Generalized string equations for double Hurwitz numbers

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Abstract

The generating function of double Hurwitz numbers is known to become a tau function of the Toda hierarchy. The associated Lax and Orlov-Schulman operators turn out to satisfy a set of generalized string equations. These generalized string equations resemble those of $c = 1$ string theory except that the Orlov-Schulman operators are contained therein in an exponentiated form. These equations are derived from a set of intertwining relations for fermion bilinears in a two-dimensional free fermion system. The intertwiner is constructed from a fermionic counterpart of the cut-and-join operator. A classical limit of these generalized string equations is also obtained. The so called Lambert curve emerges in a specialization of its solution. This seems to be another way to derive the spectral curve of the random matrix approach to Hurwitz numbers.

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1 Introduction

Hurwitz numbers count the topological types of finite ramified coverings of a given Riemann surface [1]. Some ten years ago, Hurwitz numbers of coverings of the Riemann sphere $\mathbf{CP}^1$ turned out to be related to Hodge integrals on

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the Deligne-Mumford moduli space $\bar{\mathcal{M}}_{g,n}$ of marked stable curves [2, 3, 4] and Gromov-Witten invariants of $\mathbf{CP}^1$ [5, 6, 7, 8]. These observations led to various developments and applications, such as new proofs [11, 12, 13] of Witten's conjecture [9] (Kontsevich's theorem [10]) on two-dimensional topological gravity, in other words, integrals of ψ-classes on $\bar{\mathcal{M}}_{g,n}$. Recently, a set of new recursion relations for Hurwitz numbers were derived [14, 15, 16] as an analogue of Eynard and Orantin's "topological recursion relations" [17].

These developments in the last decade are also more or less connected with integrable hierarchies of the KdV, KP and Toda type (see reviews by the Kyoto group [18, 19, 20] for basic knowledge on those integrable systems). Firstly, several Toda-like equations show up in relation with Gromov-Witten invariants of $\mathbf{CP}^1$. Secondly, the Witten conjecture is formulated in the language of the KdV hierarchy and its Virasoro symmetries. Subsequent studies [21, 22, 23] on the Witten conjecture from the Hurwitz side revealed a connection with the KP hierarchy as well.

A master integrable system in this sense is the (two-dimensional) Toda hierarchy [24]. As pointed out by Okounkov [6], a generating function of "double Hurwitz numbers" gives a special solution of the Toda hierarchy. Being a tau function of the Toda hierarchy, this generating function has two sets of independent variables $\boldsymbol{t} = (t_1, t_2, \cdots)$ and $\bar{\boldsymbol{t}} = (\bar{t}_1, \bar{t}_2, \cdots)$. By specializing part of these variables to particular values, the generating function of "simple Hurwitz numbers" are recovered. The aforementioned Toda-like equations and KP hierarchy may be thought of as equations satisfied by those specialized tau functions.

The goal of this paper is to derive the *generalized string equations*

$$L = Qe^{-\beta/2}\bar{L}e^{\beta\bar{M}}, \quad \bar{L}^{-1} = Qe^{\beta/2}L^{-1}e^{\beta M}$$

for this special solution of the Toda hierarchy, and to examine implications thereof. Here β and Q are parameters prepared along with $\boldsymbol{t}$ and $\bar{\boldsymbol{t}}$. $L, \bar{L}$ and $M, \bar{M}$ are, respectively, the Lax and Orlov-Schulman operators, all being one-dimensional difference operators. Compared with the cases of two-dimensional quantum gravity [25] and $c = 1$ string theory [26, 27, 28, 29, 30, 31], these generalized string equations have a notable new feature. Namely, they contain the exponentials $e^{\beta M}$ and $e^{\beta\bar{M}}$ of the Orlov-Schulman operators. This is in sharp contrast with the conventional string equations that are linear in M and $\bar{M}$ (except for the the case of a deformed version of $c = 1$ string theory [30], in which the string equations are quadratic in M and $\bar{M}$).

The exponentials of M and $\bar{M}$ are related to the "quantum torus algebra" in a two-dimensional free fermion system. The same Lie algebraic structure plays an important role in the melting crystal model of five-dimensional gauge theory as well [32, 33]. We borrow some technical ideas developed therein. The generalized string equations are derived from algebraic relations (referred to as "intertwining relations") among fermion bilinears. A clue therein is a special fermion bilinear that corresponds to the so called "cut-and-join operator" [34].

The generalized string equations also have a *classical limit* of the form

$$\mathcal{L} = Q\bar{\mathcal{L}}e^{\beta\bar{\mathcal{M}}}, \quad \bar{\mathcal{L}}^{-1} = Q\mathcal{L}^{-1}e^{\beta\mathcal{M}},$$

where $\mathcal{L}, \mathcal{M}, \bar{\mathcal{L}}, \bar{\mathcal{M}}$ are "long-wave" or "dispersionless" limits [20] of the Lax and Orlov-Schulman operators. Conceptually, the classical limit amounts to the genus-zero part of Hurwitz numbers. Similarity with $c = 1$ string theory becomes even more obvious in this limit (which amounts to the genus-0 part of string amplitudes). The generalized string equations of $c = 1$ string theory [26, 27, 28, 29] have a classical limit of the form

$$\mathcal{L} = \bar{\mathcal{L}}\bar{\mathcal{M}}, \quad \bar{\mathcal{L}}^{-1} = \mathcal{L}^{-1}\mathcal{M}.$$

Thus, roughly speaking, the linear terms $\mathcal{M}, \bar{\mathcal{M}}$ therein are now replaced by the exponential terms $e^{\beta\mathcal{M}}, e^{\beta\bar{\mathcal{M}}}$. Thanks to this similarity, one can apply the method developed for solving the generalized string equations of $c = 1$ string theory [29] to construct a solution of the foregoing equations in the form of power series of t_k's and $\bar{t}_k$'s. This construction simplifies when either t_k's or $\bar{t}_k$'s are specialized to particular values for which the tau function reduces to the generating function of simple Hurwitz numbers. Remarkably, we encounter therein the equation

$$x = ye^y$$

of the "Lambert curve" that lies in the heart of the new recursion relations [14, 15, 16]. This should not be a coincidence. Presumably, the classical limit of the generalized string equations will be another way to derive the "spectral curve" in the random matrix approach [15, 35, 36].

This paper is organized as follows. Section 2 reviews the notion of Hurwitz numbers and their generating functions. Schur functions and their special values are used to interpret the generating functions as tau functions. Section 3 prepares technical tools from a two-dimensional free fermion system. We introduce relevant fermion bilinears, and recall Okounkov's result on a fermionic representation of the generating function of double Hurwitz numbers [6]. Section 4 presents the generalized string equations. We start from intertwining relations of fermion bilinears, and translate those relations to the generalized string equations. Section 5 formulates the classical limit of the generalized string equations. The classical limit is first derived by heuristic consideration, then justified by showing that the associated tau function has an $\hbar$-expansion. Section 6 is devoted to solving this classical limit of the generalized string equations. The solution is constructed in much the same way as that of the generalized string equations for $c = 1$ string theory. The Lambert curve shows up in a specialization of this solution.

2 Generating functions of Hurwitz numbers

In this section, we use various notions and formulae on partitions, Young diagrams and Schur functions. They are mostly borrowed from Macdonald's book [37].

2.1 Hurwitz numbers of Riemann sphere

Let us consider finite ramified coverings of $\mathbf{CP}^1$. Two coverings $\pi : \Gamma \to \mathbf{CP}^1$ and $\pi' : \Gamma' \to \mathbf{CP}^1$ are said to be topologically equivalent if π and $\pi' s$ are connected by a homeomorphism $\phi : \Gamma \to \Gamma'$ as $\pi = \pi' \circ \phi$. Let $[\pi]$ denote the equivalence class of the covering π. Note that Γ can be a disconnected surface.

Given a positive integer d and a set of distinct points $P_1, \cdots, P_r$ of $\mathbf{CP}^1$, there are only a finite number of nonequivalent d-fold coverings that are unramified over all points other than $P_1, \cdots, P_r$. Those nonequivalent coverings can be further classified by the ramification data of sheets over $P_1, \cdots, P_r$. These data are given by conjugacy classes $C_1, \cdots, C_r$ of the d-th symmetric group S_d.

A conjugacy class C of S_d is determined by the cycle type

$$\mu = (\mu_1, \cdots, \mu_l), \quad \mu_1 \geq \cdots \geq \mu_l, \quad |\mu| = \mu_1 + \cdots + \mu_l = d$$

of a representative $\sigma \in S_d$ of C, e.g.,

$$\sigma = (1, \cdots, \mu_1)(\mu_1 + 1, \cdots, \mu_1 + \mu_2) \cdots (\mu_1 + \cdots + \mu_{l-1}, \cdots, d),$$

where $(j_1, \cdots, j_m)$ denotes the cyclic permutation sending $j_1 \to j_2 \to \cdots \to j_m \to j_1$. The cycle type thus becomes a partition of d, and has another expression

$$\mu = (1^{m_1} 2^{m_2} \cdots)$$

with the numbers m_i of i-cycles. Let $C(\mu)$ denote the conjugacy class determined by a partition μ of d.

A cyclic permutation $(j_1, \cdots, j_m)$ of length m represents a cyclic covering of degree m realized., e.g., by the Riemann sheets of $(z-a)^{1/m}$ above the point a. We use the cycle type $\mu = (\mu_1, \cdots, \mu_l)$ as ramification data above a point of $\mathbf{CP}^1$ to show that the sheets above that point locally look like a disjoint union of l cyclic coverings of degree $\mu_1, \mu_2, \cdots, \mu_l$.

The Hurwitz number $H_d(C_1, \cdots, C_r)$, also denoted by $H_d(\mu^{(1)}, \cdots, \mu^{(r)})$ where $\mu^{(k)}$'s are the cycle types of C_k's, is defined to be the sum

$$H_d(C_1, \cdots, C_r) = \sum_{[\pi]} \frac{1}{|\mathrm{Aut}(\pi)|}$$

of the weights $1/|\mathrm{Aut}(\pi)|$ over all equivalent classes $[\pi]$ of *possibly disconnected* coverings π that are ramified over the r points $P_1, \cdots, P_r$ with the ramification data $C_1, \cdots, C_r$. $\mathrm{Aut}(\pi)$ denotes the group of automorphisms of π. As the notation suggests, $H_d(C_1, \cdots, C_r)$ does not depend on the position of $P_1, \cdots, P_r$.

The Hurwitz numbers can be determined by a genuinely group theoretical method (Burnside's theorem [38]). This leads to the beautiful formula

$$H_d(\mu^{(1)}, \cdots, \mu^{(r)}) = \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!}\right)^2 \prod_{k=1}^{r} f_\lambda(\mu^{(k)}), \tag{1}$$

where the sum is over all partitions λ of d. Let us explain the notations used in this formula.

Firsly, $\dim \lambda$ denotes the dimension $\dim V_\lambda$ of the irreducible representation $(\rho_\lambda, V_\lambda)$ of S_d determined by λ. In other words, $\dim \lambda$ is the number of all standard tableaux of shape λ, and can be calculated by the so called hook length formula

$$\dim \lambda = \frac{d!}{\prod_{(i,j)\in\lambda} h(i,j)}, \tag{2}$$

where $h(i,j)$ denotes the length of the hook cornered at the cell $(i,j) \in \lambda$. Note that partitions are identified with the associated Young diagrams. As an immediate consequence, $\dim \lambda$ turns out to be symmetric under the transpose (or conjugate) $\lambda \mapsto {}^{\mathrm{t}}\lambda$ of the partition, namely,

$$\dim {}^{\mathrm{t}}\lambda = \dim \lambda. \tag{3}$$

Secondly, $f_\lambda(\mu)$ denotes the value $f_\lambda(C(\mu))$ of the class function

$$f_\lambda(C) = \frac{\chi_\lambda(C)}{\dim \lambda}|C| \tag{4}$$

on S_d evaluated at $C = C(\mu)$. χ_λ denotes the character $\mathrm{Tr}_{V_\lambda}\,\rho_\lambda$. The cardinality $|C(\mu)|$ of $C(\mu) \subset S_d$ can be written as

$$|C(\mu)| = \frac{d!}{z_\mu}, \quad z_\mu = \prod_{i \geq 1} m_i! i^{m_i}. \tag{5}$$

The values of $f_\lambda(C)$ for the simplest two cases $C = (1^d), (1^{d-2}2)$ can be explicitly calculated as

$$f_\lambda(1^d) = 1, \quad f_\lambda(1^{d-2}2) = \frac{\kappa_\lambda}{2} \tag{6}$$

where

$$\kappa_\lambda = \sum_{i=1}^{l} \lambda_i(\lambda_i - 2i + 1) = \sum_{i=1}^{l} \left(\left(\lambda_i - i + \frac{1}{2} \right)^2 - \left(-i + \frac{1}{2} \right)^2 \right).$$

This number has another useful expression of the form

$$\kappa_\lambda = 2 \sum_{(i,j) \in \lambda} (i - j), \tag{7}$$

which implies, e.g., the anti-symmetric property

$$\kappa_{{}^t\lambda} = -\kappa_\lambda. \tag{8}$$

2.2 Simple Hurwitz numbers

Let us review the notion of simple Hurwitz numbers. They are the Hurwitz numbers such that the types of all but one ramification points $P_1, \cdots, P_r$ are restricted to $1^{d-2}2$. The exceptional ramification point P_{r+1} can have an arbitrary cycle type μ. The Hurwitz numbers of this type

$$H_d(\underbrace{1^{d-2}2, \cdots, 1^{d-2}2}_{r}, \mu) = \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!} \right)^2 \left(\frac{\kappa_\lambda}{2} \right)^r f_\lambda(\mu) \tag{9}$$

are called simple Hurwitz numbers. Introducing an extra (finite or infinite) set of variables $\boldsymbol{x} = (x_1, x_2, \cdots)$, one can construct a generating function of these numbers as

$$Z(\boldsymbol{x}) = \sum_{r=0}^{\infty} \sum_{d=0}^{\infty} \sum_{|\mu|=d} H_d(\underbrace{1^{d-2}2, \cdots, 1^{d-2}2}_{r}, \mu) \frac{\beta^r}{r!} Q^d p_\mu,$$

where p_μ's are the monomials

$$p_\mu = p_{\mu_1} p_{\mu_2} \cdots$$

of the power sums

$$p_k = \sum_{i \geq 1} x_i^k, \quad k = 1, 2, \cdots .$$

We now substitute (9) in the definition of $Z(\boldsymbol{x})$ and use the Frobenius formula

$$\sum_{|\mu|=d} \frac{\chi_\lambda(C(\mu))}{z_\mu} p_\mu = s_\lambda(\boldsymbol{x}) \tag{10}$$

to rewrite the sum over μ into a Schur function times numerical factors. The numerical factors partly cancel with some other factors. The generating function thus reduces to

$$Z(\boldsymbol{x}) = \sum_{\lambda \in \mathcal{P}} \frac{\dim \lambda}{|\lambda|!} e^{\beta \kappa_\lambda / 2} Q^{|\lambda|} s_\lambda(\boldsymbol{x}). \tag{11}$$

$Z(\boldsymbol{x})$ turns out to be a tau function of the KP hierarchy [18, 19]. To see this, we change the variables from $\boldsymbol{x}$ to the standard time variables $\boldsymbol{t} = (t_1, t_2, \cdots)$ of the KP hierarchy via the power sums as

$$t_k = \frac{p_k}{k} = \frac{1}{k} \sum_{i \geq 1} x_i^k$$

and consider the Schur functions $s_\lambda(\boldsymbol{x})$ as functions of $\boldsymbol{t}$. It is convenient to use Zinn-Justin's notation $s_\lambda[\boldsymbol{t}]$ for the latter [39].

Actually, $s_\lambda[\boldsymbol{t}]$ can be redefined directly. Let us recall the Jacobi-Trudi formula

$$s_\lambda(\boldsymbol{x}) = \det(h_{\lambda_i - i + j}(\boldsymbol{x}))_{i,j=1}^n, \tag{12}$$

where λ_i's are parts of λ (which are assumed to be equal to 0 for $i > n$, namely, $\lambda = (\lambda_1, \cdots, \lambda_n, 0, \cdots)$), and $h_m(\boldsymbol{x})$'s are the complete symmetric functions defined by the generating function

$$\sum_{m=0}^{\infty} h_m(\boldsymbol{x}) z^m = \prod_{i \geq 1} (1 - x_i z)^{-1}.$$

Let $h_m[\boldsymbol{t}]$ denotes $h_m(\boldsymbol{x})$ considered as a function of $\boldsymbol{t}$. $h_m[\boldsymbol{t}]$'s can be redefined by the generating function

$$\sum_{m=0}^{\infty} h_m[\boldsymbol{t}] z^m = \exp\left(\sum_{k=1}^{\infty} t_k z^k\right).$$

Consequently, $s_\lambda[\boldsymbol{t}]$ can be expressed as

$$s_\lambda[\boldsymbol{t}] = \det(h_{\lambda_i - i + j}[\boldsymbol{t}])_{i,j=1}^{n}. \tag{13}$$

One can thus replace $s_\lambda(\boldsymbol{x})$ by $s_\lambda[\boldsymbol{t}]$ to obtain the generating function

$$Z[\boldsymbol{t}] = \sum_{\lambda\in\mathcal{P}} \frac{\dim\lambda}{|\lambda|!} e^{\beta\kappa_\lambda/2} Q^{|\lambda|} s_\lambda[\boldsymbol{t}]. \tag{14}$$

Identifying $Z[\boldsymbol{t}]$ as a KP tau function requires some more consideration. Several methods are known in the literature [12, 21, 22, 23]. One can explain this fact in the context of the Toda hierarchy as well. A clue is the formula

$$\frac{\dim\lambda}{|\lambda|!} = s_\lambda[1, 0, 0, \cdots] \tag{15}$$

that can can be derived, e.g., from the Frobenius formula (10) by letting $p_1 = 1$ and $p_k = 0$ for $k > 1$. Consequently, $Z[\boldsymbol{t}]$ can be rewritten as

$$Z[\boldsymbol{t}] = \sum_{\lambda\in\mathcal{P}} e^{\beta\kappa_\lambda/2} Q^{|\lambda|} s_\lambda[\boldsymbol{t}] s_\lambda[1, 0, \cdots]. \tag{16}$$

This function is a specialization of the generating function $Z[\boldsymbol{t}, \bar{\boldsymbol{t}}]$ of double Hurwitz numbers introduced below. $Z[\boldsymbol{t}, \bar{\boldsymbol{t}}]$ is a tau function of the Toda hierarchy (or, rather, the two-component KP hierarchy[18], because the lattice coordinate of the Toda lattice is absent here). It is well known [24] that any Toda (or two-component KP) tau function is also a tau function of the KP hierarchy with respect to one of the two sets of variables. This implies that $Z[\boldsymbol{t}]$ is a KP tau function.

2.3 Double Hurwitz numbers

Let us choose yet another point P_0 of $\mathbf{CP}^1$ of an arbitrary ramification type $\bar{\mu}$ in addition to the $r+1$ points in the case of simple Hurwitz numbers. The Hurwitz numbers of this type

$$H_d(\bar{\mu}, \underbrace{1^{d-2}2, \cdots, 1^{d-2}2}_{r}, \mu) = \sum_{|\lambda|=d} \left(\frac{\dim\lambda}{d!}\right)^2 \left(\frac{\kappa_\lambda}{2}\right)^r f_\lambda(\mu) f_\lambda(\bar{\mu}) \tag{17}$$

are called double Hurwitz numbers. To construct a generating function of these numbers, we introduce a new set of variables $\bar{\boldsymbol{x}} = (\bar{x}_1, \bar{x}_2, \cdots)$, their power sums

$$\bar{p}_k = \sum_{i\geq 1} \bar{x}_i^k$$

and their monomials

$$\bar{p}_\lambda = \bar{p}_{\lambda_1}\bar{p}_{\lambda_2}\cdots$$

along with the variables in the case of simple Hurwitz numbers. Again, with the aid of the Frobenius formula (10), the generating function

$$Z(\boldsymbol{x}, \bar{\boldsymbol{x}}) = \sum_{r=0}^{\infty}\sum_{d=0}^{\infty}\sum_{|\mu|=|\bar{\mu}|=d} H_d(\bar{\mu}, \underbrace{1^{d-2}2, \cdots, 1^{d-2}2}_{r}, \mu)\frac{\beta^r}{r!}Q^d p_\mu \bar{p}_\mu$$

can be converted to a sum over all partitions as

$$Z(\boldsymbol{x}, \bar{\boldsymbol{x}}) = \sum_{\lambda\in\mathcal{P}} e^{\beta\kappa_\lambda/2}Q^{|\lambda|}s_\lambda(\boldsymbol{x})s_\lambda(\bar{\boldsymbol{x}}). \tag{18}$$

We now introduce the two sets $\boldsymbol{t} = (t_1, t_2, \cdots)$ and $\bar{\boldsymbol{t}} = (\bar{t}_1, \bar{t}_2, \cdots)$ of time variables as

$$t_k = \frac{p_k}{k}, \quad \bar{t}_k = -\frac{\bar{p}_k}{k}$$

and consider the generating function

$$Z[\boldsymbol{t}, \bar{\boldsymbol{t}}] = \sum_{\lambda\in\mathcal{P}} e^{\beta\kappa_\lambda/2}Q^{|\lambda|}s_\lambda[\boldsymbol{t}]s_\lambda[-\bar{\boldsymbol{t}}] \tag{19}$$

of the new variables. $Z[\boldsymbol{t}, \bar{\boldsymbol{t}}]$ is a tau function of the Toda hierarchy at a point of the lattice [6] [1]. Reversing the sign of $\bar{\boldsymbol{t}}$ is conventional in the formulation of the Toda hierarchy [24] (cf. the fermionic representation of Toda tau functions reviewed in Section 3.4). Any Toda tau function thereby becomes a tau function of the two-component KP hierarchy. Let us note that the the roles of $\boldsymbol{t}$ and $\bar{\boldsymbol{t}}$ in $Z[\boldsymbol{t}, \bar{\boldsymbol{t}}]$ can be interchanged by virtue of the identities

$$s_\lambda[\boldsymbol{t}] = (-1)^{|\lambda|}s_{{}^t\lambda}[-\boldsymbol{t}], \quad s_\lambda[-\bar{\boldsymbol{t}}] = (-1)^{|\lambda|}s_{{}^t\lambda}[\bar{\boldsymbol{t}}] \tag{20}$$

of the Schur functions and the property (8) of κ_λ.

[1] A generalization of this tau function was first studied by Kharchev et al. [40] in a different context.

2.4 Cut-and-join operator

The cut-and-join operator M_0 [34] may be thought of as an infinitesimal symmetry on the space of tau functions of the KP hierarchy [12, 22, 23]. In the KP time variables $\boldsymbol{t}$, the cut-and-join operator reads

$$M_0 = \frac{1}{2}\sum_{j,k=1}^{\infty}\left(klt_k t_l \frac{\partial}{\partial t_{k+l}} + (k+l)t_{k+l}\frac{\partial^2}{\partial t_k \partial t_l}\right). \tag{21}$$

The Schur functions $s_\lambda[\boldsymbol{t}]$ are eigenfunctions of this operator with eigenvalues $\kappa_\lambda/2$:

$$M_0 s_\lambda[\boldsymbol{t}] = \frac{\kappa_\lambda}{2} s_\lambda[\boldsymbol{t}] \tag{22}$$

A combinatorial proof of this fact is presented in Zhou's paper [41]. As we shall remark in Section 3, the cut-and-join operator has a fermionic counterpart [6], which leads to another proof of (22).

(22) implies the identities

$$e^{\beta\kappa_\lambda/2} s_\lambda[\boldsymbol{t}] = e^{\beta M_0} s_\lambda[\boldsymbol{t}].$$

Therefore one can use $e^{\beta M_0}$ to recover $Z[\boldsymbol{t}]$ and $Z[\boldsymbol{t},\bar{\boldsymbol{t}}]$ from their "initial values" at $\beta = 0$ as

$$Z[\boldsymbol{t}] = e^{\beta M_0} Z[\boldsymbol{t}]|_{\beta=0}, \quad Z[\boldsymbol{t},\bar{\boldsymbol{t}}] = e^{\beta M_0} Z[\boldsymbol{t},\bar{\boldsymbol{t}}]|_{\beta=0}.$$

$Z[\boldsymbol{t}]_{\beta=0}$ and $Z[\boldsymbol{t},\bar{\boldsymbol{t}}]|_{\beta=0}$ can be calculated by the Cauchy identity

$$\sum_{\lambda\in\mathcal{P}} s_\lambda[\boldsymbol{t}] s_\lambda[-\bar{\boldsymbol{t}}] = \exp\left(-\sum_{k=1}^{\infty} k t_k \bar{t}_k\right) \tag{23}$$

and the weighted homogeneity

$$s_\lambda[ct_1, c^2 t_2, \cdots] = c^{|\lambda|} s_\lambda[t_1, t_2, \cdots] \tag{24}$$

of Schur functions as

$$Z[\boldsymbol{t}]|_{\beta=0} = \sum_{\lambda\in\mathcal{P}} Q^{|\lambda|} s_\lambda[\boldsymbol{t}] s_\lambda[1,0,0,\cdots] = e^{Qt_1}$$

and

$$Z[\boldsymbol{t},\bar{\boldsymbol{t}}]|_{\beta=0} = \sum_{\lambda\in\mathcal{P}} Q^{|\lambda|} s_\lambda[\boldsymbol{t}] s_\lambda[-\bar{\boldsymbol{t}}] = \exp\left(-\sum_{k=1}^{\infty} Q^k k t_k \bar{t}_k\right).$$

One can thus derive the well known formula [12, 22, 23]

$$Z[\boldsymbol{t}] = e^{\beta M_0} e^{Q t_1} \tag{25}$$

and its extension

$$Z[\boldsymbol{t}, \bar{\boldsymbol{t}}] = e^{\beta M_0} \exp\left(-\sum_{k=1}^{\infty} Q^k k t_k \bar{t}_k \right) \tag{26}$$

to double Hurwitz numbers.

3 Fermionic representation of tau function

3.1 Two-dimensional free fermion system

Let us introduce two-dimensional complex free fermion fields

$$\psi(z) = \sum_{n\in\mathbf{Z}} \psi_n z^{-n-1}, \quad \psi^*(z) = \sum_{n\in\mathbf{Z}} \psi_n^* z^{-n}.$$

Note that we follow the notations of our previous work [32, 33] to use integers rather than half-integers for the labels of Fourier modes ψ_n, ψ_n^*. The Fourier modes satisfy the anti-commutation relations

$$\psi_m \psi_n^* + \psi_n^* \psi_m = \delta_{m+n,0}, \quad \psi_m \psi_n + \psi_n \psi_m = 0, \quad \psi_m^* \psi_n^* + \psi_n^* \psi_m^* = 0.$$

The Fock space $\mathcal{H}$ of ket vectors and its dual space $\mathcal{H}^*$ of bra vectors are decomposed to charge-s sectors $\mathcal{H}_s, \mathcal{H}_s^*$, $s \in \mathbf{Z}$. Let $\langle s|$ and $|s\rangle$ denote the ground states in $\mathcal{H}_s$ and $\mathcal{H}_s^*$, namely,

$$\langle s| = \langle -\infty| \cdots \psi_{s-1}^* \psi_s^*, \quad |s\rangle = \psi_{-s}\psi_{-s+1} \cdots | -\infty\rangle,$$

which satisfy the annihilation conditions

$$\begin{aligned} &\psi_n|s\rangle = 0 \quad \text{for } n \ge -s, \quad \psi_n^*|s\rangle = 0 \quad \text{for } n \ge s+1, \\ &\langle s|\psi_n = 0 \quad \text{for } n \le -s-1, \quad \langle s|\psi_n^* = 0 \quad \text{for } n \le s. \end{aligned}$$

Excited states can be labelled by partitions $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n, 0, 0, \cdots)$ of arbitrary length as

$$\begin{aligned} |\lambda, s\rangle &= \psi_{-\lambda_1 - s} \cdots \psi_{-\lambda_n - s + n - 1} \psi_{s-n+1}^* \cdots \psi_s^* |s\rangle, \\ \langle \lambda, s| &= \langle s| \psi_{-s} \cdots \psi_{-s+n-1} \psi_{\lambda_n + s - n + 1}^* \cdots \psi_{\lambda_1 + s}^*. \end{aligned}$$

$|\lambda, s\rangle$ and $\langle\lambda, s|$ represent a state in which the semi-infinite subset $\{\lambda_i + s - i + 1\}_{i=1}^{\infty}$ (sometimes referred to as the "Maya diagram") of the set $\mathbf{Z}$ of all "energy levels" are occupied by particles. These vectors give dual bases of of $\mathcal{H}_s$ and $\mathcal{H}_s^*$ in the sense that

$$\langle\lambda, r|\mu, s\rangle = \delta_{\lambda\mu}\delta_{rs}. \tag{27}$$

The normal ordered fermion bilinears

$$:\psi_{-i}\psi_j^*: = \psi_{-i}\psi_j^* - \langle 0|\psi_{-i}\psi_j^*|0\rangle, \quad i, j \in \mathbf{Z},$$

span the one-dimensional central extension $\widehat{\mathrm{gl}}(\infty)$ of the Lie algebra $\mathrm{gl}(\infty)$ of infinite matrices [18, 19]. $\mathrm{gl}(\infty)$ consists of infinite matrices $A = (a_{ij})_{i,j\in\mathbf{Z}}$ of "finite-band type", namely, there is a positive integer N (depending on A) such that $a_{ij} = 0 \quad \text{if } |i - j| > N$. For such a matrix $A \in \mathrm{gl}(\infty)$, the fermion bilinear

$$\widehat{A} = \sum_{i,j\in\mathbf{Z}} a_{ij}{:}\psi_{-i}\psi_j^*{:}$$

becomes a well-defined linear operator on the Fock space, and preserves the charge, namely,

$$\langle\lambda, r|\widehat{A}|\mu, s\rangle = 0 \quad \text{if } r \neq s. \tag{28}$$

Moreover, for two such matrices $A, B \in \mathrm{gl}(\infty)$, the associated operators $\widehat{A}, \widehat{B}$ satisfy the commutation relation

$$[\widehat{A}, \widehat{B}] = \widehat{[A, B]} + \gamma(A, B) \tag{29}$$

with the c-number cocycle term

$$\gamma(A, B) = \mathrm{Tr}(A_{+-}B_{-+} - B_{+-}A_{-+}), \tag{30}$$

where $A_{\pm\mp}$ and $B_{\pm\mp}$ denote the following quarter blocks of A, B:

$$A_{+-} = (a_{ij})_{i>0,\, j\le 0}, \quad A_{-+} = (a_{ij})_{i\le 0,\, j>0},$$
$$B_{+-} = (b_{ij})_{i>0,\, j\le 0}, \quad B_{-+} = (b_{ij})_{i\le 0,\, j>0}.$$

3.2 Special fermion bilinears

The following fermion bilinears are building blocks of our Toda tau function:

$$J_m = \sum_{n\in\mathbf{Z}} {:}\psi_{-n+m}\psi_n^*{:}, \quad m \in \mathbf{Z},$$
$$L_0 = \sum_{n\in\mathbf{Z}} n{:}\psi_{-n}\psi_n^*{:}, \quad W_0 = \sum_{n\in\mathbf{Z}} n^2{:}\psi_{-n}\psi_n^*{:}.$$

J_m's span a U(1) current algebra. L_0 and W_0 are zero-modes of Virasoro and $W^{(3)}$ algebras. These fermion bilinears are associated with infinite matrices as

$$J_m = \widehat{\Lambda^m}, \quad L_0 = \widehat{\Delta}, \quad W_0 = \widehat{\Delta^2}, \tag{31}$$

where Δ and Λ are infinite matrices of the form

$$\Delta = (i\delta_{ij}), \quad \Lambda = (\delta_{i+1,j}).$$

Let $J_\pm[\boldsymbol{t}]$, $\boldsymbol{t} = (t_1, t_2, \cdots)$, denote the special linear combinations

$$J_+[\boldsymbol{t}] = \sum_{k=1}^{\infty} t_k J_k, \quad J_-[\boldsymbol{t}] = \sum_{k=1}^{\infty} t_k J_{-k}$$

of J_m's. Their exponentials act on the ground states $\langle s|$, $|s\rangle$ as

$$\langle s|e^{J_+[\boldsymbol{t}]} = \sum_{\lambda\in\mathcal{P}} \langle \lambda, s|s_\lambda[\boldsymbol{t}], \quad e^{J_-[\boldsymbol{t}]}|s\rangle = \sum_{\lambda\in\mathcal{P}} s_\lambda[\boldsymbol{t}]|\lambda, s\rangle, \tag{32}$$

yielding Schur functions as matrix elements [18, 19]:

$$s_\lambda[\boldsymbol{t}] = \langle \lambda, s|e^{J_+[\boldsymbol{t}]}|s\rangle = \langle s|e^{J_-[\boldsymbol{t}]}|\lambda, s\rangle. \tag{33}$$

Unlike other J_m's, J_0 is diagonal with respect to $|\lambda, s\rangle$'s:

$$\langle \lambda, s|J_0|\mu, s\rangle = \delta_{\lambda\mu} s. \tag{34}$$

L_0 and W_0, too, are diagonal. The diagonal matrix elements can be calculated as follows.

Lemma 1.

$$\langle \lambda, s|L_0|\mu, s\rangle = \delta_{\lambda\mu}\left(|\lambda| + \frac{s(s+1)}{2}\right), \tag{35}$$

$$\langle \lambda, s|W_0|\mu, s\rangle = \delta_{\lambda\mu}\left(\kappa_\lambda + (2s+1)|\lambda| + \frac{s(s+1)(2s+1)}{6}\right). \tag{36}$$

Proof. Assuming that $s \geq 0$, one can calculate the diagonal matrix elements as

$$\begin{aligned}
\langle \lambda, s|L_0|\lambda, s\rangle &= \sum_{i=1}^{\infty}(\lambda_i + s - i + 1) - \sum_{i=1}^{\infty}(-i+1) \quad \text{(heuristic expression)}\\
&= \sum_{i=1}^{\infty}((\lambda_i + s - i + 1) - (s - i + 1)) + \sum_{k=0}^{s} k \quad \text{(re-summed)}\\
&= |\lambda| + \frac{s(s+1)}{2}
\end{aligned}$$

and

$$\begin{aligned}\langle\lambda, s|W_0|\lambda, s\rangle &= \sum_{i=1}^{\infty}(\lambda_i + s - i + 1)^2 - \sum_{i=1}^{\infty}(-i+1)^2 \quad \text{(heuristic expression)}\\ &= \sum_{i=1}^{\infty}(\lambda_i + s - i + 1)^2 - (s - i + 1)^2) + \sum_{k=0}^{s} k^2 \quad \text{(re-summed)}\\ &= \kappa_\lambda + (2s+1)|\lambda| + \frac{s(s+1)(2s+1)}{6}.\end{aligned}$$

In the case where $s < 0$, we have only to replace the intermediate sums over k as

$$\sum_{k=0}^{s} k \to -\sum_{k=s+1}^{0} k, \quad \sum_{k=0}^{s} k^2 \to -\sum_{k=s+1}^{0} k^2,$$

ending up with the same final expression of the matrix elements. □

3.3 Toda Tau function for double Hurwitz numbers

A general tau function of the Toda hierarchy has the fermionic expression

$$\tau(s, \boldsymbol{t}, \bar{\boldsymbol{t}}) = \langle s|e^{J_+[\boldsymbol{t}]} g e^{-J_-[\bar{\boldsymbol{t}}]}|s\rangle,$$

where g is an element of $\widehat{\mathrm{GL}}(\infty)$ [42]. Inserting the aforementioned expansion (32), one can expand this function as

$$\tau(s, \boldsymbol{t}, \bar{\boldsymbol{t}}) = \sum_{\lambda,\mu\in\mathcal{P}} \langle\lambda, s|g|\mu, s\rangle s_\lambda[\boldsymbol{t}] s_\mu[-\bar{\boldsymbol{t}}]. \tag{37}$$

Following Okounkov [6], we now consider the special case where

$$g = e^{\beta W_0/2} Q^{L_0} = Q^{L_0} e^{\beta W_0/2}. \tag{38}$$

Theorem 1. *The tau function determined by (38) can be expanded as*

$$\tau(s, \boldsymbol{t}, \bar{\boldsymbol{t}}) = e^{\beta s(s+1)(2s+1)/12} Q^{s(s+1)/2} \sum_{\lambda\in\mathcal{P}} e^{\beta\kappa_\lambda/2} (e^{\beta(s+1/2)} Q)^{|\lambda|} s_\lambda[\boldsymbol{t}] s_\lambda[-\bar{\boldsymbol{t}}]. \tag{39}$$

Proof. The properties (35) and (36) of L_0 and W_0 imply that g is also diagonalized on the basis $|\lambda, s\rangle$ of the Fock space. The diagonal matrix elements take such a form as

$$\langle\lambda, s|g|\lambda, s\rangle = \exp\left(\frac{\beta}{2}\left(\kappa_\lambda + (2s+1)|\lambda| + \frac{s(s+1)(2s+1)}{6}\right)\right) Q^{|\lambda|+s(s+1)/2}.$$

The tau function in question can be thereby expanded as (39) shows. □

This is a restatement of Okounkov's result [6]. One can rewrite (39) as

$$\tau(s,\boldsymbol{t},\bar{\boldsymbol{t}}) = e^{\beta s(s+1)(2s+1)/12} Q^{s(s+1)/2} Z_{\beta, e^{\beta(s+1/2)}Q}[\boldsymbol{t},\bar{\boldsymbol{t}}], \tag{40}$$

where $Z_{\beta,Q}[\boldsymbol{t},\bar{\boldsymbol{t}}]$ denotes the generating function (19) with the parameters β and Q being explicitly indicated. Thus, apart from some numerical factors depending on s, the tau function coincides with the generating function of double Hurwitz numbers. Note that the s-dependence shows up in the generating function as the multiplier $e^{\beta(s+1/2)}$ of the parameter Q.

Let us conclude this section with a few comments on the cut-and-join operator (21). The cut-and-join operator corresponds to the fermion bilinear

$$M_0 = \frac{1}{2}\sum_{n\in\mathbf{Z}} \left(n-\frac{1}{2}\right)^2 {:}\psi_{-n}\psi_n^*{:} = \frac{W_0}{2} - \frac{L_0}{2} + \frac{J_0}{8}. \tag{41}$$

One can readily see from (34), (35) and (36) that this operator acts on $|\lambda,s\rangle$'s as

$$M_0|\lambda,s\rangle = \left(\frac{\kappa_\lambda}{2} + s|\lambda| + \frac{4s^3-s}{24}\right)|\lambda,s\rangle. \tag{42}$$

In particular, $|\lambda\rangle = |\lambda,0\rangle$ is an eigenstate with eigenvalue $\kappa_\lambda/2$. Note that s-dependent extra terms show up in the charge-s sector.

Fermion bilinears of this type can be converted to differential (or "bosonic") operators by the boson-fermion correspondence [18, 19]. In a generating functional form, the normal-ordered product

$${:}\psi(z)\psi^*(w){:} = \psi(z)\psi^*(w) - \frac{1}{z-w} \quad (|z|<|w|)$$

of the fermion fields corresponds to the two-variable vertex operator

$$\begin{aligned} &X(z,w)\\ &= \frac{1}{z-w}\left(\left(\frac{z}{w}\right)^s \exp\left(\sum_{k=1}^\infty t_k(z^k-w^k)\right)\exp\left(-\sum_{k=1}^\infty \frac{z^{-k}-w^{-k}}{k}\frac{\partial}{\partial t_k}\right)-1\right)\end{aligned}$$

as

$$\langle s|e^{J_+[\boldsymbol{t}]}{:}\psi(z)\psi^*(w){:} = X(z,w)\langle s|e^{J_+[\boldsymbol{t}]}. \tag{43}$$

A similar relation holds for $e^{-J_-[\boldsymbol{t}]}|s\rangle$ and leads to bosonization with respect to $\bar{\boldsymbol{t}}$ [20], though we omit details here.

$\Gamma(z,w)$ can be expanded in powers of $z-w$, and the coefficients of this expansion give a bosonic representation of fermion bilinears. For L_0, J_0 and W_0, this bosonic representation reads

$$L_0 = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_k} + \frac{s(s+1)}{2}, \quad J_0 = s \tag{44}$$

and

$$W_0 = \sum_{k,l=1}^{\infty} \left(k l t_k t_l \frac{\partial}{\partial t_{k+l}} + (k+l) t_{k+l} \frac{\partial^2}{\partial t_k \partial t_l} \right) + (2s+1) \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_k} + \frac{s(s+1)(2s+1)}{6}. \tag{45}$$

(41) is thus bosonized as

$$M_0 = \frac{1}{2} \sum_{k,l=1}^{\infty} \left(k l t_k t_l \frac{\partial}{\partial t_{k+l}} + (k+l) t_{k+l} \frac{\partial^2}{\partial t_k \partial t_l} \right) + s \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_k} + \frac{4s^3 - s}{24}. \tag{46}$$

In the charge-0 sector, this reduces to the cut-and-join operator (21).

4 Generalized string equations for double Hurwitz numbers

4.1 Notations for difference operators

Building blocks of the Lax formalism of the Toda hierarchy are one-dimensional difference operators in the lattice coordinate s [24]. Those operators are linear combinations of the shift operators $e^{n\partial_s}$, $e^{n\partial_s} f(s) = f(s+n)$. Although a genuine difference operator is a finite linear combination

$$A = \sum_{n=M}^{N} a_n(s) e^{n\partial_s} \quad \text{(operator of } [M,N]\text{-type)}$$

of the shift operators, one can consider a semi-infinite linear combination of the form

$$A = \sum_{n=-\infty}^{N} a_n(s) e^{n\partial_s} \quad \text{(operator of } (-\infty,N] \text{ type)}$$

or

$$A=\sum_{n=M}^{\infty}a_n(s)e^{n\partial_s}\quad\text{(operator of }[M,\infty)\text{ type)}$$

as well, which amount to pseudo-differential operators in the Lax formalism of the KP hierarchy [18]. Let $(\quad)_{\geq 0}$ and $(\quad)_{<0}$ denote the truncation operation

$$(A)_{\geq 0}=\sum_{n\geq 0}a_n(s)e^{n\partial_s},\quad (A)_{<0}=\sum_{n<0}a_n(s)e^{n\partial_s}.$$

Difference operators are in one-to-one correspondence with $\mathbf{Z}\times\mathbf{Z}$ matrices. Firstly, the n-th shift operator $e^{n\partial_s}$ corresponds to the shift matrix

$$\Lambda^n=(\delta_{i,j-n})_{i,j\in\mathbf{Z}}.$$

Secondly, the multiplication operator $a(s)$ amounts to the diagonal matrix

$$\operatorname{diag}(a(s))=(a(i)\delta_{ij})_{i,j\in\mathbf{Z}}.$$

In particular, the multiplication operator s corresponds to

$$\Delta=\operatorname{diag}(s)=(i\delta_{ij})_{i,j\in\mathbf{Z}}.$$

Consequently, a general difference operator

$$A=A(s,e^{\partial_s})=\sum_n a_n(s)e^{n\partial_s}$$

is converted to the infinite matrix

$$A(\Delta,\Lambda)=\sum_n\operatorname{diag}(a_n(s))\Lambda^n=\sum_n(a_n(i)\delta_{i,j-n})_{i,j\in\mathbf{Z}}.$$

Occasionally, it might be more convenient to write a difference operator in an anti-normal-ordered form as

$$B(e^{\partial_s},s)=\sum_n e^{n\partial_s}b_n(s).$$

In that case, the corresponding infinite matrix reads

$$B(\Lambda,\Delta)=\sum_n\Lambda^n\operatorname{diag}(b_n(s)).$$

4.2 Lax and Orlov-Schulman operators

The Lax formalism of the Toda hierarchy uses two Lax operators $L, \bar{L}$ of type $(-\infty, 1]$ and $[1, \infty)$. Actually, from the point of view of symmetry, it is better to consider L and $\bar{L}^{-1}$ rather than L and $\bar{L}$. These operators admit freedom of gauge transformations. In the gauge where L is *monic* (namely, the leading coefficients is equal to 1), L and $\bar{L}^{-1}$ can be expressed as

$$L = e^{\partial_s} + \sum_{n=1}^{\infty} u_n e^{(1-n)\partial_s},$$

$$\bar{L}^{-1} = \bar{u}_0 e^{-\partial_s} + \sum_{n=1}^{\infty} \bar{u}_n e^{(n-1)\partial_s}.$$

The coefficients u_n and $\bar{u}_n$ are functions of s and the time variables $\boldsymbol{t}, \bar{\boldsymbol{t}}$, and written as $u_n(s, \boldsymbol{t}, \bar{\boldsymbol{t}})$ and $\bar{u}_n(s, \boldsymbol{t}, \bar{\boldsymbol{t}})$ if we do not suppress the independent variables. L and $\bar{L}$ satisfy the Lax equations

$$\begin{aligned} \frac{\partial L}{\partial t_n} &= [B_n, L], \quad \frac{\partial L}{\partial \bar{t}_n} = [\bar{B}_n, L], \\ \frac{\partial \bar{L}}{\partial t_n} &= [B_n, \bar{L}], \quad \frac{\partial \bar{L}}{\partial \bar{t}_n} = [\bar{B}_n, \bar{L}], \end{aligned} \tag{47}$$

where B_n and $\bar{B}_n$ are defined as

$$B_n = (L^n)_{\geq 0}, \quad \bar{B}_n = (\bar{L}^{-n})_{<0}.$$

To formulate the generalized string equations, we need another pair of difference operators $M, \bar{M}$, namely, the Orlov-Schulman operators [44]. These operators, too, satisfy the Lax equations

$$\begin{aligned} \frac{\partial M}{\partial t_n} &= [B_n, M], \quad \frac{\partial M}{\partial \bar{t}_n} = [\bar{B}_n, M], \\ \frac{\partial \bar{M}}{\partial t_n} &= [B_n, \bar{M}], \quad \frac{\partial \bar{M}}{\partial \bar{t}_n} = [\bar{B}_n, \bar{M}] \end{aligned} \tag{48}$$

of the same form as the Lax operators do, and are related to the Lax operators by the twisted canonical commutation relations

$$[L, M] = L, \quad [\bar{L}, \bar{M}] = \bar{L}. \tag{49}$$

By "generalized string equations" we mean equations of the form

$$C(L, M) = \bar{C}(\bar{L}, \bar{M}), \tag{50}$$

where $C(L, M)$ and $\bar{C}(\bar{L}, \bar{M})$ are (possibly infinite) linear combinations of monomials of L, M and $\bar{L}, \bar{M}$ with constant coefficients. The following lemma [30, 31] explains an origin of generalized string equations.

Lemma 2. *If the fermion bilinears $\widehat{C(\Lambda,\Delta)}$ and $\widehat{\bar{C}(\Lambda,\Delta)}$ are intertwined by a $\widehat{GL}(\infty)$ element g as*

$$\widehat{C(\Lambda,\Delta)}g = g\widehat{\bar{C}(\Lambda,\Delta)}, \tag{51}$$

then the Lax and Orlov-Schulman operators satisfy (50).

4.3 Intertwining relations

Let us now consider the case of the solution determined by the $\widehat{GL}(\infty)$ element (38). We seek intertwining relations in the following special form:

$$J_k g = g\widehat{\bar{C}(\Lambda,\Delta)}, \quad \widehat{C(\Lambda,\Delta)}g = gJ_{-k}, \quad k = 1, 2, \cdots .$$

Lemma 3. *J_m's are transformed by the adjoint action of Q^{L_0} and $e^{\beta W_0/2}$ as*

$$\begin{aligned} Q^{L_0} J_m Q^{-L_0} &= Q^{-m} J_m, \\ e^{-\beta W_0/2} J_m e^{\beta W_0/2} &= e^{-\beta m^2/2} \sum_{n\in\mathbf{Z}} e^{\beta mn} {:}\psi_{-n+m}\psi_n^*{:}. \end{aligned} \tag{52}$$

Proof. Let us note the fundamental commutation relations

$$[L_0, \psi_n] = -n\psi_n, \quad [L_0, \psi_n^*] = -n\psi_n^*$$

and

$$[W_0, \psi_n] = n^2\psi_n, \quad [W_0, \psi_n^*] = -n^2\psi_n^*$$

that follow from the definition of L_0 and W_0. These commutation relations can be exponentiated as

$$Q^{L_0}\psi_n Q^{-L_0} = Q^{-n}\psi_n, \quad Q^{L_0}\psi_n^* Q^{-L_0} = Q^{-n}\psi_n^*$$

and

$$e^{-\beta W_0/2}\psi_n e^{\beta W_0/2} = e^{-\beta n^2/2}\psi_n, \quad e^{-\beta W_0/2}\psi_n^* e^{\beta W_0/2} = e^{\beta n^2/2}\psi_n^*.$$

Consequently, the exponentiated operators Q^{L_0} and $e^{\beta W_0/2}$ act on the basis ${:}\psi_{-i}\psi_j^*{:}$ of gl(∞) as

$$Q^{L_0}{:}\psi_{-i}\psi_j^*{:}Q^{-L_0} = Q^{i-j}{:}\psi_{-i}\psi_j^*{:}$$

and

$$e^{-\beta W_0/2}{:}\psi_{-i}\psi_j^*{:}e^{\beta W_0/2} = e^{-\beta(i^2-j^2)/2}{:}\psi_{-i}\psi_j^*{:}.$$

One can thereby derive (52) as

$$Q^{L_0} J_m Q^{-L_0} = \sum_{n\in\mathbf{Z}} Q^{(n-m)-n} {:}\psi_{-n+m}\psi_n^*{:} = Q^{-m} J_m$$

and

$$\begin{aligned} e^{-\beta W_0/2} J_m e^{\beta W_0/2} &= \sum_{n\in\mathbf{Z}} e^{-\beta((n-m)^2-n^2)/2} {:}\psi_{-n+m}\psi_n^*{:} \\ &= e^{-\beta m^2/2} \sum_{n\in\mathbf{Z}} e^{\beta mn} {:}\psi_{-n+m}\psi_n^*{:}. \end{aligned}$$

□

Lemma 4. *$J_{\pm k}$'s are connected with the fermion bilinears $\widehat{\Lambda^k e^{\beta k\Delta}}$ and $\widehat{\Lambda^{-k} e^{\beta k\Delta}}$ by the $\widehat{\mathrm{GL}}(\infty)$ element (38) as*

$$J_k g = g Q^k e^{-\beta k^2/2} \widehat{\Lambda^k e^{\beta k\Delta}}, \tag{53}$$

$$g J_{-k} = Q^k e^{\beta k^2/2} \widehat{\Lambda^{-k} e^{\beta k\Delta}} g. \tag{54}$$

Proof. Using the relations (52) in the previous lemma, one can calculate $g^{-1} J_k g$ as

$$\begin{aligned} g^{-1} J_k g &= e^{-\beta W_0/2} Q^{-L_0} J_k Q^{L_0} e^{\beta W_0/2} \\ &= Q^k e^{-\beta W_0/2} J_k e^{\beta W_0/2} \\ &= Q^k e^{-\beta k^2/2} \sum_{n\in\mathbf{Z}} e^{\beta kn} {:}\psi_{-n+k}\psi_n^*{:}. \end{aligned}$$

Since the last sum can be rewritten as

$$\sum_{n\in\mathbf{Z}} e^{\beta kn} {:}\psi_{-n+k}\psi_n^*{:} = \widehat{\Lambda^k e^{\beta k\Delta}},$$

the first intertwining relation (53) follows. In the same way, one can calculate $g J_{-k} g^{-1}$ as

$$\begin{aligned} g J_{-k} g^{-1} &= e^{\beta W_0/2} Q^{L_0} J_{-k} Q^{-L_0} e^{-\beta W_0/2} \\ &= Q^k e^{\beta W_0/2} J_{-k} e^{-\beta W_0/2} \\ &= Q^k e^{\beta k^2/2} \sum_{n\in\mathbf{Z}} e^{\beta kn} {:}\psi_{-n-k}\psi_n^*{:}, \end{aligned}$$

which implies the second intertwining relation (54). □

4.4 Generalized string equations

Theorem 2. *The Lax and Orlov-Schulman operators of the tau function (39) satisfy the generalized string equations*

$$L^k = Q^k e^{-\beta k^2/2} \bar{L}^k e^{\beta k \bar{M}}, \quad \bar{L}^{-k} = Q^k e^{\beta k^2/2} L^{-k} e^{\beta k M} \tag{55}$$

for $k = 1, 2, \cdots$. Moreover, these equations can be derived from the first two ($k = 1$) equations

$$L = Q e^{-\beta/2} \bar{L} e^{\beta \bar{M}}, \quad \bar{L}^{-1} = Q e^{\beta/2} L^{-1} e^{\beta M}. \tag{56}$$

Proof. The first part is a consequence of (53) and (54). Let us show the second part. The k-th power of the first equation of (56) reads

$$L^k = Q^k e^{-\beta k/2} (\bar{L} e^{\beta \bar{M}})^k.$$

The commutation equation of $\bar{L}$ and $\bar{M}$ in (49) implies that

$$[\bar{M}, \cdots, [\bar{M}, [\bar{M}, \bar{L}]] \cdots] \text{ (k-fold commutator) } = (-1)^k \bar{L}$$

for $k = 1, 2, \cdots$, so that

$$e^{\beta \bar{M}} \bar{L} e^{-\beta \bar{M}} = \bar{L} + \sum_{k=1}^{\infty} \frac{\beta^k}{k!} [\bar{M}, \cdots, [\bar{M}, [\bar{M}, \bar{L}]] \cdots] = e^{-\beta} \bar{L}.$$

Using this relation repeatedly, one can move $e^{\beta \bar{M}}$'s in $(\bar{L} e^{\beta \bar{M}})^k$ to the rightmost position as

$$\begin{aligned}
(\bar{L} e^{\beta \bar{M}})^k &= \bar{L} e^{\beta \bar{M}} (\bar{L} e^{\beta \bar{M}})^{k-1} e^{\beta \bar{M}} \\
&= e^{-\beta k} \bar{L}^2 e^{\beta \bar{M}} (\bar{L} e^{\beta \bar{M}})^{k-2} e^{2\beta \bar{M}} \\
&= e^{-\beta k - \beta(k-1)} \bar{L}^3 e^{\beta \bar{M}} (e^{\beta \bar{M}} \bar{L})^{k-3} e^{3\beta \bar{M}} \\
&= \cdots \\
&= e^{-\beta k(k-1)/2} \bar{L}^k e^{\beta k \bar{M}}.
\end{aligned}$$

Thus the first equation of (55) follows. The second equation of (55), too, can be derived from (56) in the same way. □

Thus, in contrast with two-dimensional quantum gravity [25] and $c = 1$ string theory [26, 27, 28, 29, 30, 31], the generalized string equations contain the exponential terms $e^{\beta M}, e^{\beta \bar{M}}$. These terms stem from the fermion bilinears $\widehat{\Lambda^{\pm k} e^{\beta k \Delta}}$ in (53) and (54). Ferimion blinears of a similar form are also used in the study of integrable structures of the melting crystal model [32, 33]. A common algebraic background of these fermion bilinears is the quantum torus algebra (with parameter q) spanned by the infinite matrices

$$v_m^{(k)} = q^{-km/2}\Lambda^m q^{k\Delta}, \quad k, m \in \mathbf{Z}. \tag{57}$$

In the melting crystal model [32, 33], a central extension of this Lie algebra is realized by the fermion bilinears

$$V_m^{(k)} = q^{-km}\sum_{n\in\mathbf{Z}} q^{kn}{:}\psi_{-n+m}\psi_n^*{:}. \tag{58}$$

The Lax and Orlov-Schulman operators give (two copies of) yet another kind of realization by the difference operators

$$V_m^{(k)} = q^{-km/2}L^m q^{kM} \tag{59}$$

and

$$\bar{V}_m^{(k)} = q^{-km/2}\bar{L}^m q^{k\bar{M}}. \tag{60}$$

If $q = e^\beta$, the exponentials $e^{\beta M}, e^{\beta\bar{M}}$ belong to this Lie algebra.

5 Classical limit of generalized string equations

5.1 Dispersionless Toda hierarchy

In a naive sense [43], the classical limit of the Toda hierarchy can be obtained by replacing the shift operator e^{∂_s} by a new variable p. The difference operators $L, M, \bar{L}, \bar{M}$ thus turn into Laurent series of p of the form

$$\begin{aligned}
\mathcal{L} &= p + \sum_{n=1}^\infty u_n p^{1-n}, \\
\bar{\mathcal{L}}^{-1} &= \bar{u}_0 p^{-1} + \sum_{n=1}^\infty \bar{u}_n p^{n-1}, \\
\mathcal{M} &= \sum_{n=1}^\infty n t_n \mathcal{L}^n + s + \sum_{n=1}^\infty v_n \mathcal{L}^{-n}, \\
\bar{\mathcal{M}} &= -\sum_{n=1}^\infty n\bar{t}_n \bar{\mathcal{L}}^{-n} + s + \sum_{n=1}^\infty \bar{v}_n \bar{\mathcal{L}}^n
\end{aligned}$$

that are referred to as the “Lax and Orlov-Schulman functions”.

As difference operators are replaced by Laurent series, commutators of difference operators turn into Poisson brackets by the rule

$$[e^{\partial_s}, s] = e^{\partial_s} \ \rightarrow \ \{p, s\} = s.$$

Accordingly, Poisson brackets of functions of p and s are defined as

$$\{F, G\} = p\left(\frac{\partial F}{\partial p}\frac{\partial G}{\partial s} - \frac{\partial F}{\partial s}\frac{\partial G}{\partial p}\right).$$

The Lax equations and the twisted canonical commutation relations are re-defined with respect to the Poisson bracket as

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial t_n} &= \{\mathcal{B}_n, \mathcal{L}\}, \quad \frac{\partial \mathcal{L}}{\partial \bar{t}_n} = \{\bar{\mathcal{B}}_n, \mathcal{L}\}, \\
\frac{\partial \bar{\mathcal{L}}}{\partial t_n} &= \{\mathcal{B}_n, \bar{\mathcal{L}}\}, \quad \frac{\partial \bar{\mathcal{L}}}{\partial \bar{t}_n} = \{\bar{\mathcal{B}}_n, \bar{\mathcal{L}}\}, \\
\frac{\partial \mathcal{M}}{\partial t_n} &= \{\mathcal{B}_n, \mathcal{M}\}, \quad \frac{\partial \mathcal{M}}{\partial \bar{t}_n} = \{\bar{\mathcal{B}}_n, \mathcal{M}\}, \\
\frac{\partial \bar{\mathcal{M}}}{\partial t_n} &= \{\mathcal{B}_n, \bar{\mathcal{M}}\}, \quad \frac{\partial \bar{\mathcal{M}}}{\partial \bar{t}_n} = \{\bar{\mathcal{B}}_n, \bar{\mathcal{M}}\}
\end{aligned} \tag{61}$$

and

$$\{\mathcal{L}, \mathcal{M}\} = \mathcal{L}, \quad \{\bar{\mathcal{L}}, \bar{\mathcal{M}}\} = \bar{\mathcal{L}}. \tag{62}$$

$\mathcal{B}_n$ and $\bar{\mathcal{B}}_n$ are defined by seemingly the same formulae

$$\mathcal{B}_n = (\mathcal{L}^n)_{\geq 0}, \quad \bar{\mathcal{B}}_n = (\bar{\mathcal{L}}^{-n})_{<0}$$

as in the previous case, but the notations $(\quad)_{\geq 0}$ and $(\quad)_{<0}$ now stand for projection operators on the space of Laurent series, namely,

$$\left(\sum_n a_n p^n\right)_{\geq 0} = \sum_{n \geq 0} a_n p^n, \quad \left(\sum_n a_n p^n\right)_{<0} = \sum_{n<0} a_n p^n.$$

These equations are fundamental constituents of the “dispersionless Toda hierarchy”.

5.2 $\hbar$-dependent Toda hierarchy and generalized string equations

The foregoing procedure replacing $e^{\partial_s} \to p$ can be justified as a kind of classical limit in an $\hbar$-dependent formulation of the Toda hierarchy [44].

In the $\hbar$-dependent formulation, e^{∂_s} is replaced by $e^{\hbar\partial_s}$. The "Planck constant" $\hbar$ thus plays the role of lattice spacing. The Lax and Orlov-Schulman operators are expanded in powers of $e^{\hbar\partial_s}$ as

$$\begin{aligned}
L &= e^{\hbar\partial_s} + \sum_{n=1}^{\infty} u_n e^{(1-n)\hbar\partial_s}, \\
\bar{L}^{-1} &= \bar{u}_0 e^{-\hbar\partial_s} + \sum_{n=1}^{\infty} \bar{u}_n e^{(n-1)\hbar\partial_s}, \\
M &= \sum_{n=1}^{\infty} n t_n L^n + s + \sum_{n=1}^{\infty} v_n L^{-n}, \\
\bar{M} &= -\sum_{n=1}^{\infty} n \bar{t}_n \bar{L}^{-n} + s + \sum_{n=1}^{\infty} \bar{v}_n \bar{L}^n.
\end{aligned}$$

The Lax equations and the twisted canonical commutation relations take an $\hbar$-dependent form as

$$\begin{aligned}
\hbar\frac{\partial L}{\partial t_n} &= [B_n, L], \quad \hbar\frac{\partial L}{\partial \bar{t}_n} = [\bar{B}_n, L], \\
\hbar\frac{\partial \bar{L}}{\partial t_n} &= [B_n, \bar{L}], \quad \hbar\frac{\partial \bar{L}}{\partial \bar{t}} = [\bar{B}_n, \bar{L}], \\
\hbar\frac{\partial M}{\partial t_n} &= [B_n, M], \quad \hbar\frac{\partial M}{\partial \bar{t}_n} = [\bar{B}_n, M], \\
\hbar\frac{\partial \bar{M}}{\partial t_n} &= [B_n, \bar{M}], \quad \hbar\frac{\partial \bar{M}}{\partial \bar{t}_n} = [\bar{B}_n, \bar{M}]
\end{aligned} \tag{63}$$

and

$$[L, M] = \hbar L, \quad [\bar{L}, \bar{M}] = \hbar \bar{L}. \tag{64}$$

If the coefficient $u_n, \bar{u}_n, v_n, \bar{v}_n$ (which are functions of $\hbar, s, \boldsymbol{t}, \bar{\boldsymbol{t}}$) have a smooth classical limit as

$$u_n^{(0)} = \lim_{\hbar\to 0} u_n, \quad \bar{u}_n^{(0)} = \lim_{\hbar\to 0} \bar{u}_n, \quad v_n^{(0)} = \lim_{\hbar\to 0} v_n, \quad \bar{v}_n^{(0)} = \lim_{\hbar\to 0} \bar{v}_n, \tag{65}$$

one can define the Lax and Orlov-Schulman functions as

$$\mathcal{L} = p + \sum_{n=1}^{\infty} u_n^{(0)} p^{1-n},$$

$$\bar{\mathcal{L}}^{-1} = \bar{u}_0^{(0)} p^{-1} + \sum_{n=1}^{\infty} \bar{u}_n^{(0)} p^{n-1},$$

$$\mathcal{M} = \sum_{n=1}^{\infty} n t_n \mathcal{L}^n + s + \sum_{n=1}^{\infty} v_n^{(0)} \mathcal{L}^{-n},$$

$$\bar{\mathcal{M}} = -\sum_{n=1}^{\infty} n \bar{t}_n \bar{\mathcal{L}}^{-n} + s + \sum_{n=1}^{\infty} \bar{v}_n^{(0)} \bar{\mathcal{L}}^n.$$

These Lax and Orlov-Schulman functions satisfy the Lax equations (61) and the twisted canonical Poisson relations (62).

In this $\hbar$-dependent formulation, generalized string equations (50) are modified as

$$C(L, \hbar^{-1} M) = \bar{C}(\bar{L}, \hbar^{-1} \bar{M}), \tag{66}$$

namely, M and $\bar{M}$ are multiplied by $\hbar^{-1}$ [30, 31]. Let us explain the underlying mechanism briefly. The Lax and Orlov-Schulman operators are connected with the matrices Λ and Δ by the so called "dressing operators" [44]. The twisted canonical commutation relations are thereby derived from the commutation relation

$$[\Lambda, \Delta] = \Lambda$$

of these matrices. In the $\hbar$-dependent formulation, they take the form

$$[L, \hbar^{-1} M] = \hbar^{-1} M, \quad [\bar{L}, \hbar^{-1} \bar{M}] = \hbar^{-1} \bar{M},$$

which are nothing but (64). Thus it is $\hbar^{-1} M$ and $\hbar^{-1} \bar{M}$ rather than M and $\bar{M}$ that correspond to Δ and show up in generalized string equations.

5.3 Classical limit of generalized string equations

Let us turn to the case of double Hurwitz numbers. To derive a classical limit, we have to introduce $\hbar$ therein. At least formally, this can be done by rescaling the space-time variables as

$$s \to \hbar^{-1} s, \quad \boldsymbol{t} \to \hbar^{-1} \boldsymbol{t}, \quad \bar{\boldsymbol{t}} \to \hbar^{-1} \bar{\boldsymbol{t}}. \tag{67}$$

The $\hbar$-independent Toda hierarchy is thereby converted to the $\hbar$-dependent form. Actually, it is rare that the rescaled Lax and Orlov-Schulman operators satisfy the condition (65). To achieve a meaningful (and nontrivial) classical limit, one should start from a carefully chosen *$\hbar$-dependent* solution of the *$\hbar$-independent* Toda hierarchy.

In the language of the tau function [44], a meaningful classical limit is obtained from an $\hbar$-dependent tau function $\tau(\hbar, s, \boldsymbol{t}, \bar{\boldsymbol{t}})$ such that the rescaled tau function

$$\tau_\hbar(s, \boldsymbol{t}, \bar{\boldsymbol{t}}) = \tau(\hbar, \hbar^{-1}s, \hbar^{-1}\boldsymbol{t}, \hbar^{-1}\bar{\boldsymbol{t}})$$

behaves as

$$\log \tau_\hbar(s, \boldsymbol{t}, \bar{\boldsymbol{t}}) = \hbar^{-2}\mathcal{F}(s, \boldsymbol{t}, \bar{\boldsymbol{t}}) + O(\hbar^{-1}). \tag{68}$$

$\mathcal{F} = \mathcal{F}(s, \boldsymbol{t}, \bar{\boldsymbol{t}})$ is called the "free energy" because of its relation to random matrices and topological field theories [45, 46]. If the rescaled tau function has this asymptotic form, the associated "wave functions"

$$\begin{aligned}
\Psi_\hbar(s, \boldsymbol{t}, \bar{\boldsymbol{t}}, z) &= \frac{\tau_\hbar(s, \boldsymbol{t} - \hbar[z^{-1}], \bar{\boldsymbol{t}})}{\tau_\hbar(s, \boldsymbol{t}, \bar{\boldsymbol{t}})} z^{\hbar^{-1}s} e^{\hbar^{-1}\xi(\boldsymbol{t}, z)},\\
\bar{\Psi}_\hbar(s, \boldsymbol{t}, \bar{\boldsymbol{t}}, z) &= \frac{\tau_\hbar(s + \hbar, \boldsymbol{t}, \bar{\boldsymbol{t}} - \hbar[z])}{\tau_\hbar(s, \boldsymbol{t}, \bar{\boldsymbol{t}})} z^{\hbar^{-1}s} e^{\hbar^{-1}\xi(\bar{\boldsymbol{t}}, z^{-1})},\\
[z] &= \left(z, z^2/2, \cdots, z^k/k, \cdots\right), \quad \xi(\boldsymbol{t}, z) = \sum_{k=1}^\infty t_k z^k
\end{aligned}$$

have the "WKB" form

$$\begin{aligned}
\Psi_\hbar(s, \boldsymbol{t}, \bar{\boldsymbol{t}}, z) &= \exp\left(\hbar^{-1}S(s, \boldsymbol{t}, \bar{\boldsymbol{t}}, z) + O(1)\right),\\
\bar{\Psi}_\hbar(s, \boldsymbol{t}, \bar{\boldsymbol{t}}, z) &= \exp\left(\hbar^{-1}\bar{S}(s, \boldsymbol{t}, \bar{\boldsymbol{t}}, z) + O(1)\right)
\end{aligned} \tag{69}$$

and satisfy a set of auxiliary linear equations. The phase functions $S(s, \boldsymbol{t}, \bar{\boldsymbol{t}}, z)$ and $\bar{S}(s, \boldsymbol{t}, \bar{\boldsymbol{t}}, z)$ satisfy the associated Hamilton-Jacobi equations, which can be converted to the dispersionless Lax equations (61) and the Poisson relations (62) (see the review [20] for details).

An appropriate $\hbar$-dependent reformulation of the tau function (39) of double Hurwitz numbers can be found by the following heuristic consideration. As we move into the $\hbar$-dependent formulation, the generalized string equations (56) are modified as

$$L = Qe^{-\beta/2}\bar{L}e^{\beta\hbar^{-1}\bar{M}}, \quad \bar{L}^{-1} = Qe^{\beta/2}L^{-1}e^{\beta\hbar^{-1}M}.$$

Obviously, these equations do not have a limit as $\hbar \to 0$. If, however, the parameter β is simultaneously rescaled as

$$\beta \to \hbar\beta, \tag{70}$$

the generalized string equations are further modified as

$$L = Qe^{-\hbar\beta/2}\bar{L}e^{\beta\bar{M}}, \quad \bar{L}^{-1} = Qe^{\hbar\beta/2}L^{-1}e^{\beta M}, \tag{71}$$

and have a meaningful classical limit of the form

$$\mathcal{L} = Q\bar{\mathcal{L}}e^{\beta\bar{\mathcal{M}}}, \quad \bar{\mathcal{L}}^{-1} = Q\mathcal{L}^{-1}e^{\beta\mathcal{M}}. \tag{72}$$

5.4 Existence of $\hbar$-expansion

To justify the foregoing heuristic derivation of the classical limit (72) of the generalized string equations, let us show that the tau function (39) with β rescaled as (70) does satisfy the condition (68).

Recall the expression (40) of the tau function. After rescaling $s, \boldsymbol{t}, \bar{\boldsymbol{t}}$ and β as (67) and (70), this expression is modified as

$$\tau_\hbar(s, \boldsymbol{t}, \bar{\boldsymbol{t}}) = e^{\hbar^{-2}\beta s(s+\hbar)(2s+\hbar)/12}Q^{\hbar^{-2}s(s+\hbar)/2}Z_{\hbar\beta, e^{\beta(s+\hbar/2)}Q}[\hbar^{-1}\boldsymbol{t}, \hbar^{-1}\bar{\boldsymbol{t}}]. \tag{73}$$

Therefore it is sufficient to show that the logarithm of $Z_{\hbar\beta,Q}[\hbar^{-1}\boldsymbol{t}, \hbar^{-1}\bar{\boldsymbol{t}}]$ has an $\hbar$-expansion of the form

$$\log Z_{\hbar\beta,Q}[\hbar^{-1}\boldsymbol{t}, \hbar^{-1}\bar{\boldsymbol{t}}] = \hbar^{-2}F_0 + F_1 + \hbar^2 F_2 + \cdots + \hbar^{2n}F_n + \cdots, \tag{74}$$

where $F_0, F_1, \cdots$ are analytic functions of $(\beta, Q, \boldsymbol{t}, \bar{\boldsymbol{t}})$ in a common domain. The free energy is then given by

$$\mathcal{F} = \frac{\beta s^3}{6} + \frac{s^2 \log Q}{2} + F_0(\beta, e^{\beta s}Q, \boldsymbol{t}, \bar{\boldsymbol{t}}). \tag{75}$$

(74) is a generalization of the well known topological expansion for simple Hurwitz numbers. It is common in the literature that this kind of expansion is explained by a combination of topological and combinatorial consideration (see, e.g., Section 4.2 of Mironov and Morozov [22], Section 2.1 of Bouchard and Mariño [14] and Section 2.2 of Borot et al. [15]). We take another approach based on the cut-and-join operator (21).

Theorem 3. $\log Z_{\hbar\beta,Q}[\hbar^{-1}\boldsymbol{t}, \hbar^{-1}\bar{\boldsymbol{t}}]$ *has an $\hbar$-expansion of the form (74).*

Proof. Let $F = F(\hbar, \beta, Q, \boldsymbol{t}, \bar{\boldsymbol{t}})$ denote the left hand side of (74) multiplied by $\hbar^2$. By (26), $e^{\hbar^{-2}F}$ can be expressed as

$$e^{\hbar^{-2}F} = e^{\hbar\beta M_0(\hbar)} \exp\left(-\hbar^{-2}\sum_{k=1}^{\infty} Q^k k t_k \bar{t}_k\right),$$

where $M_0(\hbar)$ denotes the rescaled cut-and-join operator

$$M_0(\hbar) = \frac{1}{2}\sum_{j,k=1}^{\infty}\left(\hbar^{-1} k l t_k t_l \frac{\partial}{\partial t_{k+l}} + \hbar(k+l)t_{k+l}\frac{\partial^2}{\partial t_k \partial t_l}\right).$$

Therefore $e^{\hbar^{-2}F}$ satisfies the differential equation

$$\frac{\partial e^{\hbar^{-2}F}}{\partial \beta} = \hbar M_0(\hbar) e^{\hbar^{-2}F}$$

with respect to β. This equation can be further converted to the differential equation

$$\frac{\partial F}{\partial \beta} = \frac{1}{2}\sum_{j,k=1}^{\infty} k l t_k t_l \frac{\partial F}{\partial t_{k+l}} + \frac{1}{2}\sum_{k,l=1}^{\infty}(k+l)t_{k+l}\left(\hbar^2 \frac{\partial^2 F}{\partial t_k \partial t_l} + \frac{\partial F}{\partial t_k}\frac{\partial F}{\partial t_l}\right)$$

for F. This equation is supplemented by the the initial condition

$$F|_{\beta=0} = -\sum_{k=1}^{\infty} Q^k k t_k \bar{t}_k.$$

We now seek a solution of this initial value problem in the form of a (formal) power series of $\hbar^2$:

$$F = F_0 + \hbar^2 F_1 + \cdots + \hbar^{2n} F_n + \cdots.$$

This reduces to solving the differential equations

$$\frac{\partial F_n}{\partial \beta} = \frac{1}{2}\sum_{k,l=1}^{\infty} k l t_k t_l \frac{\partial F_n}{\partial t_{k+l}} + \frac{1}{2}\sum_{k,l=1}^{\infty}(k+l)t_{k+l}\frac{\partial^2 F_{n-1}}{\partial t_k \partial t_l} + \frac{1}{2}\sum_{k,l=1}^{\infty}(k+l)t_{k+l}\sum_{m=0}^{n}\frac{\partial F_m}{\partial t_k}\frac{\partial F_{n-m}}{\partial t_l} \quad (76)$$

for $n = 0, 1, 2, \cdots$ under the initial conditions

$$F_n|_{\beta=0} = -\delta_{n0}\sum_{k=1}^{\infty} Q^k k t_k \bar{t}_k. \quad (77)$$

F_n's are thereby recursively determined, and become analytic functions of $(\beta, Q, \boldsymbol{t}, \bar{\boldsymbol{t}})$ in a common domain of definition (because the differential equations other than the first one for $n = 0$ are linear with respect to F_n). Since the initial value problem for F has a unique solution, the power series solution $F = F_0 + \hbar^2 F_1 + \cdots$ should coincide with the left hand side of (74). □

One can thus confirm the expected asymptotic form (68) of the the rescaled tau function (73). Let us stress that this is also a proof for the case of simple Hurwitz numbers. To consider that case, one has only to set $\bar{t}_k = -\delta_{k1}$ in the initial condition (77). Let us also point out that the main part F_0 of the free energy is determined by the $n = 0$ part of (76) and (77), namely,

$$\frac{\partial F_0}{\partial \beta} = \frac{1}{2}\sum_{k,l=1}^{\infty} klt_k t_l \frac{\partial F_0}{\partial t_{k+l}} + \frac{1}{2}\sum_{k,l=1}^{\infty}(k+l)t_{k+l}\frac{\partial F_0}{\partial t_k}\frac{\partial F_0}{\partial t_l} \tag{78}$$

and

$$F_0|_{\beta=0} = -\sum_{k=1}^{\infty} Q^k k t_k \bar{t}_k. \tag{79}$$

It will be interesting to apply the diagramatic technique of Mironov and Morozov [22] to these equations.

6 Solution of classical limit of generalized string equations

6.1 Comparison with generalized string equations of $c = 1$ string theory

Our goal in this section is to solve the generalized string equations (72) and to derive some implications thereof. To this end, it is instructive to compare these equations with the generalized string equations of $c = 1$ string theory [26, 27, 28, 29].

In the classical limit, the generalized string equations of $c = 1$ string theory read

$$\mathcal{L} = \bar{\mathcal{L}}\bar{\mathcal{M}}, \quad \bar{\mathcal{L}}^{-1} = \mathcal{L}^{-1}\mathcal{M}. \tag{80}$$

Let us mention that the same equations play a central role in a problem of complex analysis and its applications to interface dynamics [47, 48, 49].

Actually, it is the equation

$$\{\mathcal{L}, \bar{\mathcal{L}}^{-1}\} = 1 \tag{81}$$

rather than (80) that is referred to as a "string equation" in these applications. Note that one can readily derive (81) from (80). In the same sense, one can derive the equation

$$\{\log \mathcal{L}, \log \bar{\mathcal{L}}^{-1}\} = \beta \tag{82}$$

from (72) as a counterpart of (80) for double Hurwitz numbers.

(81) and (82) resemble the string equation (or the Douglas equation)

$$[Q, P] = 1 \tag{83}$$

and its classical limit

$$\{Q, P\} = 1 \tag{84}$$

in two-dimensional quantum gravity [50, 51, 52]. Q and P in (83) are one-dimensional differential operators of the form

$$Q = \partial_x^n + u_2\partial_x^{n-2} + \cdots + u_n, \quad P = \partial_x^m + v_2\partial_x^{m-2} + \cdots + v_n.$$

In the classical limit, ∂_x is replaced by a variable p with the Poisson bracket

$$\{p, x\} = 1,$$

and Q and P are polynomials of the form

$$\mathcal{Q} = p^n + u_2 p^{n-2} + \cdots + u_n, \quad \mathcal{P} = p^m + v_2 p^{m-2} + \cdots + v_n.$$

In this setting, $\mathcal{Q}$ and $\mathcal{P}$ may be thought of as coordinates of the spectral curve (parametrized by p) in the sense of Eynard and Orantin [17]. Namely, when x and other deformation variables (time variables of the underlying KP hierarchy) are fixed to special values, $\mathcal{Q}$ and $\mathcal{P}$ satisfy a defining equation $f(X, Y) = 0$ of the spectral curve as

$$f(\mathcal{Q}, \mathcal{P}) = 0.$$

Although the KP and Toda hierarchies are different in nature, the last remark seems to suggest that one may think of an equation of the form

$$f(\mathcal{L}, \bar{\mathcal{L}}^{-1}) = 0$$

as the spectral curve in the present setting. This observation is partly supported by the fact that such a curve is derived as the spectral curve in the random matrix approach to $c = 1$ string theory and interface dynamics [53, 54, 55].

Bearing these remarks in mind, let us turn to the issue of solving the generalized string equations (72). These equations, like (80), are a kind of "nonlinear Riemann-Hilbert problems". One can use the method developed for solving (80) [29] to construct a solution of (72) as power series of $\boldsymbol{t}$ and $\bar{\boldsymbol{t}}$. As it turns out, (72) can be treated in much the same way apart from technical complications.

6.2 Decomposition of equations

Let us convert (72) to the logarithmic form

$$\begin{aligned}\log(\mathcal{L}p^{-1}) &= \log Q - \log(\bar{\mathcal{L}}^{-1}p) + \beta\bar{\mathcal{M}},\\ \log(\bar{\mathcal{L}}^{-1}p) &= \log Q - \log(\mathcal{L}p^{-1}) + \beta\mathcal{M}.\end{aligned} \tag{85}$$

Since $\mathcal{L}p^{-1}$ and $\bar{\mathcal{L}}^{-1}p$ are Laurent series of the form

$$\mathcal{L}p^{-1} = 1 + \sum_{n=1}^{\infty} u_n p^{-n}, \quad \bar{\mathcal{L}}^{-1}p = \bar{u}_0 + \sum_{n=1}^{\infty} \bar{u}_n p^n$$

with nonzero leading terms, one can expand the logarithm as

$$\log(\mathcal{L}p^{-1}) = \sum_{n=1}^{\infty} \alpha_n p^{-n}, \quad \log(\bar{\mathcal{L}}^{-1}p) = \log \bar{u}_0 + \sum_{n=1}^{\infty} \bar{\alpha}_n p^n,$$

where

$$\begin{gathered}\alpha_n = u_n + (\text{polynomial of } u_1, \cdots, u_{n-1}),\\ \bar{\alpha}_n = \bar{u}_0^{-1}\bar{u}_n + (\text{polynomial of } \bar{u}_0^{-1}\bar{u}_1, \cdots, \bar{u}_0^{-1}\bar{u}_{n-1}).\end{gathered}$$

We now substitute

$$\begin{gathered}\bar{\mathcal{M}} = -\sum_{k=1}^{\infty} k\bar{t}_k\bar{\mathcal{L}}^{-k} + s + \sum_{n=1}^{\infty} \bar{v}_n\bar{\mathcal{L}}^{n},\\ \mathcal{M} = \sum_{k=1}^{\infty} kt_k\mathcal{L}^{k} + s + \sum_{n=1}^{\infty} v_n\mathcal{L}^{-n}\end{gathered}$$

in (85) and expand both hand sides in powers of p. This leads to an infinite set of equations for the coefficients $u_n, \bar{u}_n, v_n, \bar{v}_n$ of $\mathcal{L}, \mathcal{M}, \bar{\mathcal{L}}, \bar{\mathcal{M}}$ as follows.

Equating the coefficients of p^{-n}, $n = 0, 1, \cdots$, in both hand sides of the first equation of (85) gives the equations

$$0 = \log Q - \log \bar{u}_0 + \beta s - \beta \sum_{k=1}^{\infty} k\bar{t}_k (\bar{\mathcal{L}}^{-k})_0, \tag{86}$$

$$\alpha_n = -\beta \sum_{k=n}^{\infty} k\bar{t}_k (\bar{\mathcal{L}}^{-k})_{-n}, \quad n = 1, 2, \cdots, \tag{87}$$

where $(\bar{\mathcal{L}}^{-k})_{-n}$ stands for the coefficient of p^{-n} in $\bar{\mathcal{L}}^{-k}$. In the same way, the coefficients of p^n, $n = 0, 1, 2, \cdots$, in the second equation of (85) give the equations

$$\log \bar{u}_0 = \log Q + \beta s + \beta \sum_{k=1}^{\infty} kt_k (\mathcal{L}^k)_0, \tag{88}$$

$$\bar{\alpha}_n = \beta \sum_{k=n}^{\infty} kt_k (\mathcal{L}^k)_n, \quad n = 1, 2, \cdots, \tag{89}$$

where $(\mathcal{L}^k)_n$ denotes the coefficient of p^n in $\mathcal{L}^k$. Since

$$(\mathcal{L}^k)_n = (\bar{\mathcal{L}}^{-k})_{-n} = 0 \quad \text{for } k < n,$$

the range of k in the sums of (87) and (89) is limited to $k \geq n$. Note that one can use the formal residue notation

$$\text{res} \left(\sum_n a_n p^n dp \right) = a_{-1}$$

to express $\left(\mathcal{L}^k\right)_n$ and $\left(\bar{\mathcal{L}}^{-k}\right)_{-n}$ as

$$(\mathcal{L}^k)_n = \text{res}\left(\mathcal{L}^k p^{-n} d\log p\right), \quad (\bar{\mathcal{L}}^{-k})_{-n} = \text{res}\left(\bar{\mathcal{L}}^{-k} p^n d\log p\right).$$

Such an expression turns out to be useful in the subsequent consideration. Since v_n's and $\bar{v}_n$'s are absent, (86) – (89) are equations for u_n's and $\bar{u}_b$'s only.

The remaining part of (85) determine v_n's and $\bar{v}_n$'s. To see this, it is more convenient to expand (85) in powers of $\mathcal{L}$ and $\bar{\mathcal{L}}$ rather than of p. Extracting the coefficients of $\bar{\mathcal{L}}^n$ from the first equation of (72) and those of $\mathcal{L}^{-n}$ from the second equation yields the equations

$$0 = -\,\text{res}\left(\log(\bar{\mathcal{L}}^{-1}p)\bar{\mathcal{L}}^{-n} d\log \bar{\mathcal{L}}\right) + \beta \bar{v}_n, \tag{90}$$

$$0 = -\,\text{res}\left(\log(\mathcal{L}p^{-1})\mathcal{L}^{n} d\log \mathcal{L}\right) + \beta v_n \tag{91}$$

for $n = 1, 2, \cdots$. Thus v_n's and $\bar{v}_n$'s are determined by $\mathcal{L}$ and $\bar{\mathcal{L}}$ as

$$v_n = \beta^{-1} \operatorname{res} \left(\log(\mathcal{L}p^{-1})\mathcal{L}^n d \log \mathcal{L} \right),$$
$$\bar{v}_n = \beta^{-1} \operatorname{res} \left(\log(\bar{\mathcal{L}}^{-1}p)\bar{\mathcal{L}}^{-n} d \log \bar{\mathcal{L}} \right).$$

(85) can be thus decomposed into the infinite set of equations (86) – (89), (90) and (91). The next task is to show that they do have a solution.

6.3 Solution of equations

As we observed above, one can think of (86) – (89) as equations for u_n's and $\bar{u}_n$'s. We want to construct these functions as power series of ($\boldsymbol{t}$ and $\bar{\boldsymbol{t}}$ with coefficients depending on s. If u_k's and $\bar{u}_k$ are thus constructed, v_k and $\bar{v}_k$'s are determined by (90) and (91).

Let us first examine (86) and (88). Since

$$\alpha_n = u_n + (\text{monomials of higher degrees in } u_1, \cdots, u_{n-1}),$$
$$\bar{\alpha}_n = \bar{u}_0^{-1}\bar{u}_n + (\text{monomials of higher degrees in } \bar{u}_0^{-1}\bar{u}_, \cdots, \bar{u}_0^{-1}\bar{u}_{n-1}),$$

u_n and $\bar{u}_n$, $n = 1, 2, \cdots$, show up on the left hand side of these equations linearly. The right hand side consists of terms that are multiplied by t_k's and $\bar{t}_k$'s. Therefore these equations yield a huge system of recursion relations for the coefficients of power series expansion of u_n's and $\bar{u}_n$'s.

Note that $\bar{u}_0$, which remains to be determined, is contained in the coefficients of the power series expansion of u_n and $\bar{u}_n$. We need another equation to determine $\bar{u}_0$. Actually, there are two equations (86) and (88) rather than just one. This puzzle is resolved as follows [2].

Lemma 5. *If (87) and (89) are satisfied, then (86) and (88) are equivalent, and reduces to the equation*

$$\log \bar{u}_0 = \log Q + \beta s + \sum_{k=1}^{\infty} k \alpha_k \bar{\alpha}_k. \tag{92}$$

Proof. Let us use the formal residue notation to express the terms $(\mathcal{L}^k)_0$ in (86) as

$$(\mathcal{L}^k)_0 = \operatorname{res}(\mathcal{L}^k d \log p).$$

[2] A similar result holds for the generalized string equations (80) of $c = 1$ string theory. This fills a logical gap left in our previous paper [29].

Since the identity

$$0 = \operatorname{res}(\mathcal{L}^k d\log\mathcal{L}) = \operatorname{res}\left(\mathcal{L}^k p\frac{\partial\log\mathcal{L}}{\partial p}d\log p\right)$$

holds for $k \geq 1$, this expression of $(\mathcal{L}^k)_0$ can be further rewritten as

$$\begin{aligned}(\mathcal{L}^k)_0 &= \operatorname{res}\left(\mathcal{L}^k(1 - p\frac{\partial\log\mathcal{L}}{\partial p})d\log p\right)\\ &= -\operatorname{res}\left(\mathcal{L}^k\frac{\partial\log(\mathcal{L}p^{-1})}{\partial p}dp\right).\end{aligned}$$

By substituting $\mathcal{L}p^{-1} = \alpha_1 p^{-1} + \alpha_2 p^{-2} + \cdots$, the right hand side can be expanded as

$$\begin{aligned}\text{RHS} &= \alpha_1\operatorname{res}(\mathcal{L}^k p^{-2}dp) + 2\alpha_2\operatorname{res}(\mathcal{L}^k p^{-3}dp) + \cdots\\ &= \alpha_1(\mathcal{L}^k)_1 + 2\alpha_2(\mathcal{L}^k)_2 + \cdots.\end{aligned}$$

Since $(\mathcal{L}^k)_n = 0$ for $n > k$, this expansion terminates at the k-th term. One can thus obtain the identity

$$(\mathcal{L}^k)_0 = \alpha_1(\mathcal{L}^k)_1 + 2\alpha_2(\mathcal{L}^k)_2 + \cdots + k\alpha_k(\mathcal{L}^k)_k.$$

In the same way, one can derive the identity

$$(\bar{\mathcal{L}}^{-k})_0 = \bar{\alpha}_1(\bar{\mathcal{L}}^{-k})_{-1} + 2\bar{\alpha}_2(\bar{\mathcal{L}}^{-k})_{-2} + \cdots + k\bar{\alpha}_k(\bar{\mathcal{L}}^{-k})_{-k}.$$

By virtue of these identities, one can rewrite the two sums in (86) and (88) as

$$\begin{aligned}\sum_{k=1}^{\infty} kt_k(\mathcal{L}^k)_0 &= \sum_{k=1}^{\infty} kt_k\left(\alpha_1(\mathcal{L}^k)_1 + 2\alpha_2(\mathcal{L}^k)_2 + \cdots + k\alpha_k(\mathcal{L}^k)_k\right)\\ &= \alpha_1\sum_{k=1}^{\infty} kt_k(\mathcal{L}^k)_1 + 2\alpha_2\sum_{k=1}^{\infty} kt_k(\mathcal{L}^k)_2 + \cdots\\ &= \beta^{-1}\sum_{n=1}^{\infty} n\alpha_n\bar{\alpha}_n\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{\infty} k\bar{t}_k(\bar{\mathcal{L}}^{-k})_0 &= \sum_{k=1}^{\infty} k\bar{t}_k\left(\bar{\alpha}_1(\bar{\mathcal{L}}^{-k})_1 + 2\bar{\alpha}_2(\bar{\mathcal{L}}^{-k})_2 + \cdots + k\bar{\alpha}_k(\bar{\mathcal{L}}^{-k})_k\right)\\ &= \bar{\alpha}_1\sum_{k=1}^{\infty} k\bar{t}_k(\bar{\mathcal{L}}^{-k})_1 + 2\bar{\alpha}_2\sum_{k=1}^{\infty} k\bar{t}_k(\bar{\mathcal{L}}^{-k})_2 + \cdots\\ &= -\beta^{-1}\sum_{n=1}^{\infty} n\bar{\alpha}_n\alpha_n.\end{aligned}$$

Note that (89) and (87) have been used to derive the last lines. Thus (86) and (88) turn out to reduce to the same equation (92). □

We can thus use (92) in place of (86) and (88). Adding this equation to (87) and (89), we obtain a full system of equations that determine the power series expansion of u_n, $\bar{u}_n$ and $\bar{u}_0$ recursively. We can readily see from (92) that $\bar{u}_0$ is a power series of the form

$$\log \bar{u}_0 = \log Q + \beta s + (\text{terms of positive orders in } \boldsymbol{t}, \bar{\boldsymbol{t}}). \tag{93}$$

On the other hand, since

$$\begin{gathered}(\mathcal{L}^n)_n = 1, \quad (\bar{\mathcal{L}}^{-n})_{-n} = \bar{u}_0^n, \\ (\mathcal{L}^k)_n = (\text{polynomial in } u_1, \cdots, u_{k-n}) \quad \text{for } k > n, \\ (\bar{\mathcal{L}}^{-k})_{-n} = \bar{u}_0^k \times (\text{polynomial in } \bar{u}_0^{-1}\bar{u}_1, \cdots, \bar{u}_0^{-1}\bar{u}_{k-n}) \quad \text{for } k > n,\end{gathered}$$

u_n and $\bar{u}_n$ are power series of the form

$$\begin{aligned} u_n &= -\beta n \bar{t}_n \bar{u}_0^n + (\text{terms of higher orders in } \boldsymbol{t}, \bar{\boldsymbol{t}}), \\ \bar{u}_n &= \beta n t_n \bar{u}_0 + (\text{terms of higher orders in } \boldsymbol{t}, \bar{\boldsymbol{t}}). \end{aligned} \tag{94}$$

This power series solution of (72) (which is unique by construction) is homogeneous just like the solution of the generalized string equations (80) for $c = 1$ string theory [29]. This is a consequence of invariance of the the string equations under the scaling transformations

$$\begin{gathered} t_n \to c^{-n} t_n, \quad \bar{t}_n \to c^n \bar{t}_n, \quad s \to s, \quad p \to cp, \\ u_n \to c^n u_n, \quad \bar{u}_n \to c^{-n} \bar{u}_n, \quad v_n \to c^n v_n, \quad \bar{v}_n \to c^{-n} \bar{v}_n. \end{gathered} \tag{95}$$

In summary, we have observed the following:

Theorem 4. *The generalized string equations (72) have a unique solution that has power series expansion with respect to $(\boldsymbol{t}, \bar{\boldsymbol{t}})$ as shown in (93) and (94). This solution is homogeneous with respect to the scaling transformation (95).*

6.4 Solutions at special values of $\boldsymbol{t}, \bar{\boldsymbol{t}}$

The foregoing construction of solution simplifies to some extent when $\boldsymbol{t}$ and $\bar{\boldsymbol{t}}$ take special values. Of particular interest are the following two cases:

(i) t_k's are free, and $\bar{t}_k$'s are restricted to $\bar{t}_k = \bar{t}_1 \delta_{k1}$,

(ii) $\bar{t}_k$'s are restricted to $\bar{t}_k = \bar{t}_1 \delta_{k1}$, and t_k's are free.

They amount to restricting the generating function $Z[\boldsymbol{t},\bar{\boldsymbol{t}}]$ of double Hurwitz numbers to generating functions of simple Hurwitz numbers. Since these two cases are essentially equivalent, let us consider (i) only.

In the case of (i), (87) implies that α_n vanishes for $n>1$ and that the only non-vanishing component is given by

$$\alpha_1 = u_1 = -\beta\bar{t}_1\bar{u}_0.$$

Thus $\mathcal{L}$ simplifies as

$$\mathcal{L} = pe^{\alpha_1 p^{-1}} = pe^{-\beta\bar{t}_1\bar{u}_0 p^{-1}}, \tag{96}$$

and $(\mathcal{L}^k)_n$ can be written explicitly as

$$(\mathcal{L}^k)_n = \frac{(ku_1)^{k-n}}{(k-n)!} = \frac{(-k\beta\bar{t}_1\bar{u}_0)^{k-n}}{(k-n)!}. \tag{97}$$

$\bar{u}_n$, $n=1,2,\cdots$, are thereby recursively determined by (89) as a function of t_k's and $\bar{u}_0$. $\bar{u}_0$ is determined by (88), which now takes an explicit form as

$$\log\bar{u}_0 = \log Q + \beta s + \beta\sum_{k=1}^{\infty} kt_k\frac{(-k\beta\bar{t}_1\bar{u}_0)^k}{k!}. \tag{98}$$

v_n and $\bar{v}_n$'s, too, have more or less explicit formulae, though we omit details.

This result shows a remarkable feature. Namely, (96) resembles the defining equation $x = ye^y$ of Lambert's W-function $y = W(x)$ if $\mathcal{L}^{-1}$ and p^{-1} are identified with x and y. The so called Lambert curve is defined by this equation on the (x,y)-plane, and plays a fundamental role in the recent studies on Hurwitz numbers [14, 15, 16, 35, 36].

This analogy becomes more precise when t_k's, too, are specialized to $t_k = 0$, $k=1,2,\cdots$. In that case, $\bar{u}_0$ is explicitly determined as

$$\bar{u}_0 = Qe^{\beta s}.$$

Moreover, (89) implies that $\bar{\alpha}_n$ vanishes for all n, hence

$$\bar{\mathcal{L}}^{-1} = \bar{u}_0 p^{-1}. \tag{99}$$

(96) thereby turns into the equation

$$\mathcal{L} = \bar{u}_0\bar{\mathcal{L}}e^{-\beta\bar{t}_1\bar{\mathcal{L}}^{-1}} \tag{100}$$

for $\mathcal{L}$ and $\bar{\mathcal{L}}$. In view of the remarks in the beginning of this section, it seems likely that this equation can be identified with the spectral curve for simple Hurwitz numbers.

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