<table>
<thead>
<tr>
<th>Title</th>
<th>BHK interpretation and Bilateralism</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Onishi, Takuro</td>
</tr>
<tr>
<td>Citation</td>
<td>(2012)</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2012-03-14</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/154584">http://hdl.handle.net/2433/154584</a></td>
</tr>
<tr>
<td>Rights</td>
<td>Copyright: Takuro Onishi. This is not the published version. Please cite only the published version. この論文は出版社版ではありません。引用の際には出版社版をご確認ご利用ください。</td>
</tr>
<tr>
<td>Type</td>
<td>Presentation</td>
</tr>
<tr>
<td>Textversion</td>
<td>author</td>
</tr>
</tbody>
</table>

Kyoto University
BHK interpretation and Bilateralism

大西琢朗  Takuro Onishi

14 March 2012 PhilLogMath
Bilateralism

Meaning = Assertion and Denial

- Inferentialism and proof-theoretic semantics.
- Smiley, Rumfitt, Restall, Ripley, etc.

vs. Unilateralism: Constructive PTS (BHK, Dummett, Prawitz) and its dual version

Two unilateralist semantics fit together?
  - Assertion & denial, proof & dual proof
  - How to show the fit between two meanings?
Introduction

Two unilateralist semantics

Bi-Intuitionist Logic

Bilateralist validity
Introduction

Two unilateralist semantics

Bi-Intuitionist Logic

Bilateralist validity
BHK interpretation

- A proof of $A \land B$ is a pair $\langle a, b \rangle$ consisting of a proof $a$ of $A$ and a proof $b$ of $B$.
- A proof of $A \lor B$ is a pair $\langle i, x \rangle$ such that $i = 0$ and $x$ is a proof of $A$, or $i = 1$ and $x$ is a proof of $B$.
- A proof of $A \rightarrow B$ is a construction that transforms any proof of $A$ into a proof of $B$.
- $\bot$ has no proof.
BHK interpretation and PTS
(Dummett 1991; Prawitz 2006; Schroeder-Heister 2006)

- Explanation of conditions of assertion in terms of the primitive notion of proof (or construction).
- Suitable for Intuitionist Logic.

**Definition**
A sequent $A_1, \ldots, A_n \vdash B$ is $i$-valid if there is a construction that transforms any list $a_1, \ldots, a_n$ of proofs of $A_1, \ldots, A_n$ into a proof of $B$.

**Proposition (correctness)**
If $A_1, \ldots, A_n \vdash B$ is derivable in NJ (LJ), then it is $i$-valid.
Dual BHK

- Another unilateralist semantics
- In terms of dual proofs

\implies Dual-BHK and Dual-Intuitionist Logic
**LJ for \textit{Int}**

- "Singleton on the right"

\[
\begin{align*}
\Gamma, \bot & \vdash C \quad (\bot L) \\
\frac{\Gamma \vdash C}{\Gamma, A \vdash C} & \quad \text{(Weakening)} \\
\frac{\Gamma, A_i \vdash C}{\Gamma, A_0 \land A_1 \vdash C} & \quad \text{(\land L)} \\
\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \lor B \vdash C} & \quad \text{(\lor L)} \\
\frac{\Gamma \vdash A \quad \Gamma' \vdash C}{\Gamma, \Gamma', A \to B \vdash C} & \quad \text{(\to L)} \\
\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \to B} & \quad \text{(\to R)}
\end{align*}
\]

- \( \neg A := A \to \bot \).
Introduction

Two unilateralist semantics

Bi-Intuitionist Logic

Bilateralist validity

**LDJ for dual-Int**
*(cf. Czermak 1977; Urbas 1996; Goré 2000)*

- "Singleton on the left", \( \leftarrow \): subtraction, exclusion

\[
\begin{align*}
A \vdash A \quad \text{(Id)} & \quad C \vdash \top, \Delta \quad \text{(\( \bot \)L)} & \quad \frac{C \vdash A, \Delta \quad A \vdash \Delta'}{C \vdash \Delta, \Delta'} \quad \text{(Cut)} \\
C \vdash \Delta \quad \text{(Weakening)} & \quad C \vdash A, A, \Delta \quad \text{(Contraction)} \\
\frac{A_i \vdash \Delta}{A_0 \land A_1 \vdash \Delta} \quad \text{(\( \land \)L)} & \quad \frac{C \vdash A, \Delta \quad C \vdash B, \Delta}{C \vdash A \land B, \Delta} \quad \text{(\( \land \)R)} \\
\frac{A \vdash \Delta \quad B \vdash \Delta}{A \lor B \vdash \Delta} \quad \text{(\( \lor \)L)} & \quad \frac{C \vdash A_i, \Delta}{C \vdash A_0 \lor A_1, \Delta} \quad \text{(\( \lor \)R)} \\
A \vdash B, \Delta \quad \text{(\( \leftarrow \) L)} & \quad \frac{C \vdash A, \Delta \quad B \vdash \Delta'}{C \vdash A \leftarrow B, \Delta, \Delta'} \quad \text{(\( \leftarrow \) R)}
\end{align*}
\]

- \( \sim A := \top \leftarrow A \).
**Dual-BHK interpretation**
*(cf. Wansing 2010)*

- A dual proof of $A \land B$ is a pair $\langle i, x \rangle$ such that $i = 0$ and $x$ is a dual proof of $A$, or $i = 1$ and $x$ is a dual proof of $B$.
- A dual proof of $A \lor B$ is a pair $\langle a, b \rangle$ consisting of a dual proof $a$ of $A$ and a dual proof $b$ of $B$.
- A dual proof of $A \leftarrow B$ is a construction that transforms any dual proof of $B$ into a dual proof of $A$.
- $\top$ has no dual proof.
Dual-BHK interpretation

- Explanation of conditions of denial in terms of dual proof (yet another kind of construction).
- Suitable for Dual Intuitionist Logic.

Definition
A sequent $A \vdash B_1, \ldots, B_m$ is *d-valid* if there is a construction that transforms any list of dual proofs of $B_1, \ldots, B_m$ into a dual proof of $A$.

Proposition (correctness)
If $A \vdash B_1, \ldots, B_m$ is derivable in LDJ, then it is *d-valid*. 
Problems

• BHK and Dual-BHK: two unilateralist semantics.
  💚 How do they agree or disagree on the meaning?

• Bi-Intuitionist Logic: A logic with the features of Intuitionist and Dual Intuitionist Logic.
Introduction

Two unilateralist semantics

Bi-Intuitionist Logic

Bilateralist validity
Model theory for Bilnt

Definition (Language)

\( \mathcal{L} := \{\land, \lor, \rightarrow, \leftrightarrow\} \).

Let \( \bot := p \leftarrow p \) and \( \top := p \rightarrow p \) for some fixed atom \( p \).
And define \( \neg A := A \rightarrow \bot \) and \( \sim A := \top \leftarrow A \).

Definition (Model)

A Bilnt model is a triple \( \langle W, \leq, V \rangle \) where

- \( W \) : a non-empty set (of possible worlds)
- \( \leq : \) a reflexive and transitive relation on \( W \)
- \( V : \text{Atom} \rightarrow 2^W \), a valuation which is persistent, i.e.

\[
(\forall w, w' \in W)(w \in V(p) \text{ and } w \leq w' \Rightarrow w' \in V(p)).
\]
Model theory for Bilnt (cont.)

Definition
Given a Bilnt model $\langle W, \leq, V \rangle$, write $w \models p$ for $w \in V(p)$. The relation $\models$ extends as follows:

- $w \models A \land B$ if $w \models A$ and $w \models B$
- $w \models A \lor B$ if $w \models A$ or $w \models B$
- $w \models A \rightarrow B$ if $(\forall v \geq w)(v \models A \Rightarrow v \models B)$
- $w \models A \leftarrow B$ if $(\exists v \leq w)(v \models A \text{ and } v \not\models B)$
Model theory for Bilnt (cont.)

Proposition (Persistence)
$\models$ satisfies the persistence condition, i.e. for any formula $A$,

$$(\forall w, w' \in W)(w \models A \text{ and } w \leq w' \Rightarrow w' \models A).$$

Definition (Validity)
For any formula $A$ and $B$, we define

$A \models B \iff_{def.} \text{for any Bilnt model } \langle W, \leq, V \rangle \text{ and any } w \in W, \text{ if } w \models A \text{ then } w \models B.$
Characteristic validity and invalidity

Recall $\bot = p \leftrightarrow p$, $\top = p \rightarrow p$ and

- $\neg A := A \rightarrow \bot$ (intuitionist negation)
- $\sim A := \top \leftrightarrow A$ (dual intuitionist negation)

- $\top \not\models A \lor \neg A$ but $\top \models A \lor \sim A$.
- $A \land \neg A \models \bot$ but $A \land \sim A \not\models \bot$.
- $\not\models (A \rightarrow (B \lor C)) \rightarrow ((A \rightarrow B) \lor C)$
- $\not\models (A \land (B \leftrightarrow C)) \rightarrow ((A \land B) \leftrightarrow C)$.
Display Calculus $\delta$-Bilnt
(Belnap 1982; Wansing 2010)

- A generalization of sequent calculus
- A sequent $X \vdash Y$ consists of structures $X$ and $Y$ instead of sequences, multisets or sets of formulas.

Definition (structures)

The set of structures for $\delta$-Bilnt is defined by:

$$X ::= A \mid I \mid X \circ X \mid X \bullet X.$$  

Intuitively, $I$ represents an empty structure and $\circ$ ($\bullet$) corresponds to a comma on LHS (RHS).
Logical rules for $\delta$-Bilnt

$\frac{A \circ B \vdash Y}{A \land B \vdash Y}$ \hspace{1cm} $\frac{X \vdash A \quad Y \vdash B}{X \circ Y \vdash A \land B}$  
$(\land L)$ \hspace{1cm} $(\land R)$

$\frac{A \vdash X \quad B \vdash Y}{A \lor B \vdash X \bullet Y}$ \hspace{1cm} $\frac{X \vdash A \bullet B}{X \vdash A \lor B}$
$(\lor L)$ \hspace{1cm} $(\lor R)$

$\frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X \circ Y}$ \hspace{1cm} $\frac{X \vdash A \circ B}{X \vdash A \rightarrow B}$
$(\rightarrow L)$ \hspace{1cm} $(\rightarrow R)$

$\frac{A \bullet B \vdash Y}{A \leftarrow B \vdash Y}$ \hspace{1cm} $\frac{X \vdash A \quad B \vdash Y}{X \bullet Y \vdash A \leftarrow B}$
$(\leftarrow L)$ \hspace{1cm} $(\leftarrow R)$
Definition (display rules)

The display calculus $\delta$-BiInt has the following display rules:

\[
\begin{align*}
X \bowtie Y & \vdash Z & Z & \vdash X \cdot Y \\
X & \vdash Y \bowtie Z & Z \cdot X & \vdash Y
\end{align*}
\]

Intuitively,

\[
\begin{align*}
A \land B & \vdash C \\
A & \vdash B \to C \\
B \land A & \vdash C
\end{align*}
\]

\[
\begin{align*}
C & \vdash A \lor B \\
C \leftarrow A & \vdash B \\
C & \vdash B \lor A
\end{align*}
\]
Structural rules for $\delta$-Bilnt

\[
p \vdash p \text{ (Id)}
\]

\[
\frac{X \vdash Y}{X \circ I \vdash Y}
\]

\[
\frac{X \vdash Y}{X \vdash Y \bullet I}
\]

\[
\frac{X \vdash Y}{X \circ Z \vdash Y} \text{ (lm)}
\]

\[
\frac{X \vdash Y}{X \vdash Y \bullet Z} \text{ (rm)}
\]

\[
\frac{X \circ X \vdash Y}{X \vdash Y} \text{ (lc)}
\]

\[
\frac{X \vdash Y \bullet Y}{X \vdash Y} \text{ (rc)}
\]

\[
\frac{(X \circ Y) \circ Z \vdash W}{X \circ (Y \circ Z) \vdash W} \text{ (la)}
\]

\[
\frac{W \vdash (X \bullet Y) \bullet Z}{W \vdash X \bullet (Y \bullet Z)} \text{ (ra)}
\]
Display property

Definition (antecedent and succedent part)
Given a sequent $S = X \vdash Y$, we define

- $X$ is AP (an antecedent part) of $S$;
- $Y$ is SP (a succedent part) of $S$;
- $(W \circ Z)$ is AP $\Rightarrow W, Z$ are AP;
- $(W \circ Z)$ is SP $\Rightarrow W$ is AP and $Z$ is SP;
- $(W \bullet Z)$ is AP $\Rightarrow W$ is AP and $Z$ is SP;
- $(W \bullet Z)$ is SP $\Rightarrow W, Z$ are SP.
Proposition (display property)
For any sequent $S = X \vdash Y$ and any substructure $Z$ of it, we can display the occurrence of $Z$, i.e. there is a sequent $S'$ such that:

- $S$ and $S'$ are interderivable by means of display rules only,
- If $Z$ is AP of $S$, then $S'$ is of the form $Z \vdash Y'$ and
- If $Z$ is SP of $S$, then $S'$ is of the form $X' \vdash Z$.

Theorem (Cut elimination)
Cut is eliminable from any derivation in $\delta$-$\text{BiInt} + \text{Cut}$.
**Completeness**

**Definition**

The translations $\tau_1$ and $\tau_2$ from structures into formulas are defined inductively as:

\[
\begin{align*}
\tau_1(\lambda) & := \lambda & \tau_2(\lambda) & := \lambda \\
\tau_1(\top) & := \top(= p \rightarrow p) & \tau_2(\top) & := \bot(= p \leftarrow p) \\
\tau_1(X \circ Y) & := \tau_1(X) \land \tau_1(Y) & \tau_2(X \circ Y) & := \tau_1(X) \rightarrow \tau_2(Y) \\
\tau_1(X \cdot Y) & := \tau_1(X) \leftarrow \tau_2(Y) & \tau_2(X \cdot Y) & := \tau_2(X) \lor \tau_2(Y)
\end{align*}
\]

**Theorem (Completeness)**

$X \vdash Y$ is derivable in $\delta$-Bilnt if and only if $\tau_1(X) \models \tau_2(Y)$. 
Logical rules for $\delta$-Bilnt (revised)

\[
\frac{A_i \vdash Y}{A_0 \land A_1 \vdash Y} \quad (\land L) \quad \frac{X \vdash A \quad X \vdash B}{X \vdash A \land B} \quad (\land R)
\]

\[
\frac{A \vdash Y \quad B \vdash Y}{A \lor B \vdash Y} \quad (\lor L) \quad \frac{X \vdash A_i}{X \vdash A_0 \lor A_1} \quad (\lor R)
\]

\[
\frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X \circ Y} \quad (\rightarrow L) \quad \frac{X \vdash A \circ B}{X \vdash A \rightarrow B} \quad (\rightarrow R)
\]

\[
\frac{A \bullet B \vdash Y}{A \leftarrow B \vdash Y} \quad (\leftarrow L) \quad \frac{X \vdash A \quad B \vdash Y}{X \bullet Y \vdash A \leftarrow B} \quad (\leftarrow R)
\]
Introduction

Two unilateralist semantics

Bi-Intuitionist Logic

Bilateralist validity
Introduction

Two unilateralist semantics

Bi-Intuitionist Logic

Bilateralist validity

Fit?

- Proofs and dual proofs coexist in $\delta$-BiInt.
- Two kind of meanings fit together in $\delta$-BiInt?
- BHK and Dual-BHK must be extended to interpret $\leftarrow$ and $\rightarrow$. 
BHK interpretation extended
(cf. Wansing 2010)

- A proof of $\Lambda \land B$ is a pair $\langle a, b \rangle$ consisting of a proof $a$ of $\Lambda$ and a proof $b$ of $B$.
- A proof of $\Lambda \lor B$ is a pair $\langle i, x \rangle$ such that $i = 0$ and $x$ is a proof of $\Lambda$, or $i = 1$ and $x$ is a proof of $B$.
- A proof of $\Lambda \rightarrow B$ is a construction that transforms any proof of $\Lambda$ into a proof of $B$.
- A proof of $\Lambda \leftarrow B$ is a pair $\langle a, b \rangle$ consisting of a proof $a$ of $\Lambda$ and a dual proof $b$ of $B$. 
Dual-BHK interpretation extended
(cf. Wansing 2010)

- A dual proof of $A \land B$ is a pair $\langle i, x \rangle$ such that $i = 0$ and $x$ is a dual proof of $A$, or $i = 1$ and $x$ is a dual proof of $B$.
- A dual proof of $A \lor B$ is a pair $\langle a, b \rangle$ consisting of a dual proof $a$ of $A$ and a dual proof $b$ of $B$.
- A dual proof of $A \leftarrow B$ is a construction that transforms any dual proof of $B$ into a dual proof of $A$.
- A dual proof of $A \rightarrow B$ is a pair $\langle a, b \rangle$ consisting of a proof $a$ of $A$ and a dual proof $b$ of $B$. 
i-validity fails

A rule in δ-BiInt that is not i-valid.

\[
\frac{Z \bullet X \vdash Y}{Z \vdash X \bullet Y} \quad \overset{\sim}{=} \quad \frac{A \leftarrow B \vdash C}{A \vdash B \lor C}
\]

- ● on RHS: sequents become multiple-conclusion.
- Impossible to interpret it as intuitionist’s disjunction with disjunction property.
- Proof is not preserved from LHS to RHS.
- At most impossibility of dual proof of RHS

The same applies to d-validity.
Bilateralist reading of sequents  
(cf. Restall 2005)

**Definition**

A ⊢ B is **b-valid** if it is not the case that A has a proof and B has a dual proof.

- to assert A and to deny B is to make a mistake;
- if A has a proof then B can’t have a dual proof;
- if B has a dual proof then A can’t have a proof.

⇒ Two criteria of fit between proof & dual proof
Criterion 1: Identity

\( A \vdash A \)

- A can’t have both a proof and a dual proof.
- No clash between proofs and dual proofs.
- No overlap between assertion and denial.
- Established directly by BHK and Dual-BHK on the assumptions \( p \vdash p \).
- Derivable in \( \delta\text{-BiInt} \).
Criterion 2: Cut

\[
\frac{X \vdash A \quad A \vdash Y}{X \vdash Y} \text{ (Cut)}
\]

- \( X \vdash A \): a proof of \( X \) excludes dual-provability of \( A \)
- \( A \vdash Y \): a dual proof of \( Y \) excludes provability of \( A \), then
- \( X \vdash Y \): the proof of \( X \) clashes with the dual proof of \( Y \).

I.e. it is impossible to exclude both provability and dual-provability of \( A \) without any clash.

- No gap between assertion and denial.
Fit

- Cut & Identity: criteria of fit between proofs & dual proofs.
  - Agreement on meanings between BHK and dual-BHK

- No problem with Identity. How about Cut?
- Seems difficult to establish directly by BHK and dual-BHK.
- Cut elimination for $\delta$-BiInt tells us something?
Failure of $b$-validity

Almost all rules in $\delta$-BiInt are $b$-valid except for:

\[
\frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X \circ Y} \quad (\rightarrow L) \quad \frac{X \vdash A \quad B \vdash Y}{X \bullet Y \vdash A \leftarrow B} \quad (\leftarrow R)
\]

A proof of $A \rightarrow B$ requires a proof of $A$. But $X \vdash A$ gives at most impossibility of dual proof of $A$.

To make them $b$-valid:

\[
\frac{B \vdash Y}{A \rightarrow B \vdash A \circ Y} \quad (\rightarrow L') \quad \frac{X \vdash A}{X \bullet B \vdash A \leftarrow B} \quad (\leftarrow R')
\]
Revision of rules and \textbf{Cut}

New rules are equivalent to the original through \textbf{Cut}:

\[
\begin{align*}
\frac{\text{B} \vdash Y}{\frac{\text{A} \rightarrow \text{B} \vdash \text{A} \circ Y}{} } \quad (\rightarrow \text{L}')
\end{align*}
\]

\[
\begin{align*}
\frac{\text{X} \vdash \text{A}}{\frac{\text{A} \vdash \text{A} \rightarrow \text{B} \circ Y}{} } \quad \frac{\text{A} \vdash \text{A} \rightarrow \text{B} \circ Y}{} \quad \frac{\text{A} \rightarrow \text{B} \vdash \text{A} \circ Y}{} \quad \frac{\text{A} \vdash \text{A} \rightarrow \text{B} \circ Y}{} \quad (\rightarrow \text{L})
\end{align*}
\]

Let \textbf{Cut}' denote \textbf{Cut} of this form (and the dual form for $\leftarrow$ \textbf{R}').
Weak Cut-elimination

\[
\delta \text{-BiInt}' = \text{the system with } \rightarrow \text{L'} \text{ and } \leftarrow \text{R'}
\]

- Every rule is \(b\)-valid.
- \(\delta \text{-BiInt}' + \text{Cut}' \cong \delta \text{-BiInt}\)

Fact (cf. Schroeder-Heister forthcoming)

\(\delta \text{-BiInt}' + \text{Cut} \cong \delta \text{-BiInt}' + \text{Cut}' \not\cong \delta \text{-BiInt}'\).

- \(\delta \text{-BiInt}'\): \(b\)-valid but \(\text{Cut}'\) is not admissible.
- \(\delta \text{-BiInt}\): \(\text{Cut}\) is admissible but not \(b\)-valid.
Example of failure of $\text{Cut}'$

\[
\begin{align*}
\text{p} \vdash \text{p} & \quad \Rightarrow \text{L'} \quad \text{r} \vdash \text{r} \\
\text{p} \vdash \text{p} \lor \text{q} & \quad \Rightarrow \text{r} \vdash \text{p} \lor \text{q} \circ \text{r} \\
\text{p} \vdash \text{p} \lor \text{q} & \quad \Rightarrow (\text{p} \lor \text{q}) \vdash \text{r} \circ \text{r} \quad \Rightarrow \text{Cut'}
\end{align*}
\]
\[ \frac{\{
X \vdash A \quad A \vdash Y\}}{X \nvdash Y} \quad \text{(Cut)} \]

Failure of Cut indicates:

- there may be a combination of a proof of \(X\) and a dual proof of \(Y\) such that:
  - the former excludes dual-provability of \(A\),
  - the latter excludes provability of \(A\),
  - but they don’t cause any clash in \(\delta\)-BiInt\(^\prime\).

A gap between proof and dual proof.
Conclusion

Under Bilateralist reading of sequents,

- Cut & Identity: Criteria of fit between two aspects
- Failure of Cut’: a gap between proof and dual proof.
- Cut admissibility in $\delta$-BiInt:
  - Proof and dual proof are adjusted implicitly.
  - $\rightarrow L$ and $\leftarrow R$ are where the adjustment occurs.
Relativized $b$-validity

To make this observation more precise, $b$-validity (and extensions of (dual-)BHK) should be defined more carefully:

Definition (Atomic base)
An atomic base is a pair $\langle s, t \rangle$ of two sets of atomic formulas with $s \cap t = \emptyset$. Define the relation $\models$ between atomic bases and signed atomic formulas as:

\[
\langle s, t \rangle \models +p \iff p \in s; \\
\langle s, t \rangle \models -p \iff p \in t.
\]
**Relativized \( b \)-validity**

**Definition**

\( \models \) extends as follows:

\[
\begin{align*}
\langle s, t \rangle \models +T & \quad \Leftrightarrow \quad \langle s, t \rangle \not\models +\bot \\
\langle s, t \rangle \models +(A \land B) & \quad \Leftrightarrow \quad \langle s, t \rangle \models +A \text{ and } \langle s, t \rangle \models +B \\
\langle s, t \rangle \models +(A \lor B) & \quad \Leftrightarrow \quad \langle s, t \rangle \models +A \text{ or } \langle s, t \rangle \models +B \\
\langle s, t \rangle \models +(A \to B) & \quad \Leftrightarrow \quad \forall s'. \subseteq s. \langle s', t \rangle \models +A \text{ implies } \langle s', t \rangle \models +B \\
\langle s, t \rangle \models +(A \leftrightarrow B) & \quad \Leftrightarrow \quad \exists s' \subseteq s \\exists t' \subseteq t. \langle s', t' \rangle \models +A \text{ and } \langle s', t' \rangle \models -B
\end{align*}
\]

Dually for \( \langle s, t \rangle \models -A \). Especially,

\[
\begin{align*}
\langle s, t \rangle \models -(A \to B) & \quad \Leftrightarrow \quad \exists t' \subseteq t \\exists s' \subseteq s. \langle s', t' \rangle \models +A \text{ and } \langle s', t' \rangle \models -B
\end{align*}
\]
Relativized $\mathfrak{b}$-validity

Proposition

(1) Persistence: 
- $\langle s, t \rangle \vdash +A$ and $s' \supseteq s$ implies $\langle s', t \rangle \vdash +A$.
- $\langle s, t \rangle \vdash -A$ and $t' \supseteq t$ implies $\langle s, t' \rangle \vdash -A$.

(2) Irrelevance: 
- If $\langle s, t \rangle \vdash +A$, then for any $t'$, $\langle s, t' \rangle \vdash +A$.
- If $\langle s, t \rangle \vdash -A$, then for any $s'$, $\langle s', t \rangle \vdash +A$.

(3) Consistency: It is not the case that $\langle s, t \rangle \vdash +A$ and $\langle s, t \rangle \vdash -A$. 
Relativized \( b \)-validity

**Definition (\( b \)-validity)**

A sequent \( \Gamma \vdash \Delta \) is \( \langle s, t \rangle \)-valid if it is *not* the case that

\[
\langle s, t \rangle \not\vdash +\Gamma \quad \text{and} \quad \langle s, t \rangle \vdash -\Delta.
\]

\( \Gamma \vdash \Delta \) is \( b \)-valid if it is \( \langle s, t \rangle \)-valid for any \( \langle s, t \rangle \).

**Proposition (Soundness)**

If \( \Gamma \vdash \Delta \) is derivable in \( \delta \)-BiInt’', then it is \( b \)-valid.

**Proposition**

Cut’, Intuitionist LEM, \( (\Gamma \rightarrow (\Delta \lor \epsilon)) \rightarrow ((\Delta \rightarrow \Delta) \lor \epsilon) \) etc. are not \( b \)-valid.
References (1)


References (2)


