{i+1} \\ e_2^{i+1} \end{pmatrix} \right)$.

We use the same notation for any sublattice $\mathfrak{M}_\mathcal{F} \subset M_\mathcal{F}$ similarly. Here and in the following, we consider only sublattices that are $\mathcal{S} \otimes_{\mathcal{Z}_p} \mathcal{F}$-modules.

Finally, for any sublattice $\mathfrak{M}_\mathcal{F} \subset M_\mathcal{F}$ with a chosen basis $\{e_1^1, e_2^1\}$, we take $B = (B_i)_{1 \leq i \leq n} \in GL_2(\mathcal{F}((u)))^n$, the module generated by the entries of $\left( B_i \left( \begin{pmatrix} e_1^1 \\ e_2^1 \end{pmatrix} \right) \right)$ with the basis given by these entries is denoted by $B \cdot \mathfrak{M}_\mathcal{F}$. Note that $B \cdot \mathfrak{M}_\mathcal{F}$ depends on the choice of the basis of $\mathfrak{M}_\mathcal{F}$.

For each $\mathcal{O}_p$-algebra embedding $\psi : K \to \mathcal{K}$, we put $v_\psi = 1$ and set $v = (v_\psi)$. Then $\mathcal{H}^{\mathfrak{M}_{\psi}, 0}$ is the moduli space of Kisin modules with $p$-adic Hodge type $v$. The rational points of $\mathcal{H}^{\mathfrak{M}_{\psi}, 0}$ are described as in the following.

**Proposition 1.1.** If $\mathcal{F}'$ is a finite extension of $\mathcal{F}$, the elements of $\mathcal{H}^{\mathfrak{M}_{\psi}, 0}(\mathcal{F}')$ naturally correspond to free $k[[u]] \otimes_{\mathcal{F}_p} \mathcal{F}'$-submodules $\mathfrak{M}_{\mathcal{F}'} \subset M_\mathcal{F} \otimes_{\mathcal{F}} \mathcal{F}'$ of rank 2 that satisfy the following:

(1) $\mathfrak{M}_{\mathcal{F}'}$ is $\phi$-stable.
(2) For some (so any) choice of $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}'$-basis for $\mathcal{M}_{\mathbb{F}'}$, and for each $\sigma \in \text{Gal}(k/\mathbb{F}_p)$, the map
\[ \phi : \epsilon_{\sigma} \mathcal{M}_{\mathbb{F}'} \rightarrow \epsilon_{\sigma \phi}^{-1} \mathcal{M}_{\mathbb{F}'} \]
has determinant $\alpha \in \mathbb{F}'[[u]]$.  

Proof. This is [Gee, Lemma 2.2]. \qed

2. Main theorem

To prove the main theorem, in fact we prove that the non-ordinary component of $\mathcal{A}_{\mathbb{F}_{\mathcal{V}_0}}$ is rationally connected. We use the following two Lemmas to join two points by $\mathbb{P}^1$.

Lemma 2.1. Suppose $x_1, x_2 \in \mathcal{A}_{\mathbb{F}_{\mathcal{V}_0}}(\mathbb{F})$ correspond to objects $\mathcal{M}_{1,\mathbb{F}}, \mathcal{M}_{2,\mathbb{F}}$ of $(\text{Mod}/\mathcal{S})_{\mathbb{F}}$ respectively. We fix bases of $\mathcal{M}_{1,\mathbb{F}}, \mathcal{M}_{2,\mathbb{F}}$ over $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$. We assume that there is a nilpotent element $N = (N_i)_{1 \leq i \leq n}$ of $M_2(\mathbb{F}((u)))^n$ such that $\mathcal{M}_{1,\mathbb{F}} = (1 + N) \cdot \mathcal{M}_{1,\mathbb{F}}$. Let $A = (A_i)_{1 \leq i \leq n}$ be an element of $GL_2(\mathbb{F}((u)))^n$ such that $\mathcal{M}_{1,\mathbb{F}} \sim A$. If $\phi(N_i)A_iN_i+1 \in M_2(\mathbb{F}[[u]])$ for all $i$, then there is a morphism $\mathbb{P}^1 \rightarrow \mathcal{A}_{\mathbb{F}_{\mathcal{V}_0}}$ sending $0$ to $x_1$ and $1$ to $x_2$.

Proof. This is [Gee, Lemma 2.4]. \qed

Lemma 2.2. Suppose $n \geq 2$. Let $\mathcal{M}_{\mathbb{F}}$ be the object of $(\text{Mod}/\mathcal{S})_{\mathbb{F}}$ corresponding to a point $x \in \mathcal{A}_{\mathbb{F}_{\mathcal{V}_0}}(\mathbb{F})$. Fix a basis of $\mathcal{M}_{\mathbb{F}}$ over $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$. Consider $U^{(i)} = (U^{(i)}_{j})_{1 \leq j \leq n} \in GL_2(\mathbb{F}((u)))^n$ such that $U^{(i)}_1 = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ and $U^{(i)}_j = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for all $j \neq i$. If $U^{(i)} : \mathcal{M}_{\mathbb{F}}$ is $\phi$-stable, it corresponds to a point $x' \in \mathcal{A}_{\mathbb{F}_{\mathcal{V}_0}}(\mathbb{F})$, and there is a morphism $\mathbb{P}^1 \rightarrow \mathcal{A}_{\mathbb{F}_{\mathcal{V}_0}}$ sending $0$ to $x$ and $1$ to $x'$. If $(U^{(i)})^{-1} : \mathcal{M}_{\mathbb{F}}$ is $\phi$-stable, it corresponds to a point $x'' \in \mathcal{A}_{\mathbb{F}_{\mathcal{V}_0}}(\mathbb{F})$, and there is a morphism $\mathbb{P}^1 \rightarrow \mathcal{A}_{\mathbb{F}_{\mathcal{V}_0}}$ sending $0$ to $x$ and $1$ to $x''$.

Proof. This is [Ima, Lemma 2.3]. \qed

To prove the main theorem, it suffices to show the following theorem. The strategy of the proof is the same as in [Ima], and we focus on the changed points in the case $p = 2$.

Theorem 2.3. Let $\mathbb{F}'$ be a finite extension of $\mathbb{F}$. Suppose $x_1, x_2 \in \mathcal{A}_{\mathbb{F}_{\mathcal{V}_0}}(\mathbb{F}')$ correspond to objects $\mathcal{M}_{1,\mathbb{F}}, \mathcal{M}_{2,\mathbb{F}}$ of $(\text{Mod}/\mathcal{S})_{\mathbb{F}'}$ respectively. If $\mathcal{M}_{1,\mathbb{F}}$ and $\mathcal{M}_{2,\mathbb{F}}$ are both non-ordinary, then $x_1$ and $x_2$ lie on the same connected component of $\mathcal{A}_{\mathbb{F}_{\mathcal{V}_0}}$.

Proof. When $n = 1$, this was proved in [Kis], and we did not use the condition $p > 2$ in the proof. If $e < p - 1$, then $\mathcal{A}_{\mathbb{F}_{\mathcal{V}_0}}(\mathbb{F}')$ is one point by [Ray, Theorem 3.3.3]. So we may assume $n \geq 2$ and $e \geq p - 1$. Furthermore, replacing $V_2$ by $V_2 \otimes_{\mathbb{F}_p} \mathbb{F}'$, we may assume $\mathbb{F} = \mathbb{F}'$.

In the case where $V_2$ is reducible, the proof of [Ima, Theorem 2.4] goes on, even if $p = 2$. So, by a base change, we may assume that $V_2$ is absolutely irreducible. As in the proof of [Ima, Theorem 2.4], we can prove that, after extending the field $\mathbb{F}$, there exists a basis such that
\[
M_{\mathbb{F}} \sim \begin{pmatrix} \alpha_1 \& 0 \& u^{e_1} \\ 0 \& u^{e_1} \& 0 \end{pmatrix} \cdot \alpha_2 \begin{pmatrix} u^{e_2} \& 0 \& 0 \\ 0 \& u^{e_2} \& 0 \end{pmatrix} \cdots \cdots \alpha_n \begin{pmatrix} u^{e_n} \& 0 \& 0 \\ 0 \& 0 \& u^{e_n} \end{pmatrix}.
\]
where \( \alpha_i \in \mathbb{F} \), \( 0 \leq s_i, t_i \leq e \), \( s_i + t_i = e \) and \( |s_i - t_i| \leq p + 1 \) for all \( i \). Note that we have proved that we may assume \( |s_i - t_i| \leq p + 1 \) for all \( i \) in the last paragraph of [Ima, p. 1197].

Let \( \mathcal{M}_{\mathcal{F}, 0} \) be the \( \mathbb{F}_p \)-module generated by the basis giving the above matrix expression. Then \( \mathcal{M}_{\mathcal{F}, 0} \) satisfies the condition in Proposition 1.1. We take the point \( x_0 \) of \( \mathcal{F}_{\mathcal{F}, 0}(\mathbb{F}) \) corresponding to \( \mathcal{M}_{\mathcal{F}, 0} \). We are going to prove that \( x_0 \) and \( x_1 \) lie on the same connected component. We can prove that \( x_0 \) and \( x_2 \) lie on the same connected component by the same argument.

By the Iwasawa decomposition and the determinant conditions, we can take \( B = (B_i)_{1 \leq i \leq n} \in GL_2(\mathbb{F}((u))) \) such that \( \mathcal{M}_{1, \mathcal{F}} = B \cdot \mathcal{M}_{0, \mathcal{F}} \) and \( B_i = \left( \begin{array}{cc} u_i & v_i \\ 0 & u_i \end{array} \right) \) for \( a_i \in \mathbb{Z} \) and \( v_i \in \mathbb{F}((u)) \). Then we put \( r_i = v_i(u_i) \). Now we have

\[
\phi(B_1) \left( \begin{array}{cc} 0 & u_1^r \\ u_1 & 0 \end{array} \right) B_1^{-1} = \left( \begin{array}{cc} \phi(v_1)u_1^{a_1} & u_1^{a_1} - \phi(v_1)v_1^r \\ v_1^r & -v_1^{a_1} \end{array} \right),
\]

\[
\phi(B_2) \left( \begin{array}{cc} u_1^r & 0 \\ 0 & u_1 \\ \end{array} \right) B_2^{-1} = \left( \begin{array}{cc} u_1^{a_1} - \phi(v_1)v_1^r & -u_1^{a_1} + v_1^r \\ v_1^r & u_1^{a_1} - \phi(v_1)v_1^r \end{array} \right)
\]

for \( 2 \leq i \leq n \). On the right-hand sides, every component of the matrices is integral because \( \mathcal{M}_{1, \mathcal{F}} \) is \( \phi \)-stable.

First, we consider the case \( t_1 + pa_1 + a_2 > e \). In this case,

\[
(pr_1 + t_1 + a_2) + (r_2 + t_1 + pa_1) = e, \quad s_1 - pa_1 - a_2 = pr_1 + r_2 + t_1 < 0
\]

by the \( \phi \)-stability and the determinant conditions of \( \mathcal{M}_{1, \mathcal{F}} \). We have \( a_1 > r_1 \), because \( t_1 + pa_1 + a_2 > e \geq pr_1 + t_1 + a_2 \). Similarly, we have \( a_2 > r_2 \), because \( t_1 + pa_1 + a_2 > e \geq r_2 + t_1 + pa_1 \).

We consider the following operations:

\[
a_i \sim a_i - 1, \quad v_i \sim uv_i, \quad \text{if it preserves the } \phi \text{-stability of } B \cdot \mathcal{M}_{0, \mathcal{F}}.
\]

These operations replace \( x_1 \) by a point that lies on the same connected component as \( x_1 \) by Lemma 2.2. We prove that we can continue these operations until we get to the situation where \( t_1 + pa_1 + a_2 \leq e \). In other words, we reduce the problem to the case \( t_1 + pa_1 + a_2 \leq e \). If we can continue the operations endlessly, we get to the situation where \( t_1 + pa_1 + a_2 \leq e \), because the conditions \( s_i - pa_i + a_{i+1} \geq 0 \) for \( 2 \leq i \leq n \) exclude that both \( a_1 \) and \( a_2 \) remain bounded below. Suppose we cannot continue the operations. This is equivalent to the following condition:

\[
s_n - pa_n + a_1 = 0 \quad \text{or} \quad r_2 + t_1 + pa_1 \leq p - 1,
\]

\[
pr_1 + t_1 + a_2 = 0 \quad \text{or} \quad t_2 + pa_2 - a_3 \leq p - 1,
\]

\[
s_i - pa_i - a_i = 0 \quad \text{or} \quad t_i + pa_i - a_{i+1} \leq p - 1 \quad \text{for each } 3 \leq i \leq n.
\]

If \( e \geq p \), there are only the following two cases, because \( (pr_1 + t_1 + a_2) + (r_2 + t_1 + pa_1) = e \) and \( (s_i - pa_i + a_{i+1}) + (t_i + pa_i - a_{i+1}) = e \) for \( 2 \leq i \leq n \).

Case 1: \( pr_1 + t_1 + a_2 = 0, \ s_i - pa_i + a_{i+1} = 0 \) for \( 2 \leq i \leq n \). Case 2: \( r_2 + t_1 + pa_1 \leq p - 1, \ t_1 + pa_i - a_{i+1} \leq p - 1 \) for \( 2 \leq i \leq n \).

If \( e = p - 1 \), clearly it is in Case 2.

In the Case 1, we have a contradiction as in the proof of [Ima, Theorem 2.4]. So we may assume that it is in the Case 2.
Then we can show that
\[ r_i < a_i, \quad pr_i + t_i - a_{i+1} = r_{i+1} + s_i - pa_i < 0 \quad \text{for } 2 \leq i \leq n \]
as in the proof of [Ima, Theorem 2.4]. Combining these equations with \( s_1 - pa_1 - a_2 = pr_1 + r_2 + t_1 \), we get
\[
- (p^n + 1)r_1 = (p^n + 1)a_1 + (s_n - t_n) + p(s_{n-1} - t_{n-1}) + \cdots + p^{n-3}(s_3 - t_3) + p^{n-2}(s_2 - t_2) - p^{n-1}(s_1 - t_1),
\]
\[
- (p^n + 1)r_2 = (p^n + 1)a_2 - (s_1 - t_1) - p(s_n - t_n) - \cdots - p^{n-3}(s_4 - t_4) - p^{n-2}(s_3 - t_3) - p^{n-1}(s_2 - t_2),
\]
\[
- (p^n + 1)r_3 = (p^n + 1)a_3 + (s_2 - t_2) - p(s_1 - t_1) - \cdots - p^{n-3}(s_5 - t_5) - p^{n-2}(s_4 - t_4) - p^{n-1}(s_3 - t_3),
\]
\[
\vdots
\]
\[
- (p^n + 1)r_n = (p^n + 1)a_n + (s_{n-1} - t_{n-1}) + p(s_{n-2} - t_{n-2}) + \cdots + p^{n-3}(s_2 - t_2) - p^{n-2}(s_1 - t_1) - p^{n-1}(s_n - t_n).
\]
As \( |s_i - t_i| \leq p + 1 \) and
\[
(p + 1) + p(p + 1) + \cdots + p^{n-1}(p + 1) = \left( \frac{p^n - 1}{p - 1} \right)(p + 1) < 3(p^n + 1),
\]
we get \(-a_i - 2 \leq r_i \leq -a_i + 2\). If \( e = p \), as \( |s_i - t_i| \leq p \) and
\[
p + p^2 + \cdots + p^n = \left( \frac{p^n - 1}{p - 1} \right)p < 2(p^n + 1),
\]
we get \(-a_i - 1 \leq r_i \leq -a_i + 1\). If \( e = p - 1 \), as \( |s_i - t_i| \leq p - 1 \) and
\[
(p - 1) + p(p - 1) + \cdots + p^{n-1}(p - 1) = \left( \frac{p^n - 1}{p - 1} \right)(p - 1) < (p^n + 1),
\]
we get \(-a_i = r_i\).
As \( r_2 + t_1 + pa_1 \leq p - 1 \), we have
\[
pa_1 \leq t_1 + pa_1 \leq p - 1 - r_2 \leq a_2 + p + 1.
\]
For \( 2 \leq i \leq n \), as \( t_i + pa_i - a_{i+1} \leq p - 1 \), we have
\[
pa_i \leq t_i + pa_i \leq a_{i+1} + p - 1.
\]
Take an index \( i_0 \) such that \( a_{i_0} \) is the greatest. If \( 2 \leq i_0 \leq n \), we get \( a_{i_0} \leq 1 \) by \( pa_{i_0} \leq a_{i_0 + 1} + p - 1 \leq a_{i_0} + p - 1 \). If \( i_0 = 1 \) and \( a_1 \geq 3 \), we get \( a_2 \geq 3 \), by \( pa_1 \leq a_2 + p + 1 \), and this contradicts the case where \( 2 \leq i_0 \leq n \). So, if \( i_0 = 1 \), we have \( a_1 \leq 2 \). Combining \(-a_i - 2 \leq r_i \) and \( r_i < a_i \), we get \( a_i \geq 0 \). Hence \( 0 \leq a_i \leq 2 \) and \( 0 \leq a_i \leq 1 \) for \( 2 \leq i \leq n \).
First, we assume \( a_2 = 0 \). Now we have \(-2 \leq r_2 \leq -1 \). Comparing \( t_1 + pa_1 + a_2 > e \) with \( r_2 + t_1 + pa_1 \leq p - 1 \), we get \( e \leq p - 2 - r_2 \). If \( r_2 = -2 \), we get \( e \leq p \). Then we have \(-a_2 - 1 \leq r_2 \), and this is a contradiction. If \( r_2 = -1 \), we get \( e \leq p - 1 \). Then we have \(-a_2 = r_2 \), and this is a contradiction.
Next, we assume \( a_2 = 1 \). As \( 0 \leq t_i + pa_i - a_{i+1} \leq p - 1 \) for \( 2 \leq i \leq n \), we have \( a_i = 1 \) for all \( i \) and \( t_i = 0 \) for \( 2 \leq i \leq n \). As \( r_2 + pa_1 + t_1 \leq p - 1 \), we have \( r_2 \leq -1 \).
As \( p r_2 + t_2 - a_3 = r_3 + s_2 - p a_2 \), we have \( r_1 = p r_2 + p - e - 1 \leq -e - 1 \). If \( e \geq p + 1 \), then \( -a_3 - 2 \leq r_1 \) and \( r_3 \leq -e - 1 \leq -4 \). This is a contradiction. If \( e = p \), then \( -a_3 - 1 \leq r_3 \) and \( r_3 \leq -e - 1 \leq -3 \). This is a contradiction. If \( e = p - 1 \), then \( -a_3 = r_3 \) and \( r_3 \leq -e - 1 \leq -2 \). This is a contradiction.

Thus we may assume \( t_1 + p a_1 + a_2 \leq e \). We put \( \mathcal{M}_{3,F} = \left( \begin{pmatrix} u^{-a_1} & 0 \\ 0 & u^{a_1} \end{pmatrix} \right)_i \mathcal{M}_{0,F} \), then

\[
\mathcal{M}_{3,F} \sim \left( \begin{array}{ccc} 0 & u^{s_1-pa_1-a_2} & 0 \\ u_{t_1+pa_1+a_2} & 0 & 0 \\ 0 & u_{t_2+pa_2-a_3} & 0 \\ \vdots & \vdots & \vdots \\ 0 & u_{t_i+pa_i-a_{i+1}} & 0 \end{array} \right),
\]

and \( \mathcal{M}_{1,F} = \left( \begin{pmatrix} 1 & v_i u^{-a_i} \\ 0 & 1 \end{pmatrix} \right)_i \mathcal{M}_{3,F} \). Note that \( \mathcal{M}_{3,F} \) satisfies the conditions of Proposition 1.1, and let \( x_3 \) be the point of \( \mathcal{R}^N_{V_{p,0}} \) corresponding to \( \mathcal{M}_{3,F} \). If we put \( N_i = \left( \begin{pmatrix} 0 & 0 \\ 0 & v_i u^{-a_i} \end{pmatrix} \right) \), then

\[
\phi(N_i) \left( \begin{array}{ccc} 0 & u^{s_1-pa_1-a_2} & 0 \\ u_{t_1+pa_1+a_2} & 0 & 0 \\ 0 & u_{t_2+pa_2-a_3} & 0 \end{array} \right) N_2 = \left( \begin{array}{ccc} 0 & \phi(v_i) v_2 u^{t_1} & 0 \\ 0 & 0 & 0 \end{array} \right),
\]

and

\[
\phi(N_i) \left( \begin{array}{ccc} 0 & u^{s_i-pa_i+a_{i+1}} & 0 \\ u_{t_i+pa_i-a_{i+1}} & 0 & 0 \end{array} \right) N_{i+1} = 0
\]

for \( 2 \leq i \leq n \). Here we have \( v_n \phi(v_i) v_2 u^{t_1} \geq 0 \), because \( s_1 - pa_1 - a_2 \geq 0 \) and \( v_n \left( u^{s_1-pa_1-a_2} - \phi(v_i) v_2 u^{t_1} \right) \geq 0 \). Hence \( x_1 \) and \( x_3 \) lie on the same connected component by Lemma 2.1.

We are going to compare \( \mathcal{M}_{0,F} \) and \( \mathcal{M}_{3,F} \). First, we treat the case \( e \geq p \). We consider the operations that decrease \( |a_i| \) by 1 for an index \( i \) keeping the condition of \( \phi \)-stability. By Lemma 2.2, these operations do not affect which of the connected components \( x_3 \) lies on. We prove that we can continue the operations until we have \( a_i = 0 \) for all \( i \), that is, \( x_0 \) and \( x_3 \) lie on the same connected component. Suppose that we cannot continue the operations and there is some nonzero \( a_i \). The condition of \( \phi \)-stability is equivalent to

\[
C_1 : 0 \leq s_1 - pa_1 - a_2 \leq e, \quad C_2 : 0 \leq s_2 - pa_2 + a_3 \leq e, \\
\ldots, C_n : 0 \leq s_n - pa_n + a_1 \leq e.
\]

Note that if \( a_i \neq 0 \) or \( a_{i+1} \neq 0 \), we can decrease \( |a_i| \) or \( |a_{i+1}| \) keeping \( C_i \), because \( e \geq p \).

We put

\[
c_i = \sharp \{ i \leq j \leq i + 1 \mid we \ can \ decrease \ |a_j| \ keeping \ C_i \},
\]

and claim that \( \sharp \{ j \mid a_j \neq 0 \} = \sum_{i=1}^{n} c_i \). First, if \( a_i \neq 0 \), we have \( c_{i-1} \geq 1 \) and \( c_i \geq 1 \) from the above remark. So we have \( \sharp \{ j \mid a_j \neq 0 \} \leq \sum_{i=1}^{n} c_i \). Second, we count \( a_i \neq 0 \) in not both of \( C_{i-1} \) and \( C_i \), because we cannot continue the operations. So we have \( \sharp \{ j \mid a_j \neq 0 \} \geq \sum_{i=1}^{n} c_i \). Hence we have equality. From this equality, we have \( a_i \neq 0 \) and \( c_i = 1 \) for all \( i \). For \( 2 \leq i \leq n \), we have \( a_i a_{i+1} > 0 \) because \( c_i = 1 \). So we have \( a_1 a_2 > 0 \), but this contradicts \( c_1 = 1 \).
In the case $e = p - 1$. We have $|pa_1 + a_2| \leq p - 1$ by $C_1$, and $|pa_i - a_{i+1}| \leq p - 1$ by $C_i$ for $2 \leq i \leq n$. Summing up these inequalities after multiplying some $p$-powers so that we can eliminate $a_j$ for $j \neq i$, we get $|(p^n + 1)a_i| \leq p^n - 1$. So we have $a_i = 0$ for all $i$.

Hence $x_0$ and $x_3$ lie on the same connected component. This completes the proof. \hfill $\square$

References


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