A REMARK ON THE C_2-COFINITENESS CONDITION ON VERTEX ALGEBRAS

TOMOYUKI ARAKAWA

Abstract. We show that a finitely strongly generated, non-negatively graded vertex algebra V is C_2-cofinite if and only if it is lisse in the sense of Beilinson, Feigin and Mazur [BFM]. This shows that the C_2-cofiniteness is indeed a natural finiteness condition.

1. Introduction

The purpose of this note is to clarify the equivalence of the two finiteness conditions on vertex algebras.

One of them is the finiteness condition introduced by Zhu [Zhu96], which is now called the *C_2-cofiniteness condition*. This condition has been assumed in many significant theories of vertex operator algebras, such as [Zhu96, DLM00, Miy04, NT05, Hua08b, Hua08a]. However the original definition looks rather technical, and hence it has been often considered as a mere technical condition.

The other is the finiteness condition defined by Beilinson, Feigin and Mazur [BFM] in the case of the Virasoro (vertex) algebra, which is called *lisse*. The definition is as follows. Let $\mathcal{L}$ be the Virasoro (Lie) algebra, $U(\mathcal{L})$ its universal enveloping algebra. There is a natural increasing filtration of $U(\mathcal{L})$ in the Lie theory, called the *standard filtration*, and the associated graded algebra $\operatorname{gr} U(\mathcal{L})$ of $U(\mathcal{L})$ is isomorphic to the symmetric algebra $S(\mathcal{L})$ of $\mathcal{L}$. Let M be a finitely generated $\mathcal{L}$-module, $\{\Gamma_p M\}$ a good filtration (i.e., a filtration compatible with the standard filtration of $\mathcal{L}$ such that the associated graded space $\operatorname{gr}^\Gamma M$ is finitely generated over $S(\mathcal{L})$). The *singular support* $SS(M)$ of M is the support of the $S(\mathcal{L})$-module $\operatorname{gr}^\Gamma M$, which is known to be independent of the choice of a good filtration. A $\mathcal{L}$-module M is called lisse if $\dim SS(M) = 0$. In the case that M is a highest weight representation of $\mathcal{L}$ then M is lisse if and only if any element of $\mathcal{L}$ acts locally nilpotently on $\operatorname{gr}^\Gamma M$. This implies that lisse representations are natural analogues of finite-dimensional representations.

Let V be a finitely strongly generated, non-negatively graded vertex algebra. The notion of singular supports can be naturally extended to V by using the canonical filtration introduced by Li [Li05]. More precisely, define the *associated variety* X_M of a V-module M as the support of the $V/C_2(V)$-module $M/C_2(M)$. Then the singular support of M can be naturally defined as a subscheme of the infinite jet scheme of X_M. Having defined the associated variety and the singular support, the equivalence of the C_2-cofiniteness condition and the lisse condition easily follows

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from the fact that the jet scheme of a zero-dimensional variety is zero-dimensional (see Theorems 3.3.3, 3.3.4).

We note that in the case that V is a Virasoro vertex algebra and M is a V-module, the Li filtration of M is slightly different from the Lie theoretic filtration defined by considering it as a module over the Virasoro (Lie) algebra $\mathcal{L}$. Nevertheless M is lisse if and only if it is so in the sense of [BFM] provided that M is a highest weight representation of $\mathcal{L}$ (see Proposition 3.4.2). In the case that V is an affine vertex algebra associated with a Lie algebra the Li filtration is essentially the same as the Lie theoretic filtration, although it is a *decreasing* filtration (see Propositions 2.6.1, 2.7.1).

In a separate paper [Ara] we prove the C_2-cofiniteness for a large family of W-algebras using the methods in this note, including all the (non-principal) exceptional W-algebras recently discovered by Kac and Wakimoto [KW08].

Notation. The ground field will be $\mathbb{C}$ throughout the note .

2. Graded Poisson vertex algebras associated with vertex algebras

2.1. **Vertex algebras and their modules.** A (quantum) *field* on a vector space V is a formal series

$$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \in (\operatorname{End} V)[[z, z^{-1}]]$$

such that $a_{(n)}v = 0$ with $n \gg 0$ for any $v \in V$. Let $Fields(V)$ denote the space of all fields on V.

A *vertex algebra* is a vector space V equipped with the following data:

- a linear map $Y(?, z) : V \to Fields(V)$, $a \mapsto a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$,
- a vector $\mathbf{1} \in V$, called the *vacuum vector*,
- a linear operator $T : V \to V$, called the *translation operator*.

These data are subjected to satisfy the following axioms:

(i) $(Ta)(z) = \partial_z a(z)$,
(ii) $\mathbf{1}(z) = \mathrm{id}_V$,
(iii) $a(z)\mathbf{1} \in V[[z]]$ and $a_{(-1)}\mathbf{1} = a$,
(iv)

$$\sum_{j\geq 0} \binom{m}{j} (a_{(n+j)}b)_{(m+k-j)} = \sum_{j\geq 0} (-1)^j \binom{n}{j} \left(a_{(m+n-j)} b_{(k+j)} - (-1)^n b_{(n+k-j)} a_{(m+j)} \right).$$

A *Hamiltonian* of V is a semisimple operator H on V satisfying

$$[H, a_{(n)}] = -(n + 1)a_{(n)} + (Ha)_{(n)} \tag{1}$$

for all $a \in V$, $n \in \mathbb{Z}$. A vertex algebra equipped with a Hamiltonian H is called *graded*. Let $V_\Delta = \{a \in V; Ha = \Delta a\}$ for $\Delta \in \mathbb{C}$, so that $V = \bigoplus_{\Delta\in\mathbb{C}} V_\Delta$. For $a \in V_\Delta$, Δ is called the *conformal weight* of a and denoted by Δ_a. We have

$$a_{(n)}b \in V_{\Delta_a+\Delta_b-n-1}$$

for homogeneous elements $a, b \in V$.

Throughout the note V is assumed to be $\frac{1}{r_0}\mathbb{Z}_{\geq 0}$-graded for some $r_0 \in \mathbb{N}$, that is, V is graded and $V_\Delta = 0$ for $\Delta \notin \frac{1}{r_0}\mathbb{Z}_{\geq 0}$.

A graded vertex algebra V is called *conformal* if there exists a vector ω (called the *conformal vector*) and $c_V \in \mathbb{C}$ (called the *central charge*) satisfying

$$\omega_{(0)} = T, \quad \omega_{(1)} = H,$$
$$[\omega_{(m+1)}, \omega_{(n+1)}] = (m-n)\omega_{(m-n+1)} + \frac{(m^3-m)}{12}\delta_{m+n,0}c_V.$$

A $\mathbb{Z}$-graded conformal vertex algebra is also called a *vertex operator algebra*.

A vertex algebra V is called *finitely strongly generated* if there exists a finitely many elements $a^1, \dots, a^r$ such that V is spanned by the elements of the form

$$a^{i_1}_{(-n_1)} \dots a^{i_r}_{(-n_r)}\mathbf{1} \tag{2}$$

with $r \geq 0$, $n_i \geq 1$.

A *module* over a vertex algebra V is a vector space M together with a linear map

$$Y^M(?, z) : V \to Fields(M), \quad a \mapsto a^M(z) = \sum_{n \in \mathbb{Z}} a^M_{(n)} z^{-n-1},$$

which satisfies the following axioms:

(i) $\mathbf{1}^M(z) = \mathrm{id}_M$,

(ii)
$$\sum_{j \geq 0} \binom{m}{j} (a_{(n+j)}b)^M_{(m+k-j)}$$
$$= \sum_{j \geq 0} (-1)^j \binom{n}{j} \left(a^M_{(m+n-j)} b^M_{(k+j)} - (-1)^n b^M_{(n+k-j)} a^M_{(m+j)} \right).$$

A V-module M is called *graded* if there is a compatible semisimple action of H on M, that is, $M = \bigoplus_{d \in \mathbb{C}} M_d$, where $M_d = \{m \in M; Hm = dm\}$, and $[H, a^M_{(n)}] = -(n+1)a^M_{(n)} + (Ha)^M_{(n)}$ for all $a \in V$. We have $a^M_{(n)} M_d \subset M_{d+\Delta_a-n-1}$ for a homogeneous element $a \in V$. If V is conformal and M is a V-module on which $\omega^M_{(1)}$ acts locally finite, then M is graded by the semisimplification of $\omega^M_{(1)}$.

If no confusion can arise we write simply $a_{(n)}$ for $a^M_{(n)}$.

Throughout the note a V-module M is assumed to be lower truncated, that is, M is graded and there exists a finites subset $\{d_1, \dots, d_r\} \subset \mathbb{C}$ such that $M_d \neq 0$ unless $d \in d_i + \frac{1}{r_0}\mathbb{Z}_{\geq 0}$ for some i.

For a V-module M, set

$$C_2(M) = \mathrm{span}_{\mathbb{C}}\{a_{(-2)}m; a \in V, m \in M\}.$$

Then

$$C_2(M) = \mathrm{span}_{\mathbb{C}}\{a_{(-n)}m; a \in V, m \in M,\ n \geq 2\}$$

by the axiom (i) of vertex algebras. A V-module M is called C_2-*cofinite* if $\dim M/C_2(M) < \infty$, and V is called C_2-cofinite if it is C_2-cofinite as a module over itself.

Suppose that V is conformal. Then a V-module M is called C_1-*cofinite* [Li99] if $M/C_1(M)$ is finite-dimensional, where

$$C_1(M) = \mathrm{span}_{\mathbb{C}}\{\omega_{(0)}m, a_{(-1)}m; a \in \bigoplus_{\Delta>0} V_\Delta, m \in M\}.$$

If $V_0 = \mathbb{C}\mathbf{1}$, then $C_2(V)$ is a subspace of $C_1(V)$ ([Li99, Remark 3.2]).

2.2. **Vertex Poisson algebras and their modules.** A vertex algebra V is said to be *commutative* if $a_{(n)} = 0$ in $\operatorname{End} V$ for all $n \geq 0$, $a \in V$. This is equivalent to that

$$[a_{(m)}, b_{(n)}] = 0 \quad \text{for } a, b \in V,\ m, n \in \mathbb{Z}.$$

A commutative vertex algebra is the same as a differential algebra (=a unital commutative algebra with a derivation) [Bor86]: the multiplication is given by $a \cdot b = a_{(-1)}b$ for $a, b \in V$; The derivation is given by the translation operator T.

A commutative vertex algebra V is called a *vertex Poisson algebra* [FBZ04] if it is also equipped with a linear operation

$$Y_-(?, z) : V \to \mathrm{Hom}(V, z^{-1}V[z^{-1}]), \quad a \mapsto a_-(z),$$

such that

$$(Ta)_{(n)} = -na_{(n-1)}, \tag{3}$$

$$a_{(n)}b = \sum_{j\geq 0}(-1)^{n+j+1}\frac{1}{j!}T^j(b_{(n+j)}a), \tag{4}$$

$$[a_{(m)}, b_{(n)}] = \sum_{j\geq 0}\binom{m}{j}(a_{(j)}b)_{(m+n-j)}, \tag{5}$$

$$a_{(n)}(b \cdot c) = (a_{(n)}b) \cdot c + b \cdot (a_{(n)}c) \tag{6}$$

for $a, b, c \in V$ and $n, m \geq 0$. Here, by abuse of notation, we have set

$$a_-(z) = \sum_{n\geq 0} a_{(n)}z^{-n-1}.$$

Below, for an element a of a vertex Poisson algebra, we will denote by $a_{(n)}$ with $n \leq -1$ the Fourier coefficient of z^{-n-1} in the field $Y(a, z) = a(z)$, and by $a_{(n)}$ with $n \geq 0$ the Fourier coefficient of z^{-n-1} in $Y_-(a, z) = a_-(z)$.

A vertex Poisson algebra V is called *graded* if there exists a semisimple operator H on V satisfying (1) for all $n \in \mathbb{Z}$.

A *module* over a vertex Poisson algebra V is a module M over V as an associative algebra together with a linear map

$$Y_-^M(?, z) : V \to \mathrm{Hom}(M, z^{-1}M[z^{-1}]), \quad a \mapsto a_-^M(z) = \sum_{n\geq 0} a_{(n)}^M z^{-n-1},$$

such that

$$\begin{aligned}
&\mathbf{1}^M(z) = \mathrm{id}_M,\\
&(Ta)_{(n)}^M = -na_{(n-1)}^M,\\
&[a_{(m)}^M, b_{(n)}^M] = \sum_{i\geq 0}\binom{m}{i}(a_{(i)}b)_{(m+n-i)}^M,\\
&a_{(n)}^M(b \cdot m) = (a_{(n)}b) \cdot m + b \cdot (a_{(n)}^M m). \qquad (7)
\end{aligned}$$

for $a, b \in V$, $m \in M$ and $n \geq 0$.

A module M over a graded vertex Poisson algebra V is called *graded* if there exists a semisimple operator H^M on M satisfying

$$[H^M, a^M_{(n)}] = -(n+1)a^M_{(n)} + (Ha)^M_{(n)} \tag{8}$$

for all $a \in V$, $n \in \mathbb{Z}$.

If no confusion can arise below we write $a_{(n)}$ for $a^M_{(n)}$ and H for H^M.

We call M a *differential V-module* if it is equipped with the linear action of T satisfying

$$[T, a_{(n)}] = -na_{(n-1)} \quad \text{for all } n \in \mathbb{Z}.$$

2.3. **Functions on jet schemes of affine Poisson varieties are vertex Poisson algebras.** For a scheme X of finite type we denote by X_m be the jet scheme of order m of X and by X_∞ the infinite jet scheme of X.

Let us recall the definition of jet schemes. For general theory of jet schemes see e.g., [EM09]. The connection between jet schemes and chiral algebras goes back to Beilinson and Drinfeld [BD04]. The scheme X_m is determined by its functor of points: for every commutative $\mathbb{C}$-algebra A, there is a bijection

$$\operatorname{Hom}(\operatorname{Spec} A, X_m) \cong \operatorname{Hom}(\operatorname{Spec} A[t]/(t^{m+1}), X).$$

If $m > n$, we have projections $X_m \to X_n$. This yields a projective system $\{X_m\}$ of schemes, and the infinite jet scheme X_∞ is the projective limit $\varprojlim_m X_m$ in the category of schemes.

For an affine scheme $X = \operatorname{Spec} R$, the jet scheme X_m is explicitly described. Choose a presentation $R = \mathbb{C}[x^1, \dots, x^r]/\langle f_1, \dots, f_s\rangle$. Define new variables $x^j_{(-i)}$ for $i = 1, \dots, m+1$ and a derivation T of the ring $\mathbb{C}[x^j_{(-i)}; i = 1, 2, \dots, m+1,\ j = 1, \dots, r]$ by setting

$$Tx^j_{(-i)} = \begin{cases} ix^j_{(-i-1)} & \text{for } i \leq m, \\ 0 & \text{for } i = m+1. \end{cases}$$

Identify x^j with $x^j_{(-1)}$ and set

$$R_m = \mathbb{C}[x^j_{(-i)}; i = 1, \dots, m+1,\ j = 1, \dots, r]/\langle T^j f_i; i = 1, \dots, s,\ j = 0, \dots, m+1\rangle,$$

and we have $X_m \cong \operatorname{Spec} R_m$.

Let R_∞ denotes the differential algebra obtained from R_m by taking the limit $m \to \infty$:

$$R_\infty = \mathbb{C}[x^j_{(-i)}; i = 1, 2, \dots,\ j = 1, \dots,]/\langle T^j f_i; i = 1, \dots, s, j = 0, \dots, \rangle. \tag{9}$$

Then $X_\infty \cong \operatorname{Spec}(R_\infty)$.

Proposition 2.3.1. *Let R be a Poisson algebra. Then there is a unique vertex Poisson algebra structure on R_∞ such that*

$$u_{(n)}v = \begin{cases} \{u, v\} & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases}$$

for $u, v \in R \subset R_\infty$.

Proof. Set

$$u_{(n)}(T^l v) = \begin{cases} \frac{l!}{(l-n)!} T^{l-n}\{u,v\} & \text{if } l \geq n, \\ 0 & \text{else} \end{cases} \tag{10}$$

for $a, b \in R$. This extends to a well-defined linear map

$$R \to \operatorname{Der}(R_\infty)[[z^{-1}]]z^{-1}, \quad u \mapsto u_-(z) = \sum_{n \in \geq 0} u_{(n)} z^{-n-1}, \tag{11}$$

where $\operatorname{Der}(R_\infty)$ is the space of derivations on R_∞. It is straightforward to check that

$$[T, u_-(z)] = \partial_z u_-(z) \quad \text{for } u \in R,$$

$$u_{(n)}v = \sum_{j \geq 0} (-1)^{n+j+1} \frac{1}{j!} T^j (v_{(n+j)}u) \quad \text{for } u, v \in R,$$

$$[u_{(m)}, v_{(n)}] = \sum_{j \geq 0} \binom{m}{j} (u_{(j)}v)_{(m+n-j)} \quad \text{for } u, v \in R. \tag{12}$$

The map (11) can be extended to the linear map

$$R_\infty \to \operatorname{Der}(R_\infty)[[z^{-1}]]z^{-1}, \quad a \mapsto a_-(z) = \sum_{n \geq 0} a_{(n)} z^{-n-1} \tag{13}$$

by setting

$$a_-(z) T^k u = \operatorname{Sing}(e^{zT} (-\partial_z)^k u_-(-z) a)$$

for $a \in R_\infty$, $u \in R$, $k \geq 0$. Here $\operatorname{Sing}(f)$ is the singular part of the formal series f ([Pri99, Li04]). (Note that the axiom (4) is equivalent to that $a_-(z)b = \operatorname{Sing}(e^{zT} b_-(-z)a)$.) We find from the argument of [Li04, Proof of Proposition 3.10] that

$$(Ta)_-(z) = \partial_z a_-(z) \quad \text{for } a \in R_\infty,$$

$$a_-(z)b = \operatorname{Sing}(e^{zT} b_-(-z)a) \quad \text{for } a, b \in R_\infty.$$

This together (12) proves that (13) defines a vertex Poisson structure on R_∞ by [Li04, Theorem 3.6]. The uniqueness statement is easily seen, cf. [Li04, Proof of Proposition 3.10]. □

The vertex Poisson algebra structure on $R_\infty = \mathbb{C}[X_\infty]$ given in Proposition 2.3.1 for an affine scheme $X = \operatorname{Spec} R$ will be called the *level* 0 *vertex Poisson algebra structure*.

Remark 2.3.2. The differential algebra R_∞ makes sense even when R is not finitely generated and Proposition 2.3.1 still holds in this case. Also, the assertion of Proposition 2.3.1 extends straightforwardly to the Poisson superalgebra cases.

2.4. **Ideals of vertex Poisson algebras.** Let V be a vertex Poisson algebra, M a V-module. A *submodule* of M is a submodule N of M as an associative algebra such that $a_{(n)}N \subset N$ for all $a \in V$, $n \geq 0$. An *ideal* of V is a submodule I of V stable under the action of T. By (4) it follows that $u_{(n)}V \subset I$ for all $u \in I$ and $n \geq 0$. Hence V/I inherits the vertex Poisson algebra from V.

Lemma 2.4.1. *Let R be a commutative $\mathbb{Q}$-algebra, ∂ a derivation on R, I a ∂-stable ideal of R. Then the radical $\sqrt{I}$ of I is stable under the action of ∂.*

Proof. ([CO78]) Let $a \in \sqrt{I}$, so that there exists $m \in \mathbb{N}$ such that $a^m \in I$. Because $\partial I \subset I$, $(\partial)^m a^m \in I$. But

$$(\partial)^m a^m \equiv m!(\partial a)^m \pmod{\sqrt{I}},$$

and therefore we get that $(\partial a)^m \in \sqrt{I}$. This gives $\partial a \in \sqrt{I}$. □

Corollary 2.4.2. (i) *Let R be a Poisson algebra, I a Poisson ideal of R. Then $\sqrt{I}$ is also a Poisson ideal.*

(ii) *Let V be a vertex Poisson algebra, I a V-submodule of V. Then $\sqrt{I}$ is also a V-submodule of V.*

(iii) *Let V be a vertex Poisson algebra, I a vertex Poisson algebra ideal of V. Then $\sqrt{I}$ is also a vertex Poisson algebra ideal of V.*

2.5. **Li filtration.** Let V be a vertex algebra. Following [Li05][1], define F^pV to be the subspace of V spanned by the vectors

$$a^1_{(-n_1-1)}\cdots a^r_{(-n_r-1)}b$$

with $a^i \in V$, $b \in V$ $n_i \in \mathbb{Z}_{\geq 0}$, $n_1+\cdots+n_r \geq p$. Then

$$V = F^0V \supset F^1V \supset \cdots, \quad \bigcap_p F^pV = 0,$$
$$TF^pV \subset F^{p+1}V,$$
$$a_{(n)}F^qV \subset F^{p+q-n-1}V \quad \text{for } a \in F^pV,\ n \in \mathbb{Z},$$
$$a_{(n)}F^qV \subset F^{p+q-n}V \quad \text{for } a \in F^pV,\ n \geq 0. \tag{14}$$

Here we have set $F^pV = V$ for $p<0$. Note that the filtration $\{F^pV\}$ is independent of the grading of V.

Let $\operatorname{gr}^F V = \bigoplus_p F^pV/F^{p+1}V$ be the associated graded vector space. The space $\operatorname{gr}^F V$ is a vertex Poisson algebra by

$$\sigma_p(a)\sigma_q(b) = \sigma_{p+q}(a_{(-1)}b)$$
$$T\sigma_p(a) = \sigma_{p+1}(Ta),$$
$$Y_-(\sigma_p(a),z)\sigma_q(b) = \sum_{n\geq 0}\sigma_{p+q-n}(a_{(n)}b)z^{-n-1},$$

where $\sigma_p : F^pV \to F^pV/F^{p+1}V$ is the principal symbol map.

The filtration $\{F^pV\}$ is called the *Li filtration* of V.

We have [Li05, Lemma 2.9]

$$F^pV = \operatorname{span}_{\mathbb{C}}\{a_{(-i-1)}b; a \in V, i \geq 1, b \in F^{p-i}V\} \quad \text{for all } p \geq 1. \tag{15}$$

In particular

$$F^1M = C_2(M).$$

Set

$$R_V = V/C_2(V) = F^0V/F^1V \subset \operatorname{gr}^F V.$$

[1]In [Li05] F^pV was denoted by E_p.

It is known by Zhu [Zhu96] that R_V is a Poisson algebra. In fact the Poisson algebra structure of R_V can be understood as the restriction of the vertex Poisson structure of $\mathrm{gr}^F V$ [Li05, Proposition 3.5]; it is given by

$$\bar{a}\cdot\bar{b} = \overline{a_{(-1)}b},$$
$$\{\bar{a},\bar{b}\} = \overline{a_{(0)}b}$$

for $a, b \in V$, where $\bar{a} = a + C_2(V)$.

By [Li05, Lemma 4.2], the embedding $R_V \hookrightarrow \mathrm{gr}^F V$ extends to the surjective homomorphism

$$(R_V)_\infty \twoheadrightarrow \mathrm{gr}^F V \tag{16}$$

of differential algebras.

Proposition 2.5.1. *The surjection (16) is a homomorphism of vertex Poisson algebras, where $(R_V)_\infty$ is equipped with the level 0 vertex Poisson algebra structure.*

Proof. From the definition we see that the vertex Poisson algebra structure coincides on the generating subspace R_V, that is, for $a, b \in R_V$, $a_{(0)}b = \{a, b\}$ and $a_{(n)}b = 0$ for all $n > 0$ in both $\mathrm{gr}\, V$ and $(R_V)_\infty$. But then [Li04, Lemma 3.3] says that the map $(R_V)_\infty \to \mathrm{gr}\, V$ must be a vertex Poisson algebra homomorphism. □

2.6. **Standard filtration vs Li filtration.** There is an another filtration $\{G_pV; p \in \frac{1}{r_0}\mathbb{Z}_{\geq 0}\}$ of V called the *standard filtration*, which is an *increasing* filtration defined also by Li [Li04][2]: choose a set $\{a^i; i \in I\}$ of homogeneous strong generators of V. Let G_pV be the linear subspace of V spanned by the vectors

$$a^{i_1}_{(-n_1)} \dots a^{i_r}_{(-n_r)}\mathbf{1} \quad \text{satisfying} \quad \Delta_{a^{i_1}} + \dots + \Delta_{a^{i_r}} \leq p, \tag{17}$$

with $r \geq 0$, $n_i \geq 1$. Then

$$G_pV \subset G_qV \text{ for } p < q, \tag{18}$$

$$V = \bigcup_p G_pV, \tag{19}$$

$$TG_pV \subset G_pV, \tag{20}$$

$$a_{(n)}G_qV \subset G_{p+q}V, \quad \text{for } a \in G_pV,\ n \in \mathbb{Z}, \tag{21}$$

$$a_{(n)}G_qV \subset G_{p+q-1}V, \quad \text{for } a \in G_pV,\ n \in \mathbb{Z}_{\geq 0}. \tag{22}$$

It follows that $\mathrm{gr}^G V = \bigoplus_{p \in \frac{1}{r_0}\mathbb{Z}_{\geq 0}} G_pV/G_{p-1}V$ is naturally a vertex Poisson algebra, where we have set $G_pV = 0$ for $p < 0$.

By [Li04, Thoerem 4.14] the standard filtration $\{G_pV\}$ is characterized as the finest increasing filtration of V satisfying (18)-(22) such that

$$V_\Delta \subset G_\Delta V. \tag{23}$$

In particular, it is independent of the choice of a set of strong generators of V.

Both filtrations $\{F^pV\}$ and $\{G_pV\}$ are stable under the action of the Hamiltonian. Let $F^pV_\Delta = V_\Delta \cap F^pV$, $G_pV_\Delta = V_\Delta \cap G_pV$, so that $F^pV = \bigoplus_\Delta F^pV_\Delta$, $G_pV = \bigoplus_\Delta G_pV_\Delta$.

[2] In [Li04] it is assumed that V is $\mathbb{Z}_{\geq 0}$-graded and $V_0 = \mathbb{C}$, but this condition can be easily relaxed, as we can see from Proposition 2.6.1.

Proposition 2.6.1. *We have*

$$F^pV_\Delta = G_{\Delta-p}V_\Delta$$

for all p and Δ. Moreover, the linear isomorphism

$$\operatorname{gr}^F V \xrightarrow{\sim} \operatorname{gr}^G V$$

is an isomorphism of vertex Poisson algebras.

Proof. The second assertion is easily seen from the first. So let us prove the first assertion. First, we have by (23) $V_\Delta = G_\Delta V_\Delta$, namely,

$$F^0V_\Delta = G_\Delta V_\Delta.$$

Next we show the inclusion $F^pV_\Delta \subset G_{\Delta-p}V_\Delta$ by induction on $p \geq 0$. Let $p > 0$. By (15), F^pV_Δ is spanned by the elements

$$a_{(-i-1)}b, \quad \text{with } a \in V_{\Delta_a},\ b \in F^{p-i}V_{\Delta_b},\ i \geq 1,\ \Delta_a + \Delta_b + i = \Delta. \tag{24}$$

By the induction hypothesis $F^{p-i}V_{\Delta_b} \subset G_{\Delta_b-p+i}V_{\Delta_b}$ for $i \geq 1$. Because $a \in V_{\Delta_a} \subset G_{\Delta_a}V$, for the vector $a_{(-i-1)}b$ of the form (24) we have

$$a_{(-i-1)}b \in a_{(-i-1)}G_{\Delta_b-p+i}V_{\Delta_b} \subset G_{\Delta_a+\Delta_b-p+i}V_\Delta = G_{\Delta-p}V_\Delta.$$

Hence $F^pV_\Delta \subset G_{\Delta-p}V_\Delta$.

It remains to show the opposite inclusion $G_pV_\Delta \subset F^{\Delta-p}V_\Delta$. We prove that any element v of G_pV_Δ of the form (17) belongs to $F^{\Delta-p}V_\Delta$ by induction on $r \geq 0$. For $r = 0$ this is obvious. So let $r > 0$. Then $v = a^{i_1}_{(-n_1)}w$ with $w = a^{i_2}_{(-n_2)} \dots a^{i_r}_{(-n_r)}\mathbf{1}$, $n_i \geq 1$, $\sum \Delta_{a^i} \leq p$, $\Delta_{a^{i_1}} + \Delta_w + n - 1 = \Delta$, where each a^i is homogeneous. Because $w \in G_{p-\Delta_{a^{i_1}}}V$, the induction hypothesis gives that $w \in F^{\Delta_{a^{i_1}}+\Delta_w-p}V$. Hence $a^{i_1}_{(-n)}w \in F^{\Delta_{a^1}+\Delta_w-p+n-1}V_\Delta = F^{\Delta-p}V_\Delta$. This completes the proof. □

From Proposition 2.6.1 we get the following well-known fact [GN03].

Corollary 2.6.2. *Let $\{a^i; i \in I\}$ be a set of homogeneous vectors of V. Then the following are equivalent:*

(i) *$\{a^i; i \in I\}$ strongly generates V.*
(ii) *$\{\bar{a}^i; i \in I\}$ generates R_V.*

In particular V is finitely strongly generated if and only if R_V is finitely generated.

Proof. Suppose that $\{a^i; i \in I\}$ strongly generates V. By Proposition 2.6.1 $C_2(V) = F^1V$ is spanned by the vectors of the form (17) with $n_1 + \dots + n_r - r \geq 1$, proving that $\{\bar{a}^i; i \in I\}$ generates R_V.

Conversely, suppose that $\{\bar{a}^i; i \in I\}$ generates R_V. Then by (16) $\{\bar{a}^i\}$ generates $\operatorname{gr}^F V$ as a differential algebra. As the principal symbol map gives the isomorphism $V \xrightarrow{\sim} \operatorname{gr}^F V$ of vector spaces, it follows that $\{a^i\}$ strongly generates V. □

2.7. **Example: universal affine vertex algebras.** Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ the Kac-Moody affinization of $\mathfrak{g}$ with the central element K. For $k \in \mathbb{C}$, define

$$V^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}_k,$$

where $\mathbb{C}_k$ is the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which $\mathfrak{g}[t]$ acts trivially and K acts as the multiplication by k.

There is a unique vertex algebra structure in $V^k(\mathfrak{g})$ such that $\mathbf{1} = 1\otimes 1$ is the vacuum vector and

$$Y(x_{(-1)}\mathbf{1}, z) = x(z) := \sum_{n\in\mathbb{Z}} x_{(n)} z^{-n-1} \quad \text{for } x \in \mathfrak{g},$$

where $x_{(n)} = x\otimes t^n$. The vertex algebra $V^k(\mathfrak{g})$ is called the *universal affine vertex algebra associated with* $\mathfrak{g}$ *at level* k (see [Kac98, 4.7], [FBZ04, 2.4.2]).

The vertex algebra $V^k(\mathfrak{g})$ is conformal by the Sugawara construction, provided that $k \neq -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$.

Proposition 2.7.1. *For any $k \in \mathbb{C}$ we have the following.*

(i) *$R_{V^k(\mathfrak{g})} \cong \mathbb{C}[\mathfrak{g}^*]$ as Poisson algebras, where $\mathfrak{g}^*$ is equipped with the Kirillov-Kostant Poisson structure.*
(ii) *$\operatorname{gr}^F V^k(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}^*_\infty]$ as vertex Poisson algebras, where $\mathbb{C}[\mathfrak{g}^*_\infty]$ is equipped with the level 0 vertex Poisson structure.*

In particular $R_{V^k(\mathfrak{g})}$ and $\operatorname{gr}^F V^k(\mathfrak{g})$ are independent of $k \in \mathbb{C}$.

Proof. Although Proposition is well-known (see e.g., [DLM02, DSK05, DSK06]), we include the proof for completeness.

The vertex algebra $V^k(\mathfrak{g})$ is naturally $\mathbb{Z}_{\geq 0}$-graded (see e.g., [Kac98, Example 4.9b]). We consider the corresponding standard filtration $\{G_p V^k(\mathfrak{g})\}$.

We have $V^k(\mathfrak{g}) \cong U(\mathfrak{g}[t^{-1}]t^{-1})$ as vectors spaces. Under this isomorphism $G_p V^k(\mathfrak{g})$ gets identified with $U_p(\mathfrak{g}[t^{-1}]t^{-1})$, where $\{U_p(\mathfrak{g}[t^{-1}]t^{-1})\}$ is the standard filtration of the enveloping algebra $U(\mathfrak{g}[t^{-1}]t^{-1})$ in the Lie theory, that is, $U_p(\mathfrak{g}[t^{-1}]t^{-1})$ is the linear span of products of at most p elements in $\mathfrak{g}[t^{-1}]t^{-1}$. It follows that $\operatorname{gr}^G V^k(\mathfrak{g}) \cong S(\mathfrak{g}[t^{-1}]t^{-1})$ as commutative rings, and thus $\operatorname{gr}^F V^k(\mathfrak{g}) \cong S(\mathfrak{g}[t^{-1}]t^{-1})$ as commutative rings by Proposition 2.6.1.

Now $S(\mathfrak{g}[t^{-1}]t^{-1})$ is naturally isomorphic to $S(\mathfrak{g})_\infty = \mathbb{C}[\mathfrak{g}^*_\infty]$ as commutative rings, and we get the isomorphism

$$\Phi : \mathbb{C}[\mathfrak{g}^*_\infty] \xrightarrow{\sim} \operatorname{gr}^F V^k(\mathfrak{g}),$$

which is easily seen to be an isomorphism of differential algebras. It remains to prove that Φ is a homomorphism of vertex Poisson algebras. For this, it is sufficient to check that $\Phi(x_{(n)}) = \Phi(x)_{(n)}$ only for $x \in \mathfrak{g}$ by [Li04, Lemma 3.3]. For $n > 0$ this follows immediately from the definition, and for $n = 0$ this is equivalent to (i) of Proposition, which is easy to see. □

3. Associated varieties of vertex algebras and their modules

For the rest of the paper we will assume that V to be finitely strongly generated. Thus, in particular, R_V is finitely generated (Corollary 2.6.2).

3.1. **Filtration of V-modules.** Let M be a V-module. A *compatible filtration* $\{\Gamma^p M\}$ of M is a decreasing filtration $M = \Gamma^0 M \supset \Gamma^1 M \supset \cdots$ satisfying

$$\begin{aligned} & a_{(n)}\Gamma^q M \subset \Gamma^{p+q-n-1} M, \quad \text{for } a \in F^p V,\ n \in \mathbb{Z}, \\ & a_{(n)}\Gamma^q M \subset \Gamma^{p+q-n} M \quad \text{for } a \in F^p V,\ n \geq 0, \\ & H\Gamma^q M \subset \Gamma^q M, \end{aligned} \tag{25}$$

where $\{F^p V\}$ is the Li filtration of V. The associated graded space $\operatorname{gr}^\Gamma M = \bigoplus \Gamma^p M/\Gamma^{p+1} M$ is naturally a module over the vertex Poisson algebra $\operatorname{gr}^F V$. Here we have set $\Gamma^p M = M$ for $p < 0$. Note that each subspace $\Gamma^p M/\Gamma^{p+1}M$ of $\operatorname{gr}^\Gamma M$ is a submodule over $R_V = V/C_2(V)$.

A compatible filtration $\{\Gamma^p M\}$ is *good*[3] if it is separated (i.e. $\bigcap \Gamma^p M = 0$) and $\operatorname{gr}^\Gamma M$ is finitely generated over the ring $\operatorname{gr} V$.

Remark 3.1.1. Let $\{\Gamma^p M\}$ be a compatible filtration of a V-module M. Then $\bigcap_p \Gamma^p M$ is a submodule of M. Hence, if M is simple, we have either $\bigcap \Gamma^p M = 0$ or $\bigcap \Gamma^p M = M$.

Lemma 3.1.2. *Let V be a conformal vertex algebra, M a V-module, $\{\Gamma^p M\}$ a compatible filtration. Then $\operatorname{gr}^\Gamma M$ is a differential $\operatorname{gr}^F V$-module.*

Proof. Let ω be the Virasoro element of V. Then the derivation $\Gamma^p M/\Gamma^{p+1} \to \Gamma^{p-1}/\Gamma^p M$, $\sigma_p(m) \to \sigma_{p-1}(\omega_{(0)}m)$, defines an action of T with the desired property. □

Let $F^p M$ be a subspace of M spanned by the vectors

$$a^1_{(-n_1-1)} \cdots a^r_{(-n_r-1)} m$$

with $a^i \in V$, $m \in M$, $n_i \in \mathbb{Z}_{\geq 0}$, $n_1 + \cdots + n_r \geq p$. Then $\{F^p M\}$ is a compatible filtration [Li05]. It is separated because M is lower truncated by our assumption, see (the proof of) [Li05, Lemma 2.1.4]. The filtration $\{F^p M\}$ is called the *Li filtration* of a V-module M. Note that

$$F^1 M = C_2(M). \tag{26}$$

The $\operatorname{gr} V$-module structure of $\operatorname{gr}^F M$ gives a R_V-module structure on $M/C_2(M)$:

$$\bar{a}.\bar{m} = \overline{a_{(-1)}m}, \quad \text{for } a \in V,\ m \in M,$$

where $\bar{a} = a + C_2(V)$, $\bar{m} = m + C_2(M)$. Note that $M/C_2(M)$ is also a module over R_V viewed as a Lie algebra by the assignment $\bar{a} \mapsto L_{\bar{a}}$, where

$$L_{\bar{a}}\bar{m} = \overline{a_{(0)}m}, \quad \text{for } a \in V,\ m \in M.$$

These two actions are compatible in the sense that

$$L_{\bar{a}}\bar{b}.\bar{m} = \{\bar{a}, \bar{b}\}.\bar{m} + \bar{b}.L_{\bar{a}}\bar{m}. \tag{27}$$

Lemma 3.1.3 ([Li05, Lemma 4.2])**.** *For a V-module M, the $\operatorname{gr}^F V$-module $\operatorname{gr}^F M$ is generated by the subspace $M/C_2(M) = F^0 M/F^1 M$.*

Lemma 3.1.4. *Let $\{a^i; i \in I\}$ be a set of strong generators of V. Then*

$$C_2(M) = \operatorname{span}_{\mathbb{C}}\{a^i_{(-n)}m; i \in I,\ n \geq 2,\ m \in M\}.$$

Proof. Let $\{F^p C_2(V)\}$ be the induced filtration. Then we have $\operatorname{gr}^F C_2(V) = \bigoplus_{p \geq 1} F^p M/F^{p+1}M$ and

$$\operatorname{gr}^F C_2(M) = \operatorname{span}_{\mathbb{C}}\{\bar{a}^i_{(-n)}\bar{m}; i \in I, n \geq 2, m \in M\}$$

by (16), Corollary 2.6.2, and Lemma 3.1.3, where $\bar{m}$ is the image of $m \in M$ in $\operatorname{gr}^F M$. This proves the assertion. □

We call M *finitely strongly generated over V* if $M/C_2(M)$ is finitely generated over R_V.

---

[3]The definition of a "good" filtration in this note is different form the one given in [Li04].

Lemma 3.1.5. *A V-module M is finitely strongly generated if and only if there exists a good filtration of M.*

Proof. Suppose that $M/C_2(M)$ is finitely generated over R_V. Then the Li filtration of M is good by Lemma 3.1.3.

Let us show the opposite direction. Suppose that there exists a good filtration $\{\Gamma^p M\}$ of M, so that $\operatorname{gr}^\Gamma M$ is finitely generated over $\operatorname{gr}^F V$. Let $v_1, \dots, v_r$ elements of M such that there images $\bar{v}_1, \dots, \bar{v}_r$ generate $\operatorname{gr}^\Gamma M$ as a $\operatorname{gr}^F V$-module. Then it follows that M is spanned by the vectors of the form $(T^{j_1}a_1)\dots(T^{j_r}a_r)v_i$ with $a_s \in V$, $j_s \in \mathbb{Z}_{\geq 0}$, $s = 1, \dots, r$. This means that the images of v_i's generate $\operatorname{gr}^F M$ as well, and the assertion follows. □

Lemma 3.1.6. *Suppose that V is conformal and $V_0 = \mathbb{C}\mathbf{1}$. Then for a V-module M, M is finitely strongly generated over V if and only if M is C_1-cofinite.*

Proof. Observe that the grading on V induces a grading on R_V such that

$$R_V = \bigoplus_{\Delta \in \frac{1}{r_0}\mathbb{Z}_{\geq 0}} (R_V)_\Delta, \quad (R_V)_0 = \mathbb{C}. \tag{28}$$

The inclusion $C_2(M) \hookrightarrow C_1(M)$ gives the surjection

$$\eta : M/C_2(M) \twoheadrightarrow M/C_1(M).$$

As easily seen η is a homomorphism of R_V-modules, where $M/C_1(M)$ is considered as a trivial R_V-module, and we have $\ker \eta = R_V^* (M/C_2(M))$, where R_V^* is the argumentation ideal of R_V:

$$R_V^* = \bigoplus_{\Delta > 0} (R_V)_\Delta. \tag{29}$$

□

3.2. **Associated varieties of vertex algebras and their modules.**

Lemma 3.2.1. *Let M be a V-module.*

(i) *The annihilator $\operatorname{Ann}_{R_V}(M/C_2(M))$ of $M/C_2(M)$ in R_V is a Poisson ideal of R_V.*
(ii) *Let $\{\Gamma^p M\}$ be a compatible filtration. Then $\operatorname{Ann}_{(R_V)_\infty}(\operatorname{gr}^\Gamma M)$ is a $(R_V)_\infty$-submodule of $(R_V)_\infty$. If V is conformal, then $\operatorname{Ann}_{(R_V)_\infty}(\operatorname{gr}^\Gamma M)$ is a vertex Poisson algebra ideal of $(R_V)_\infty$.*

Proof. (i) follows from (27). (ii) By (6), $\operatorname{Ann}_{\operatorname{gr}^F V}(\operatorname{gr}^\Gamma M)$ is a submodule of $\operatorname{gr}^F V$. Because $(R_V)_\infty$ acts on $\operatorname{gr}^\Gamma M$ via the surjective homomorphism $\Phi : (R_V)_\infty \to \operatorname{gr}^F V$ of vertex Poisson algebras, $\operatorname{Ann}_{(R_V)_\infty}(\operatorname{gr}^\Gamma M) = \Phi^{-1}(\operatorname{Ann}_{\operatorname{gr}^F V}(\operatorname{gr}^\Gamma M))$ is a submodule of $(R_V)_\infty$. If V is conformal then $\operatorname{gr}^\Gamma M$ is a differential $\operatorname{gr}^F V$-module by Lemma 3.1.2. It follows that $\operatorname{Ann}_{\operatorname{gr}^F V}(\operatorname{gr}^\Gamma M)$ is an ideal and hence so is $\operatorname{Ann}_{(R_V)_\infty}(\operatorname{gr}^\Gamma M)$. □

Define the *associated variety* X_V of V by

$$X_V = \operatorname{Spec} R_V.$$

More generally, for a finitely strongly generated V-module M, define the associated variety X_M of M by

$$\begin{aligned} X_M &= \operatorname{supp}_{R_V}(M/C_2(M)) \\ &= \{\mathfrak{p} \in \operatorname{Spec} R_V ; \mathfrak{p} \supset \operatorname{Ann}_{R_V}(M/C_2(M))\}. \end{aligned}$$

By Lemma 3.2.1, X_M is a Poisson subvariety of X_V.

The following assertion is clear.

Lemma 3.2.2. *Let M be a finitely strongly generated V-module. Then the following are equivalent:*

(i) *M is C_2-cofinite.*
(ii) $\dim X_M = 0$.

Remark 3.2.3. Let V be as in Lemma 3.1.6. Then by (28) the variety X_V is conical. If M is a graded V-module then X_M is also conical. Hence

$$M \text{ is } C_2\text{-cofinite} \iff X_M = \{0\} \quad \text{(as topological spaces)}.$$

This is equivalent to that, for any homogeneous element $a \in V$ with $\Delta_a > 0$, $a + C_2(V)$ acts nilpotently on $M/C_2(M)$.

3.3. **Singular support of V-modules.** Let M be a finitely strongly generated V-module, $\{\Gamma^p M\}$ a good filtration of M. Define the *singular support* $SS(M)$ of M by

$$\begin{aligned} SS(M) &= \operatorname{supp}_{(R_V)_\infty}(\operatorname{gr}^\Gamma M) \\ &= \{p \in \operatorname{Spec}(R_V)_\infty ; \mathfrak{p} \supset \operatorname{Ann}_{(R_V)_\infty}(\operatorname{gr}^\Gamma M)\}. \end{aligned}$$

Then $SS(M)$ is a closed subscheme of the infinite jet scheme $(X_V)_\infty = \operatorname{Spec}(R_V)_\infty$. It is well-known that $SS(M)$ is independent of the choice of a good filtration of M.

Let

$$\pi_m : (X_V)_\infty \to (X_V)_m$$

be the natural projection.

Lemma 3.3.1. *Let M be a finitely strongly generated V-module.*

(i) *We have $X_M = \pi_0(SS(M))$.*
(ii) *If V is conformal then $SS(M) \subset (X_M)_\infty$.*

Proof. We may take the Li filtration $\{F^p M\}$ as a good filtration. By Lemma 3.1.3,

$$\operatorname{Ann}_{\operatorname{gr}^F V}(\operatorname{gr}^F M) \cap R_V = \operatorname{Ann}_{R_V}(M/C_2(M)) \tag{30}$$

Hence $X_M = \pi_0(SS(M))$. Next assume that V is conformal. By Lemma 3.2.1 $\operatorname{Ann}_{(R_V)_\infty}(\operatorname{gr}^F M)$ is T-stable. Thus from (30) it follows that $\operatorname{Ann}_{\operatorname{gr}^F V}(\operatorname{gr}^F M)$ contains the defining ideal of $(X_M)_\infty$, that is, the minimal T-stable ideal of $\mathbb{C}[(X_V)_\infty]$ containing $\operatorname{Ann}_{R_V}(M/C_2(M))$. This shows that $SS(M) \subset (X_M)_\infty$. □

A V-module M is called *lisse* if M is finitely strongly generated and $SS(M)$ is zero-dimensional, or equivalently, $\pi_m(SS(M))$ is zero-dimensional for all $m \geq 0$. A vertex algebra V is called lisse if it is lisse as a module over itself.

Remark 3.3.2. Suppose that V is $\mathbb{Q}_{\geq 0}$-graded and $V_0 = \mathbb{C}$. Then as in Remark 3.2.3 we see that a C_1-cofinite V-module M is lisse if and only if $SS(M) = \{0\}$. or equivalently, for any homogeneous element a with $\Delta_a > 0$, its principal symbol $\sigma_p(a)$ acts nilpotently on in $\operatorname{gr}^F M$.

Theorem 3.3.3. *The following are equivalent.*

(i) *V is C_2-cofinite.*
(ii) *V is lisse.*

Proof. The direction (ii) $\Rightarrow$ (i) immediately follows from Lemma 3.3.1. The direction (i) $\Rightarrow$ (ii) follows from the fact that $SS(V) \subset (X_V)_\infty$ and the jet scheme of 0-dimensional variety is 0-dimensional. □

The same assertion holds for the modules provided that $\operatorname{gr}^\Gamma M$ is a differential $\operatorname{gr} V$-module for a good grading $\{\Gamma^p M\}$. This is the case when V is conformal:

Theorem 3.3.4. *Suppose that V is conformal and M is a finitely strongly generated V-module. Then the following are equivalent.*

(i) *M is C_2-cofinite, or equivalently, $\dim X_M = 0$.*
(ii) *M is lisse.*

Proof. By Lemma 3.1.2 the assertion follows from in the same manner as Theorem 3.3.3. □

3.4. **Example: Virasoro vertex algebras.** Let $\mathcal{L} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}\mathbf{c}$ be the Virasoro algebra, with the commutation relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{(m^3-m)}{12}\delta_{m+n,0}\mathbf{c},$$
$$[\mathbf{c}, \mathcal{L}] = 0.$$

Define the subalgebras $\mathcal{L}_{\geq 0} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}\mathbf{c}$, $\mathcal{L}_{<0} = \bigoplus_{n<0} \mathbb{C}L_n$, so that $\mathcal{L} = \mathcal{L}_{<0} \oplus \mathcal{L}_{\geq 0}$. For $c \in \mathbb{C}$, denote by $\mathbb{C}_c$ the one-dimensional representation of $\mathcal{L}_{\geq 0}$ on which L_n acts trivially and $\mathbf{c}$ acts as the multiplication by c. Define $M_c = U(\mathcal{L}) \otimes_{U(\mathcal{L}_{\geq 0})} \mathbb{C}_c$. Then $L_{-1}(1\otimes 1) \in M_c$ is annihilated by all L_n with $n > 0$. Set

$$Vir^c = M_c / U(\mathcal{L})L_{-1}(1\otimes 1).$$

As is well-known there is a unique vertex algebra structure on Vir^c such that the image of $1\otimes 1$ is the vacuum vector $\mathbf{1}$ and

$$Y(L_{-2}\mathbf{1}, z) = L(z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2}.$$

The vertex algebra Vir^c is called the *universal Virasoro vertex algebra with central charge c*. Any $\mathcal{L}$-module with central charge c (i.e., $\mathbf{c}$ acts as the multiplication by c) on which $L(z)$ is a field can be considered as a Vir^c-module.

We have

$$R_{Vir^c} \cong \mathbb{C}[x] \tag{31}$$

(with the trivial Poisson structure), where x is the image of $L_{-2}\mathbf{1}$. Let N_c its unique maximal submodule of Vir_c, and $Vir_c = Vir^c/N_c$ its unique simple quotient.

Proposition 3.4.1. *The following are equivalent.*

(i) *Vir_c is C_2-cofinite.*
(ii) *Vir^c is reducible.*
(iii) *$c = 1 - 6(p-q)^2/pq$ for some $p, q \in \mathbb{Z}_{\geq 2}$ such that $(p,q) = 1$. (These are precisely the central charges of the minimal series representations of $\mathcal{L}$.)*

Proof. It is known that the image of N_c in R_{Vir^c} is nonzero if $N_c \neq 0$ (see e.g. [Wan93, Lemma 4.2 and Lemma 4.3] or [GK07, Proposition 4.3.2]). Therefore $X_{Vir_c} = \{0\}$ if and only if Vir^c is not irreducible. This happens if and only if the central charge is of the form in (iii) ([Kac74, FF84, GK07]). □

We now compare the C_2-cofiniteness condition with the zero singular support condition of Beilinson, Feigin and Mazur [BFM]. Let $\{U_p(\mathcal{L})\}$ be the standard filtration of $U(\mathcal{L})$ as in Introduction. Then the associated graded algebra $\operatorname{gr} U(\mathcal{L})$ is isomorphic to the symmetric algebra $S(\mathcal{L})$. Let M be a highest weight representation of $\mathcal{L}$ with central charge c, v the highest weight vector of M. Then $\{U_p(\mathcal{L})v\}$ defines an increasing filtration of M compatible with the standard filtration of $U(\mathcal{L})$. (Note that $U_p(\mathcal{L})v = U_p(\mathcal{L}_{<0})v$.) Let $\operatorname{gr}^{Lie} M$ denote the associated graded space $\bigoplus_p U_p(\mathcal{L})v/U_{p+1}(\mathcal{L})v$. Then $\operatorname{gr}^{Lie} M$ is an $S(\mathcal{L})$-module generated by the image of v. The singular support $SS^{BFM}(M)$ defined in [BFM] is by definition the support $\operatorname{supp}_{S(\mathcal{L})} \operatorname{gr}^{Lie}(M)$. Clearly, we have $SS^{BFM}(M) \subset \mathcal{L}^*_{<0}$.

Proposition 3.4.2. *Let M be a highest weight representation of $\mathcal{L}$ with central charge c. Then the following are equivalent.*

(i) *M is C_2-cofinite.*
(ii) *$SS^{BFM}(M) = \{0\}$.*

Proof. First, by Lemma 3.1.4 we have

$$C_2(M) = \operatorname{span}_{\mathbb{C}}\{L_{-n}m; m \in M, n \geq 3\}$$

because Vir^c is strongly generated by $L_{-2}\mathbf{1}$. It follows that $M/C_2(M)$ is spanned by the images of the vectors of the form $L_{-2}^m L_{-1}^n v$ with $m, n \geq 0$, where v is the highest weight vector of M. In particular $M/C_2(M)$ is generated by the image of the vectors $L_{-1}^n v$ with $n \geq 0$ over R_{Vir^c} by (31).

Suppose that $SS^{BFM}(M) = \{0\}$. We then have $L_{-1}^p v \in U_{p-1}(\mathcal{L}_{<0})v$ for a sufficiently large p. By considering the L_0-eigenvalue we find that this is equivalent to $L_{-1}^p v \in \sum_{n \geq 2} L_{-n} U(\mathcal{L}_{<0})v$. This happens if and only if $M/C_2(M)$ is finitely generated over R_{Vir^c}. We also have $L_{-2}^p v \in U_{p-1}(\mathcal{L}_{<0})v$ for a sufficiently large p. By taking account of the L_0-eigenvalue it follows that this is equivalent to $L_{-2}^p v \in \sum_{n \geq 3} L_{-n} U(\mathcal{L}_{<0})v = C_2(M)$. Therefore M is C_2-cofinite.

Conversely, suppose that M is C_2-cofinite. From the above argument it follows that L_{-1} and L_{-2} act nilpotently on $\operatorname{gr}^{Lie} M$. Then $SS^{BFM}(M)$ must be $\{0\}$, because L_{-1} and L_{-2} generates $\mathcal{L}_{<0}$ and $\sqrt{\operatorname{Ann}_{S(\mathcal{L}_{<0})} \operatorname{gr}^{Lie} M}$ is involtive by Gabber's theorem. □

By Proposition 3.4.2 a result of [BFM] can be read as follows.

Theorem 3.4.3 ([BFM])**.** *Let M be an irreducible highest weight representation of $\mathcal{L}$ with central charge c. Then the following are equivalent.*

(i) *M is C_2-cofinite.*
(ii) *M is isomorphic to one of the minimal series representations of $\mathcal{L}$.*

3.5. **Example: affine vertex algebras.** Let $\mathfrak{g}$, $\widehat{\mathfrak{g}}$, $V^k(\mathfrak{g})$ be as in Example 2.7.

For a weight λ of $\widehat{\mathfrak{g}}$, let $L(\lambda)$ be the irreducible (graded) representation of $\widehat{\mathfrak{g}}$ of highest weight λ. If λ is of level k, that is, if $\lambda(K) = k$, then $L(\lambda)$ can be naturally considered as a (simple) module over $V^k(\mathfrak{g})$. In particular $L(k\Lambda_0)$ (in the notation of [Kac90]) is isomorphic to the unique simple graded quotient of $V^k(\mathfrak{g})$.

Let $\lambda(K) = k$. Because

$$C_2(L(\lambda)) = \mathrm{span}_{\mathbb{C}}\{x_{(-n)}m; x \in \mathfrak{g},\ m \in L(\lambda),\ n \geq 2\}$$

by Lemma 3.1.4, it follows that $L(\lambda)$ is finitely strongly generated over $V^k(\mathfrak{g})$ if and only if the highest weight vector of $L(\lambda)$ generates a finite-dimensional $\mathfrak{g}$-module, or equivalently, the restriction of λ to the Cartan subalgebra of $\mathfrak{g}$ is integral dominant. This happens if and only if $L(\lambda)$ is a direct sum of finite-dimensional $\mathfrak{g}$-modules. In this case $X_{L(\lambda)}$ is a $\mathrm{Ad}\, G$-invariant, conic, Poisson subvariety of $\mathfrak{g}^*$.

We give a simple proof of the following well-known assertion (cf. [Zhu96]).

Proposition 3.5.1. *Let λ be a weight of $\widehat{\mathfrak{g}}$, and set $k = \lambda(K)$. Then $L(\lambda)$ is a C_2-cofinite $V^k(\mathfrak{g})$-module if it is an integrable representation of $\widehat{\mathfrak{g}}$. In particular the simple affine vertex algebra $L(k\Lambda_0)$ is C_2-cofinite if $k \in \mathbb{Z}_{\geq 0}$.*

Proof. Suppose that $L(\lambda)$ is integrable and set $J = \mathrm{Ann}_{R_{V^k(\mathfrak{g})}} L(\lambda)/C_2(L(\lambda))$. Let x_α is any root vector of $\mathfrak{g}$. Because $(x_\alpha)_{(-1)}$ acts locally nilpotently on $L(\lambda)$, its image $x_\alpha \in \mathbb{C}[\mathfrak{g}^*] = R_{V^k(\mathfrak{g})}$ belongs to $\sqrt{J}$. As $\sqrt{J}$ is an Poisson ideal of $\mathbb{C}[\mathfrak{g}^*]$ by Corollary 2.4.2 and Lemma 3.2.1, $\sqrt{J}$ contains $\{\mathfrak{g}, x_\alpha\} = \mathfrak{g}$, proving that $X_{L(\lambda)} = \{0\}$. □

One can show that the converse of Proposition 3.5.1 is also true, that is, $L(\lambda)$ is C_2-cofinite if and only if it is integrable, see [DM06, Ara].

References

[Ara] T. Arakawa. Associated varieties of modules over Kac-Moody algebras and C_2-cofiniteness of W-algebras. *preprint.* arXiv:1004.1554[math.QA].

[BD04] Alexander Beilinson and Vladimir Drinfeld. *Chiral algebras*, Vol. 51 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[BFM] A. Beilinson, B. Feigin, and B Mazur. Introduction to algebraic field theory on curves. *preprint.*

[Bor86] Richard E. Borcherds. Vertex algebras, Kac-Moody algebras, and the Monster. *Proc. Nat. Acad. Sci. U.S.A.*, Vol. 83, No. 10, pp. 3068–3071, 1986.

[CO78] William Casselman and M. Scott Osborne. The restriction of admissible representations to $\mathfrak{n}$. *Math. Ann.*, Vol. 233, No. 3, pp. 193–198, 1978.

[DLM00] Chongying Dong, Haisheng Li, and Geoffrey Mason. Modular-invariance of trace functions in orbifold theory and generalized Moonshine. *Comm. Math. Phys.*, Vol. 214, No. 1, pp. 1–56, 2000.

[DLM02] Chongying Dong, Haisheng Li, and Geoffrey Mason. Vertex Lie algebras, vertex Poisson algebras and vertex algebras. In *Recent developments in infinite-dimensional Lie algebras and conformal field theory (Charlottesville, VA, 2000)*, Vol. 297 of *Contemp. Math.*, pp. 69–96. Amer. Math. Soc., Providence, RI, 2002.

[DM06] Chongying Dong and Geoffrey Mason. Integrability of C_2-cofinite vertex operator algebras. *Int. Math. Res. Not.*, pp. Art. ID 80468, 15, 2006.

[DSK05] Alberto De Sole and Victor G. Kac. Freely generated vertex algebras and non-linear Lie conformal algebras. *Comm. Math. Phys.*, Vol. 254, No. 3, pp. 659–694, 2005.

[DSK06] Alberto De Sole and Victor G. Kac. Finite vs affine W-algebras. *Japan. J. Math.*, Vol. 1, No. 1, pp. 137–261, 2006.

[EM09] Lawrence Ein and Mircea Mustaţă. Jet schemes and singularities. In *Algebraic geometry—Seattle 2005. Part 2*, Vol. 80 of *Proc. Sympos. Pure Math.*, pp. 505–546. Amer. Math. Soc., Providence, RI, 2009.

[FBZ04] Edward Frenkel and David Ben-Zvi. *Vertex algebras and algebraic curves*, Vol. 88 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2004.

[FF84] B. L. Feĭgin and D. B. Fuchs. Verma modules over the Virasoro algebra. In *Topology (Leningrad, 1982)*, Vol. 1060 of *Lecture Notes in Math.*, pp. 230–245. Springer, Berlin, 1984.

[GK07] Maria Gorelik and Victor Kac. On simplicity of vacuum modules. *Adv. Math.*, Vol. 211, No. 2, pp. 621–677, 2007.

[GN03] Matthias R. Gaberdiel and Andrew Neitzke. Rationality, quasirationality and finite W-algebras. *Comm. Math. Phys.*, Vol. 238, No. 1-2, pp. 305–331, 2003.

[Hua08a] Yi-Zhi Huang. Rigidity and modularity of vertex tensor categories. *Commun. Contemp. Math.*, Vol. 10, No. suppl. 1, pp. 871–911, 2008.

[Hua08b] Yi-Zhi Huang. Vertex operator algebras and the Verlinde conjecture. *Commun. Contemp. Math.*, Vol. 10, No. 1, pp. 103–154, 2008.

[Kac74] V. G. Kac. Infinite-dimensional Lie algebras, and the Dedekind η-function. *Funkcional. Anal. i Priložen.*, Vol. 8, No. 1, pp. 77–78, 1974.

[Kac90] Victor G. Kac. *Infinite-dimensional Lie algebras*. Cambridge University Press, Cambridge, third edition, 1990.

[Kac98] Victor Kac. *Vertex algebras for beginners*, Vol. 10 of *University Lecture Series*. American Mathematical Society, Providence, RI, second edition, 1998.

[KW08] Victor G. Kac and Minoru Wakimoto. On rationality of W-algebras. *Transform. Groups*, Vol. 13, No. 3-4, pp. 671–713, 2008.

[Li99] Haisheng Li. Some finiteness properties of regular vertex operator algebras. *J. Algebra*, Vol. 212, No. 2, pp. 495–514, 1999.

[Li04] Haisheng Li. Vertex algebras and vertex Poisson algebras. *Commun. Contemp. Math.*, Vol. 6, No. 1, pp. 61–110, 2004.

[Li05] Haisheng Li. Abelianizing vertex algebras. *Comm. Math. Phys.*, Vol. 259, No. 2, pp. 391–411, 2005.

[Miy04] Masahiko Miyamoto. Modular invariance of vertex operator algebras satisfying C_2-cofiniteness. *Duke Math. J.*, Vol. 122, No. 1, pp. 51–91, 2004.

[NT05] Kiyokazu Nagatomo and Akihiro Tsuchiya. Conformal field theories associated to regular chiral vertex operator algebras. I. Theories over the projective line. *Duke Math. J.*, Vol. 128, No. 3, pp. 393–471, 2005.

[Pri99] Mirko Primc. Vertex algebras generated by Lie algebras. *J. Pure Appl. Algebra*, Vol. 135, No. 3, pp. 253–293, 1999.

[Wan93] Weiqiang Wang. Rationality of Virasoro vertex operator algebras. *Internat. Math. Res. Notices*, No. 7, pp. 197–211, 1993.

[Zhu96] Yongchang Zhu. Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.*, Vol. 9, No. 1, pp. 237–302, 1996.

Department of Mathematics, Nara Women's University, Nara 630-8506, JAPAN

Current address: Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, JAPAN

E-mail address: `arakawa@kurims.kyoto-u.ac.jp`