BHK interpretation and Bilateralism

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Bilateralism

Meaning = Assetion and Denial

- Inferentialism and proof-theoretic semantics.
- Smiley, Rumfitt, Restall, Ripley, etc.

vs. Unilateralism: Constructive PTS (BHK, Dummett, Prawitz) and its dual version

♡ Two unilateralist semantics fit together?

- Assertion & denial, proof & dual proof
- How to show the fit between two meanings?

BHK interpretation

- A proof of $A \wedge B$ is a pair $\langle a, b \rangle$ consisting of a proof a of A and a proof b of B.
- A proof of $A \vee B$ is a pair $\langle i, x \rangle$ such that $i = 0$ and x is a proof of A, or $i = 1$ and x is a proof of B.
- A proof of $A \to B$ is a construction that transforms any proof of A into a proof of B.
- $\bot$ has no proof.

BHK interpretation and PTS
(Dummett 1991; Prawitz 2006; Schroeder-Heister 2006)

- Explanation of conditions of assertion in terms of the primitive notion of proof (or construction).
- Suitable for Intuitionist Logic.

Definition

A sequent $A_1, \ldots, A_n \vdash B$ is i-*valid* if there is a construction that transforms any list $a_1, \ldots, a_n$ of proofs of $A_1, \ldots, A_n$ into a proof of B.

Proposition (correctness)

If $A_1, \ldots, A_n \vdash B$ is derivable in NJ (LJ), then it is i-valid.

Dual BHK

- Another unilateralist semantics
- In terms of dual proofs

$\Rightarrow$ Dual-BHK and Dual-Intuitionist Logic

LJ for Int

- "Singleton on the right"

$$A \vdash A \ (\mathrm{Id}) \quad \Gamma, \bot \vdash C \ (\bot L) \quad \frac{\Gamma \vdash A \quad A, \Gamma' \vdash C}{\Gamma, \Gamma' \vdash C} \ (\mathrm{Cut})$$

$$\frac{\Gamma \vdash C}{\Gamma, A \vdash C} \ (Weakening) \quad \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C} \ (\mathrm{Contraction})$$

$$\frac{\Gamma, A_i \vdash C}{\Gamma, A_0 \wedge A_1 \vdash C} \ (\wedge L) \quad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \ (\wedge R)$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C} \ (\vee L) \quad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_0 \vee A_1} \ (\vee R)$$

$$\frac{\Gamma \vdash A \quad \Gamma', B \vdash C}{\Gamma, \Gamma', A \to B \vdash C} \ (\to L) \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \ (\to R)$$

- $\neg A := A \to \bot$.

LDJ for dual-Int
(cf. Czermak 1977; Urbas 1996; Goré 2000)

- "Singleton on the left", $\leftarrow$: subtraction, exclusion

$$A \vdash A \ (\mathrm{Id}) \quad C \vdash \top, \Delta \ (\bot L) \quad \frac{C \vdash A, \Delta \quad A \vdash \Delta'}{C \vdash \Delta, \Delta'} \ (\mathrm{Cut})$$

$$\frac{C \vdash \Delta}{C \vdash A, \Delta} \ (\mathit{Weakening}) \quad \frac{C \vdash A, A, \Delta}{C \vdash A, \Delta} \ (\mathrm{Contraction})$$

$$\frac{A_i \vdash \Delta}{A_0 \wedge A_1 \vdash \Delta} \ (\wedge L) \quad \frac{C \vdash A, \Delta \quad C \vdash B, \Delta}{C \vdash A \wedge B, \Delta} \ (\wedge R)$$

$$\frac{A \vdash \Delta \quad B \vdash \Delta}{A \vee B \vdash \Delta} \ (\vee L) \quad \frac{C \vdash A_i, \Delta}{C \vdash A_0 \vee A_1, \Delta} \ (\vee R)$$

$$\frac{A \vdash B, \Delta}{A \leftarrow B \vdash \Delta} \ (\leftarrow L) \quad \frac{C \vdash A, \Delta \quad B \vdash \Delta'}{C \vdash A \leftarrow B, \Delta, \Delta'} \ (\leftarrow R)$$

- $\sim A := \top \leftarrow A$.

Dual-BHK interpretation
(cf. Wansing 2010)

- A dual proof of $A \wedge B$ is a pair $\langle i, x \rangle$ such that $i = 0$ and x is a dual proof of A, or $i = 1$ and x is a dual proof of B.
- A dual proof of $A \vee B$ is a pair $\langle a, b \rangle$ consisting of a dual proof a of A and a dual proof b of B.
- A dual proof of $A \leftarrow B$ is a construction that transforms any dual proof of B into a dual proof of A.
- $\top$ has no dual proof.

Dual-BHK interpretation

- Explanation of conditions of denial in terms of dual proof (yet another kind of construction).
- Suitable for Dual Intuitionist Logic.

Definition

A sequent $A \vdash B_1, \ldots, B_m$ is d-*valid* if there is a construction that transforms any list of dual proofs of $B_1, \ldots, B_m$ into a dual proof of A.

Proposition (correctness)

If $A \vdash B_1, \ldots, B_m$ is derivable in LDJ, then it is d-valid.

Problems

- BHK and Dual-BHK: two unilateralist semantics.

♡ How do they agree or disagree on the meaning?

- Bi-Intuitionist Logic: A logic with the features of Intuitionist and Dual Intuitionist Logic.

Model theory for BiInt

Definition (Language)

$$\mathcal{L} := \{\wedge, \vee, \to, \leftarrow\}.$$

Let $\bot := p \leftarrow p$ and $\top := p \to p$ for some fixed atom p.
And define $\neg A := A \to \bot$ and $\sim A := \top \leftarrow A$.

Definition (Model)

A BiInt model is a triple $\langle W, \leqslant, V \rangle$ where

- W : a non-empty set (of possible worlds)
- $\leqslant$: a reflexive and transitive relation on W
- $V : Atom \to 2^W$, a valuation which is persistent, i.e.

$$(\forall w, w' \in W)\big(w \in V(p) \text{ and } w \leqslant w' \ \Rightarrow\ w' \in V(p)\big).$$

Model theory for BiInt (cont.)

Definition

Given a BiInt model $\langle W, \leqslant, V\rangle$, write $w \models p$ for $w \in V(p)$. The relation $\models$ extends as follows:

$$w \models A \land B \text{ if } w \models A \text{ and } w \models B$$

$$w \models A \lor B \text{ if } w \models A \text{ or } w \models B$$

$$w \models A \to B \text{ if } (\forall v \geqslant w)(v \models A \Rightarrow v \models B)$$

$$w \models A \leftarrow B \text{ if } (\exists v \leqslant w)(v \models A \text{ and } v \not\models B)$$

Model theory for BiInt (cont.)

Proposition (Persistence)

$\models$ satisfies the persistence condition, i.e. for any formula A,

$$(\forall w, w' \in W)\big(w \models A \text{ and } w \leqslant w' \;\Rightarrow\; w' \models A\big).$$

Definition (Validity)

For any formula A and B, we define

$A \models B \;\Leftrightarrow_{\mathit{def.}}$ *for any BiInt model* $\langle W, \leqslant, V\rangle$ *and any* $w \in W$, *if* $w \models A$ *then* $w \models B$.

Characteristic validity and invalidity

Recall $\bot = p \leftarrow p$, $\top = p \rightarrow p$ and

$$\neg A := A \rightarrow \bot \qquad \text{(intuitionist negation)}$$

$$\sim A := \top \leftarrow A \qquad \text{(dual intuitionist negation)}$$

- $\top \not\models A \vee \neg A$ but $\top \models A \vee \sim A$.
- $A \wedge \neg A \models \bot$ but $A \wedge \sim A \not\models \bot$.
- $\not\models (A \rightarrow (B \vee C)) \rightarrow ((A \rightarrow B) \vee C)$
- $\not\models (A \wedge (B \leftarrow C)) \rightarrow ((A \wedge B) \leftarrow C)$.

Display Calculus δ-BiInt (Belnap 1982; Wansing 2010)

- A generalization of sequent calculus
- A sequent $X \vdash Y$ consists of structures X and Y instead of sequences, multisets or sets of formulas.

Definition (structures)

The set of structures for δ-BiInt is defined by:

$$X ::= A \mid \mathbf{I} \mid X \circ X \mid X \bullet X.$$

Intuitively, $\mathbf{I}$ represents an empty structure and $\circ$ ($\bullet$) corresponds to a comma on LHS (RHS).

Logical rules for δ-BiInt

$$\frac{A \circ B \vdash Y}{A \wedge B \vdash Y}\,(\wedge L) \qquad \frac{X \vdash A \quad Y \vdash B}{X \circ Y \vdash A \wedge B}\,(\wedge R)$$

$$\frac{A \vdash X \quad B \vdash Y}{A \vee B \vdash X \bullet Y}\,(\vee L) \qquad \frac{X \vdash A \bullet B}{X \vdash A \vee B}\,(\vee R)$$

$$\frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X \circ Y}\,(\rightarrow L) \qquad \frac{X \vdash A \circ B}{X \vdash A \rightarrow B}\,(\rightarrow R)$$

$$\frac{A \bullet B \vdash Y}{A \leftarrow B \vdash Y}\,(\leftarrow L) \qquad \frac{X \vdash A \quad B \vdash Y}{X \bullet Y \vdash A \leftarrow B}\,(\leftarrow R)$$

Display rules

Definition (display rules)

The display calculus δ-BiInt has the following display rules:

$$\frac{\dfrac{X \circ Y \vdash Z}{X \vdash Y \circ Z}}{Y \circ X \vdash Z} \qquad \frac{\dfrac{Z \vdash X \bullet Y}{Z \bullet X \vdash Y}}{Z \vdash Y \bullet X}$$

Intuitively,

$$\frac{\dfrac{A \wedge B \vdash C}{A \vdash B \to C}}{B \wedge A \vdash C} \qquad \frac{\dfrac{C \vdash A \vee B}{C \leftarrow A \vdash B}}{C \vdash B \vee A}$$

Structural rules for δ-BiInt

$$p \vdash p \ (\mathrm{Id})$$

$$\frac{X \vdash Y}{\overline{\overline{X \circ \mathbf{I} \vdash Y}}} \qquad \frac{X \vdash Y}{\overline{\overline{X \vdash Y \bullet \mathbf{I}}}}$$

$$\frac{X \vdash Y}{X \circ Z \vdash Y}\,(lm) \qquad \frac{X \vdash Y}{X \vdash Y \bullet Z}\,(rm)$$

$$\frac{X \circ X \vdash Y}{X \vdash Y}\,(lc) \qquad \frac{X \vdash Y \bullet Y}{X \vdash Y}\,(rc)$$

$$\frac{(X \circ Y) \circ Z \vdash W}{X \circ (Y \circ Z) \vdash W}\,(la) \qquad \frac{W \vdash (X \bullet Y) \bullet Z}{W \vdash X \bullet (Y \bullet Z)}\,(ra)$$

Display property

Definition (antecedent and succedent part)

Given a sequent $S = \mathrm{X} \vdash \mathrm{Y}$, we define

- X is AP (an antecedent part) of S;
- Y is SP (a succedent part) of S;
- $(W \circ \mathrm{Z})$ is AP $\Rightarrow$ W, Z are AP;
- $(W \circ \mathrm{Z})$ is SP $\Rightarrow$ W is AP and Z is SP;
- $(W \bullet \mathrm{Z})$ is AP $\Rightarrow$ W is AP and Z is SP;
- $(W \bullet \mathrm{Z})$ is SP $\Rightarrow$ W, Z are SP.

Display property

Proposition (display property)

For any sequent $S = X \vdash Y$ and any substructure Z of it, we can display the occurrence of Z, i.e. there is a sequent S' such that:

- S and S' are interderivable by means of display rules only,
- If Z is AP of S, then S' is of the form $Z \vdash Y'$ and
- If Z is SP of S, then S' is of the form $X' \vdash Z$.

Theorem (Cut elimination)

Cut is eliminable from any derivation in $\delta\text{-}\mathrm{BiInt} + \mathrm{Cut}$.

Completeness

Definition

The translations τ_1 and τ_2 from structures into formulas are defined inductively as:

$$
\begin{aligned}
\tau_1(A) &:= A & \tau_2(A) &:= A \\
\tau_1(\mathbf{I}) &:= \top (= p \to p) & \tau_2(\mathbf{I}) &:= \bot (= p \leftarrow p) \\
\tau_1(X \circ Y) &:= \tau_1(X) \wedge \tau_1(Y) & \tau_2(X \circ Y) &:= \tau_1(X) \to \tau_2(Y) \\
\tau_1(X \bullet Y) &:= \tau_1(X) \leftarrow \tau_2(Y) & \tau_2(X \bullet Y) &:= \tau_2(X) \vee \tau_2(Y)
\end{aligned}
$$

Theorem (Completeness)

$X \vdash Y$ is derivable in δ-BiInt if and only if $\tau_1(X) \models \tau_2(Y)$.

Logical rules for δ-BiInt (revised)

$$\frac{A_i \vdash Y}{A_0 \wedge A_1 \vdash Y}\,(\wedge L) \qquad \frac{X \vdash A \quad X \vdash B}{X \vdash A \wedge B}\,(\wedge R)$$

$$\frac{A \vdash Y \quad B \vdash Y}{A \vee B \vdash Y}\,(\vee L) \qquad \frac{X \vdash A_i}{X \vdash A_0 \vee A_1}\,(\vee R)$$

$$\frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X \circ Y}\,(\rightarrow L) \qquad \frac{X \vdash A \circ B}{X \vdash A \rightarrow B}\,(\rightarrow R)$$

$$\frac{A \bullet B \vdash Y}{A \leftarrow B \vdash Y}\,(\leftarrow L) \qquad \frac{X \vdash A \quad B \vdash Y}{X \bullet Y \vdash A \leftarrow B}\,(\leftarrow R)$$

Introduction

Two unilateralist semantics

Bi-Intuitionist Logic

Bilateralist validity

Fit?

- Proofs and dual proofs coexist in δ-$\mathtt{BiInt}$.
- Two kind of meanings fit together in δ-$\mathtt{BiInt}$?
- BHK and Dual-BHK must be extended to interpret $\leftarrow$ and $\rightarrow$.

BHK interpretation extended (cf. Wansing 2010)

- A proof of $A \wedge B$ is a pair $\langle a, b\rangle$ consisting of a proof a of A and a proof b of B.
- A proof of $A \vee B$ is a pair $\langle i, x\rangle$ such that $i = 0$ and x is a proof of A, or $i = 1$ and x is a proof of B.
- A proof of $A \rightarrow B$ is a construction that transforms any proof of A into a proof of B.
- A proof of $A \leftarrow B$ is a pair $\langle a, b\rangle$ consisting of a proof a of A and a dual proof b of B.

Dual-BHK interpretation extended (cf. Wansing 2010)

- A dual proof of $A \wedge B$ is a pair $\langle i, x \rangle$ such that $i = 0$ and x is a dual proof of A, or $i = 1$ and x is a dual proof of B.
- A dual proof of $A \vee B$ is a pair $\langle a, b \rangle$ consisting of a dual proof a of A and a dual proof b of B.
- A dual proof of $A \leftarrow B$ is a construction that transforms any dual proof of B into a dual proof of A.
- A dual proof of $A \rightarrow B$ is a pair $\langle a, b \rangle$ consisting of a proof a of A and a dual proof b of B.

$\mathfrak{i}$-validity fails

A rule in δ-$\mathfrak{BiInt}$ that is not $\mathfrak{i}$-valid.

$$\frac{Z \bullet X \vdash Y}{Z \vdash X \bullet Y} \quad \cong \quad \frac{A \leftarrow B \vdash C}{A \vdash B \vee C}$$

- $\bullet$ on RHS: sequents become multiple-conclusion.
- Impossible to interpret it as intuitionist's disjunction with disjunction property.
- Proof is not preserved from LHS to RHS.
- At most *impossibility of dual proof* of RHS

The same applies to $\mathfrak{d}$-validity.

Bilateralist reading of sequents (cf. Restall 2005)

Definition

$A \vdash B$ is b-valid if it is not the case that A has a proof and B has a dual proof.

- to assert A and to deny B is to make a mistake;
- if A has a proof then B can't have a dual proof;
- if B has a dual proof then A can't have a proof.

$\Rightarrow$ Two criteria of fit between proof & dual proof

Criterion 1: Identity

$$A \vdash A$$

- A can't have both a proof and a dual proof.
- No clash between proofs and dual proofs.
- No overlap between assertion and denial.

- Established directly by BHK and Dual-BHK on the assumptions $p \vdash p$.
- Derivable in δ-BiInt.

Criterion 2: Cut

$$\frac{X \vdash A \quad A \vdash Y}{X \vdash Y}\,(\mathrm{Cut})$$

- $X \vdash A$: a proof of X excludes dual-provability of A
- $A \vdash Y$: a dual proof of Y excludes provability of A, then
- $X \vdash Y$: the proof of X clashes with the dual proof of Y.

I.e. it is impossible to exclude both provability and dual-provability of A without any clash.

- No gap between assertion and denial.

Fit

- Cut & Identity: criteria of fit between proofs & dual proofs.
  - Agreement on meanings between BHK and dual-BHK
- No problem with Identity. How about Cut?
- Seems difficult to establish directly by BHK and dual-BHK.
- Cut elimination for δ-$\mathtt{BiInt}$ tells us something?

Failure of $\mathfrak{b}$-validity

Almost all rules in δ-$\mathtt{BiInt}$ are $\mathfrak{b}$-valid *except for*:

$$\frac{X \vdash A \quad B \vdash Y}{A \to B \vdash X \circ Y}\,(\to L) \qquad \frac{X \vdash A \quad B \vdash Y}{X \bullet Y \vdash A \leftarrow B}\,(\leftarrow R)$$

A proof of $A \to B$ requires a *proof* of A. But $X \vdash A$ gives at most impossibility of dual proof of A.

To make them $\mathfrak{b}$-valid:

$$\frac{B \vdash Y}{A \to B \vdash A \circ Y}\,(\to L') \qquad \frac{X \vdash A}{X \bullet B \vdash A \leftarrow B}\,(\leftarrow R')$$

Revision of rules and Cut

New rules are equivalent to the original through Cut:

$$\dfrac{X \vdash A \quad \dfrac{\dfrac{B \vdash Y}{A \to B \vdash A \circ Y}\,(\to L')}{\overline{\overline{A \vdash A \to B \circ Y}}}}{\dfrac{X \vdash A \to B \circ Y}{\overline{\overline{A \to B \vdash X \circ Y}}}}\,(\mathrm{Cut}) \qquad \dfrac{A \vdash A \quad B \vdash Y}{A \to B \vdash A \circ Y}\,(\to L)$$

Let Cut' denote Cut of this form (and the dual form for $\leftarrow R'$).

Weak Cut-elimination

- $\delta\text{-BiInt}'$ = the system with $\rightarrow \mathrm{L}'$ and $\leftarrow \mathrm{R}'$
  - Every rule is b-valid.
  - $\delta\text{-BiInt}' + \mathrm{Cut}' \cong \delta\text{-BiInt}$

Fact (cf. Schroeder-Heister *forthcoming*)

$\delta\text{-BiInt}' + \mathrm{Cut} \cong \delta\text{-BiInt}' + \mathrm{Cut}' \not\cong \delta\text{-BiInt}'$.

- $\delta\text{-BiInt}'$: b-valid but Cut' is not admissible.
- $\delta\text{-BiInt}$: Cut is admissible but not b-valid.

Example of failure of Cut'

$$\dfrac{\dfrac{p \vdash p}{p \vdash p \lor q} \qquad \dfrac{\dfrac{r \vdash r}{(p \lor q) \to r \vdash p \lor q \circ r}\,(\to \mathrm{L}')}{p \lor q \vdash (p \lor q) \to r \circ r}}{p \vdash (p \lor q) \to r \circ r}\,(\mathrm{Cut}')$$

$$\frac{X \vdash A \quad A \vdash Y}{X \nvdash Y}\,(\mathrm{Cut})$$

Failure of Cut indicates:

- there may be a combination of a proof of X and a dual proof of Y such that:
- the former excludes dual-provability of A,
- the latter excludes provability of A,
- but they don't cause any clash in δ-BiInt'.

A gap between proof and dual proof.

Conclusion

Under Bilateralist reading of sequents,

- Cut & Identity: Criteria of fit between two aspects
- Failure of Cut': a gap between proof and dual proof.
- Cut admissibility in δ-BiInt:
    - Proof and dual proof are adjusted implicitly.
    - $\rightarrow \mathrm{L}$ and $\leftarrow \mathrm{R}$ are where the adjustment occurs.

Relativized $\mathfrak{b}$-validity

To make this observation more precise, $\mathfrak{b}$-validity (and extensions of (dual-)BHK) should be defined more carefully:

Definition (Atomic base)

An atomic base is a pair $\langle s, t \rangle$ of two sets of atomic formulas with $s \cap t = \emptyset$. Define the relation $\Vdash$ between atomic bases and signed atomic formulas as:

$$\langle s, t \rangle \Vdash +p \quad \Leftrightarrow \quad p \in s;$$
$$\langle s, t \rangle \Vdash -p \quad \Leftrightarrow \quad p \in t.$$

Relativized b-validity

Definition

$\Vdash$ extends as follows:

$$\langle s,t\rangle \Vdash +\top \qquad \langle s,t\rangle \nVdash +\bot$$

$$\begin{array}{rcl}
\langle s,t\rangle \Vdash +(A \wedge B) & \Leftrightarrow & \langle s,t\rangle \Vdash +A \text{ and } \langle s,t\rangle \Vdash +B \\
\langle s,t\rangle \Vdash +(A \vee B) & \Leftrightarrow & \langle s,t\rangle \Vdash +A \text{ or } \langle s,t\rangle \Vdash +B \\
\langle s,t\rangle \Vdash +(A \rightarrow B) & \Leftrightarrow & \forall s' \supseteq s.\ \langle s',t\rangle \Vdash +A \text{ implies } \langle s',t\rangle \Vdash +B \\
\langle s,t\rangle \Vdash +(A \leftarrow B) & \Leftrightarrow & \exists s' \subseteq s\ \exists t' \supseteq t. \\
 & & \quad \langle s',t'\rangle \Vdash +A \text{ and } \langle s',t'\rangle \Vdash -B
\end{array}$$

Dually for $\langle s,t\rangle \Vdash -A$. Especially,

$$\begin{array}{rcl}
\langle s,t\rangle \Vdash -(A \rightarrow B) & \Leftrightarrow & \exists t' \subseteq t\ \exists s' \supseteq s. \\
 & & \quad \langle s',t'\rangle \Vdash +A \text{ and } \langle s',t'\rangle \Vdash -B
\end{array}$$

Relativized b-validity

Proposition

(1) Persistence:

- $\langle s, \mathrm{t}\rangle \Vdash +A$ and $s' \supseteq s$ implies $\langle s', \mathrm{t}\rangle \Vdash +A$.
- $\langle s, \mathrm{t}\rangle \Vdash -A$ and $\mathrm{t}' \supseteq \mathrm{t}$ implies $\langle s, \mathrm{t}'\rangle \Vdash -A$.

(2) Irrelevance:

- If $\langle s, \mathrm{t}\rangle \Vdash +A$, then for any t', $\langle s, \mathrm{t}'\rangle \Vdash +A$.
- If $\langle s, \mathrm{t}\rangle \Vdash -A$, then for any s', $\langle s', \mathrm{t}\rangle \Vdash +A$.

(3) Consistency: It is not the case that $\langle s, \mathrm{t}\rangle \Vdash +A$ and $\langle s, \mathrm{t}\rangle \Vdash -A$.

Relativized b-validity

Definition (b-validity)

A sequent $A \vdash B$ is $\langle s, t\rangle$-valid if it is *not* the case that

$$\langle s, t\rangle \Vdash +A \text{ and } \langle s, t\rangle \Vdash -B.$$

$A \vdash B$ is b-valid if it is $\langle s, t\rangle$-valid for any $\langle s, t\rangle$.

Proposition (Soundness)

If $X \vdash Y$ is derivable in $\delta\text{-}\mathrm{BiInt}'$, then it is b-valid.

Proposition

Cut', Intuitionist LEM, $(A \to (B \vee C)) \to ((A \to B) \vee C)$ etc. are not b-valid.

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