BHK interpretation and Bilateralism

大西琢朗 Takuro Onishi

14 March 2012 PhilLogMath
Bilateralism

\[
\text{Meaning} = \text{Assertion and Denial}
\]

- Inferentialism and proof-theoretic semantics.
- Smiley, Rumfitt, Restall, Ripley, etc.

vs. Unilateralism: Constructive PTS (BHK, Dummett, Prawitz) and its dual version

♡ Two unilateralist semantics fit together?
  - Assertion & denial, proof & dual proof
  - How to show the fit between two meanings?
<table>
<thead>
<tr>
<th>Introduction</th>
<th>Two unilateralist semantics</th>
<th>Bi-Intuitionist Logic</th>
<th>Bilateralist validity</th>
</tr>
</thead>
</table>

**Introduction**

**Two unilateralist semantics**

**Bi-Intuitionist Logic**

**Bilateralist validity**
Introduction

Two unilateralist semantics

Bi-Intuitionist Logic

Bilateralist validity
BHK interpretation

- A proof of $A \land B$ is a pair $\langle a, b \rangle$ consisting of a proof $a$ of $A$ and a proof $b$ of $B$.
- A proof of $A \lor B$ is a pair $\langle i, x \rangle$ such that $i = 0$ and $x$ is a proof of $A$, or $i = 1$ and $x$ is a proof of $B$.
- A proof of $A \rightarrow B$ is a construction that transforms any proof of $A$ into a proof of $B$.
- $\bot$ has no proof.
BHK interpretation and PTS
(Dummett 1991; Prawitz 2006; Schroeder-Heister 2006)

- Explanation of conditions of assertion in terms of the primitive notion of proof (or construction).
- Suitable for Intuitionist Logic.

Definition
A sequent $A_1, \ldots, A_n \vdash B$ is i-valid if there is a construction that transforms any list $a_1, \ldots, a_n$ of proofs of $A_1, \ldots, A_n$ into a proof of $B$.

Proposition (correctness)
If $A_1, \ldots, A_n \vdash B$ is derivable in NJ (LJ), then it is i-valid.
Dual BHK

- Another unilateralist semantics
- In terms of dual proofs

⇒ Dual-BHK and Dual-Intuitionist Logic
LJ for Int

- "Singleton on the right"

\[ \Gamma, \bot \vdash C \quad (\bot L) \quad \frac{\Gamma \vdash A}{\Gamma, \Gamma' \vdash C} \quad (Cut) \]

\[ \frac{\Gamma \vdash C}{\Gamma, A \vdash C} \quad (Weakening) \quad \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C} \quad (Contraction) \]

\[ \frac{\Gamma, A_i \vdash C}{\Gamma, A_0 \land A_1 \vdash C} \quad (\land L) \quad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \land B} \quad (\land R) \]

\[ \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \lor B \vdash C} \quad (\lor L) \quad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_0 \lor A_1} \quad (\lor R) \]

\[ \frac{\Gamma \vdash A \quad \Gamma', B \vdash C}{\Gamma, \Gamma', A \rightarrow B \vdash C} \quad (\rightarrow L) \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \quad (\rightarrow R) \]

- \( \neg A := A \rightarrow \bot \).
LDJ for dual-Int

• ”Singleton on the left”, \(\leftarrow\): subtraction, exclusion

\[
A \vdash A \quad \text{(Id)} \\
C \vdash \top, \Delta \quad \text{(\(\bot\)L)} \\
\frac{C \vdash \Delta}{C \vdash A, \Delta} \quad \text{(Weakening)} \\
\frac{C \vdash A, A, \Delta}{C \vdash A, \Delta} \quad \text{(Contraction)} \\
\frac{C \vdash A, \Delta}{C \vdash A_i, \Delta} \quad \text{(\(\land\)L)} \\
\frac{A \vdash \Delta \quad B \vdash \Delta}{A \lor B \vdash \Delta} \quad \text{(\(\lor\)L)} \\
\frac{C \vdash B, \Delta}{C \vdash A \leftarrow B, \Delta} \quad \text{(\(\leftarrow\)L)} \\
\frac{C \vdash A, \Delta \quad B \vdash \Delta'}{C \vdash A \leftarrow B, \Delta, \Delta'} \quad \text{(\(\leftarrow\)R)}
\]

• \(\sim A := \top \leftarrow A\).
Dual-BHK interpretation
(cf. Wansing 2010)

- A dual proof of $A \land B$ is a pair $\langle i, x \rangle$ such that $i = 0$ and $x$ is a dual proof of $A$, or $i = 1$ and $x$ is a dual proof of $B$.
- A dual proof of $A \lor B$ is a pair $\langle a, b \rangle$ consisting of a dual proof $a$ of $A$ and a dual proof $b$ of $B$.
- A dual proof of $A \rightarrow B$ is a construction that transforms any dual proof of $B$ into a dual proof of $A$.
- $\top$ has no dual proof.
Dual-BHK interpretation

- Explanation of conditions of denial in terms of dual proof (yet another kind of construction).
- Suitable for Dual Intuitionist Logic.

**Definition**
A sequent $A \vdash B_1, \ldots, B_m$ is d-valid if there is a construction that transforms any list of dual proofs of $B_1, \ldots, B_m$ into a dual proof of $A$.

**Proposition (correctness)**
If $A \vdash B_1, \ldots, B_m$ is derivable in LDJ, then it is d-valid.
Problems

• BHK and Dual-BHK: two unilateralist semantics.
  How do they agree or disagree on the meaning?

• Bi-Intuitionist Logic: A logic with the features of Intuitionist and Dual Intuitionist Logic.
Introduction

Two unilateralist semantics

Bi-Intuitionist Logic

Bilateralist validity
Model theory for Bilnt

Definition (Language)

\[ \mathcal{L} := \{ \land, \lor, \to, \leftrightarrow \}. \]

Let \( \bot := p \leftrightarrow p \) and \( \top := p \to p \) for some fixed atom \( p \).
And define \( \neg A := A \to \bot \) and \( \sim A := \top \leftarrow A \).

Definition (Model)

A Bilnt model is a triple \( \langle W, \leq, V \rangle \) where

- \( W \) : a non-empty set (of possible worlds)
- \( \leq \) : a reflexive and transitive relation on \( W \)
- \( V : \text{Atom} \rightarrow 2^W \), a valuation which is persistent, i.e.

\[ (\forall w, w' \in W)(w \in V(p) \text{ and } w \leq w' \Rightarrow w' \in V(p)). \]
Model theory for Bilnt (cont.)

Definition
Given a Bilnt model $\langle W, \leq, V \rangle$, write $w \models p$ for $w \in V(p)$. The relation $\models$ extends as follows:

- $w \models A \land B$ if $w \models A$ and $w \models B$
- $w \models A \lor B$ if $w \models A$ or $w \models B$
- $w \models A \rightarrow B$ if $\forall v \geq w(v \models A \Rightarrow v \models B)$
- $w \models A \leftrightarrow B$ if $\exists v \leq w(v \models A \text{ and } v \not\models B)$
Proposition (Persistence)
\[ \models \text{satisfies the persistence condition, i.e. for any formula } A, \]
\[ (\forall w, w' \in W)(w \models A \text{ and } w \leq w' \implies w' \models A). \]

Definition (Validity)
For any formula \( A \) and \( B \), we define
\[ A \models B \iff_{\text{def.}} \text{for any Bilnt model } \langle W, \leq, V \rangle \text{ and any } \]
\[ w \in W, \text{ if } w \models A \text{ then } w \models B. \]
Characteristic validity and invalidity

Recall $\perp = p \Leftarrow p$, $\top = p \rightarrow p$ and

$$\neg A := A \rightarrow \perp$$

(intuitionist negation)

$$\sim A := \top \Leftarrow A$$

(dual intuitionist negation)

- $\top \not\models A \vee \neg A$ but $\top \models A \vee \sim A$.
- $A \land \neg A \models \perp$ but $A \land \sim A \not\models \perp$.
- $\not\models (A \rightarrow (B \lor C)) \rightarrow ((A \rightarrow B) \lor C)$
- $\not\models (A \land (B \Leftarrow C)) \rightarrow ((A \land B) \Leftarrow C)$. 

Display Calculus $\delta$-Bilnt
(Belnap 1982; Wansing 2010)

- A generalization of sequent calculus
- A sequent $X \vdash Y$ consists of *structures* $X$ and $Y$ instead of sequences, multisets or sets of formulas.

**Definition (structures)**

The set of structures for $\delta$-Bilnt is defined by:

$$X ::= A \mid I \mid X \circ X \mid X \bullet X.$$  

Intuitively, $I$ represents an empty structure and $\circ$ ($\bullet$) corresponds to a comma on LHS (RHS).
Logical rules for \(\delta\)-BilInt

\[
\frac{A \circ B \vdash Y}{A \land B \vdash Y} \quad (\land L) \quad \frac{X \vdash A \quad Y \vdash B}{X \circ Y \vdash A \land B} \quad (\land R)
\]

\[
\frac{A \vdash X \quad B \vdash Y}{A \lor B \vdash X \bullet Y} \quad (\lor L) \quad \frac{X \vdash A \bullet B}{X \vdash A \lor B} \quad (\lor R)
\]

\[
\frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X \circ Y} \quad (\rightarrow L) \quad \frac{X \vdash A \circ B}{X \vdash A \rightarrow B} \quad (\rightarrow R)
\]

\[
\frac{A \bullet B \vdash Y}{A \leftarrow B \vdash Y} \quad (\leftarrow L) \quad \frac{X \vdash A \quad B \vdash Y}{X \bullet Y \vdash A \leftarrow B} \quad (\leftarrow R)
\]
Definition (display rules)

The display calculus $\delta$-BiInt has the following display rules:

\[
\begin{align*}
&X \circ Y \vdash Z \\
&\frac{X \vdash Y \circ Z}{Y \circ X \vdash Z} \quad \\
&Z \vdash X \cdot Y \\
&\frac{Z \cdot X \vdash Y}{Z \vdash Y \cdot X}
\end{align*}
\]

Intuitively,

\[
\begin{align*}
&A \land B \vdash C \\
&\frac{A \vdash B \rightarrow C}{B \land A \vdash C} \quad \\
&C \vdash A \lor B \\
&\frac{C \leftarrow A \vdash B}{C \vdash B \lor A}
\end{align*}
\]
Structural rules for $\delta$-Bilnt

\[ p \vdash p \quad \text{(Id)} \]

\[ \frac{X \vdash Y}{X \circ I \vdash Y} \quad \frac{X \vdash Y}{X \vdash Y \bullet I} \]

\[ X \vdash Y \quad \text{(lm)} \quad X \vdash Y \quad \text{(rm)} \]

\[ \frac{X \circ X \vdash Y}{X \vdash Y} \quad \frac{X \vdash Y \bullet Y}{X \vdash Y} \quad \frac{X \vdash Y \bullet Y}{X \vdash Y} \quad \text{(rc)} \]

\[ \frac{(X \circ Y) \circ Z \vdash W}{X \circ (Y \circ Z) \vdash W} \quad \frac{W \vdash (X \bullet Y) \bullet Z}{W \vdash X \bullet (Y \bullet Z)} \quad \text{(ra)} \]
Display property

Definition (antecedent and succedent part)

Given a sequent $S = X \vdash Y$, we define

- $X$ is AP (an antecedent part) of $S$;
- $Y$ is SP (a succedent part) of $S$;
- $(W \circ Z)$ is AP $\Rightarrow W, Z$ are AP;
- $(W \circ Z)$ is SP $\Rightarrow W$ is AP and $Z$ is SP;
- $(W \bullet Z)$ is AP $\Rightarrow W$ is AP and $Z$ is SP;
- $(W \bullet Z)$ is SP $\Rightarrow W, Z$ are SP.
Display property

Proposition (display property)
For any sequent \( S = X \vdash Y \) and any substructure \( Z \) of it, we can display the occurrence of \( Z \), i.e. there is a sequent \( S' \) such that:
- \( S \) and \( S' \) are interderivable by means of display rules only,
- If \( Z \) is AP of \( S \), then \( S' \) is of the form \( Z \vdash Y' \) and
- If \( Z \) is SP of \( S \), then \( S' \) is of the form \( X' \vdash Z \).

Theorem (Cut elimination)
Cut is eliminable from any derivation in \( \delta\text{-BiInt} + \text{Cut} \).
Completeness

Definition
The translations $\tau_1$ and $\tau_2$ from structures into formulas are defined inductively as:

$$
\begin{align*}
\tau_1(A) &:= A \\
\tau_1(I) &:= \top(= p \rightarrow p) \\
\tau_1(X \circ Y) &:= \tau_1(X) \land \tau_1(Y) \\
\tau_1(X \bullet Y) &:= \tau_1(X) \leftarrow \tau_2(Y)
\end{align*}
$$

$$
\begin{align*}
\tau_2(A) &:= A \\
\tau_2(I) &:= \perp(= p \leftarrow p) \\
\tau_2(X \circ Y) &:= \tau_1(X) \rightarrow \tau_2(Y) \\
\tau_2(X \bullet Y) &:= \tau_2(X) \lor \tau_2(Y)
\end{align*}
$$

Theorem (Completeness)
$X \vdash Y$ is derivable in $\delta$-Bilnt if and only if $\tau_1(X) \models \tau_2(Y)$. 

Logical rules for $\delta$-Bilnt (revised)

$\frac{A_i \vdash Y}{A_0 \land A_1 \vdash Y}$ \hspace{1em} ($\land L$) \hspace{1em} $\frac{X \vdash A \quad X \vdash B}{X \vdash A \land B}$ \hspace{1em} ($\land R$)

$\frac{A \vdash Y \quad B \vdash Y}{A \lor B \vdash Y}$ \hspace{1em} ($\lor L$) \hspace{1em} $\frac{X \vdash A_i}{X \vdash A_0 \lor A_1}$ \hspace{1em} ($\lor R$)

$\frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X \circ Y}$ \hspace{1em} ($\rightarrow L$) \hspace{1em} $\frac{X \vdash A \circ B}{X \vdash A \rightarrow B}$ \hspace{1em} ($\rightarrow R$)

$\frac{A \bullet B \vdash Y}{A \leftarrow B \vdash Y}$ \hspace{1em} ($\leftarrow L$) \hspace{1em} $\frac{X \vdash A \quad B \vdash Y}{X \bullet Y \vdash A \leftarrow B}$ \hspace{1em} ($\leftarrow R$)
Introduction

Two unilateralist semantics

Bi-Intuitionist Logic

Bilateralist validity
Fit?

- Proofs and dual proofs coexist in δ-BiInt.
- Two kind of meanings fit together in δ-BiInt?
- BHK and Dual-BHK must be extended to interpret ← and →.
BHK interpretation extended
(cf. Wansing 2010)

- A proof of $A \land B$ is a pair $\langle a, b \rangle$ consisting of a proof $a$ of $A$ and a proof $b$ of $B$.
- A proof of $A \lor B$ is a pair $\langle i, x \rangle$ such that $i = 0$ and $x$ is a proof of $A$, or $i = 1$ and $x$ is a proof of $B$.
- A proof of $A \rightarrow B$ is a construction that transforms any proof of $A$ into a proof of $B$.
- A proof of $A \leftarrow B$ is a pair $\langle a, b \rangle$ consisting of a proof $a$ of $A$ and a dual proof $b$ of $B$. 
Dual-BHK interpretation extended
(cf. Wansing 2010)

- A dual proof of $A \land B$ is a pair $\langle i, x \rangle$ such that $i = 0$ and $x$ is a dual proof of $A$, or $i = 1$ and $x$ is a dual proof of $B$.
- A dual proof of $A \lor B$ is a pair $\langle a, b \rangle$ consisting of a dual proof $a$ of $A$ and a dual proof $b$ of $B$.
- A dual proof of $A \leftarrow B$ is a construction that transforms any dual proof of $B$ into a dual proof of $A$.
- A dual proof of $A \rightarrow B$ is a pair $\langle a, b \rangle$ consisting of a proof $a$ of $A$ and a dual proof $b$ of $B$. 
\textbf{i-validity fails}

A rule in \(\delta\)-BiInt that is not i-valid.

\[
\frac{Z \bullet X \vdash Y}{Z \vdash X \bullet Y} \quad \approx \quad \frac{A \leftarrow B \vdash C}{A \vdash B \lor C}
\]

- \(\bullet\) on RHS: sequents become multiple-conclusion.
- Impossible to interpret it as intuitionist’s disjunction with disjunction property.
- Proof is not preserved from LHS to RHS.
- At most \textit{impossibility of dual proof} of RHS

The same applies to d-validity.
Bilateralist reading of sequents  
(cf. Restall 2005)

Definition

\( A \vdash B \) is b-valid if it is not the case that \( A \) has a proof and \( B \) has a dual proof.

- to assert \( A \) and to deny \( B \) is to make a mistake;
- if \( A \) has a proof then \( B \) can’t have a dual proof;
- if \( B \) has a dual proof then \( A \) can’t have a proof.

\( \Rightarrow \) Two criteria of fit between proof & dual proof
**Criterion 1: Identity**

\[ A \vdash A \]

- A can’t have both a proof and a dual proof.
- No clash between proofs and dual proofs.
- No overlap between assertion and denial.
- Established directly by BHK and Dual-BHK on the assumptions \( p \vdash p \).
- Derivable in \( \delta\text{-BiInt} \).
**Criterion 2: Cut**

\[
\frac{X \vdash A \quad A \vdash Y}{X \vdash Y} \quad \text{(Cut)}
\]

- \(X \vdash A\): a proof of \(X\) excludes dual-provability of \(A\)
- \(A \vdash Y\): a dual proof of \(Y\) excludes provability of \(A\), then
- \(X \vdash Y\): the proof of \(X\) clashes with the dual proof of \(Y\).

I.e. it is impossible to exclude both provability and dual-provability of \(A\) without any clash.

- No gap between assertion and denial.
Fit

- Cut & Identity: criteria of fit between proofs & dual proofs.
  - Agreement on meanings between BHK and dual-BHK
- No problem with Identity. How about Cut?
- Seems difficult to establish directly by BHK and dual-BHK.
- Cut elimination for $\delta$-BiInt tells us something?
Failure of $b$-validity

Almost all rules in $\delta$-BiInt are $b$-valid except for:

$$
\frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X \circ Y} \quad (\rightarrow L) \quad \frac{X \vdash A \quad B \vdash Y}{X \bullet Y \vdash A \leftarrow B} \quad (\leftarrow R)
$$

A proof of $A \rightarrow B$ requires a proof of $A$. But $X \vdash A$ gives at most impossibility of dual proof of $A$.

To make them $b$-valid:

$$
\frac{B \vdash Y}{A \rightarrow B \vdash A \circ Y} \quad (\rightarrow L') \quad \frac{X \vdash A}{X \bullet B \vdash A \leftarrow B} \quad (\leftarrow R')
$$
New rules are equivalent to the original through $\text{Cut}$:

\[
\begin{align*}
B & \vdash Y \\
A \rightarrow B & \vdash A \circ Y \\
X & \vdash A \\
A & \vdash A \rightarrow B \circ Y \\
& \quad (\text{Cut}) \\
& \quad (\rightarrow L') \\
& \quad (\rightarrow L)
\end{align*}
\]

Let $\text{Cut}'$ denote $\text{Cut}$ of this form (and the dual form for $\leftarrow R'$).
Weak Cut-elimination

- $\delta$-BiInt' = the system with $\rightarrow L'$ and $\leftarrow R'$
  - Every rule is b-valid.
  - $\delta$-BiInt' + Cut' $\cong$ $\delta$-BiInt

Fact (cf. Schroeder-Heister forthcoming)

$\delta$-BiInt' + Cut $\cong$ $\delta$-BiInt' + Cut' $\not\cong$ $\delta$-BiInt'.

- $\delta$-BiInt': b-valid but Cut' is not admissible.
- $\delta$-BiInt: Cut is admissible but not b-valid.
Example of failure of *Cut’*

\[ \begin{align*} 
\text{Premise: } & p \\ 
\text{Premise: } & p \lor q \\
\text{Conclusion: } & (p \lor q) \rightarrow r
\end{align*} \]

\[ \frac{r \vdash r}{(\rightarrow L')} \]

\[ \frac{(p \lor q) \rightarrow r \vdash p \lor q \circ r}{(Cut')} \]

\[ \frac{p \vdash p \lor q}{(Cut')} \]

\[ \frac{p \vdash (p \lor q) \rightarrow r \circ r}{(Cut')} \]
\[
\frac{X \vdash A \quad A \vdash Y}{X \nvdash Y} \quad \text{(Cut)}
\]

Failure of Cut indicates:

- there may be a combination of a proof of \( X \) and a dual proof of \( Y \) such that:
  - the former excludes dual-provability of \( A \),
  - the latter excludes provability of \( A \),
  - but they don’t cause any clash in \( \delta\text{-BiInt} \).

A gap between proof and dual proof.
Conclusion

Under Bilateralist reading of sequents,

- Cut & Identity: Criteria of fit between two aspects
- Failure of Cut': a gap between proof and dual proof.
- Cut admissibility in δ-BiInt:
  - Proof and dual proof are adjusted implicitly.
  - → L and ← R are where the adjustment occurs.
Relativized $b$-validity

To make this observation more precise, $b$-validity (and extensions of (dual-)BHK) should be defined more carefully:

**Definition (Atomic base)**

An atomic base is a pair $\langle s, t \rangle$ of two sets of atomic formulas with $s \cap t = \emptyset$. Define the relation $\vdash$ between atomic bases and signed atomic formulas as:

$$\langle s, t \rangle \vdash +p \iff p \in s;$$

$$\langle s, t \rangle \vdash -p \iff p \in t.$$
**Relativized \( \mathfrak{b} \)-validity**

**Definition**

\( \models \) extends as follows:

\[
\begin{align*}
\langle s, t \rangle \models +T & \quad \iff \quad \langle s, t \rangle \not\models +\bot \\
\langle s, t \rangle \models +(A \land B) & \quad \iff \quad \langle s, t \rangle \models +A \text{ and } \langle s, t \rangle \models +B \\
\langle s, t \rangle \models +(A \lor B) & \quad \iff \quad \langle s, t \rangle \models +A \text{ or } \langle s, t \rangle \models +B \\
\langle s, t \rangle \models +(A \rightarrow B) & \quad \iff \quad \forall s' \supseteq s. \langle s', t \rangle \models +A \text{ implies } \langle s', t \rangle \models +B \\
\langle s, t \rangle \models +(A \leftarrow B) & \quad \iff \quad \exists s' \subseteq s \exists t' \supseteq t. \\
& \qquad \langle s', t' \rangle \models +A \text{ and } \langle s', t' \rangle \models +B
\end{align*}
\]

Dually for \( \langle s, t \rangle \models -A \). Especially,

\[
\begin{align*}
\langle s, t \rangle \models -(A \rightarrow B) & \quad \iff \quad \exists t' \subseteq t \exists s' \supseteq s. \\
& \quad \langle s', t' \rangle \models +A \text{ and } \langle s', t' \rangle \models +B
\end{align*}
\]
Relativized $b$-validity

Proposition

(1) Persistence:
- $\langle s, t \rangle \vdash +A$ and $s' \supseteq s$ implies $\langle s', t \rangle \vdash +A$.
- $\langle s, t \rangle \vdash -A$ and $t' \supseteq t$ implies $\langle s, t' \rangle \vdash -A$.

(2) Irrelevance:
- If $\langle s, t \rangle \vdash +A$, then for any $t'$, $\langle s, t' \rangle \vdash +A$.
- If $\langle s, t \rangle \vdash -A$, then for any $s'$, $\langle s', t \rangle \vdash +A$.

(3) Consistency: It is not the case that $\langle s, t \rangle \vdash +A$ and $\langle s, t \rangle \vdash -A$. 
Relativized $b$-validity

**Definition ($b$-validity)**

A sequent $A \vdash B$ is $\langle s, t \rangle$-valid if it is not the case that $\langle s, t \rangle \not\models +A$ and $\langle s, t \rangle \not\models -B$.

$A \vdash B$ is $b$-valid if it is $\langle s, t \rangle$-valid for any $\langle s, t \rangle$.

**Proposition (Soundness)**

If $X \vdash Y$ is derivable in $\delta$-BiInt', then it is $b$-valid.

**Proposition**

Cut', Intuitionist LEM, $(A \rightarrow (B \lor C)) \rightarrow ((A \rightarrow B) \lor C)$ etc. are not $b$-valid.
References (1)

References (2)


