State cycles which represent the canonical class of Lee's homology of a knot

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Abstract

For a diagram of a knot, Lee associated a complex which is called Lee's complex. We introduce the notion of a state cycle of Lee's complex, which is a certain cycle of Lee's complex. We describe state cycles which represent the canonical class of Lee's homology of a knot. As a corollary, we give the shaper slice-Bennequin inequality for the Rasmussen invariant of a knot in the viewpoint of cycles of Lee's complex.

Keywords: Rasmussen invariant, Lee's homology, canonical class, state cycle, homogeneous diagram

1. Introduction

In [16], Rasmussen introduced a smooth concordance invariant of a knot K, now called the Rasmussen invariant $s(K)$, which is defined by cycles of Lee's complex. There are many computation results on the Rasmussen invariant. For example, see [3], [5], [6], [7], [8], [10], [11], [12], [13], [18] and [19]. However very little is known on cycles of Lee's complex. Our goal is to simplify the computation of the Rasmussen invariant by studying cycles of Lee's complex.

In this paper, we introduce the notion of state cycles for Lee's complex. We describe state cycles which represent the canonical class of Lee's homology of a knot (Theorem 3.6 and Lemma 6.1). The definition of the canonical class of Lee's homology of a knot is given in Remark 2.3. As a corollary, we give a new proof of the sharper slice-Bennequin inequality for the Rasmussen invariant (Theorem 5.4) in the viewpoint of cycles of Lee's complex, which was first proved by Kawamura [7]. In Section 7, we consider the Rasmussen invariant of the pretzel knot of type $(3, -5, -7)$, denoted by $P(3, -5, -7)$. Let D be the standard pretzel diagram of $P(3, -5, -7)$. Then we explicitly give a cycle of Lee's complex of D which determine the Rasmussen invariant of $P(3, -5, -7)$. Here we do not use Rudolph's theory to determine the Rasmussen invariant of $P(3, -5, -7)$.

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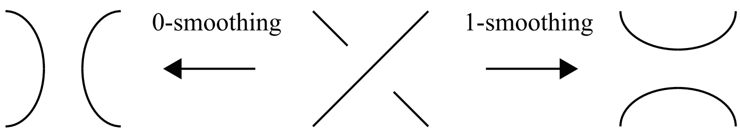

Figure 1: 0- and 1-smoothings

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2. Review of Lee's homology of a knot

Lee [9] constructed a homology theory which is closely related to Khovanov homology theory. We review the results in [9].

2.1. The construction of Lee's homology of a knot

In this subsection, we recall the construction of Lee's homology of a knot.

Let K be a knot, D a diagram of K, $c_1, \cdots, c_n$ the crossings of D and $n_-(D)$ the number of negative crossings of D. A *state* $s = (s_1, \cdots, s_n)$ for D is a vertex of the n-dimensional cube $[0,1]^n$, that is, an element of $\{0,1\}^n$. The *grading* of s is the sum $\sum_{i=1}^n s_i - n_-(D)$ and denote it by $|s|$. A *0-smoothing* and a *1-smoothing* are local moves on a link diagram as in Figure 1. We denote by D_s the loops which are obtained from D by applying s_i-smoothing at c_i $(i = 1, \cdots, n)$ and by $|D_s|$ the number of components of D_s. Let $V = \mathbb{Q}[x]/(x^2-1)$ be a vector space, which is spanned by 1 and x. The object of Lee's complex is defined as follows:

$$C^i_{Lee}(D) = \bigoplus_{s\in\{0,1\}^n :\, |s|=i} V^{\otimes |D_s|} \quad \text{and} \quad C^*_{Lee}(D) = \bigoplus_{i\in\mathbb{Z}} C^i_{Lee}(D).$$

The multiplication $m : V \otimes V \to V$ and the comultiplication $\Delta : V \to V \otimes V$ are defined by

$$\begin{aligned} m(1\otimes 1) = m(x\otimes x) = 1, &\quad \Delta(1) = 1\otimes x + x\otimes 1, \\ m(1\otimes x) = m(x\otimes 1) = x, &\quad \Delta(x) = x\otimes x + 1\otimes 1. \end{aligned}$$

Let $\xi = (\xi_1, \cdots, \xi_i, \cdots, \xi_n)$ be an edge of the n-dimensional cube $[0,1]^n$, that is, an element of $\{0,*,1\}^n$ with just one $*$. Suppose that $\xi_i = *$. Then we define to be $|\xi| = \xi_1 + \cdots + \xi_{i-1}$, $\xi(0) = (\xi_1, \cdots, \xi_{i-1}, 0, \xi_{i+1}, \cdots, \xi_n)$, $\xi(1) = (\xi_1, \cdots, , \xi_{i-1}, 1, \xi_{i+1}, \cdots, \xi_n)$ and $\xi(*) = i$. For example, suppose that $n = 5$ and $\xi = (1, 1, *, 0, 1)$. Then $|\xi| = 2$, $\xi(0) = (1, 1, 0, 0, 1)$, $\xi(1) = (1, 1, 1, 0, 1)$ and $\xi(*) = 3$.

For an edge ξ, we associate the cobordism S_ξ from $D_{\xi(0)}$ to $D_{\xi(1)}$ as follows: we remove a neighborhood of the $\xi(*)$-th crossing, assign a product cobordism, and fill the saddle cobordism between the 0- and 1-smoothings around the $\xi(*)$-th crossing. The cobordism is either of the following two types: (i) two circles of $D_{\xi(0)}$ merge into one circle of $D_{\xi(1)}$, or (ii) one circle of $D_{\xi(0)}$ splits into two circles of $D_{\xi(1)}$. Furthermore, we associate the map $d_\xi \colon V^{\otimes |D_{\xi(0)}|} \longrightarrow V^{\otimes |D_{\xi(1)}|}$ as follows: the homeomorphism d_ξ is induced by the map m if the cobordism S_ξ is of type (i) and by the map Δ if the cobordism S_ξ is of type (ii). Note that we set d_ξ to be the identity on the tensor factors corresponding to the loops that do not participate. For an element $x \in V^{\otimes |D_s|} \subset C^i_{Lee}(D)$, we define d^i as follows,

$$d^i(x) = \sum_{\xi \in \{0,*,1\}^n \colon \xi(0)=s} (-1)^{|\xi|} d_\xi(x),$$

where s is a state for D. Let d be $\bigoplus_{i\in\mathbb{Z}} d^i$. We obtain $d^2 = 0$. The complex $C^*_{Lee}(D) = (C^*_{Lee}(D),\ d)$ is called Lee's complex. The Lee's homology of K, $H^*_{Lee}(K)$, is defined to be the homology group of $C^*_{Lee}(D)$. By the following lemma, $H^*_{Lee}(K)$ does not depend on the choice of diagrams of K.

Lemma 2.1 ([9])**.** *Let D and D' be diagrams of a knot K. Then $C^*_{Lee}(D)$ and $C^*_{Lee}(D')$ are chain homotopic.*

2.2. The basis of Lee's homology of a knot

Lee's homology of a knot is very simple as a vector space. Lee [9] showed that $\dim H^*_{Lee}(K) = 2$ and described a basis of Lee's homology of a knot K. In this subsection, we explain these results.

It is useful to use the basis $\{a, b\}$ for V, where $a = 1 + x$ and $b = 1 - x$[1]. With respect to this basis, we have

$$\begin{aligned} &m(a \otimes a) = 2a,\ m(b \otimes b) = 2b, && \Delta(a) = a \otimes a,\\ &m(a \otimes b) = 0,\ m(b \otimes a) = 0, && \Delta(b) = -b \otimes b. \end{aligned}$$

For a state s for D, we define *col*(D_s) to be the set of coloring maps from the components of D_s to V. Note that an element of $\mathrm{col}(D_s)$ is naturally identified with an element of $V^{\otimes |D_s|} \subset C^{|s|}_{Lee}(D)$. Hereafter we always identify an element of $\mathrm{col}(D_s)$ with the element of $V^{\otimes |D_s|} \subset C^{|s|}_{Lee}(D)$. We call an element of $\mathrm{col}(D_s)$ an *enhanced state*.

Let o be the orientation of D and s_o the state for D corresponding to o, that is, the state whose i-th element is 0 if the sign of c_i is positive and 1 if the sign of c_i is negative. Then, by definition, D_{s_o} are the Seifert circles and $|s_o| = 0$. Let $f_o(D) \in \mathrm{col}(D_{s_o})$ be the

[1] Lee defined to be $a = x + 1$ and $b = x - 1$ and Rasmussen used this notation. Our convention is suitable for our purpose since $a + b = 2$. For example, our convention helps us state Lemma 3.3.

enhanced state whose values of any adjacent Seifert circles are a and b respectively and the outer most right-handed Seifert circle is a and the outer most left-handed Seifert circle is b (see Figure 4). Let $\bar{o}$ be the reversed orientation of D. Then $f_o(D)$ and $f_{\bar{o}}(D)$ are cycles of $C^0_{Lee}(D)$ and we obtain the following.

Theorem 2.2 ([9])**.** *Let K be a knot. Then*

$$H^i_{Lee}(K) = \begin{cases} \mathbb{Q} \oplus \mathbb{Q} & i = 0, \\ 0 & i \neq 0. \end{cases}$$

Furthermore, a basis of $H^0_{Lee}(K)$ consists of $[f_o(D)]$ and $[f_{\bar{o}}(D)]$ for a diagram D of K.

Remark 2.3. The two cycles $f_o(D)$ and $f_{\bar{o}}(D)$ are determined up to multiplication of 2^c, where c is an integer (see [9]). Therefore we call $[f_o(D)]$ and $[f_{\bar{o}}(D)]$ *canonical classes* of $H^*_{Lee}(K)$. In particular, we call $[f_o(D)]$ the canonical class [2] of $H^*_{Lee}(K)$. Here after, we simply denote $f_o(D)$ by f_o and $f_{\bar{o}}(D)$ by $f_{\bar{o}}$, respectively.

3. State cycles which represent canonical classes

Elliott [4] introduced the notion of a state cycle for the Khovanov complex. In this section, we introduce the notion of a state cycle for Lee's complex, which is a certain cycle of $C^0_{Lee}(D)$. We describe some state cycles which represent the same element of Lee's homology of a knot (Lemmas 3.3 and 3.4) and state the main result (Theorem 3.6).

Let D be a diagram of a knot. A Seifert circle for D is *strongly negative* if signs of the adjacent crossings to it are all negative. Figure 2 (the first figure from the right) may help us understand the definition. We define $col_o(D_{s_o})$ to be the set which consists of enhanced states $g \in \mathrm{col}(D_{s_o})$ such that $g(l) = f_o(l)$ for any Seifert circle l which is not strongly negative. Now we prove the following lemma.

Lemma 3.1. *Any enhanced state $g \in col_o(D_{s_o})$ is a cycle of $C^0_{Lee}(D)$ i.e. $d^0(g) = 0$.*

Proof. Recall that

$$d^0(g) = \sum_{\xi \in \{0,*,1\}^n :\, \xi(0) = s_o} (-1)^{|\xi|} d_\xi(g).$$

Let ξ be an edge of the n-dimensional cube $[0,1]^n$ with $\xi(0) = s_o$. Now we prove that $d_\xi(g) = 0$. Let $\alpha_{\xi(*)}$ be the trace of the crossing $c_{\xi(*)}$ of D. Then adjacent Seifert circles to $\alpha_{\xi(*)}$ are not strongly negative since $c_{\xi(*)}$ is positive. Then, by definition, the values of g of adjacent Seifert circles to $\alpha_{\xi(*)}$ are same as that of f_o, that is, a and b, respectively. Therefore $d_\xi(g) = 0$. This implies that $d^0(g) = 0$. □

[2] Note that a canonical class of $H^*_{Lee}(K)$ implies $[f_o(D)]$ or $[f_{\bar{o}}(D)]$ and the canonical class of $H^*_{Lee}(K)$ implies $[f_o(D)]$.

According to Lemma 3.1, we call an enhanced state of $col_o(D_{s_o})$ a *state cycle*. A typical state cycle is f_o. The following example demonstrates three state cycles which represent the same element of Lee's homology.

Example 3.2. Let D be the standard pretzel diagram of $P(3,-3,-3)$ and number the crossings of D from 1 to 9, see the first figure from the left in Figure 2. Figure 2 also illustrates signs of crossings of D, the Seifert circles for D and strongly negative Seifert circles for D.

Recall that f_o is a state cycle, see Figure 4. Let g and $h \in C^{-1}_{Lee}(D)$ be the enhanced states as in Figure 3. Then we can see that $f_o - d^{-1}(g)$ and $f_o - d^{-1}(g) - d^{-1}(h)$ are also state cycles as in Figure 4. Therefore $[f_o]$ has, at least, three representatives $f_o, f_o - d^{-1}(g)$ and $f_o - d^{-1}(g) - d^{-1}(h)$ which are state cycles, and Figure 5 illustrates this fact.

The first equality in Figure 5 is generalized as follows:

Lemma 3.3. *Let* $\overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}$ *be a state cycle whose values of adjacent strongly negative Seifert circles are* a *and* b*. Then*

$$[\ \overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}\ ] = [\ \overset{2}{\bigcirc}\cdots\overset{b}{\bigcirc}\ ] = [\ \overset{a}{\bigcirc}\cdots\overset{2}{\bigcirc}\ ],$$

where $\overset{2}{\bigcirc}\cdots\overset{b}{\bigcirc}$ *and* $\overset{a}{\bigcirc}\cdots\overset{2}{\bigcirc}$ *are the state cycles such that* $\overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}$ *and* $\overset{2}{\bigcirc}\cdots\overset{b}{\bigcirc}$ *differ by a single value of the Seifert circle from* a *to* 2 *and* $\overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}$ *and* $\overset{a}{\bigcirc}\cdots\overset{2}{\bigcirc}$ *differ by a single value of the Seifert circle from* b *to* 2*.*

Proof. Let i be a positive integer such that the trace of the crossing c_i is the dotted arc in $\overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}$. Let ξ be the edge of the n-dimensional cube $[0,1]^n$ such that $\xi(*) = i$ and $\xi(1) = s_o$.

Let $\overset{b}{\bigcirc\!\!\bigcirc} \in \mathrm{col}(D_{\xi(0)})(\subset C^{-1}_{Lee}(D))$ be the enhanced state such that values of $\overset{b}{\bigcirc\!\!\bigcirc}$ and $\overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}$ agree for the Seifert circles which are not the adjacent strongly negative Seifert circles. Then one can see that $d^{-1}(\overset{b}{\bigcirc\!\!\bigcirc}) = d_\xi(\overset{b}{\bigcirc\!\!\bigcirc}) = (-1)^{|\xi|+1}\ \overset{b}{\bigcirc}\cdots\overset{b}{\bigcirc}$. Therefore $[\ \overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}\ ] = [\ \overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc} + (-1)^{|\xi|+1}d^{-1}(\overset{b}{\bigcirc\!\!\bigcirc})\ ] = [\ \overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc} + \overset{b}{\bigcirc}\cdots\overset{b}{\bigcirc}\ ] = [\ \overset{2}{\bigcirc}\cdots\overset{b}{\bigcirc}\ ]$.

Let $\overset{a}{\bigcirc\!\!\bigcirc} \in \mathrm{col}(D_{\xi(0)})(\subset C^{-1}_{Lee}(D))$ be the enhanced state such that values of $\overset{a}{\bigcirc\!\!\bigcirc}$ and $\overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}$ agree for the Seifert circles which are not the adjacent strongly negative Seifert circles. Then one can see that $d^{-1}(\overset{a}{\bigcirc\!\!\bigcirc}) = d_\xi(\overset{a}{\bigcirc\!\!\bigcirc}) = (-1)^{|\xi|}\ \overset{a}{\bigcirc}\cdots\overset{a}{\bigcirc}$. Therefore $[\ \overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}\ ] = [\ \overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc} + (-1)^{|\xi|}d^{-1}(\overset{a}{\bigcirc\!\!\bigcirc})\ ] = [\ \overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc} + \overset{a}{\bigcirc}\cdots\overset{a}{\bigcirc}\ ] = [\ \overset{a}{\bigcirc}\cdots\overset{2}{\bigcirc}\ ]$. $\square$

The second equality in Figure 5 is generalized as follows:

Lemma 3.4. *Let* $\overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}$ *be a state cycle whose values of two adjacent Seifert circles are* a *and* b *such that the left sided Seifert circle in* $\overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}$ *is strongly negative and the right sided Seifert circle in* $\overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}$ *is not strongly negative. Then*

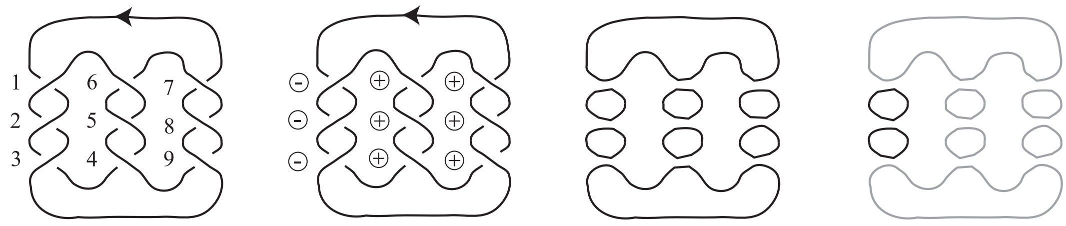

Figure 2: the first figure from the left is the standard pretzel diagram D of $P(3,-3,-3)$ and the second one illustrates signs of crossings of D. The third one denotes the Seifert circles for D and the last one illustrates strongly negative Seifert circles for D. Here, black circles represent strongly negative Seifert circles.

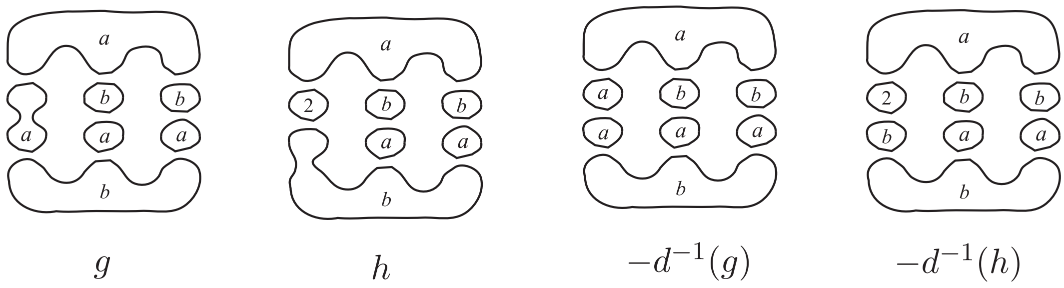

Figure 3: two enhanced state g and $h \in C^{-1}_{Lee}(D)$ and its images $-d^{-1}(g)$ and $-d^{-1}(h) \in C^{0}_{Lee}(D)$

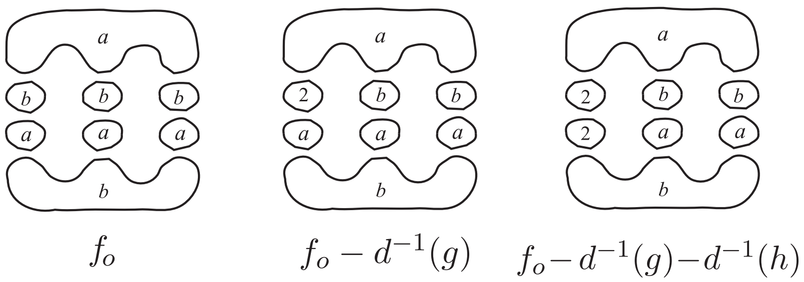

Figure 4: three state cycles which represent the same element

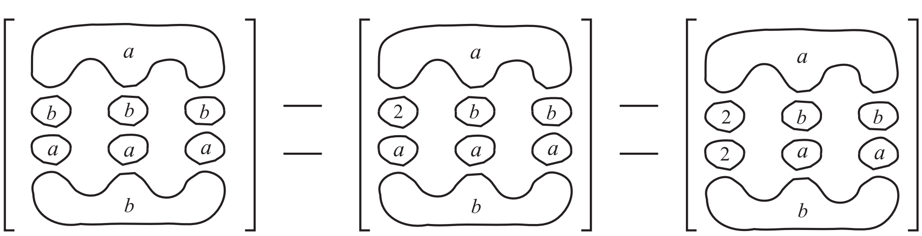

Figure 5: three homology classes are the same.

$$[\ \overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}\ ] = [\ \overset{2}{\bigcirc}\cdots\overset{b}{\bigcirc}\ ],$$

where $\overset{2}{\bigcirc}\cdots\overset{b}{\bigcirc}$ *the state cycle whose values of the left sided Seifert circle in* $\overset{2}{\bigcirc}\cdots\overset{b}{\bigcirc}$ *is* 2 *and whose values of the other Seifert circles coincide with that of* $\overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}$.

Let $\overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}$ *be a state cycle whose values of the adjacent Seifert circles are* a *and* b *such that the left sided Seifert circle in* $\overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}$ *is not strongly negative and the right sided Seifert circle in* $\overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}$ *is strongly negative. Then*

$$[\ \overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}\ ] = [\ \overset{a}{\bigcirc}\cdots\overset{2}{\bigcirc}\ ],$$

where $\overset{a}{\bigcirc}\cdots\overset{2}{\bigcirc} \in \mathrm{col}_o(D_{s_o})$ *the enhanced state whose values of the left sided Seifert circle in* $\overset{a}{\bigcirc}\cdots\overset{2}{\bigcirc}$ *is* 2 *and whose values of the other Seifert circles coincide with that of* $\overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}$.

Proof. We only prove the first half of this theorem. Let i be a positive integer such that the trace of the crossing c_i is the dotted arc in $\overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}$. Let ξ be an edge of the n-dimensional cube $[0,1]^n$ such that $\xi(*) = i$ and $\xi(1) = s_o$. Let $\overset{b}{\bigcirc\!\!\bigcirc} \in \mathrm{col}(D_{\xi(0)})(\subset C^{-1}_{Lee}(D))$ be the enhanced state such that values of $\overset{b}{\bigcirc\!\!\bigcirc}$ and $\overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}$ agree for the Seifert circles which are not the two strongly negative Seifert circles. Then one can easily see that $d^{-1}(\overset{b}{\bigcirc\!\!\bigcirc}) = d_\xi(\overset{b}{\bigcirc\!\!\bigcirc}) = (-1)^{|\xi|+1}\ \overset{b}{\bigcirc}\cdots\overset{b}{\bigcirc}$. Therefore $[\ \overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}\ ] = [\ \overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc} + (-1)^{|\xi|+1} d^{-1}(\overset{b}{\bigcirc\!\!\bigcirc})\ ] = [\ \overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc} + \overset{b}{\bigcirc}\cdots\overset{b}{\bigcirc}\ ] = [\ \overset{2}{\bigcirc}\cdots\overset{b}{\bigcirc}\ ]$. We can prove the later half similarly. $\square$

Let f_2 be the state cycle such that $f_2(l) = 2$ for strongly negative Seifert circles l. Then $[f_2]$ is determined up to multiplication of 2^c, where c is an integer (see Remark 2.3). Example 3.2 implies that $[f_o] = [f_2]$ for the standard pretzel diagram of $P(3,-3,-3)$. Another example is the following.

Example 3.5. Figure 6 illustrates, from the left, a diagram D of $P(1,3,3)$ and signs of crossings of D [3], the Seifert circles for D and strongly negative Seifert circles for D. Then, by Lemmas 3.3 and 3.4, we obtain $[f_o] = [f_2]$. See Figures 7, 8 and 9. In particular, we used Lemma 3.4 to show the second equality in Figure 9

Examples 3.2 and 3.5 give us the idea of a proof of the following theorem.

Theorem 3.6. *Let* D *be a non-negative diagram of a knot. Then* $[f_o] = [f_2]$.

The proof is given in the next section.

4. A graph-theoretical argument

In this section, we prove Theorem 3.6 by a graph-theoretical argument.

[3] We also number the crossings of D from 1 to 9 arbitrarily.

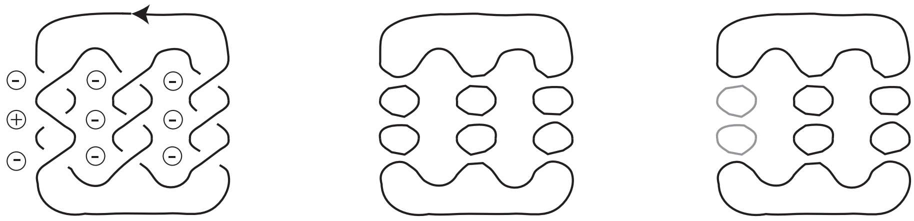

Figure 6: a diagram D of $P(1,3,3)$ and signs of its crossings, the Seifert circles for D and strongly negative Seifert circles for D. Here, black circles represent strongly negative Seifert circles.

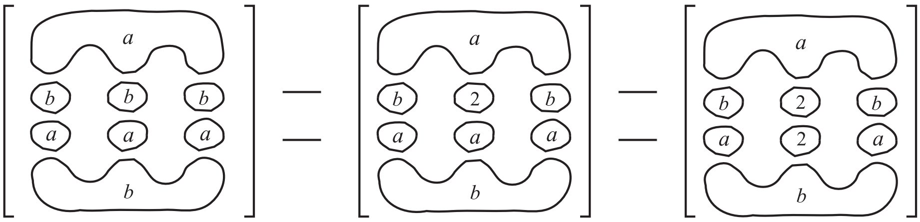

Figure 7: the homology class of f_o

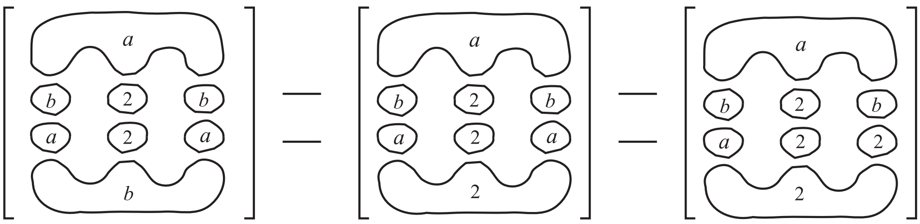

Figure 8: the homology class of f_o

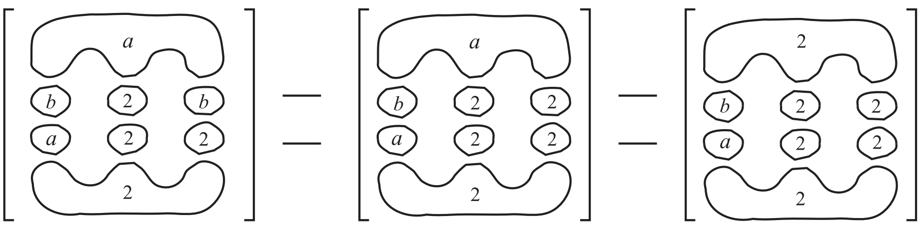

Figure 9: the homology class of f_o

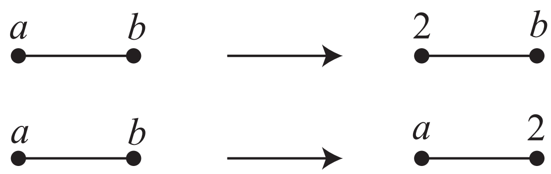

Figure 10: move1s

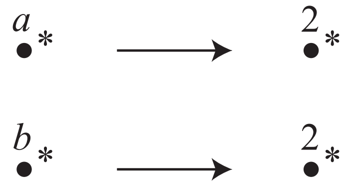

Figure 11: move2s

4.1. Colorings of vertices of a graph

In this subsection, we study colorings of vertices of a graph.

Let G be a graph. We denote by $V(G)$ the vertices of G. We define $col(V(G))$ to be the set of maps $V(G) \longrightarrow V$, where V is the two dimensional vector space which is spanned by a and b. We call an element of $\mathrm{col}(V(G))$ a *coloring* of $V(G)$. For $c \in V$, we associate a map $f_c : V(G) \longrightarrow V$ such that $f_c(v) = c$ for any $v \in V(G)$. By abuse of notation, we simply denote the map f_c by c. Suppose that G is a bipartite graph and (X, Y) the bipartition. A coloring $f \in \mathrm{col}(V(G))$ is *canonical* if $f\big|_X = a$ and $f\big|_Y = b$ or $f\big|_X = b$ and $f\big|_Y = a$.

Now we define a local move on $\mathrm{col}(V(G))$. A *move*1 is a local move on colorings which change one of values of adjacent vertices as in Figure 10.

Lemma 4.1. *For a connected bipartite graph G, let f be a canonical coloring of $V(G)$, v a vertex of G and g_v the coloring of $V(G)$ such that $f(v) = g_v(v)$ and $g_v\big|_{V(G)\setminus v} = 2$. Then f and g_v are related by a sequence of move1s.*

Proof. The proof is reduced to the case where G is a tree by taking its spanning tree of G. Thus we suppose that G is a tree. We prove the lemma by induction on the number n of $V(G)$. If n is one, the lemma is true. We suppose that the lemma is true for $n = N \geq 1$. Suppose that G be a graph such that the number of $V(G)$ is $N + 1$. Then we choose a leaf l of G which is not v. By using a move1 once, we obtain the coloring h such that $h(l) = 2$ (and $h(u) = f(u)$ for the other vertices u). Now we consider the subgraph G' of G such that $V(G') = V(G) \setminus l$ and $E(G') = E(G) \setminus e$, where e is the edge which is incident to l. Note that the number of $V(G')$ is N and $h\big|_{V(G')}$ is a canonical coloring of $V(G')$. Using the assumption of the induction, $h\big|_{V(G')}$ and $g_v\big|_{V(G')}$ are related by a sequence of move1s and we see that f and g_v are related by a sequence of move1s. Therefore this lemma is also true for $n = N + 1$. $\square$

A graph is *marked* if, at least, one of edges of each connected component of the graph is marked. The top lefts in Figures 12 and 13 are examples of marked graphs. Let G be a marked graph. We define another local move on $\mathrm{col}(V(G))$. A *move*2 is a local move on colorings which change a value of a marked vertex as in Figure 11.

Lemma 4.2. *For a marked bipartite graph G, let f be a canonical coloring of $V(G)$ and h the constant coloring of $V(G)$ (i.e. $h = 2$). Then f and h are related by a sequence of move1s and move2s.*

Proof. Suppose that G is connected. Let v be a marked vertex of G and $g_v \in \mathrm{col}(V(G))$ the coloring such that $f(v) = g_v(v)$ and $g_v\big|_{V(G)\setminus v} = 2$. By Lemma 4.1, f and g_v are related by a sequence of move1s. By applying a move2 to v, we obtain h. From the construction, f and h are related by a sequence of move1s and a move2. If G is not connected, we obtain h from f by a sequence of move1s and move2s by the same argument component-wisely. $\square$

4.2. Colorings of vertices of a strongly negative Seifert graph

In this subsection, we introduce a marked graph which is derived from a diagram of a knot, and prove Theorem 3.6.

Let D be a diagram of a knot. The Seifert graph $G(D)$ of D is constructed as follows: for each Seifert circle for D, we associate a vertex of $G(D)$ and two vertices of $G(D)$ are connected by an edge if there is a crossing of D whose adjacent two Seifert circles are corresponding to the two vertices.

Here we suppose that D has a strongly negative Seifert circle. We denote by $O_<(D)$ the number of strongly negative Seifert circles for D. A vertex of $G(D)$ is *strongly negative* if the corresponding Seifert circle is strongly negative. We can associate a marked graph for D : let $G_<(D)$ be the induced graph by the strongly negative vertices. We give a mark to a vertex v of $G_<(D)$ if there exists a non-strongly negative vertex which is adjacent to v. Then $G_<(D)$ is the marked graph. A state cycle of $\mathrm{col}_o(D_{s_o})$ is identified with a coloring of $\mathrm{col}(V(G_<(D)))$(we just consider a strongly negative Seifert circle as a vertex of $G_<(D)$, see Figures 12 and 13). Therefore we obtain the following map.

$$\Phi\colon \mathrm{col}_o(D_{s_o}) \longrightarrow \mathrm{col}(V(G_<(D))).$$

Note that the map Φ is bijective.

Lemma 4.3. *Let D be a diagram of a knot with $O_<(D) > 0$. Let $\overset{a}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{b}{\bullet}$ be a coloring of $V(G(D))$ and $\overset{2}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{b}{\bullet}$ and $\overset{a}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{2}{\bullet}$ the colorings which are obtained from $\overset{a}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{b}{\bullet}$ by a move1, respectively. Then*

$$\left[\Phi^{-1}\left(\overset{a}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{b}{\bullet}\right)\right] = \left[\Phi^{-1}\left(\overset{2}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{b}{\bullet}\right)\right] = \left[\Phi^{-1}\left(\overset{a}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{2}{\bullet}\right)\right].$$

Proof. It is immediately obtained from Lemma 3.3. $\square$

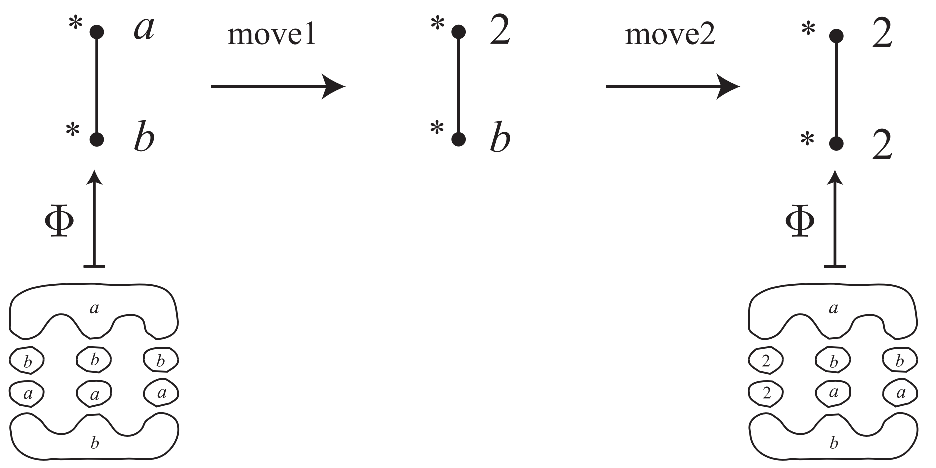

Figure 12: an example of Φ from Example 3.2

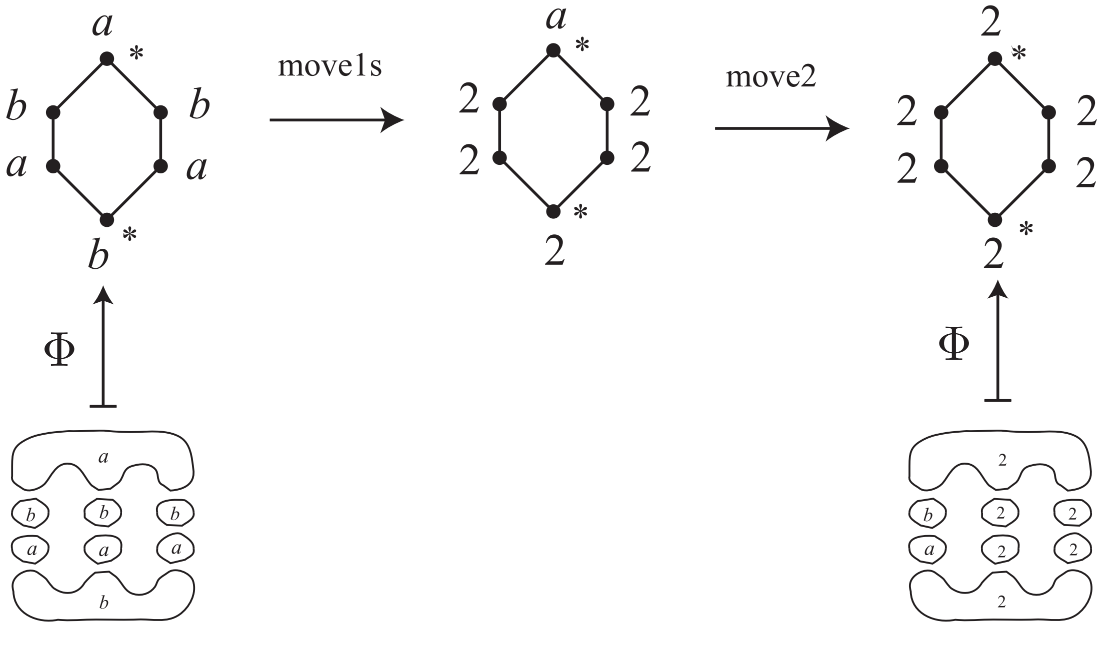

Figure 13: an example of Φ from Example 3.5

Lemma 4.4. *Let D be a diagram of a knot with $O_<(D) > 0$.*
(1) Let $\overset{a}{\bullet}*$ be a coloring which is obtained from $\Phi(f_o)$ by a sequence of move1s and move2s and $\overset{2}{\bullet}*$ the coloring which is obtained from $\overset{a}{\bullet}*$ by a move2. Then

$$\left[\Phi^{-1}\left(\overset{a}{\bullet}*\right)\right] = \left[\Phi^{-1}\left(\overset{2}{\bullet}*\right)\right].$$

(2) Let $\overset{b}{\bullet}*$ be a coloring which is obtained from $\Phi(f_o)$ by a sequence of move1s and move2s and $\overset{2}{\bullet}*$ the coloring which is obtained from $\overset{b}{\bullet}*$ by a move2. Then

$$\left[\Phi^{-1}\left(\overset{b}{\bullet}*\right)\right] = \left[\Phi^{-1}\left(\overset{2}{\bullet}*\right)\right].$$

Proof. (1) Since $\overset{a}{\bullet}*$ is a coloring which is obtained from $\Phi(f_o)$ by a sequence of move1s and move2s, we can denote $\Phi^{-1}\left(\overset{a}{\bullet}*\right)$ by $\overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}$, where the left sided Seifert circle in $\overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}$ is corresponding to $\overset{a}{\bullet}*$ and the left sided Seifert circle in $\overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}$ is not strongly negative. By Lemma 3.4,

$$\left[\Phi^{-1}\left(\overset{a}{\bullet}*\right)\right] = \left[\ \overset{a}{\bigcirc}\cdots\overset{b}{\bigcirc}\ \right] = \left[\ \overset{2}{\bigcirc}\cdots\overset{b}{\bigcirc}\ \right] = \left[\Phi^{-1}\left(\overset{2}{\bullet}*\right)\right].$$

(2) We can prove by the same argument. ☐

Proof of Theorem 3.6. If $O_<(D) = 0$, then $f_o = f_2$. Therefore $[f_o] = [f_2]$. Now we assume that $O_<(D) > 0$. Then we obtain the signed bipartite graph $G_<(D)$ and the map

$$\Phi\colon \mathrm{col}_o(D_{s_o}) \longrightarrow \mathrm{col}(V(G_<(D))).$$

Note that $\Phi(f_o)$ is a canonical coloring of $G_<(D)$ and $\Phi(f_2) = 2$. Here we note that the notion of a canonical coloring is only defined for a bipartite graph.

By Lemma 4.2, $\Phi(f_o)$ and $\Phi(f_2)$ are related by a sequence of move1s and move2s. By Lemmas 4.3 and 4.4, $[f_o] = [f_2]$. ☐

5. Estimations for the Rasmussen invariant of a knot

In this section, we recall the definition of the Rasmussen invariant of a knot and give a new proof of a refinement of the slice-Bennequin inequality for the Rasmussen invariant by Kawamura [7].

For a diagram D of a knot K, a filtration of $C^*_{Lee}(D)$ is defined as follows: We define a grading p on V by setting $p(1) = 1$ and $p(x) = -1$ and extend it to $V^{\otimes n}$ by $p(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \sum_{i=1}^n p(v_i)$. Next we define a filtration grading q for a monomial v of $C^i_{Lee}(D)$ by $q(v) = p(v) + i + \omega(D)$, where $\omega(D)$ is the writhe of D and extend it to a non-zero element v of $C^i_{Lee}(D)$ by $q(v) = \min\{q(v_j) \big| v = \sum_{j=1}^n v_j, \text{where } v_j \text{ is a monomial}\}$. Let

$$\mathcal{F}^i C^*_{Lee}(D) = \{v \in C^*_{Lee}(D) \setminus \{0\} \big| q(v) \geq i\} \cup \{0\}.$$

Then $\{\mathcal{F}^i C^*_{Lee}(D)\}_{i=-\infty}^{\infty}$ is a filtration of $C^*_{Lee}(D)$. Note that one can easily check that $q(v)$ is always odd. Therefore $\mathcal{F}^{2i}C^*_{Lee}(D) = \mathcal{F}^{2i+1}C^*_{Lee}(D)$. Rasmussen showed the following.

Lemma 5.1 ([16]). *Let D and D' be diagrams of a knot. Then $C^*_{Lee}(D)$ and $C^*_{Lee}(D')$ are filtered chain homotopic.*

By this lemma, we can also define a filtration grading q of $H^*_{Lee}(K)$ by

$$q(x) = \max\{q(y) \big| x = [y], y \in C^*_{Lee}(D)\}^4,$$

where $x \in H^*_{Lee}(K) \setminus \{0\}$. Let $\mathcal{F}^i = \{x \in H^*_{Lee}(K) \setminus \{0\} \big| q(x) \geq i\} \cup \{0\}$. Then $\{\mathcal{F}^i\}_{i=-\infty}^{\infty}$ is a filtration of $H^*_{Lee}(K)$. For this filtration, Rasmussen also showed the following.

Theorem 5.2 ([16]). *Let D be a diagram of a knot K. Then*

$$\mathbb{Q} \oplus \mathbb{Q} \simeq H^0_{Lee}(K) = H^*_{Lee}(K) = \cdots = \mathcal{F}^{q_{\min}} \supsetneq \mathcal{F}^{\frac{q_{\max}-q_{\min}}{2}} = \mathcal{F}^{q_{\max}} \supsetneq \mathcal{F}^{q_{\max}+1} = 0, \textit{and}$$

$$q_{\min} = q([f_o]) = q([f_{\bar{o}}]),$$

where $q_{\max} = \max\{q(x) \big| x \in H^*_{Lee}(K), x \neq 0\}$ *and* $q_{\min} = \min\{q(x) \big| x \in H^*_{Lee}(K), x \neq 0\}$.

The *Rasmussen invariant* of a knot K, $s(K)$, is define to be $\dfrac{q_{\max} - q_{\min}}{2}$. Note that $s(K)$ is equal to $q\big([f_o]\big) + 1$ by Theorem 5.2. This implies that, for a diagram D of a knot K, the Rasmussen invariant is completely determined by cycles which are homotopic to f_o, and any cycle which is homotopic to f_o gives a lower bound for $s(K)$.

Inequality (5.1) is the slice-Bennequin inequality for the Rasmussen invariant which was proved by Plamenevskaya [15] and Shumakovitch [17]. For the sake of the reader, we give a proof.

Theorem 5.3 ([15] and [17]). *Let D be a diagram of a knot K. Then*

$$w(D) - O(D) + 1 = q(f_o) \leq s(K), \tag{5.1}$$

where $O(D)$ denotes the number of the Seifert circles for D.

Proof. We can easily check that $q(f_o) = w(D) - O(D)$. By the definition of the Rasmussen invariant, we obtain

$$s(K) = q([f_o]) + 1 \geq q(f_o) + 1 = w(D) - O(D) + 1.$$

□

[4] For convenience, we use the same symbol as the filtration grading q of $C^i_{Lee}(D)$.

In Theorem 5.3, if D is positive, then the equality holds (see [16]). If D is not positive, then it does not always hold the equality and, indeed, there exists many diagrams such that the equality does not hold. However, there is a refinement of the slice-Bennequin inequality for the Rasmussen invariant, the shaper slice-Bennequin inequality for the Rasmussen invariant, which was first proved by Kawamura [7]. We give another proof of the shaper slice-Bennequin inequality for the Rasmussen invariant of a knot. Furthermore, we explicitly give a cycle which gives the lower bound for the shaper slice-Bennequin inequality for the Rasmussen invariant of a knot as follows:

Theorem 5.4. *Let D be a non-negative diagram of a knot K. Then*

$$w(D) - O(D) + 1 + 2O_{<}(D) = q(f_2) + 1 \leq s(K) \tag{5.2}$$

Proof. By Theorem 3.6, $[f_o] = [f_2]$. This implies that $q([f_o]) = q([f_2])$. We can easily check that $q(f_2) = w(D) + (-O(D) + 2O_{<}(D))$. By the definition of the Rasmussen invariant, we obtain

$$s(K) = q([f_o]) + 1 = q([f_2]) + 1 \geq q(f_2) + 1 = w(D) + (-O(D) + 2O_{<}(D)) + 1.$$

This completes the proof. □

Remark 5.5. In Theorem 5.3, if D is negative, then the inequality (5.2) does not hold. This is because $G_{<}(D)$ is not a marked graph. In the next section, we consider the Rasmussen invariant of a negative knot.

6. The Rasmussen invariant of a negative knot

In this section, we study the Rasmussen invariant of a negative knot.

Let D be a negative diagram of a knot K. Then all Seifert circles of D are strongly negative. We choose a Seifert circle l for D. We define f_l to be the state cycle such that $f_l(l) = f_o(l)$ and $f_l(l') = 2$ for a Seifert circle $l' \neq l$ for D. Then we obtain the following.

Lemma 6.1. *Let D be a negative diagram of a knot and l a Seifert circle for D. Then*

$$[f_o] = [f_l].$$

Proof. The coloring $\Phi(f_o)$ is a canonical coloring of $V(G_{<}(D))$, and $\Phi(f_l)$ is the coloring of $V(G_{<}(D))$ such that

$$\Phi(f_o)(v) = \Phi(f_l)(v) \text{ and } \Phi(f_l)\big|_{V(G_{<}(D))\setminus v} = 2,$$

where v is the vertex of $V(G_{<}(D))$ which is corresponding to l. By Lemma 4.2, $\Phi(f_o)$ and $\Phi(f_l)$ are related by a sequence of move1s. By Lemma 4.3, $[f_o] = [f_l]$. □

Rasmussen showed the following.

Theorem 6.2 ([16]). *Let D be a negative diagram of a knot K. Then*

$$s(K) = -c(D) + O(D) - 1.$$

As a corollary, we obtain a cycle which determine the Rasmussen invariant of a knot which has a negative diagram.

Corollary 6.3. *Let D be a negative diagram of a knot K and l a Seifert circle for D. Then*

$$s(K) = q(f_l) + 1.$$

Proof. We can easily check that $q(f_l) = w(D) + (O(D) - 2)$. By the definition of the Rasmussen invariant and Lemma 6.1, we obtain

$$s(K) = q([f_o]) + 1 = q([f_l]) + 1 \geq q(f_l) + 1 = -c(D) + O(D) - 1.$$

By Theorem 6.2, this implies that $s(K) = q(f_l) + 1$. □

Remark 6.4. Let D be a negative diagram of a knot K. The cycle f_l is not uniquely determined for D and, of course, it depends on choice of Seifert circles l for D. Corollary 6.3 implies that there are, at least, $O(D)$ state cycles which determine $s(K)$.

7. Non-state cycles which represent canonical classes

In this section, for the standard pretzel diagram of $P(3, -5, -7)$, we explicitly give a cycle of Lee's complex which determine the Rasmussen invariant of $P(3, -5, -7)$.

Let D be a diagram of a knot. We define $O_+(D)$ to be the number of connected components of the diagram which is obtained from D by smoothing all negative crossings of D. Kawamura [8] and Lobb [13] independently obtained the following estimation for the Rasmussen invariant, which is stronger than the shaper slice-Bennequin inequality for the Rasmussen invariant.

Theorem 7.1 ([8] and [13]). *Let D be a diagram of a knot K. Then*

$$w(D) - O(D) + 1 + 2(O_+(D) - 1) \leq s(K).$$

A *homogeneous* diagram is a generalization of alternating diagrams and positive diagrams. For the definition of a homogeneous diagram, see [1]. In [1], we determine the Rasmussen invariant of a knot which has a homogeneous diagram as follows:

Theorem 7.2 ([1]). *Let D be a homogeneous diagram of a knot K. Then*

$$s(K) = w(D) - O(D) + 1 + 2(O_+(D) - 1).$$

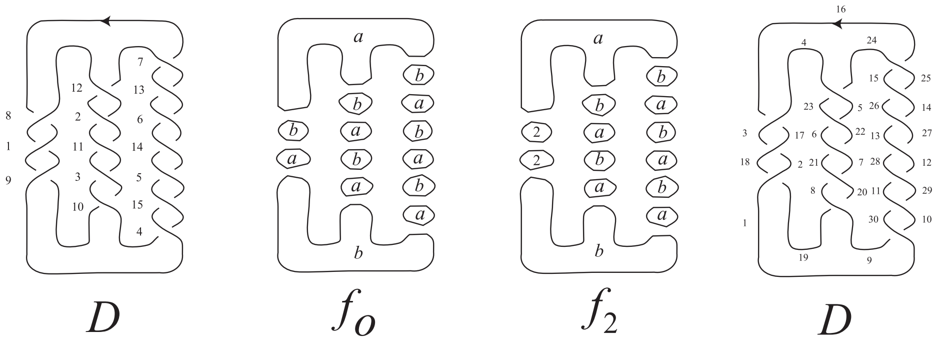

Figure 14: the first figure from the left is the standard pretzel diagram of $P(3,-5,-7)$ and the second and third ones illustrate state cycles f_o and f_2, respectively. The final one illustrates a PD-notation for D.

Therefore we are interested in knots which have no homogeneous diagrams. A typical knot which has no homogeneous diagrams is $P(3,-5,-7)$. Let D be the standard pretzel diagram of $P(3,-5,-7)$ and number the crossings of D from 1 to 15 as in Figure 14. Here we let f_3 be the cycle of $C^0_{Lee}(D)$ as in Figure 15 [5]. We prove that the cycle f_3 determine the Rasmussen invariant of $P(3,-5,-7)$ as follows:

Theorem 7.3. *Let D be the standard pretzel diagram of $P(3,-5,-7)$. Then*

$$s(P(3,-5,-7)) = q(f_3) + 1 = 2.$$

Proof. For a knot K, Rasmussen [16] showed

$$|s(K)| \leq 2g_*(K) \leq 2g(K),$$

where $g_*(K)$ and $g(K)$ denote the 4-ball genus of K and the genus of K, respectively. Since $g(P(3,-5,-7)) = 1$, we obtain that $s(P(3,-5,-7))$ is equal to -2, 0 or 2. On the other hand, it is not too difficult to see that $q(f_3) = 1$. Here, by the definition of f_3 and Theorem 3.6, we have

$$[f_o] = [f_2] = [f_3].$$

Therefore we obtain

$$s(K) = q([f_o]) + 1 = q([f_3]) + 1 \geq q(f_3) + 1 = 2.$$

This implies that $s(P(3,-5,-7)) = q(f_3) + 1 = 2$. $\square$

Remark 7.4. Let D be the standard pretzel diagram of $P(3,-5,-7)$. Then f_o and f_2 are the cycles as in Figures 14 and these cycles give the following estimations

$$-4 = q(f_o) + 1 \leq s(P(3,-5,-7)),$$

[5] The order of crossings for D is derived from the PD-notation for D of the first figure from the right in Figure 15. For more details, see [2].

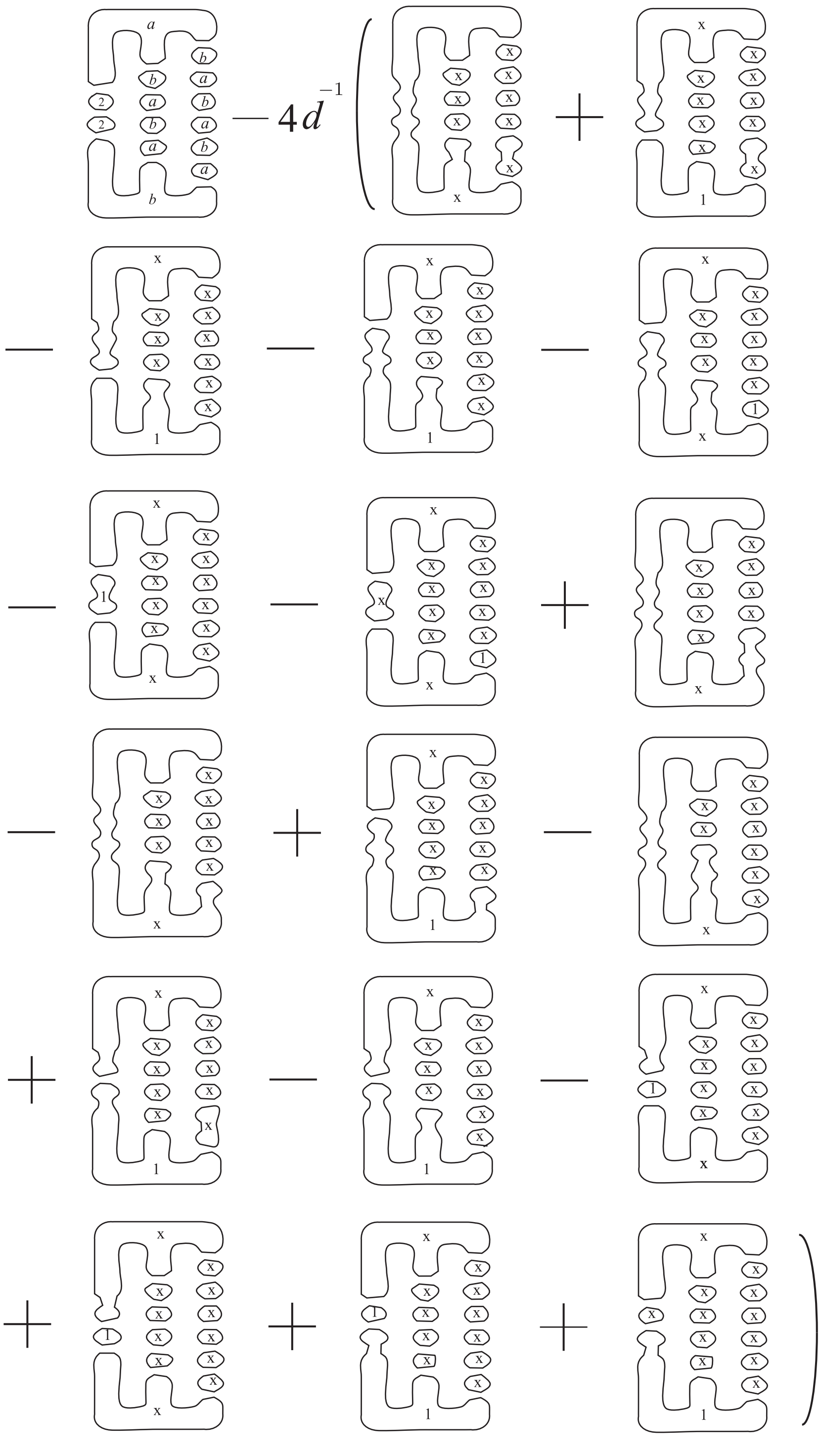

Figure 15: the definition of the cycle f_3

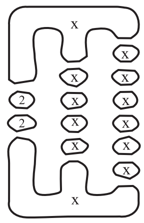

Figure 16: a state cycle which is contained in f_2

$$0 = q(f_2) + 1 \leq s(P(3, -5, -7)).$$

Note that Theorem 7.3 partially solves Problem 6.1 in [14] which was proposed by the author. To find the cycle f_3, we used a Mathematica program, which was slightly modified from Bar-Natan's one. On the other hand, we can check that $q(f_3) = 1$ by hand.

Here we briefly explain why $q(f_3) = 1$. We can see that $f_2 = g + h_1$, where g is the state cycle as in Figure 16 and h_1 is some element of $C^0_{Lee}(D)$ with $q(h_1) \geq 1$. Note that $q(g) = -1$. Let $f_3 = f_2 - 4d^{-1}(h_2)$, where h_2 is the sum of the 17 enhanced states in Figure 15. Then we can check that $4d^{-1}(h_2) = g + h_3$, where h_3 is some element of $C^0_{Lee}(D)$ with $q(h_3) \geq 1$. Therefore $q(f_3) = 1$.

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