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BHK interpretation and Bilateralism

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Bilateralism

*Meaning = Assertion and Denial*

- Inferentialism and proof-theoretic semantics.
- Smiley, Rumfitt, Restall, Ripley, etc.

vs. Unilateralism: Constructive PTS (BHK, Dummett, Prawitz) and its dual version

❤ Two unilateralist semantics fit together?
  - Assertion & denial, proof & dual proof
  - How to show the fit between two meanings?
Introduction

Two unilateralist semantics

Bi-Intuitionist Logic

Bilateralist validity
Introduction

Two unilateralist semantics

Bi-Intuitionist Logic

Bilateralist validity
BHK interpretation

- A proof of $A \land B$ is a pair $\langle a, b \rangle$ consisting of a proof $a$ of $A$ and a proof $b$ of $B$.
- A proof of $A \lor B$ is a pair $\langle i, x \rangle$ such that $i = 0$ and $x$ is a proof of $A$, or $i = 1$ and $x$ is a proof of $B$.
- A proof of $A \rightarrow B$ is a construction that transforms any proof of $A$ into a proof of $B$.
- $\bot$ has no proof.
BHK interpretation and PTS
(Dummett 1991; Prawitz 2006; Schroeder-Heister 2006)

- Explanation of conditions of assertion in terms of the primitive notion of proof (or construction).
- Suitable for Intuitionist Logic.

Definition
A sequent $A_1, \ldots, A_n \vdash B$ is $i$-valid if there is a construction that transforms any list $a_1, \ldots, a_n$ of proofs of $A_1, \ldots, A_n$ into a proof of $B$.

Proposition (correctness)
If $A_1, \ldots, A_n \vdash B$ is derivable in NJ (LJ), then it is $i$-valid.
Dual BHK

- Another unilateralist semantics
- In terms of dual proofs
  \[\Rightarrow\]  Dual-BHK and Dual-Intuitionist Logic
LJ for Int

- "Singleton on the right"

\[
\begin{align*}
A & \vdash A \text{ (Id)} & \Gamma, \bot & \vdash C \text{ (}\bot\text{L)} & \frac{\Gamma \vdash A \quad A, \Gamma' \vdash C}{\Gamma, \Gamma' \vdash C} \text{ (Cut)} \\
\frac{\Gamma \vdash C}{\Gamma, A \vdash C} \text{ (Weakening)} & & \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C} \text{ (Contraction)} \\
\frac{\Gamma, A_i \vdash C}{\Gamma, A_0 \land A_1 \vdash C} \text{ (}\land\text{L)} & & \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \land B} \text{ (}\land\text{R)} \\
\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \lor B \vdash C} \text{ (}\lor\text{L)} & & \frac{\Gamma \vdash A_i}{\Gamma \vdash A_0 \lor A_1} \text{ (}\lor\text{R)} \\
\frac{\Gamma \vdash A \quad \Gamma', B \vdash C}{\Gamma, \Gamma', A \to B \vdash C} \text{ (}\to\text{L)} & & \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \text{ (}\to\text{R)}
\end{align*}
\]

- \( \neg A := A \to \bot \).
LDJ for dual-Int

• ”Singleton on the left”, $\leftarrow$: subtraction, exclusion

\[
\begin{align*}
A & \vdash A \text{ (Id)} & C & \vdash \top, \Delta \text{ (}\perp\text{L)} & \frac{C \vdash A, \Delta}{C \vdash \Delta, \Delta'} \text{ (Cut)} \\
\frac{C \vdash \Delta}{C \vdash A, \Delta} \text{ (Weakening)} & \quad \frac{C \vdash A, A, \Delta}{C \vdash A, \Delta} \text{ (Contraction)} \\
\frac{A_i \vdash \Delta}{A_0 \land A_1 \vdash \Delta} \text{ (}\land\text{L)} & \quad \frac{C \vdash A, \Delta \quad C \vdash B, \Delta}{C \vdash A \land B, \Delta} \text{ (}\land\text{R)} \\
\frac{A \vdash \Delta \quad B \vdash \Delta}{A \lor B \vdash \Delta} \text{ (}\lor\text{L)} & \quad \frac{C \vdash A_i, \Delta}{C \vdash A_0 \lor A_1, \Delta} \text{ (}\lor\text{R)} \\
\frac{A \vdash B, \Delta}{A \leftarrow B \vdash \Delta} \text{ (}\leftarrow\text{L)} & \quad \frac{C \vdash A, \Delta \quad B \vdash \Delta'}{C \vdash A \leftarrow B, \Delta, \Delta'} \text{ (}\leftarrow\text{R)} \\
\end{align*}
\]

• $\sim A := \top \leftarrow A$. 
**Dual-BHK interpretation**

(cf. Wansing 2010)

- A dual proof of $A \land B$ is a pair $\langle i, x \rangle$ such that $i = 0$ and $x$ is a dual proof of $A$, or $i = 1$ and $x$ is a dual proof of $B$.
- A dual proof of $A \lor B$ is a pair $\langle a, b \rangle$ consisting of a dual proof $a$ of $A$ and a dual proof $b$ of $B$.
- A dual proof of $A \leftarrow B$ is a construction that transforms any dual proof of $B$ into a dual proof of $A$.
- $\top$ has no dual proof.
Dual-BHK interpretation

- Explanation of conditions of denial in terms of dual proof (yet another kind of construction).
- Suitable for Dual Intuitionist Logic.

Definition
A sequent $A \vdash B_1, \ldots, B_m$ is $d$-valid if there is a construction that transforms any list of dual proofs of $B_1, \ldots, B_m$ into a dual proof of $A$.

Proposition (correctness)
If $A \vdash B_1, \ldots, B_m$ is derivable in LDJ, then it is $d$-valid.
Problems

- BHK and Dual-BHK: two unilateralist semantics.
- How do they agree or disagree on the meaning?
- Bi-Intuitionist Logic: A logic with the features of Intuitionist and Dual Intuitionist Logic.
Introduction

Two unilateralist semantics

Bi-Intuitionist Logic

Bilateralist validity
Model theory for Bilnt

Definition (Language)

\[ \mathcal{L} := \{ \land, \lor, \rightarrow, \leftarrow \}. \]

Let \( \bot := p \leftarrow p \) and \( \top := p \rightarrow p \) for some fixed atom \( p \).

And define \( \neg A := A \rightarrow \bot \) and \( \sim A := \top \leftarrow A \).

Definition (Model)

A Bilnt model is a triple \( \langle W, \leq, V \rangle \) where

- \( W \) : a non-empty set (of possible worlds)
- \( \leq \) : a reflexive and transitive relation on \( W \)
- \( V : \text{Atom} \rightarrow 2^W \), a valuation which is persistent, i.e.

\[ (\forall w, w' \in W)(w \in V(p) \text{ and } w \leq w' \Rightarrow w' \in V(p)). \]
Model theory for Bilnt (cont.)

Definition
Given a Bilnt model $\langle W, \leq, V \rangle$, write $w \models p$ for $w \in V(p)$. The relation $\models$ extends as follows:

- $w \models A \land B$ if $w \models A$ and $w \models B$
- $w \models A \lor B$ if $w \models A$ or $w \models B$
- $w \models A \rightarrow B$ if $(\forall v \geq w)(v \models A \Rightarrow v \models B)$
- $w \models A \leftarrow B$ if $(\exists v \leq w)(v \models A \text{ and } v \not\models B)$
Model theory for Bilnt (cont.)

Proposition (Persistence)
\[ \models \text{satisfies the persistence condition, i.e. for any formula } A, \]
\[ (\forall w, w' \in W)(w \models A \text{ and } w \leq w' \implies w' \models A). \]

Definition (Validity)
For any formula A and B, we define
\[ A \models B \iff_{\text{def.}} \text{for any Bilnt model } \langle W, \leq, V \rangle \text{ and any } \]
\[ w \in W, \text{ if } w \models A \text{ then } w \models B. \]
Characteristic validity and invalidity

Recall \( \bot = p \leftarrow p, \top = p \rightarrow p \) and

\[
\neg A := A \rightarrow \bot \quad \text{(intuitionist negation)}
\]
\[
\sim A := \top \leftarrow A \quad \text{(dual intuitionist negation)}
\]

- \( \top \not\models A \lor \neg A \) but \( \top \models A \lor \sim A \).
- \( A \land \neg A \models \bot \) but \( A \land \sim A \not\models \bot \).
- \( \not\models (A \rightarrow (B \lor C))) \rightarrow ((A \rightarrow B) \lor C) \)
- \( \not\models (A \land (B \leftarrow C)) \rightarrow ((A \land B) \leftarrow C). \)
Display Calculus $\delta$-Bilnt
(Belnap 1982; Wansing 2010)

- A generalization of sequent calculus
- A sequent $X \vdash Y$ consists of structures $X$ and $Y$ instead of sequences, multisets or sets of formulas.

Definition (structures)
The set of structures for $\delta$-Bilnt is defined by:

$$X ::= A \mid I \mid X \circ X \mid X \bullet X.$$ 

Intuitively, $I$ represents an empty structure and $\circ$ ($\bullet$) corresponds to a comma on LHS (RHS).
Logical rules for $\delta$-Bilnt

$$\frac{A \circ B \vdash Y}{A \land B \vdash Y} \quad (\land L) \quad \frac{X \vdash A \quad Y \vdash B}{X \circ Y \vdash A \land B} \quad (\land R)$$

$$\frac{A \vdash X \quad B \vdash Y}{A \lor B \vdash X \bullet Y} \quad (\lor L) \quad \frac{X \vdash A \bullet B}{X \vdash A \lor B} \quad (\lor R)$$

$$\frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X \circ Y} \quad (\rightarrow L) \quad \frac{X \vdash A \circ B}{X \vdash A \rightarrow B} \quad (\rightarrow R)$$

$$\frac{A \bullet B \vdash Y}{A \leftarrow B \vdash Y} \quad (\leftarrow L) \quad \frac{X \vdash A \quad B \vdash Y}{X \bullet Y \vdash A \leftarrow B} \quad (\leftarrow R)$$
Definition (display rules)

The display calculus $\delta$-BiInt has the following display rules:

\[
\begin{align*}
X \circ Y & \vdash Z \\
X & \vdash Y \circ Z \\
Y \circ X & \vdash Z \\
Z & \vdash X \cdot Y \\
Z \cdot X & \vdash Y \\
Z & \vdash Y \cdot X
\end{align*}
\]

Intuitively,

\[
\begin{align*}
A \land B & \vdash C \\
A & \vdash B \to C \\
B \land A & \vdash C \\
C & \vdash A \lor B \\
C & \vdash A \lor B \\
B & \lor A & \vdash C
\end{align*}
\]
Structural rules for $\delta$-Bilnt

\[ p \vdash p \ (\text{Id}) \]

\[ \frac{X \vdash Y}{X \circ I \vdash Y} \quad \frac{X \vdash Y}{X \vdash Y \cdot I} \]

\[ \frac{X \vdash Y}{X \circ Z \vdash Y} \quad \frac{X \vdash Y}{X \vdash Y \cdot Z} \quad \text{(lm)} \quad \text{(rm)} \]

\[ \frac{X \circ X \vdash Y}{X \vdash Y} \quad \frac{X \vdash Y \cdot Y}{X \vdash Y} \quad \text{(lc)} \quad \text{(rc)} \]

\[ \frac{(X \circ Y) \circ Z \vdash W}{X \circ (Y \circ Z) \vdash W} \quad \frac{W \vdash (X \cdot Y) \cdot Z}{W \vdash X \cdot (Y \cdot Z)} \quad \text{(la)} \quad \text{(ra)} \]
Display property

Definition (antecedent and succedent part)
Given a sequent $S = X \vdash Y$, we define

- $X$ is AP (an antecedent part) of $S$;
- $Y$ is SP (a succedent part) of $S$;
- $(W \circ Z)$ is AP $\Rightarrow$ $W, Z$ are AP;
- $(W \circ Z)$ is SP $\Rightarrow$ $W$ is AP and $Z$ is SP;
- $(W \bullet Z)$ is AP $\Rightarrow$ $W$ is AP and $Z$ is SP;
- $(W \bullet Z)$ is SP $\Rightarrow$ $W, Z$ are SP.
Display property

Proposition (display property)
For any sequent $S = X \vdash Y$ and any substructure $Z$ of it, we can display the occurrence of $Z$, i.e. there is a sequent $S'$ such that:

- $S$ and $S'$ are interderivable by means of display rules only,
- If $Z$ is AP of $S$, then $S'$ is of the form $Z \vdash Y'$ and
- If $Z$ is SP of $S$, then $S'$ is of the form $X' \vdash Z$.

Theorem (Cut elimination)
Cut is eliminable from any derivation in $\delta$-BiInt + Cut.
Completeness

Definition
The translations $\tau_1$ and $\tau_2$ from structures into formulas are defined inductively as:

\[
\begin{align*}
\tau_1(A) &:= A \\
\tau_1(I) &:= \top (= p \to p) \\
\tau_1(X \circ Y) &:= \tau_1(X) \land \tau_1(Y) \\
\tau_1(X \cdot Y) &:= \tau_1(X) \leftarrow \tau_2(Y)
\end{align*}
\]

\[
\begin{align*}
\tau_2(A) &:= A \\
\tau_2(I) &:= \bot (= p \leftarrow p) \\
\tau_2(X \circ Y) &:= \tau_1(X) \to \tau_2(Y) \\
\tau_2(X \cdot Y) &:= \tau_2(X) \lor \tau_2(Y)
\end{align*}
\]

Theorem (Completeness)
$X \vdash Y$ is derivable in $\delta$-BilInt if and only if $\tau_1(X) \models \tau_2(Y)$. 
Logical rules for $\delta$-Bilnt (revised)

$$
\begin{align*}
\frac{A_i \vdash Y}{A_0 \land A_1 \vdash Y} & \quad (\land L) & \frac{X \vdash A}{X \vdash A \land B} & \quad (\land R) \\
\frac{A \vdash Y \quad B \vdash Y}{A \lor B \vdash Y} & \quad (\lor L) & \frac{X \vdash A_i}{X \vdash A_0 \lor A_1} & \quad (\lor R) \\
\frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X \circ Y} & \quad (\rightarrow L) & \frac{X \vdash A \circ B}{X \vdash A \rightarrow B} & \quad (\rightarrow R) \\
\frac{A \bullet B \vdash Y}{A \leftarrow B \vdash Y} & \quad (\leftarrow L) & \frac{X \vdash A \quad B \vdash Y}{X \bullet Y \vdash A \leftarrow B} & \quad (\leftarrow R)
\end{align*}
$$
Introduction

Two unilateralist semantics

Bi-Intuitionist Logic

Bilateralist validity
Fit?

- Proofs and dual proofs coexist in $\delta$-BiInt.
- Two kind of meanings fit together in $\delta$-BiInt?
- BHK and Dual-BHK must be extended to interpret $\leftarrow$ and $\rightarrow$. 
**BHK interpretation extended**
*(cf. Wansing 2010)*

- A proof of $A \land B$ is a pair $\langle a, b \rangle$ consisting of a proof $a$ of $A$ and a proof $b$ of $B$.
- A proof of $A \lor B$ is a pair $\langle i, x \rangle$ such that $i = 0$ and $x$ is a proof of $A$, or $i = 1$ and $x$ is a proof of $B$.
- A proof of $A \rightarrow B$ is a construction that transforms any proof of $A$ into a proof of $B$.
- A proof of $A \leftarrow B$ is a pair $\langle a, b \rangle$ consisting of a proof $a$ of $A$ and a dual proof $b$ of $B$. 
Introduction

Two unilateralist semantics

Bi-Intuitionist Logic

Bilateralist validity

Dual-BHK interpretation extended
(cf. Wansing 2010)

- A dual proof of $A \land B$ is a pair $\langle i, x \rangle$ such that $i = 0$ and $x$ is a dual proof of $A$, or $i = 1$ and $x$ is a dual proof of $B$.
- A dual proof of $A \lor B$ is a pair $\langle a, b \rangle$ consisting of a dual proof $a$ of $A$ and a dual proof $b$ of $B$.
- A dual proof of $A \leftrightarrow B$ is a construction that transforms any dual proof of $B$ into a dual proof of $A$.
- A dual proof of $A \rightarrow B$ is a pair $\langle a, b \rangle$ consisting of a proof $a$ of $A$ and a dual proof $b$ of $B$. 
\textbf{i-validity fails}

A rule in $\delta$-BiInt that is not i-valid.

\[
\frac{Z \cdot X \vdash Y}{Z \vdash X \cdot Y} \quad \sim \quad \frac{A \leftarrow B \vdash C}{A \vdash B \lor C}
\]

- $\bullet$ on RHS: sequents become multiple-conclusion.
- Impossible to interpret it as intuitionist’s disjunction with disjunction property.
- Proof is not preserved from LHS to RHS.
- At most \textit{impossibility of dual proof} of RHS

The same applies to d-validity.
**Bilateralist reading of sequents**
*(cf. Restall 2005)*

**Definition**

\( A \vdash B \) is **b-valid** if it is not the case that \( A \) has a proof and \( B \) has a dual proof.

- to assert \( A \) and to deny \( B \) is to make a mistake;
- if \( A \) has a proof then \( B \) can’t have a dual proof;
- if \( B \) has a dual proof then \( A \) can’t have a proof.

\( \Rightarrow \) Two criteria of fit between proof & dual proof
Criterion 1: Identity

\[ A \vdash A \]

- A can’t have both a proof and a dual proof.
- No clash between proofs and dual proofs.
- No overlap between assertion and denial.
- Established directly by BHK and Dual-BHK on the assumptions \( p \vdash p \).
- Derivable in \( \delta\)-BiInt.
Criterion 2: Cut

\[
\frac{X \vdash A \quad A \vdash Y}{X \vdash Y} \quad \text{(Cut)}
\]

- \(X \vdash A\): a proof of \(X\) excludes dual-provability of \(A\)
- \(A \vdash Y\): a dual proof of \(Y\) excludes provability of \(A\), then
- \(X \vdash Y\): the proof of \(X\) clashes with the dual proof of \(Y\).

I.e. it is impossible to exclude both provability and dual-provability of \(A\) without any clash.
- No gap between assertion and denial.
Fit

- Cut & Identity: criteria of fit between proofs & dual proofs.
  - Agreement on meanings between BHK and dual-BHK

- No problem with Identity. How about Cut?
- Seems difficult to establish directly by BHK and dual-BHK.
- Cut elimination for $\delta$-BiInt tells us something?
**Introduction**

Two unilateralist semantics

Bi-Intuitionist Logic

Bilateralist validity

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**Failure of \( b \)-validity**

Almost all rules in \( \delta \)-BiInt are \( b \)-valid *except for*:

\[
\frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X \circ Y} \quad (\rightarrow \text{L}) \quad \frac{X \vdash A \quad B \vdash Y}{X \bullet Y \vdash A \leftrightarrow B} \quad (\leftarrow \text{R})
\]

A proof of \( A \rightarrow B \) requires a *proof* of \( A \). But \( X \vdash A \) gives at most impossibility of dual proof of \( A \).

To make them \( b \)-valid:

\[
\frac{B \vdash Y}{A \rightarrow B \vdash A \circ Y} \quad (\rightarrow \text{L}') \quad \frac{X \vdash A}{X \bullet B \vdash A \leftrightarrow B} \quad (\leftarrow \text{R}')
\]
Revision of rules and \textit{Cut}

New rules are equivalent to the original through \textit{Cut}:

\[
\begin{array}{c}
B \vdash Y \\
A \rightarrow B \vdash A \circ Y \\
\hline
X \vdash A \\
X \vdash A \rightarrow B \circ Y \\
\hline
A \rightarrow B \vdash X \circ Y
\end{array}
\quad (\rightarrow \text{L'})
\]

\[
\begin{array}{c}
A \vdash A \\
A \vdash A \rightarrow B \circ Y \\
\hline
A \rightarrow B \vdash A \circ Y
\end{array}
\quad (\text{Cut})
\]

\[
\begin{array}{c}
A \rightarrow B \vdash X \circ Y \\
\hline
A \vdash A \\
\vdash B \vdash A \circ Y \\
\hline
A \rightarrow B \vdash X \circ Y
\end{array}
\quad (\rightarrow \text{L})
\]

Let \textit{Cut'} denote \textit{Cut} of this form (and the dual form for $\leftarrow \text{R'}$).
Weak Cut-elimination

- $\delta$-BiInt' = the system with $\rightarrow$ L' and $\leftarrow$ R'
  - Every rule is b-valid.
  - $\delta$-BiInt' + Cut' $\cong$ $\delta$-BiInt

Fact (cf. Schroeder-Heister *forthcoming*)

$\delta$-BiInt' + Cut $\cong$ $\delta$-BiInt' + Cut' $\not\cong$ $\delta$-BiInt'.

- $\delta$-BiInt': b-valid but Cut' is not admissible.
- $\delta$-BiInt: Cut is admissible but not b-valid.
Example of failure of $\text{Cut}'$

\[
\begin{align*}
&p \vdash p \\
\hline
&p \vdash p \lor q \\
\hline
&r \vdash r \\
\hline
&(p \lor q) \rightarrow r \vdash p \lor q \circ r \\
\hline
&p \lor q \vdash (p \lor q) \rightarrow r \circ r \\
\hline
&p \vdash (p \lor q) \rightarrow r \circ r
\end{align*}
\]

$(\rightarrow L')$ and $(\text{Cut}')$
\[
\frac{\Gamma \vdash A \quad A \vdash \Delta}{\Gamma \not\vdash \Delta} \quad (\text{Cut})
\]

Failure of Cut indicates:

- there may be a combination of a proof of \( \Gamma \) and a dual proof of \( \Delta \) such that:
  - the former excludes dual-provability of \( A \),
  - the latter excludes provability of \( A \),
  - but they don’t cause any clash in \( \delta\text{-BiInt'} \).

A gap between proof and dual proof.
Conclusion

Under Bilateralist reading of sequents,

- **Cut & Identity**: Criteria of fit between two aspects
- **Failure of Cut’**: a gap between proof and dual proof.
- **Cut admissibility in δ-BiInt**:
  - Proof and dual proof are adjusted implicitly.
  - $\rightarrow L$ and $\leftarrow R$ are where the adjustment occurs.
**Relativized b-validity**

To make this observation more precise, $b$-validity (and extensions of (dual-)BHK) should be defined more carefully:

**Definition (Atomic base)**

An atomic base is a pair $\langle s, t \rangle$ of two sets of atomic formulas with $s \cap t = \emptyset$. Define the relation $\models$ between atomic bases and signed atomic formulas as:

\[
\begin{align*}
\langle s, t \rangle & \models +p \iff p \in s; \\
\langle s, t \rangle & \models -p \iff p \in t.
\end{align*}
\]
**Relativized $b$-validity**

**Definition**

$\models$ extends as follows:

\[ \langle s, t \rangle \models +T \quad \quad \langle s, t \rangle \not\models +\bot \]

\[ \langle s, t \rangle \models +(A \land B) \iff \langle s, t \rangle \models +A \text{ and } \langle s, t \rangle \models +B \]

\[ \langle s, t \rangle \models +(A \lor B) \iff \langle s, t \rangle \models +A \text{ or } \langle s, t \rangle \models +B \]

\[ \langle s, t \rangle \models +(A \rightarrow B) \iff \forall s'. \subseteq s. \langle s', t \rangle \models +A \text{ implies } \langle s', t \rangle \models +B \]

\[ \langle s, t \rangle \models +(A \leftrightarrow B) \iff \exists s'. \subseteq s \exists t'. \supseteq t. \]

\[ \langle s', t' \rangle \models +A \text{ and } \langle s', t' \rangle \models -B \]

Dually for $\langle s, t \rangle \models -A$. Especially,

\[ \langle s, t \rangle \models -(A \rightarrow B) \iff \exists t'. \subseteq t \exists s'. \supseteq s. \]

\[ \langle s', t' \rangle \models +A \text{ and } \langle s', t' \rangle \models -B \]
Relativized $b$-validity

Proposition

(1) Persistence:
   - $\langle s, t \rangle \Vdash +A$ and $s' \supseteq s$ implies $\langle s', t \rangle \Vdash +A$.
   - $\langle s, t \rangle \Vdash -A$ and $t' \supseteq t$ implies $\langle s, t' \rangle \Vdash -A$.

(2) Irrelevance:
   - If $\langle s, t \rangle \Vdash +A$, then for any $t'$, $\langle s, t' \rangle \Vdash +A$.
   - If $\langle s, t \rangle \Vdash -A$, then for any $s'$, $\langle s', t \rangle \Vdash +A$.

(3) Consistency: It is not the case that $\langle s, t \rangle \Vdash +A$ and $\langle s, t \rangle \Vdash -A$. 
**Relativized \( b \)-validity**

**Definition (\( b \)-validity)**

A sequent \( A \vdash B \) is \( \langle s, t \rangle \)-valid if it is *not* the case that

\[
\langle s, t \rangle \not \models +A \quad \text{and} \quad \langle s, t \rangle \not \models -B.
\]

\( A \vdash B \) is \( b \)-valid if it is \( \langle s, t \rangle \)-valid for any \( \langle s, t \rangle \).

**Proposition (Soundness)**

If \( X \vdash Y \) is derivable in \( \delta \text{-BiInt}' \), then it is \( b \)-valid.

**Proposition**

\( \text{Cut}', \text{ Intuitionist LEM}, \ (A \rightarrow (B \lor C)) \rightarrow ((A \rightarrow B) \lor C) \) etc. are not \( b \)-valid.
References (1)

References (2)