BHK interpretation and Bilateralism

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**Bilateralism**

\[\text{Meaning} = \text{Assertion and Denial}\]

- Inferentialism and proof-theoretic semantics.
- Smiley, Rumfitt, Restall, Ripley, etc.

**vs.** Unilateralism: Constructive PTS (BHK, Dummett, Prawitz) and its dual version

 Loving Two unilateralist semantics fit together?
   - Assertion & denial, proof & dual proof
   - How to show the fit between two meanings?
Introduction

Two unilateralist semantics

Bi-Intuitionist Logic

Bilateralist validity
Introduction

Two unilateralist semantics

Bi-Intuitionist Logic

Bilateralist validity
BHK interpretation

- A proof of $A \land B$ is a pair $\langle a, b \rangle$ consisting of a proof $a$ of $A$ and a proof $b$ of $B$.
- A proof of $A \lor B$ is a pair $\langle i, x \rangle$ such that $i = 0$ and $x$ is a proof of $A$, or $i = 1$ and $x$ is a proof of $B$.
- A proof of $A \rightarrow B$ is a construction that transforms any proof of $A$ into a proof of $B$.
- $\bot$ has no proof.
BHK interpretation and PTS
(Dummett 1991; Prawitz 2006; Schroeder-Heister 2006)

- Explanation of conditions of assertion in terms of the primitive notion of proof (or construction).
- Suitable for Intuitionist Logic.

Definition
A sequent $A_1, \ldots, A_n \vdash B$ is $i$-valid if there is a construction that transforms any list $a_1, \ldots, a_n$ of proofs of $A_1, \ldots, A_n$ into a proof of $B$.

Proposition (correctness)
If $A_1, \ldots, A_n \vdash B$ is derivable in NJ (LJ), then it is $i$-valid.
Dual BHK

- Another unilateralist semantics
- In terms of dual proofs

⇒ Dual-BHK and Dual-Intuitionist Logic
**LJ for Int**

- "Singleton on the right"

\[
\begin{align*}
A &\vdash A \quad \text{(Id)} \\
\Gamma, \bot &\vdash C \quad \text{(} \bot \text{L)} \\
\frac{\Gamma \vdash C}{\Gamma, A \vdash C} \quad \text{(Weakening)} \\
\frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C} \quad \text{(Contraction)} \\
\frac{\Gamma, A_i \vdash C}{\Gamma, A_0 \land A_1 \vdash C} \quad \text{(}\land\text{L)} \\
\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \land B} \quad \text{(}\land\text{R)} \\
\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \lor B \vdash C} \quad \text{(}\lor\text{L)} \\
\frac{\Gamma \vdash A_i}{\Gamma \vdash A_0 \lor A_1} \quad \text{(}\lor\text{R)} \\
\frac{\Gamma \vdash A \quad \Gamma', B \vdash C}{\Gamma, \Gamma', A \to B \vdash C} \quad \text{(}\to\text{L)} \\
\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \quad \text{(}\to\text{R)}
\end{align*}
\]

- \( \neg A := A \to \bot \).
LDJ for dual-Int

- "Singleton on the left", $\iff$: subtraction, exclusion

\[
\begin{align*}
A \vdash A \quad & (\text{Id}) & C \vdash \top, \Delta \quad & (\bot L) & \frac{C \vdash \Delta, \Delta'}{C \vdash \Delta, \Delta'} \quad (\text{Cut}) \\
\frac{C \vdash \Delta}{C \vdash A, \Delta} \quad & (\text{Weakening}) & \frac{C \vdash A, A, \Delta}{C \vdash A, \Delta} \quad & (\text{Contraction}) \\
\frac{A_i \vdash \Delta}{A_0 \land A_1 \vdash \Delta} \quad & (\land L) & \frac{C \vdash A, \Delta \quad C \vdash B, \Delta}{C \vdash A \land B, \Delta} \quad & (\land R) \\
\frac{A \vdash \Delta \quad B \vdash \Delta}{A \lor B \vdash \Delta} \quad & (\lor L) & \frac{C \vdash A_i, \Delta}{C \vdash A_0 \lor A_1, \Delta} \quad & (\lor R) \\
\frac{A \vdash B, \Delta}{A \iff B \vdash \Delta} \quad & (\iff L) & \frac{C \vdash A, \Delta \quad B \vdash \Delta'}{C \vdash A \iff B, \Delta, \Delta'} \quad & (\iff R) \\
\end{align*}
\]

- $\sim A := \top \iff A$. 
Dual-BHK interpretation
(cf. Wansing 2010)

- A dual proof of $A \land B$ is a pair $\langle i, x \rangle$ such that $i = 0$ and $x$ is a dual proof of $A$, or $i = 1$ and $x$ is a dual proof of $B$.
- A dual proof of $A \lor B$ is a pair $\langle a, b \rangle$ consisting of a dual proof $a$ of $A$ and a dual proof $b$ of $B$.
- A dual proof of $A \leftarrow B$ is a construction that transforms any dual proof of $B$ into a dual proof of $A$.
- $\top$ has no dual proof.
Dual-BHK interpretation

- Explanation of **conditions of denial** in terms of dual proof (yet another kind of construction).
- Suitable for Dual Intuitionist Logic.

**Definition**
A sequent $A \vdash B_1, \ldots, B_m$ is $d$-**valid** if there is a construction that transforms any list of dual proofs of $B_1, \ldots, B_m$ into a dual proof of $A$.

**Proposition (correctness)**
If $A \vdash B_1, \ldots, B_m$ is derivable in LDJ, then it is $d$-valid.
Problems

- BHK and Dual-BHK: two unilateralist semantics.
  How do they agree or disagree on the meaning?

- Bi-Intuitionist Logic: A logic with the features of Intuitionist and Dual Intuitionist Logic.
Introduction

Two unilateralist semantics

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Bilateralist validity
Model theory for Bilnt

Definition (Language)

\[ \mathcal{L} := \{ \land, \lor, \to, \iff \} . \]

Let \( \bot := p \iff p \) and \( \top := p \to p \) for some fixed atom \( p \).
And define \( \neg A := A \to \bot \) and \( \sim A := \top \iff A \).

Definition (Model)

A Bilnt model is a triple \( \langle W, \leq, V \rangle \) where

- \( W \) : a non-empty set (of possible worlds)
- \( \leq \) : a reflexive and transitive relation on \( W \)
- \( V : \text{Atom} \to 2^W \), a valuation which is persistent, i.e.

\[ (\forall w, w' \in W)(w \in V(p) \text{ and } w \leq w' \Rightarrow w' \in V(p)) \].
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Model theory for Bilnt (cont.)

Definition
Given a Bilnt model $\langle W, \leq, V \rangle$, write $w \models p$ for $w \in V(p)$. The relation $\models$ extends as follows:

- $w \models A \land B$ if $w \models A$ and $w \models B$
- $w \models A \lor B$ if $w \models A$ or $w \models B$
- $w \models A \rightarrow B$ if $(\forall v \geq w)(v \models A \Rightarrow v \models B)$
- $w \models A \leftrightarrow B$ if $(\exists v \leq w)(v \models A$ and $v \not\models B)$
Model theory for Bilnt (cont.)

Proposition (Persistence)
\[ \models \text{ satisfies the persistence condition, i.e. for any formula } \mathcal{A}, \]
\[ (\forall w, w' \in W)(w \models \mathcal{A} \text{ and } w \leq w' \Rightarrow w' \models \mathcal{A}). \]

Definition (Validity)
For any formula \( \mathcal{A} \) and \( \mathcal{B} \), we define
\[ \mathcal{A} \models \mathcal{B} \iff \text{def. for any Bilnt model } \langle W, \leq, V \rangle \text{ and any } \]
\[ w \in W, \text{ if } w \models \mathcal{A} \text{ then } w \models \mathcal{B}. \]
Characteristic validity and invalidity

Recall \( \bot = p \leftrightarrow p, \top = p \rightarrow p \) and

\[
\neg A := A \rightarrow \bot \\
\sim A := \top \leftarrow A
\]

(intuitionist negation)

(dual intuitionist negation)

- \( \top \not\models A \lor \neg A \) but \( \top \models A \lor \sim A \).
- \( A \land \neg A \models \bot \) but \( A \land \sim A \not\models \bot \).
- \( \not\models (A \rightarrow (B \lor C)) \rightarrow ((A \rightarrow B) \lor C) \)
- \( \not\models (A \land (B \leftarrow C)) \rightarrow ((A \land B) \leftarrow C) \).
Display Calculus \( \delta \)-Bilnt
(Belnap 1982; Wansing 2010)

- A generalization of sequent calculus
- A sequent \( X \vdash Y \) consists of structures \( X \) and \( Y \) instead of sequences, multisets or sets of formulas.

**Definition (structures)**

The set of structures for \( \delta \)-Bilnt is defined by:

\[
X ::= A \mid I \mid X \circ X \mid X \bullet X.
\]

Intuitively, \( I \) represents an empty structure and \( \circ \) (\( \bullet \)) corresponds to a comma on LHS (RHS).
Logical rules for $\delta$-Bilnt

\[
\frac{A \circ B \vdash Y}{A \land B \vdash Y} \quad (\land L) \quad \frac{X \vdash A \quad Y \vdash B}{X \circ Y \vdash A \land B} \quad (\land R)
\]

\[
\frac{A \vdash X \quad B \vdash Y}{A \lor B \vdash X \bullet Y} \quad (\lor L) \quad \frac{X \vdash A \bullet B}{X \vdash A \lor B} \quad (\lor R)
\]

\[
\frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X \circ Y} \quad (\rightarrow L) \quad \frac{X \vdash A \circ B}{X \vdash A \rightarrow B} \quad (\rightarrow R)
\]

\[
\frac{A \bullet B \vdash Y}{A \leftarrow B \vdash Y} \quad (\leftarrow L) \quad \frac{X \vdash A \quad B \vdash Y}{X \bullet Y \vdash A \leftarrow B} \quad (\leftarrow R)
\]
Definition (display rules)

The display calculus $\delta$-BiInt has the following display rules:

\[
\begin{align*}
X \circ Y \vdash Z & \quad Z \vdash X \bullet Y \\
X \vdash Y \circ Z & \quad Z \bullet X \vdash Y \\
Y \circ X \vdash Z & \quad Z \vdash Y \bullet X
\end{align*}
\]

Intuitively,

\[
\begin{align*}
A \land B \vdash C & \quad C \vdash A \lor B \\
A \vdash B \rightarrow C & \quad C \leftarrow A \vdash B \\
B \land A \vdash C & \quad C \vdash B \lor A
\end{align*}
\]
Structural rules for $\delta$- Bilnt

\[
p \vdash p \quad \text{(Id)}
\]

\[
\begin{array}{ll}
\frac{X \vdash Y}{X \circ I \vdash Y} & \frac{X \vdash Y}{X \vdash Y \bullet I} \\
\frac{X \vdash Y}{X \circ Z \vdash Y} & \frac{X \vdash Y}{X \vdash Y \bullet Z} \\
\frac{X \circ X \vdash Y}{X \vdash Y} & \frac{X \vdash Y \bullet Y}{X \vdash Y} \\
\frac{(X \circ Y) \circ Z \vdash W}{X \circ (Y \circ Z) \vdash W} & \frac{W \vdash (X \bullet Y) \bullet Z}{W \vdash X \bullet (Y \bullet Z)}
\end{array}
\]

(lm) (rm) (lc) (rc) (la) (ra)
Definition (antecedent and succedent part)

Given a sequent $S = X \vdash Y$, we define

- $X$ is AP (an antecedent part) of $S$;
- $Y$ is SP (a succedent part) of $S$;
- $(W \circ Z)$ is AP $\Rightarrow W, Z$ are AP;
- $(W \circ Z)$ is SP $\Rightarrow W$ is AP and $Z$ is SP;
- $(W \bullet Z)$ is AP $\Rightarrow W$ is AP and $Z$ is SP;
- $(W \bullet Z)$ is SP $\Rightarrow W, Z$ are SP.
Display property

Proposition (display property)
For any sequent $S = X \vdash Y$ and any substructure $Z$ of it, we can display the occurrence of $Z$, i.e. there is a sequent $S'$ such that:

- $S$ and $S'$ are interderivable by means of display rules only,
- If $Z$ is AP of $S$, then $S'$ is of the form $Z \vdash Y'$ and
- If $Z$ is SP of $S$, then $S'$ is of the form $X' \vdash Z$.

Theorem (Cut elimination)
Cut is eliminable from any derivation in $\delta$-BiInt + Cut.
Completeness

Definition
The translations $\tau_1$ and $\tau_2$ from structures into formulas are defined inductively as:

\[
\begin{align*}
\tau_1(A) & := A \\
\tau_1(I) & := \top(=p \rightarrow p) \\
\tau_1(X \circ Y) & := \tau_1(X) \land \tau_1(Y) \\
\tau_1(X \bullet Y) & := \tau_1(X) \leftarrow \tau_2(Y)
\end{align*}
\]

\[
\begin{align*}
\tau_2(A) & := A \\
\tau_2(I) & := \bot(=p \leftarrow p) \\
\tau_2(X \circ Y) & := \tau_1(X) \rightarrow \tau_2(Y) \\
\tau_2(X \bullet Y) & := \tau_2(X) \lor \tau_2(Y)
\end{align*}
\]

Theorem (Completeness)
$X \vdash Y$ is derivable in $\delta$-Bilnt if and only if $\tau_1(X) \models \tau_2(Y)$.
Logical rules for $\delta$-Bilnt (revised)

\[
\begin{align*}
\frac{A_i \vdash Y}{A_0 \land A_1 \vdash Y} & \quad (\land L) & \frac{X \vdash A \quad X \vdash B}{X \vdash A \land B} & \quad (\land R) \\
\frac{A \vdash Y \quad B \vdash Y}{A \lor B \vdash Y} & \quad (\lor L) & \frac{X \vdash A_i}{X \vdash A_0 \lor A_1} & \quad (\lor R) \\
\frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X \circ Y} & \quad (\rightarrow L) & \frac{X \vdash A \circ B}{X \vdash A \rightarrow B} & \quad (\rightarrow R) \\
\frac{A \circ B \vdash Y}{A \leftarrow B \vdash Y} & \quad (\leftarrow L) & \frac{X \vdash A \quad B \vdash Y}{X \circ Y \vdash A \leftarrow B} & \quad (\leftarrow R)
\end{align*}
\]
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Bi-Intuitionist Logic

Bilateralist validity
Fit?

- Proofs and dual proofs coexist in $\delta$-BiInt.
- Two kind of meanings fit together in $\delta$-BiInt?
- BHK and Dual-BHK must be extended to interpret $\leftarrow$ and $\rightarrow$. 
BHK interpretation extended
(cf. Wansing 2010)

- A proof of \( A \land B \) is a pair \( \langle a, b \rangle \) consisting of a proof \( a \) of \( A \) and a proof \( b \) of \( B \).
- A proof of \( A \lor B \) is a pair \( \langle i, x \rangle \) such that \( i = 0 \) and \( x \) is a proof of \( A \), or \( i = 1 \) and \( x \) is a proof of \( B \).
- A proof of \( A \rightarrow B \) is a construction that transforms any proof of \( A \) into a proof of \( B \).
- A proof of \( A \leftarrow B \) is a pair \( \langle a, b \rangle \) consisting of a proof \( a \) of \( A \) and a dual proof \( b \) of \( B \).
Dual-BHK interpretation extended  
(cf. Wansing 2010)

- A dual proof of $A \land B$ is a pair $\langle i, x \rangle$ such that $i = 0$ and $x$ is a dual proof of $A$, or $i = 1$ and $x$ is a dual proof of $B$.
- A dual proof of $A \lor B$ is a pair $\langle a, b \rangle$ consisting of a dual proof $a$ of $A$ and a dual proof $b$ of $B$.
- A dual proof of $A \leftarrow B$ is a construction that transforms any dual proof of $B$ into a dual proof of $A$.
- A dual proof of $A \rightarrow B$ is a pair $\langle a, b \rangle$ consisting of a proof $a$ of $A$ and a dual proof $b$ of $B$. 
i-validity fails

A rule in $\delta$-BiInt that is not i-valid.

$$\frac{Z \cdot X \vdash Y}{Z \vdash X \cdot Y} \quad \Rightarrow \quad \frac{\text{A } \leftarrow \text{B } \vdash \text{C}}{\text{A } \vdash \text{B } \lor \text{C}}$$

- ● on RHS: sequents become multiple-conclusion.
- Impossible to interpret it as intuitionist’s disjunction with disjunction property.
- Proof is not preserved from LHS to RHS.
- At most impossibility of dual proof of RHS

The same applies to d-validity.
**Bilateralist reading of sequents**  
(cf. Restall 2005)

**Definition**

$A \vdash B$ is **b-valid** if it is not the case that $A$ has a proof and $B$ has a dual proof.

- to assert $A$ and to deny $B$ is to make a mistake;
- if $A$ has a proof then $B$ can’t have a dual proof;
- if $B$ has a dual proof then $A$ can’t have a proof.

$\Rightarrow$ Two criteria of fit between proof & dual proof
Criterion 1: Identity

1. A can’t have both a proof and a dual proof.
2. No clash between proofs and dual proofs.
3. No overlap between assertion and denial.
4. Established directly by BHK and Dual-BHK on the assumptions $p \vdash p$.
5. Derivable in $\delta$-BiInt.
Criterion 2: Cut

\[
\frac{X \vdash A \quad A \vdash Y}{X \vdash Y} \quad \text{(Cut)}
\]

- \( X \vdash A \): a proof of \( X \) excludes dual-provability of \( A \)
- \( A \vdash Y \): a dual proof of \( Y \) excludes provability of \( A \), then
- \( X \vdash Y \): the proof of \( X \) clashes with the dual proof of \( Y \).

I.e. it is impossible to exclude both provability and dual-provability of \( A \) without any clash.

- No gap between assertion and denial.
Fit

- Cut & Identity: criteria of fit between proofs & dual proofs.
  - Agreement on meanings between BHK and dual-BHK

- No problem with Identity. How about Cut?
- Seems difficult to establish directly by BHK and dual-BHK.
- Cut elimination for $\delta$-BiInt tells us something?
Almost all rules in $\delta$-BiInt are $b$-valid except for:

\[
\begin{align*}
\frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X \circ Y} (\rightarrow L) \\
\frac{X \vdash A \quad B \vdash Y}{X \bullet Y \vdash A \leftarrow B} (\leftarrow R)
\end{align*}
\]

A proof of $A \rightarrow B$ requires a proof of $A$. But $X \vdash A$ gives at most impossibility of dual proof of $A$.

To make them $b$-valid:

\[
\begin{align*}
\frac{B \vdash Y}{A \rightarrow B \vdash A \circ Y} (\rightarrow L') \\
\frac{X \vdash A}{X \bullet B \vdash A \leftarrow B} (\leftarrow R')
\end{align*}
\]
Revision of rules and **Cut**

New rules are equivalent to the original through **Cut**:

\[
\frac{B \vdash Y}{A \to B \vdash A \circ Y} \quad (\to L')
\]

\[
\frac{X \vdash A}{A \vdash A \to B \circ Y} \quad \frac{A \vdash A \to B \circ Y}{X \vdash A \to B \circ Y} \quad (\text{Cut})
\]

\[
\frac{A \vdash A \to B \circ Y}{A \to B \vdash A \circ Y} \quad (\to L)
\]

Let **Cut'** denote **Cut** of this form (and the dual form for $\leftarrow R'$).
Weak Cut-elimination

- $\delta\text{-BiInt}' = \text{the system with } \rightarrow L' \text{ and } \leftarrow R'$
  - Every rule is $b$-valid.
  - $\delta\text{-BiInt}' + \text{Cut}' \cong \delta\text{-BiInt}$

Fact (cf. Schroeder-Heister \textit{forthcoming})

$\delta\text{-BiInt}' + \text{Cut} \cong \delta\text{-BiInt}' + \text{Cut}' \not\cong \delta\text{-BiInt}'$.

- $\delta\text{-BiInt}'$: $b$-valid but $\text{Cut}'$ is not admissible.
- $\delta\text{-BiInt}$: $\text{Cut}$ is admissible but not $b$-valid.
Example of failure of $\text{Cut}'$

\[
\begin{align*}
&\quad (\rightarrow L') \\
&\begin{array}{c}
p \vdash p \quad (p \lor q) \rightarrow r \vdash p \lor q \circ r \\
p \lor q \vdash (p \lor q) \rightarrow r \circ r \\
p \vdash (p \lor q) \rightarrow r \circ r
\end{array}
\end{align*}
\]
Failure of Cut indicates:

- there may be a combination of a proof of $X$ and a dual proof of $Y$ such that:
  - the former excludes dual-provability of $A$,
  - the latter excludes provability of $A$,
  - but they don’t cause any clash in $\delta$-BiInt'.

A gap between proof and dual proof.
Conclusion

Under Bilateralist reading of sequents,

- Cut & Identity: Criteria of fit between two aspects
- Failure of Cut’: a gap between proof and dual proof.
- Cut admissibility in δ-BiInt:
  - Proof and dual proof are adjusted implicitly.
  - \( \rightarrow L \) and \( \leftarrow R \) are where the adjustment occurs.
Relativized $b$-validity

To make this observation more precise, $b$-validity (and extensions of (dual-)BHK) should be defined more carefully:

**Definition (Atomic base)**

An atomic base is a pair $\langle s, t \rangle$ of two sets of atomic formulas with $s \cap t = \emptyset$. Define the relation $\vdash$ between atomic bases and signed atomic formulas as:

\[
\langle s, t \rangle \vdash +p \iff p \in s;
\]

\[
\langle s, t \rangle \vdash -p \iff p \in t.
\]
Relativized $b$-validity

Definition

$\models$ extends as follows:

$\langle s, t \rangle \models +T \quad \quad \langle s, t \rangle \not\models +\bot$

$\langle s, t \rangle \models +(A \land B) \iff \langle s, t \rangle \models +A \land \langle s, t \rangle \models +B$

$\langle s, t \rangle \models +(A \lor B) \iff \langle s, t \rangle \models +A \lor \langle s, t \rangle \models +B$

$\langle s, t \rangle \models +(A \rightarrow B) \iff \forall s' \supseteq s. \langle s', t \rangle \models +A \text{ implies } \langle s', t \rangle \models +B$

$\langle s, t \rangle \models +(A \leftarrow B) \iff \exists s' \subseteq s \exists t' \supseteq t.
\langle s', t' \rangle \models +A \land \langle s', t' \rangle \models -B$

Dually for $\langle s, t \rangle \models -A$. Especially,

$\langle s, t \rangle \models -(A \rightarrow B) \iff \exists t' \subseteq t \exists s' \supseteq s.
\langle s', t' \rangle \models +A \land \langle s', t' \rangle \models -B$
Proposition

(1) Persistence:
- \( \langle s, t \rangle \vdash +A \) and \( s' \supseteq s \) implies \( \langle s', t \rangle \vdash +A \).
- \( \langle s, t \rangle \vdash -A \) and \( t' \supseteq t \) implies \( \langle s, t' \rangle \vdash -A \).

(2) Irrelevance:
- If \( \langle s, t \rangle \vdash +A \), then for any \( t' \), \( \langle s, t' \rangle \vdash +A \).
- If \( \langle s, t \rangle \vdash -A \), then for any \( s' \), \( \langle s', t \rangle \vdash +A \).

(3) Consistency: It is not the case that \( \langle s, t \rangle \vdash +A \) and \( \langle s, t \rangle \vdash -A \).
Relativized $b$-validity

Definition ($b$-validity)
A sequent $A \vdash B$ is $\langle s, t \rangle$-valid if it is not the case that

$$\langle s, t \rangle \not\vdash +A \text{ and } \langle s, t \rangle \not\vdash -B.$$ 

$A \vdash B$ is $b$-valid if it is $\langle s, t \rangle$-valid for any $\langle s, t \rangle$.

Proposition (Soundness)
If $X \vdash Y$ is derivable in $\delta$-BiInt', then it is $b$-valid.

Proposition
Cut’, Intuitionist LEM, $(A \rightarrow (B \lor C)) \rightarrow ((A \rightarrow B) \lor C)$ etc. are not $b$-valid.
References (1)

References (2)


