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General relativistic symmetry of electron spin torque

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Abstract In our recent paper, the electron spin torque is found to be counter-balanced by the chiral electron density. In this paper, we shall show that the origin of the chiral nature is manifest in the principle of equivalence in general relativity.

Keywords spin torque; zeta force; vorticity; chirality; principle of equivalence; general relativity

1 Introduction

The dynamics of electron spin with the realization of spin-orbit coupling has recently been of keen interest particularly in the field of spin torque transfer in spintronics; see recent reviews [1-3] and references cited therein. We have shown that the spin torque of the spin-1/2 Fermion is counter-balanced by the chiral electron density through the zeta force [4]. We have also reported some preliminary numerical data of the spin torque and zeta force for dimer of alkali atoms [5]. The special interest in this previous paper was to the spin dynamics issues in Bose-Einstein condensation; see our previous paper [5] and references cited therein. We shall show in this paper that the origin of the chiral nature is manifest in the principle of equivalence in general relativity. We invoke here the covariant formalism of general relativity equipped with vierbein (tetrad) field on curved spacetime: see documents [6-8] and references cited therein.

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We may first quickly review basic mathematics. The coordinate x with the contravariant components x^{μ} and the covariant components x_{μ} and the metric tensor $\eta_{\mu\nu}=\eta^{\mu\nu}$ of the Minkowski space, together with the inner product of two 4-vectors A and B written as $A\cdot B$ as well as the inner product of the Dirac gamma matrices γ^{μ} and a 4-vectors A written as the Dirac slash $\not{A}$ are defined as follows:

$$x^{\mu}=\left(x^{0},x^{k}\right)=\left(x^{0},x^{1},x^{2},x^{3}\right)=\left(ct,x,y,z\right)=\left(ct,\vec{r}\right)=\left(ct,\vec{x}\right) \tag{1}$$

$$x_{\mu}=\eta_{\mu\nu}x^{\nu}=\left(x_{0},x_{k}\right)=\left(x_{0},x_{1},x_{2},x_{3}\right)=\left(ct,-x,-y,-z\right)=\left(ct,-\vec{r}\right)=\left(ct,-\vec{x}\right) \tag{2}$$

$$\eta_{\mu\nu}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}=\eta^{\mu\nu},\quad \eta^{\mu\rho}\eta_{\rho\nu}=\delta^{\mu}{}_{\nu}=\begin{cases}1, & \mu=\nu\\0, & \mu\neq\nu\end{cases} \tag{3}$$

$$A\cdot B=\eta_{\mu\nu}A^{\mu}B^{\nu}=A^{0}B^{0}-\vec{A}\bullet\vec{B},\quad \vec{A}\bullet\vec{B}=A_{x}B_{x}+A_{y}B_{y}+A_{z}B_{z} \tag{4}$$

$$\not{A}=\eta_{\mu\nu}\gamma^{\mu}A^{\nu}=\gamma^{0}A^{0}-\vec{\gamma}\bullet\vec{A},\quad \vec{\gamma}\bullet\vec{A}=\gamma^{1}A_{x}+\gamma^{2}A_{y}+\gamma^{3}A_{z} \tag{5}$$

where c denotes the speed of light in vacuum and the Greek letter runs from 0 to 3 and the Latin from 1 to 3 and the Einstein summation convention is used. The Poincaré transformation $T\left(\Lambda,a\right)\in P_{+}^{\uparrow}$ consists of proper orthochronous Lorentz transformation $T\left(\Lambda,0\right)=\Lambda\in L_{+}^{\uparrow}$ and translation $T\left(0,a\right)$, which acts on x as

$$\begin{gathered}x'=\Lambda x+a,\quad x'^{\mu}=\Lambda^{\mu}{}_{\nu}x^{\nu}+a^{\mu},\quad \det\Lambda=1,\quad \Lambda^{0}{}_{0}>1\\ \left(\Lambda^{-1}\right)^{\mu}{}_{\nu}=\Lambda_{\nu}{}^{\mu}=\eta_{\nu\kappa}\Lambda^{\kappa}{}_{\lambda}\eta^{\lambda\mu}\end{gathered} \tag{6}$$

On the Minkowski space, the relativistic quantum mechanical motion of electron is described by 4-spinor. According to the spinor representation of the Poincare group $P_{+}^{\uparrow}$ [9-11] we have the linear momentum generator P^{μ} and the angular momentum generator $J^{\mu\nu}$, which is composed of mutually commutable orbital $L^{\mu\nu}$ and spin $S^{\mu\nu}$ parts with commutator $\left[a,b\right]=ab-ba$, as

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$$P^{\mu} = p^{\mu}, \quad J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}, \quad \left[L^{\kappa\lambda}, S^{\mu\nu}\right] = 0 \tag{7}$$

$$L^{k\ell} = i\hbar\left(p^k \frac{\partial}{\partial p^\ell} - p^\ell \frac{\partial}{\partial p^k}\right), \quad L^{k0} = i\hbar p^0 \frac{\partial}{\partial p^k} \tag{8}$$

$$S^{\mu\nu} = \frac{i}{4}\hbar\left[\gamma^\mu, \gamma^\nu\right] \tag{9}$$

where p^μ denotes the momentum and γ^μ the Dirac gamma matrices:

$$\gamma^\mu = \left(\gamma^0, \gamma^k\right) = \left(\gamma^0, \gamma^1, \gamma^2, \gamma^3\right) = \left(\gamma^0, \vec{\gamma}\right) \tag{10}$$

$$\gamma_\mu = \eta_{\mu\nu}\gamma^\nu = \left(\gamma_0, \gamma_k\right) = \left(\gamma_0, \gamma_1, \gamma_2, \gamma_3\right) = \left(\gamma^0, -\gamma^1, -\gamma^2, -\gamma^3\right) = \left(\gamma^0, -\vec{\gamma}\right) \tag{11}$$

The spinor $\psi(x)$ in the chiral representation $\psi_{\text{chiral}}(x)$ is constructed by the undotted spinor $\psi_R(x) = \xi^A(x)$ with right-handed chirality and the dotted spinor $\psi_L(x) = \eta_{\dot{U}}(x)$ with left-handed chirality as [12,13]

$$\psi = \psi_{\text{chiral}} = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \begin{pmatrix} \xi^A \\ \eta_{\dot{U}} \end{pmatrix} \tag{12}$$

$$\xi^A = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}, \quad \eta_{\dot{U}} = \begin{pmatrix} \eta_{\dot{1}} \\ \eta_{\dot{2}} \end{pmatrix} \tag{13}$$

The undotted and dotted capital Latin letters run from 1 to 2 and change position by using the antisymmetric matrix ε as

$$\xi_A = \xi^B \varepsilon_{BA}, \quad \eta^{\dot{U}} = \varepsilon^{\dot{U}\dot{V}} \eta_{\dot{V}} \tag{14}$$

$$\xi^A = \varepsilon^{AB} \xi_B, \quad \eta_{\dot{U}} = \eta^{\dot{V}} \varepsilon_{\dot{V}\dot{U}} \tag{15}$$

$$\varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon^{AB}, \quad \varepsilon^{\dot{U}\dot{V}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon_{\dot{U}\dot{V}} \tag{16}$$

where the Einstein summation convention is used. Using the two-dimensional irreducible representation $\lambda = \lambda(\Lambda)$ and the outer automorphism $\lambda \to \bar{\lambda} = \lambda^{\dagger -1}$ of $SL(2,\mathbf{C})$, the chiral spinor representation $D(\lambda)$ of the Poincaré group $P_+^\uparrow$ is the direct

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sum of λ and its inequivalent complex conjugate irreducible representation $\bar{\lambda} = \lambda^{\dagger -1}$ as [13]

$$\psi' = D(\lambda)\psi, \quad \det \lambda = 1 \tag{17}$$

$$D(\lambda) = \begin{pmatrix} \lambda^{A}{}_{B} & 0 \\ 0 & \bar{\lambda}_{\dot{U}}{}^{\dot{V}} \end{pmatrix}, \quad \bar{\lambda} = \lambda^{\dagger -1} \tag{18}$$

$$\xi'^{A} = \lambda^{A}{}_{B}\xi^{B}, \quad \eta'_{\dot{U}} = \bar{\lambda}_{\dot{U}}{}^{\dot{V}}\eta_{\dot{V}} \tag{19}$$

The Pauli matrix σ with the contravariant components σ^{μ} and the covariant components σ_{μ}

$$\sigma^{\mu} = \left(\sigma^{0}, \sigma^{k}\right) = \left(\sigma^{0}, \sigma^{1}, \sigma^{2}, \sigma^{3}\right) = \left(1, \sigma_{x}, \sigma_{y}, \sigma_{z}\right) = \left(1, \vec{\sigma}\right) \tag{20}$$

$$\sigma_{\mu} = \eta_{\mu\nu}\sigma^{\nu} = \left(\sigma_{0}, \sigma_{k}\right) = \left(\sigma_{0}, \sigma_{1}, \sigma_{2}, \sigma_{3}\right) = \left(1, -\sigma_{x}, -\sigma_{y}, -\sigma_{z}\right) = \left(1, -\vec{\sigma}\right) \tag{21}$$

(note the use of 1 as the unit matrix throuout in this paper) are cast into the Misner-Thorne-Wheeler (MTW) representation as [14]

$$\begin{aligned}
\left(\sigma_{0}\right)^{A\dot{U}} &= \left(\sigma^{0}\right)_{\dot{V}B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma^{0} \\
\left(\sigma_{1}\right)^{A\dot{U}} &= \left(\sigma^{1}\right)_{\dot{V}B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_{x} \\
\left(\sigma_{2}\right)^{A\dot{U}} &= \left(\sigma^{2}\right)_{\dot{V}B} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_{y} \\
\left(\sigma_{3}\right)^{A\dot{U}} &= \left(\sigma^{3}\right)_{\dot{V}B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_{z}
\end{aligned} \tag{22}$$

Then, the Lorentz transformation

$$x' = \Lambda x, \quad \sigma' = \lambda\sigma\lambda^{\dagger}, \quad \lambda = \lambda(\Lambda) \tag{23}$$

leaves the determinant of the inner product $x \cdot \sigma$

$$\left(x \cdot \sigma\right)^{A\dot{U}} = \begin{pmatrix} ct + z & x - iy \\ x + iy & ct - z \end{pmatrix} \tag{24}$$

invariant:

$$x'^{2} = \det\left(x' \cdot \sigma\right)^{A\dot{U}} = \det\left(x \cdot \sigma'\right)^{A\dot{U}} = \det\left(x \cdot \sigma\right)^{A\dot{U}} = x^{2} \tag{25}$$

using

$$\left(x'\cdot\sigma\right)^{A\dot{U}}=\left(x\cdot\sigma'\right)^{A\dot{U}} \tag{26}$$

with

$$\left(\sigma'_{\mu}\right)^{A\dot{U}}=\lambda^{A}{}_{B}\lambda^{*\dot{U}}{}_{\dot{V}}\left(\sigma_{\mu}\right)^{B\dot{V}}=\Lambda^{\nu}{}_{\mu}\left(\sigma_{\nu}\right)^{A\dot{U}} \tag{27}$$

$$\left(\sigma'^{\mu}\right)^{A\dot{U}}=\lambda^{A}{}_{B}\lambda^{*\dot{U}}{}_{\dot{V}}\left(\sigma^{\mu}\right)^{B\dot{V}}=\Lambda_{\nu}{}^{\mu}\left(\sigma^{\nu}\right)^{A\dot{U}} \tag{28}$$

Also, the Dirac gamma matrices γ^{μ} and the chiral matrix γ_5

$$\gamma_5=i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{29}$$

are given in the chiral representation using the MTW representation of the Pauli matrices as

$$\begin{aligned}\gamma^0&=\begin{pmatrix}0&\left(\sigma_0\right)^{A\dot{U}}\\ \left(\sigma^0\right)_{\dot{V}B}&0\end{pmatrix}=\begin{pmatrix}0&\sigma^0\\ \sigma^0&0\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\\ \gamma^k&=\begin{pmatrix}0&-\left(\sigma_k\right)^{A\dot{U}}\\ \left(\sigma^k\right)_{\dot{V}B}&0\end{pmatrix}=\begin{pmatrix}0&-\sigma^k\\ \sigma^k&0\end{pmatrix}\\ \gamma_5&=\begin{pmatrix}\left(\sigma^0\right)^{A}{}_{B}&0\\ 0&-\left(\sigma^0\right)_{\dot{U}}{}^{\dot{V}}\end{pmatrix}=\begin{pmatrix}\sigma^0&0\\0&-\sigma^0\end{pmatrix}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\end{aligned} \tag{30}$$

where the following MTW representation is found for the diagonal block:

$$\begin{aligned}\left(\sigma^0\right)^{A}{}_{B}&=\left(\sigma^0\right)_{\dot{U}}{}^{\dot{V}}=\sigma^0\\ \left(\sigma^1\right)^{A}{}_{B}&=\left(\sigma^1\right)_{\dot{U}}{}^{\dot{V}}=\sigma_x\\ \left(\sigma^2\right)^{A}{}_{B}&=\left(\sigma^2\right)_{\dot{U}}{}^{\dot{V}}=\sigma_y\\ \left(\sigma^3\right)^{A}{}_{B}&=\left(\sigma^3\right)_{\dot{U}}{}^{\dot{V}}=\sigma_z\end{aligned} \tag{31}$$

Using the MTW representation, the Clifford algebra of the Dirac gamma matrices with anticommutator $\{a,b\}=ab+ba$ should be

$$\left\{\gamma^{\mu},\gamma^{\nu}\right\}=2\eta^{\mu\nu}\begin{pmatrix}\left(\sigma^0\right)^{A}{}_{B}&0\\0&\left(\sigma^0\right)_{\dot{U}}{}^{\dot{V}}\end{pmatrix}=2\eta^{\mu\nu}\begin{pmatrix}1&0\\0&1\end{pmatrix}=2\eta^{\mu\nu} \tag{32}$$

Likewise, the conjugate $\psi^\dagger$, the Dirac conjugate $\bar{\psi}=\psi^\dagger\gamma^0$, and the Lorentz scalar $\bar{\psi}\psi$ are

$$\psi^\dagger=\begin{pmatrix}\xi^A\\ \eta_{\dot{U}}\end{pmatrix}^\dagger=\left(\left(\xi^A\right)^\dagger,\left(\eta_{\dot{U}}\right)^\dagger\right) \tag{33}$$

$$\bar{\psi}=\left(\left(\xi^B\right)^\dagger,\left(\eta_{\dot{V}}\right)^\dagger\right)\begin{pmatrix}0 & \left(\sigma_0\right)^{B\dot{U}}\\ \left(\sigma^0\right)_{\dot{V}A} & 0\end{pmatrix}=\left(\left(\eta_{\dot{V}}\right)^\dagger\left(\sigma^0\right)_{\dot{V}A},\left(\xi^B\right)^\dagger\left(\sigma_0\right)^{B\dot{U}}\right) \tag{34}$$

$$\bar{\psi}\psi=\left(\eta_{\dot{V}}\right)^\dagger\left(\sigma^0\right)_{\dot{V}A}\xi^A+\left(\xi^B\right)^\dagger\left(\sigma_0\right)^{B\dot{U}}\eta_{\dot{U}} \tag{35}$$

2 Electron spin density

Electron spin density and torque for it have recently been studied by Vernes-Györffy-Weinberger in terms of polarization density [2]. In our recent paper [4], we have studied the electron spin density and torque for it in terms of the bilinear covariants of the Lorentz transformation [15] using the approach of Ref. [9-11] described in the previous section, which we shall briefly review in the following subsections 2.1 and 2.2.

2.1 Bilinear covariants

Using Eqs. (9) and (30), the chiral representation of the spin angular momentum reduces to

$$\begin{aligned}S^{k\ell}&=\frac{i}{4}\hbar\left[\gamma^k,\gamma^\ell\right]=\frac{1}{2}\hbar\varepsilon_{k\ell m}S^m\\ \vec{S}&=\frac{1}{2}\hbar\gamma^0\vec{\gamma}\gamma_5=\frac{1}{2}\hbar\begin{pmatrix}\left(\vec{\sigma}\right)^A{}_B & 0\\ 0 & \left(\vec{\sigma}\right)_{\dot{U}}{}^{\dot{V}}\end{pmatrix}\end{aligned} \tag{36}$$

$$S^{k0}=\frac{1}{2}i\hbar\gamma^k\gamma^0=-\frac{1}{2}i\hbar\begin{pmatrix}\left(\sigma^k\right)^A{}_B & 0\\ 0 & -\left(\sigma^k\right)_{\dot{U}}{}^{\dot{V}}\end{pmatrix}=-i\gamma_5S^k \tag{37}$$

where $\varepsilon_{k\ell m}$ denotes the Levi-Civita symbol.

The spin density is then written in the bilinear covariant form as the axial vector (pseudovector):

$$\vec{s}(x)=\frac{1}{2}\hbar\bar{\psi}\left(x\right)\vec{\gamma}\gamma_5\psi\left(x\right)\vec{s}(x)=\psi^\dagger\left(x\right)\vec{S}\psi\left(x\right)=\frac{1}{2}\hbar\vec{\sigma}(x) \tag{38}$$

$$\vec{\sigma}(x) = \vec{\sigma}_R(x) + \vec{\sigma}_L(x) \tag{39}$$

$$\begin{aligned} \vec{\sigma}_R(x) &= \psi_R{}^{\dagger}(x)\vec{\sigma}\psi_R(x) = \left(\xi^A\right)^{\dagger}(x)\left(\vec{\sigma}\right)^A{}_B\,\xi^B(x) \\ \vec{\sigma}_L(x) &= \psi_L{}^{\dagger}(x)\vec{\sigma}\psi_L(x) = \left(\eta_{\dot{U}}\right)^{\dagger}(x)\left(\vec{\sigma}\right)_{\dot{U}}{}^{\dot{V}}\,\eta_{\dot{V}}(x) \end{aligned} \tag{40}$$

which is the spatial part of the 3rd rank antisymmetric tensor [12]. For discrete symmetry for the spin density $\vec{s}(x)$, we have charge conjugation *C*, parity *P*, time-reversal *T* symmetry as follows:

$$\mathbf{C}\vec{s}\left(x\right)\mathbf{C}^{-1} = \vec{s}\left(x\right) \tag{41}$$

$$\mathbf{P}\vec{s}\left(x\right)\mathbf{P}^{-1} = \vec{s}\left(Px\right) \tag{42}$$

$$\mathbf{T}\vec{s}\left(x\right)\mathbf{T}^{-1} = -{}^{t}\vec{s}\left(Tx\right) \tag{43}$$

with

$$P^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{44}$$

$$T^{\mu}{}_{\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{45}$$

where ${}^{t}\vec{s}$ in the right hand side of Eq. (43) means ${}^{t}\vec{\sigma}$ be used in place of $\vec{\sigma}$ in Eq. (40).

We may first examine the chiral charge, spin density and spin torque summarized in Appendices A and B (see Appendices A and B) for free electron satisfying the Dirac equation in this case as

$$\left(i\hbar\gamma^{\mu}\partial_{\mu} - mc\right)\psi\left(x\right) = 0, \quad \partial_{\mu} = \partial / \partial x^{\mu} \tag{46}$$

where m is the mass of electron. The stationary state solution with the 3rd eigenvalue $\zeta = \pm\frac{1}{2}\hbar$ of spin $S^3 = \vec{S} \bullet \vec{e}_z$ using the unit vector $\vec{e}_z$ along the 3rd axis is:

$$\psi\left(x,\zeta\right) = u\left(\vec{p},\zeta\right)e^{-\frac{i}{\hbar}x\cdot p}, \quad \left(\not{p} - mc\right)u\left(\vec{p},\zeta\right) = 0, \quad \zeta = \pm\frac{1}{2}\hbar \tag{47}$$

where [15]

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$$u(\vec{p},\frac{1}{2}\hbar)=\frac{1}{2\sqrt{p^0(p^0+mc)}}\begin{pmatrix} p^0+mc+p_z \\ p_+ \\ p^0+mc-p_z \\ -p_+ \end{pmatrix},\quad p_+=p_x+ip_y \tag{48}$$

$$u(\vec{p},-\frac{1}{2}\hbar)=\frac{1}{2\sqrt{p^0(p^0+mc)}}\begin{pmatrix} p_- \\ p^0+mc-p_z \\ -p_- \\ p^0+mc+p_z \end{pmatrix},\quad p_-=p_x-ip_y \tag{49}$$

In the rest frame attached to electron, the charge density, Eq. (A5), and the chiral spin density, Eq. (40), are then

$$\begin{aligned} N_R\left(\vec{0},\pm\frac{1}{2}\hbar\right)&=\frac{1}{2} \\ N_L\left(\vec{0},\pm\frac{1}{2}\hbar\right)&=\frac{1}{2} \end{aligned} \tag{50}$$

$$\begin{aligned} \vec{\sigma}_R(\vec{0},\pm\frac{1}{2}\hbar)&=\pm\frac{1}{2}\vec{e}_z \\ \vec{\sigma}_L(\vec{0},\pm\frac{1}{2}\hbar)&=\pm\frac{1}{2}\vec{e}_z \end{aligned} \tag{51}$$

In the inertial frame attached to observer, we have instead

$$\begin{aligned} N_R\left(\vec{p},\pm\frac{1}{2}\hbar\right)&=\frac{1}{2p^0}\left(p^0\pm p_z\right) \\ N_L\left(\vec{p},\pm\frac{1}{2}\hbar\right)&=\frac{1}{2p^0}\left(p^0\mp p_z\right) \end{aligned} \tag{52}$$

$$\begin{aligned} \vec{\sigma}_R(\vec{p},\pm\frac{1}{2}\hbar)&=\pm\frac{mc}{2p^0}\vec{e}_z+\frac{1\pm\frac{p_z}{p^0+mc}}{2p^0}\vec{p} \\ \vec{\sigma}_L(\vec{p},\pm\frac{1}{2}\hbar)&=\pm\frac{mc}{2p^0}\vec{e}_z-\frac{1\mp\frac{p_z}{p^0+mc}}{2p^0}\vec{p} \end{aligned} \tag{53}$$

where the spin-orbit coupling (polarization) appears in the chiral spin density. The polarization may be combined to give

$$\vec{s}\left(\vec{p},\pm\frac{1}{2}\hbar\right)=\frac{1}{2}\hbar\vec{\sigma}\left(\vec{p},\pm\frac{1}{2}\hbar\right)=\pm\frac{1}{2}\hbar\left(\frac{mc}{p^0}\vec{e}_z+\frac{p_z}{p^0\left(p^0+mc\right)}\vec{p}\right) \tag{54}$$

which is the well-known formula, Eq. (3.147) of [16]. The spin torque does not of course work in this case, but if electron is accelerated by the external electromagnetic field, further spin-orbit coupling, the Thomas precession and therefore the spin torque emerge to bring about the resultant further polarization as shown in Appendix C (see Appendix C).

It should be noted that chirality of a particle has nothing to do with helicity as an observable but is right (or left)-handed if its spin transforms according to a two-dimensional irreducible representation λ (or its inequivalent complex conjugate irreducible representation $\bar{\lambda} = \lambda^{\dagger -1}$) of $SL(2,\mathbf{C})$. Helicity of a particle is right (or left)-handed if its spin is parallel (or antiparallel) to its linear momentum. For massive particles, helicity is not conserved since the direction of linear momentum depends on the inertial frame of observer. For massless particles, helicity is conserved keeping direct relationship with chirality.

2.2 Spin torque

The equation of motion of electron spin is [4]

$$\frac{\partial}{\partial t}\vec{s}(x) = \vec{t}(x) + \vec{\zeta}(x) \tag{55}$$

for electron under the external electromagnetic field satisfying the Dirac equation with the covariant derivative as

$$\left(i\hbar\gamma^{\mu}D_{\mu} - mc\right)\psi\left(x\right) = 0 \tag{56}$$

$$D_{\mu} = \partial_{\mu} + i\frac{q}{\hbar c}A_{\mu}\left(x\right) \tag{57}$$

where $q = -e$ is the charge of electron and $A^{\mu}\left(x\right)$ the Abelian gauge potential.

The right hand side of Eq. (55) is composed of two terms. First, the spin torque $\vec{t}(x)$, which is given by the antisymmetric part of the nonsymmetric (symmetry-polarized) Hermitean stress tensor $\vec{\tau}^{\Pi}(x)$:

$$t^{k}(x) = -\varepsilon_{\ell nk}\tau^{\Pi\ell n}(x) \tag{58}$$

$$\tau^{\Pi k\ell}(x) = \frac{c}{2}\left(\bar{\psi}(x)\gamma^{\ell}\left(-i\hbar D^{k}\right)\psi(x) + h.c.\right) \tag{59}$$

$$\vec{\tau}^{\Pi\dagger}(x) = \vec{\tau}^{\Pi}(x) \tag{60}$$

$$\tau^{\Pi k\ell}(x) \neq \tau^{\Pi\ell k}(x);\ \text{nonsymmetric (symmetry-polarized)} \tag{61}$$

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and the zeta force $\vec{\zeta}(x)$, which is given by the gradient of the zeta potential $\phi_5(x)$ proportional to the chiral charge density $j_5^0(x)$ (see Appendix A) as follows:

$$\zeta^k(x) = -\partial_k \phi_5(x) \tag{62}$$

$$\phi_5(x) = \frac{\hbar c}{2q} j_5^0(x) = \frac{\hbar c^2}{2}\left(N_R(x) - N_L(x)\right) \tag{63}$$

For chiral spin-1/2 Fermion with the non-Abelian gauge potential, analogous equation of motion of spin has been found [4].

Electron spin density $\vec{s}(x)$ has rotational character as manifestly shown by the vorticity $\mathrm{rot}\vec{s}\left(x\right)$:

$$\mathrm{rot}\vec{s}\left(x\right) = \frac{1}{2}\left(\bar{\psi}\left(x\right)\vec{\gamma}\left(i\hbar D_0\right)\psi\left(x\right) + h.c.\right) - \vec{\Pi}\left(x\right) \tag{64}$$

In Eq. (64) is shown the kinetic momentum density $\vec{\Pi}(x)$ defined as

$$\vec{\Pi}(x) = \frac{1}{2}\left(\psi^\dagger(x)\left(i\hbar\vec{D}(x)\right)\psi(x) + h.c.\right) \tag{65}$$

This satisfies the equation of motion [4]

$$\frac{\partial}{\partial t}\vec{\Pi}(x) = \vec{F}(x) \tag{66}$$

$$\vec{F}(x) = \vec{L}(x) + \vec{\tau}^{\Pi}(x) \tag{67}$$

The force density $\vec{F}(x)$ is composed not only of the Lorentz force density $\vec{L}(x)$ but also of the tension density $\vec{\tau}^{\Pi}(x)$ which is the divergence of the symmetry-polarized stress tensor $\vec{\vec{\tau}}^{\Pi}(x)$ given in Eq. (59):

$$\vec{\tau}^{\Pi}(x) = \mathrm{div}\,\vec{\vec{\tau}}^{\Pi}(x), \quad \vec{\tau}^{\Pi k}(x) = \partial_\ell \tau^{\Pi k\ell}(x) \tag{68}$$

The stress tensor itself is not defined uniquely [17,18] since mathematically any tensor whose divergence is zero can be added to. Our stress tensor $\vec{\vec{\tau}}^{\Pi}(x)$ in Eq. (59) is defined in such a way that it appears in the equation of motion of $\vec{\Pi}(x)$ as in Eqs. (66)-(68).

The chiral partitionings of the working equations are summarized in Appendix B (see Appendix B). Examples of the spin torque are found in Appendix C: the Volkow wave function of the Dirac electron under the external time-dependent plane wave

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electromagnetic field (see Appendix C) and Appendix D: the Landau wave function of the Dirac electron under the influence of the external static uniform magnetic field (see Appendix D).

2.3 Energy-momentum tensor

In this section, we derive the spin torque, zeta force and vorticity as a consequence of the symmetry of energy-momentum tensor of gravitation. The gravitational action I_G is added to the system action I_S and made stationary:

$$\delta I = 0, \quad I = I_G + I_S \tag{69}$$

under the variation $\delta g^{\mu\nu}$ of the metric tensor $g^{\mu\nu}$:

$$I_G = \frac{c}{2\kappa}\int R\sqrt{-g}\,d^4x, \;\; \delta I_G = \frac{c}{2\kappa}\int\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu}\sqrt{-g}\,d^4x \tag{70}$$

$$I_S = \frac{1}{c}\int L\sqrt{-g}\,d^4x, \;\; \delta I_S = \frac{1}{2c}\int T_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}\,d^4x \tag{71}$$

The Einstein equation is then derived

$$G_{\mu\nu}(x) = Y_{\mu\nu}(x) \tag{72}$$

with the definition

$$G_{\mu\nu}(x) = R_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)R(x) \tag{73}$$

$$Y_{\mu\nu}(x) = -\frac{\kappa}{c^2}T_{\mu\nu}(x) \tag{74}$$

Since the Einstein tensor $G_{\mu\nu}(x)$ is symmetric, so is the energy momentum tensor

$T_{\mu\nu}(x)$:

$$G_{\mu\nu}(x) = G_{\nu\mu}(x); \text{ symmetric} \tag{75}$$

$$T_{\mu\nu}(x) = T_{\nu\mu}(x); \text{ symmetric} \tag{76}$$

Using *the tetrad formalism equipped with the principle of equivalence*, the metric tensor in any general noninertial coordinate system is given as

$$g_{\mu\nu}(x) = e^a{}_\mu(x)e^b{}_\nu(x)\eta_{ab} \tag{77}$$

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where $e^{a}{}_{\mu}(x)$ denotes the tetrad field and the Latin letters a,b,c and so on runs from 0 to 3 in this and the subsequent subsections 2.3 and 2.4. The tetrad field $e^{a}{}_{\mu}(x)$ is a coordinate vector and a Lorentz vector for the Lorentz transformation $x \to x'$ associated with the vector representation $\Lambda^{a}{}_{b}(x)$[8]:

$$e^{a}{}_{\mu}(x) \to e'^{a}{}_{\mu}(x') = \frac{\partial x^{\nu}}{\partial x'^{\mu}} e^{a}{}_{\nu}(x) \tag{78}$$

$$e^{a}{}_{\mu}(x) \to e'^{a}{}_{\mu}(x) = \Lambda^{a}{}_{b}(x) e^{b}{}_{\mu}(x) \tag{79}$$

and is parallely transported :

$$\partial_{\nu} e_{a}{}^{\lambda} + \left\{ {}_{\kappa}{}^{\lambda}{}_{\nu} \right\} e_{a}{}^{\kappa} - \gamma_{a}{}^{b}{}_{\nu} e_{b}{}^{\lambda} = 0 \tag{80}$$

In Eq. (80), we used the Levi-Civita affine connection

$$\left\{ {}_{\mu}{}^{\lambda}{}_{\nu} \right\} = \frac{1}{2} g^{\lambda\rho} \left(\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\mu\rho} - \partial_{\rho} g_{\mu\nu} \right) = \left\{ {}_{\nu}{}^{\lambda}{}_{\mu} \right\} \tag{81}$$

and spin connection

$$\gamma_{a}{}^{b}{}_{\mu} = e_{a\nu;\mu} \eta^{bc} e_{c}{}^{\nu} \tag{82}$$

where the covariant derivative is defined as

$$e_{a}{}^{\lambda}{}_{;\nu} = e_{a}{}^{\lambda}{}_{,\nu} + \left\{ {}_{\kappa}{}^{\lambda}{}_{\nu} \right\} e_{a}{}^{\kappa} \tag{83}$$

$$e_{a\lambda;\nu} = e_{a\lambda,\nu} - \left\{ {}_{\lambda}{}^{\kappa}{}_{\nu} \right\} e_{a\kappa} \tag{84}$$

with the usual partial derivative denoted as

$$f_{,\mu} = \partial_{\mu} f \tag{85}$$

In the tetrad formalism [7,8], the absolute parallelism of the tetrad field $e^{a}{}_{\mu}(x)$ is found to be

$$D^{*}{}_{\nu} e_{a}{}^{\lambda} = \partial_{\nu} e_{a}{}^{\lambda} + \Gamma^{*}{}_{\mu}{}^{\lambda}{}_{\nu} e_{a}{}^{\mu} = 0 \tag{86}$$

and the connection

$$\Gamma^{*}{}_{\mu}{}^{\lambda}{}_{\nu} = \left\{ {}_{\mu}{}^{\lambda}{}_{\nu} \right\} - e^{a}{}_{\mu} \gamma_{a}{}^{b}{}_{\nu} e_{b}{}^{\lambda} \tag{87}$$

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is used to define the torsion tensor

$$T^{*\lambda}_{\bullet\ \mu\nu} = \Gamma^{*\ \lambda}_{\mu\ \nu} - \Gamma^{*\ \lambda}_{\nu\ \mu} \tag{88}$$

and contorsion tensor

$$K^{*}_{\lambda\mu\nu} = \frac{1}{2}\left(T^{*}_{\lambda\mu\nu} - T^{*}_{\mu\lambda\nu} - T^{*}_{\nu\lambda\mu}\right) \tag{89}$$

In the tetrad formalism, the Dirac spinor field is a coordinate scalar and a Lorentz spinor [8]:

$$\psi_\alpha(x) \to \psi'_\alpha(x') = \psi_\alpha(x) \tag{90}$$

$$\psi_\alpha(x) \to \psi'_\alpha(x) = D_{\alpha\beta}\left(\Lambda(x)\right)\psi_\beta(x) \tag{91}$$

Also, what is important, the covariant derivative $D_\mu(g)$ is not only a coordinate scalar, but also a Lorentz vector, as shown in Eqs. (12.5.15)-(12.5.17) and (12.5.24) of [8]:

$$D_\mu(g) = \partial_\mu + \Gamma_\mu \tag{92}$$

$$\Gamma_\mu(x) \to \Gamma_\mu'(x) = D\left(\Lambda(x)\right)\Gamma_\mu D^{-1}\left(\Lambda(x)\right) - \left(\partial_\mu D\left(\Lambda(x)\right)\right)D^{-1}\left(\Lambda(x)\right) \tag{93}$$

The Lagrangian density for the quantum electrodynamics (QED) system under external gravity is then given as

$$L = L_{EM} + L_e \tag{94}$$

with the definition

$$L_{EM} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{16\pi}F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{95}$$

$$L_e = \frac{1}{2}c\bar{\psi}\left(i\hbar\gamma^a e_a^{\ \mu}D_\mu(g) - mc\right)\psi + h.c. \tag{96}$$

The gravitational covariant derivative $D_\mu(g)$ in Eq. (96), which satisfies Eqs. (92)-(93) under the Lorentz transformation, is concretely written as

$$\begin{aligned} D_\mu(g) &= \partial_\mu + i\frac{1}{2\hbar}\gamma_{ab\mu}J^{ab} + i\frac{q}{\hbar c}A_\mu \\ &= D_\mu + i\frac{1}{2\hbar}\gamma_{ab\mu}J^{ab} \end{aligned} \tag{97}$$

where the spin angular momentum J^{ab}

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$$J^{ab}=\frac{i\hbar}{4}\left[\gamma^a,\gamma^b\right] \tag{98}$$

is added to the covariant derivative D_μ given in Eq. (57) through the coupling with spin connection $\gamma_{ab\mu}$ given in Eq. (82). *The emergence of the spin connection is manifest as the consequence of the principle of equivalence in general relativity.*

It should be noted here that, after some manipulation we can rewrite Eq. (96) in a very significant form as follows:

$$\begin{aligned}L_e&=\frac{1}{2}c\overline{\psi}\left(i\hbar\gamma^a e_a{}^\mu D_\mu(g)-mc\right)\psi+h.c.\\&=\frac{1}{2}c\overline{\psi}\left(i\hbar\gamma^a e_a{}^\mu\partial_\mu-mc\right)\psi+h.c.-\frac{3\hbar}{4q}a_\mu j_5{}^\mu-\frac{1}{c}A_\mu j^\mu\end{aligned} \tag{99}$$

Namely, which is hidden in Eq. (96), but in this Eq. (99), minimal couplings are manifestly shown; those not only of current $j^\mu(x)$ *with photon vector potential* $A^\mu(x)$ *but also of chiral current* $j_5^\mu(x)$ *(see Appendix A) with spin coupling vector* $a^\mu(x)$ *defined as*

$$a^\mu=\frac{1}{6}\varepsilon^{\mu\nu\rho\sigma}T^*{}_{\nu\rho\sigma} \tag{100}$$

where $T^*{}_{\nu\rho\sigma}$ *is the torsion tensor given in Eq. (88) and we have used the Levi-Civita tensor:*

$$\varepsilon^{\mu\nu\rho\sigma}=\frac{1}{\sqrt{-g}}\delta^{\mu\nu\rho\sigma},\quad \delta^{0123}=1 \tag{101}$$

$$\varepsilon_{\mu\nu\rho\sigma}=\sqrt{-g}\,\delta_{\mu\nu\rho\sigma},\quad \delta_{0123}=-1 \tag{102}$$

Using the Lagrangian given in Eq. (94), the variation principle with respect to the spinor field

$$\frac{\delta}{\delta\overline{\psi}}I_S=0 \tag{103}$$

leads to the field equation

$$\left(i\hbar\gamma^a e_a{}^\mu D_\mu(g)-mc\right)\psi=0 \tag{104}$$

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Second, the variation principle with respect to the tetrad field leads to the symmetric energy-momentum tensor $T_{\mu\nu}$ and the conservation law as follows [8]:

$$\delta I_S = \delta \frac{1}{c}\int L\sqrt{-g}\,d^4x = \frac{1}{c}\int T_{\mu}^{\cdot a}\,\delta e_a{}^{\mu}\sqrt{-g}\,d^4x \tag{105}$$

$$T_{\mu\nu}\sqrt{-g} = \eta_{ab} e^b{}_{\nu} \frac{\partial}{\partial e_a{}^{\mu}} L\sqrt{-g} \tag{106}$$

$$T_{\mu\nu} = -\varepsilon^{\Pi}{}_{\mu\nu} - \tau^{\Pi}{}_{\mu\nu}\left(g\right) - \frac{1}{4\pi} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} - g_{\mu\nu}\left(L_e + L_{EM}\right) = T_{\nu\mu};\ \text{symmetric} \tag{107}$$

$$T^{\nu}{}_{\mu;\nu} = 0 \tag{108}$$

In Eq. (107), we have shown that the symmetric energy-momentum tensor $T_{\mu\nu}$ comprises not only the symmetric tensors but also polarized geometrical tensor $\varepsilon^{\Pi}{}_{\mu\nu}$ defined as

$$\begin{aligned}\varepsilon^{\Pi}{}_{\mu\nu} &= \frac{\hbar c}{4} e_{\lambda\nu} K^{*}{}_{\rho\sigma\mu} \varepsilon^{\lambda\rho\sigma\kappa} \bar{\psi}\gamma_{\kappa}\gamma_5\psi \\ &+ 2\left(\left(D^{*}_{\lambda} + T^{*\kappa}_{\cdot\ \kappa\lambda}\right) F_{\mu\nu\cdot}^{\ \ \ \lambda} + T^{*}{}_{\rho\sigma\mu} F^{\rho\sigma}_{\cdot\ \cdot\ \nu} - \frac{1}{2} T^{*}{}_{\nu\rho\sigma} F_{\mu}^{\cdot\rho\sigma}\right)\end{aligned} \tag{109}$$

with

$$F^{abc} = \frac{\hbar c}{8} \varepsilon^{dabc} \bar{\psi}\gamma_d\gamma_5\psi \tag{110}$$

and polarized stress tensor $\tau^{\Pi}{}_{\mu\nu}\left(g\right)$ with the covariant derivative $D_{\mu}\left(g\right)$ given in Eq. (97) :

$$\tau^{\Pi}{}_{\mu\nu}\left(g\right) = \frac{c}{2}\left(\bar{\psi}\gamma_{\nu}\left(-i\hbar D_{\mu}\left(g\right)\right)\psi + h.c.\right) \tag{111}$$

Now that the energy-momentum tensor $T_{\mu\nu}$ is symmetric, the antisymmetric components should cancel with each other [6]:

$$\varepsilon^{A\mu\nu} + \tau^{A\mu\nu}\left(g\right) = 0 \tag{112}$$

where

$$\varepsilon^{\Pi\mu\nu} = \varepsilon^{S\mu\nu} + \varepsilon^{A\mu\nu} \tag{113}$$

$$\varepsilon^{S\mu\nu} = \frac{1}{2}\left(\varepsilon^{\Pi\mu\nu} + \varepsilon^{\Pi\nu\mu}\right) \tag{114}$$

$$\varepsilon^{A\mu\nu} = \frac{1}{2}\left(\varepsilon^{\Pi\mu\nu} - \varepsilon^{\Pi\nu\mu}\right) \tag{115}$$

and

$$\tau^{\Pi\mu\nu}(g) = \tau^{S\mu\nu}(g) + \tau^{A\mu\nu}(g) \tag{116}$$

$$\tau^{S\mu\nu}(g) = \frac{1}{2}\left(\tau^{\Pi\mu\nu}(g) + \tau^{\Pi\nu\mu}(g)\right) \tag{117}$$

$$\tau^{A\mu\nu}(g) = \frac{1}{2}\left(\tau^{\Pi\mu\nu}(g) - \tau^{\Pi\nu\mu}(g)\right) \tag{118}$$

2.4 Weak gravitation limit

In the limit of weak gravitation field

$$e^{a}{}_{\mu} \to \delta^{a}{}_{\mu}, \quad g_{\mu\nu} \to \eta_{\mu\nu} \tag{119}$$

the equation of motion of the Dirac spinor field $\psi(x)$ is reduced from Eq. (104) to the Dirac Eq. (56) in due course. *Moreover, the antisymmetry cancelling condition of Eq. (112) is reduced to* $\frac{\partial}{\partial t}\vec{s}(x) = \vec{t}(x) + \vec{\zeta}(x)$ *and* $\mathrm{rot}\vec{s}(x) = \frac{1}{2}\left(\bar{\psi}(x)\vec{\gamma}(i\hbar D_0)\psi(x) + h.c.\right) - \vec{\Pi}(x)$. *These equations are nothing but Eqs. (55) and (64) respectively.*

3 Result and discussion

We have shown the electron spin torque, zeta force and vorticity as a consequence of the symmetry of energy-momentum tensor of gravitation. We have invoked here the covariant formalism of general relativity equipped with vierbein (tetrad) field on curved spacetime [6-8]. The symmetry we use here is hence restricted in this sense and not reflect supersymmetry. The development of the spin torque, zeta force and vorticity in the context of supersymmetry should be very interesting, since then the spin of Boson as well as Fermion can be treated in a unified manner, which is under way and will be published elsewhere.

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Appendix A

The electron current is defined by

$$j^{\mu}\left(x\right)=cq\bar{\psi}\left(x\right)\gamma^{\mu}\psi\left(x\right) \tag{A1}$$

Here we have a very interesting chiral decomposition of electron current $j^{\mu}\left(x\right)$ as

$$j^{0}(x)=cqN(x)=cq\left(N_{R}(x)+N_{L}(x)\right) \tag{A2}$$

$$\vec{j}(x)=cq\left(\vec{\sigma}_{R}(x)-\vec{\sigma}_{L}(x)\right) \tag{A3}$$

where the charge density is decomposed into the chiral parts as

$$N\left(x\right)=\psi^{\dagger}\left(x\right)\psi\left(x\right)=N_{R}\left(x\right)+N_{L}\left(x\right) \tag{A4}$$

$$\begin{aligned} N_{R}\left(x\right)&=\psi_{R}{}^{\dagger}\left(x\right)\psi_{R}\left(x\right)=\left(\xi^{A}\right)^{\dagger}(x)\xi^{A}(x) \\ N_{L}\left(x\right)&=\psi_{L}{}^{\dagger}\left(x\right)\psi_{L}\left(x\right)=\left(\eta_{\dot{U}}\right)^{\dagger}(x)\eta_{\dot{U}}(x) \end{aligned} \tag{A5}$$

and $\vec{\sigma}_{R}(x)$ and $\vec{\sigma}_{L}(x)$ are given in Eq. (41). *Namely, the spatial part of the current density is given by the difference in the chiral parts of the spin density.*

The chiral decomposition of $j^{\mu}\left(x\right)$ is also realized in a dual manner with the chiral current $j_{5}^{\mu}(x)$ defined as

$$j_{5}^{\mu}(x)=cq\bar{\psi}(x)\gamma^{\mu}\gamma_{5}\psi(x) \tag{A6}$$

Here we have the chiral decomposition of electron current $j_{5}^{\mu}(x)$ dual to $j^{\mu}\left(x\right)$ as

$$j_{5}{}^{0}(x)=cq\left(N_{R}(x)-N_{L}(x)\right) \tag{A7}$$

$$\vec{j}_5(x) = cq\vec{\sigma}(x) = cq\left(\vec{\sigma}_R(x) + \vec{\sigma}_L(x)\right) \tag{A8}$$

where Eqs. (A7-8) are dual to Eqs. (A2-3). *Namely, the chiral charge density* $j_5^{\ 0}(x)$ *is given by the difference in the chiral parts of the charge density, and the spatial part of the chiral current density* $\vec{j}_5(x)$ *is given by the spin density.*

The charge is conserved but not the chiral charge since we have no continuity equation for the latter because of the nonzero mass of electron. Actually, using Eq. (56), we have continuity equation for $j^\mu(x)$ as

$$\partial_\mu j^\mu(x) = 0 \tag{A9}$$

while for $j_5^{\ \mu}(x)$ we have residual pseudoscalar as the 4th rank antisymmetric tensor as

$$\begin{aligned}
\frac{1}{cq}\partial_\mu j_5^{\ \mu}(x) &= i\frac{2mc}{\hbar}\bar{\psi}(x)\gamma_5\psi(x) \\
&= i\frac{2mc}{\hbar}\left(-\psi_R^{\ \dagger}(x)\psi_L(x) + \psi_L^{\ \dagger}(x)\psi_R(x)\right) \\
&= i\frac{2mc}{\hbar}\left(-\left(\xi^A\right)^\dagger(x)\left(\sigma_0\right)^{A\dot{U}}\eta_{\dot{U}}(x) + \left(\eta_{\dot{U}}\right)^\dagger(x)\left(\sigma^0\right)_{\dot{U}A}\xi^A(x)\right)
\end{aligned} \tag{A10}$$

which is not zero unless m is zero.

Appendix B

The stress tensor is decomposed into the chiral parts as

$$\vec{\tau}^{\Pi}(x) = \vec{\tau}_R^{\ \Pi}(x) - \vec{\tau}_L^{\ \Pi}(x) \tag{B1}$$

$$\begin{aligned}
\tau_R^{\ \Pi k\ell}(x) &= \frac{c}{2}\left(\left(\xi^A\right)^\dagger(x)\left(\sigma^\ell\right)^A{}_B\left(-i\hbar D^k\right)\xi^B(x) + h.c.\right) \\
\tau_L^{\ \Pi k\ell}(x) &= \frac{c}{2}\left(\left(\eta_{\dot{U}}\right)^\dagger(x)\left(\sigma^\ell\right)_{\dot{U}}{}^{\dot{V}}\left(-i\hbar D^k\right)\eta_{\dot{V}}(x) + h.c.\right)
\end{aligned} \tag{B2}$$

and therefore the torque as

$$\vec{t}(x) = \vec{t}_R(x) - \vec{t}_L(x) \tag{B3}$$

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$$t_R{}^k(x) = -\varepsilon_{\ell nk}\tau_R{}^{\Pi\ell n}(x)$$
$$t_L{}^k(x) = -\varepsilon_{\ell nk}\tau_L{}^{\Pi\ell n}(x) \tag{B4}$$

and also the zeta force as

$$\hat{\vec{\zeta}}(x) = \hat{\vec{\zeta}}_R(x) - \hat{\vec{\zeta}}_L(x) \tag{B5}$$

$$\zeta_R{}^k(x) = -\partial_k\phi_{5R}(x)$$
$$\zeta_L{}^k(x) = -\partial_k\phi_{5L}(x) \tag{B6}$$

with the zeta potential

$$\phi(x) = \phi_R(x) - \phi_L(x) \tag{B7}$$

$$\phi_{5R}(x) = \frac{\hbar c^2}{2}N_R(x)$$
$$\phi_{5L}(x) = \frac{\hbar c^2}{2}N_L(x) \tag{B8}$$

The chiral parts of the kinetic momentum follows

$$\vec{\Pi}_R\left(x\right) = \vec{\Pi}_R\left(x\right) + \vec{\Pi}_L\left(x\right) \tag{B9}$$

$$\vec{\Pi}_R\left(x\right) = \frac{1}{2}\left(\left(\xi^A\right)^\dagger(x)\left(i\hbar\vec{D}(x)\right)\xi^A(x) + h.c.\right)$$
$$\vec{\Pi}_L\left(x\right) = \frac{1}{2}\left(\left(\eta_{\dot{U}}\right)^\dagger(x)\left(i\hbar\vec{D}(x)\right)\eta_{\dot{U}}(x) + h.c.\right) \tag{B10}$$

Thus, the chiral partitionings in Eq. (55) are

$$\frac{\partial}{\partial t}\left(\vec{s}_R(x) + \vec{s}_L(x)\right) = \vec{t}_R(x) + \vec{\zeta}_R(x) - \left(\vec{t}_L(x) + \vec{\zeta}_L(x)\right) \tag{B11}$$

$$\mathrm{rot}\left(\vec{s}_R\left(x\right) + \vec{s}_L\left(x\right)\right) + \left(\vec{\Pi}_R\left(x\right) + \vec{\Pi}_L\left(x\right)\right)$$
$$= \frac{1}{2}\left(\left(\xi^A\right)^\dagger(x)\left(\vec{\sigma}\right)^A{}_B\, i\hbar D_0\left(x\right)\xi^B(x) + h.c.\right)$$
$$-\frac{1}{2}\left(\left(\eta_{\dot{U}}\right)^\dagger(x)\left(\vec{\sigma}\right)_{\dot{U}}{}^{\dot{V}}\, i\hbar D_0\left(x\right)\eta_{\dot{V}}(x) + h.c.\right) \tag{B12}$$

Appendix C

The Volkov solution of the Dirac electron under a plane-wave radiation field

$$A^\mu = A^\mu\left(\phi\right), \quad \phi = k\cdot x = k^0ct - \vec{k}\bullet\vec{r}, \quad \lim_{\phi^2\to\infty}A^\mu\left(\phi\right) = 0 \tag{C1}$$

is given as [12,19]

$$\psi = \left(1 + \frac{1}{2k\cdot p}\frac{q}{c}\not{k}\not{A}\right) e^{\frac{i}{\hbar}S_0} u \tag{C2}$$

$$S_0 = -x\cdot p - \int_{-\infty}^{\phi}\left(\frac{1}{k\cdot p}\frac{q}{c}p\cdot A - \frac{1}{2k\cdot p}\left(\frac{q}{c}\right)^2 A^2\right)d\phi \tag{C3}$$

$$\left(\not{p} - mc\right)u = 0, \quad \partial\cdot u = 0 \tag{C4}$$

$$p^2 = \left(mc\right)^2 \tag{C5}$$

Let the asymptotic free boundary condition with the 3rd eigenvalue $\zeta = \pm\frac{1}{2}\hbar$ of spin $S^3 = \vec{S}\bullet\vec{e}_z$ be

$$\lim_{\phi^2\to\infty}\zeta = \pm\frac{1}{2}\hbar \tag{C6}$$

then we have

$$j^\mu\left(\vec{p}, \pm\frac{1}{2}\hbar\right) = cq\frac{1}{p^0}\left(p^\mu - \frac{q}{c}A^\mu + k^\mu\left(\frac{1}{k\cdot p}\frac{q}{c}A\cdot p - \frac{1}{2k\cdot p}\left(\frac{q}{c}\right)^2 A^2\right)\right) \tag{C7}$$

$$j_5^{\,0}\left(\vec{p}, \pm\frac{1}{2}\hbar\right) = \pm cq\left(\begin{array}{l}\dfrac{p_z}{p^0} \\ +\dfrac{1}{2k\cdot p}\dfrac{q}{c}\left(\begin{array}{l}-2A^0\left(k^0\dfrac{p_z}{p^0} - \dfrac{1}{p^0\left(p^0+mc\right)}p_z\vec{k}\bullet\vec{p} - \dfrac{mc}{p^0}k_z\right) \\ +2k^0\left(A^0\dfrac{p_z}{p^0} - \dfrac{1}{p^0\left(p^0+mc\right)}p_z\vec{A}\bullet\vec{p} - \dfrac{mc}{p^0}A_z\right)\end{array}\right) \\ -\left(\dfrac{1}{2k\cdot p}\dfrac{q}{c}\right)^2 2A^2k^0\left(k^0\dfrac{p_z}{p^0} - \dfrac{1}{p^0\left(p^0+mc\right)}p_z\vec{k}\bullet\vec{p} - \dfrac{mc}{p^0}k_z\right)\end{array}\right) \tag{C8}$$

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$$j_5^{\,1}\left(\vec{p},\pm\frac{1}{2}\hbar\right)=\pm cq\left(\begin{array}{l}\frac{1}{p^0\left(p^0+mc\right)}p_z p_x\\ +\frac{1}{2k\cdot p}\frac{q}{c}\left(\begin{array}{l}-2A_x\left(k^0\frac{p_z}{p^0}-\frac{1}{p^0\left(p^0+mc\right)}p_z\vec{k}\bullet\vec{p}-\frac{mc}{p^0}k_z\right)\\ +2k_x\left(A^0\frac{p_z}{p^0}-\frac{1}{p^0\left(p^0+mc\right)}p_z\vec{A}\bullet\vec{p}-\frac{mc}{p^0}A_z\right)\end{array}\right)\\ -\left(\frac{1}{2k\cdot p}\frac{q}{c}\right)^2 2A^2k_x\left(k^0\frac{p_z}{p^0}-\frac{1}{p^0\left(p^0+mc\right)}p_z\vec{k}\bullet\vec{p}-\frac{mc}{p^0}k_z\right)\end{array}\right)$$

(C9)

$$j_5^{\,2}\left(\vec{p},\pm\frac{1}{2}\hbar\right)=\pm cq\left(\begin{array}{l}\frac{1}{p^0\left(p^0+mc\right)}p_z p_y\\ +\frac{1}{2k\cdot p}\frac{q}{c}\left(\begin{array}{l}-2A_y\left(k^0\frac{p_z}{p^0}-\frac{1}{p^0\left(p^0+mc\right)}p_z\vec{k}\bullet\vec{p}-\frac{mc}{p^0}k_z\right)\\ +2k_y\left(A^0\frac{p_z}{p^0}-\frac{1}{p^0\left(p^0+mc\right)}p_z\vec{A}\bullet\vec{p}-\frac{mc}{p^0}A_z\right)\end{array}\right)\\ -\left(\frac{1}{2k\cdot p}\frac{q}{c}\right)^2 2A^2k_y\left(k^0\frac{p_z}{p^0}-\frac{1}{p^0\left(p^0+mc\right)}p_z\vec{k}\bullet\vec{p}-\frac{mc}{p^0}k_z\right)\end{array}\right)$$

(C10)

$$j_5^{\,3}\left(\vec{p},\pm\frac{1}{2}\hbar\right)=\pm cq\left(\begin{array}{l}\frac{mc}{p^0}+\frac{1}{p^0\left(p^0+mc\right)}p_z^{\,2}\\ +\frac{1}{2k\cdot p}\frac{q}{c}\left(\begin{array}{l}-2A_z\left(k^0\frac{p_z}{p^0}-\frac{1}{p^0\left(p^0+mc\right)}p_z\vec{k}\bullet\vec{p}-\frac{mc}{p^0}k_z\right)\\ +2k_z\left(A^0\frac{p_z}{p^0}-\frac{1}{p^0\left(p^0+mc\right)}p_z\vec{A}\bullet\vec{p}-\frac{mc}{p^0}A_z\right)\end{array}\right)\\ -\left(\frac{1}{2k\cdot p}\frac{q}{c}\right)^2 2A^2k_z\left(k^0\frac{p_z}{p^0}-\frac{1}{p^0\left(p^0+mc\right)}p_z\vec{k}\bullet\vec{p}-\frac{mc}{p^0}k_z\right)\end{array}\right)$$

(C11)

Assume then for simplicity, first, radiation field propagates along the 3rd axis associated with the electric field along the 1st axis and the magnetic field in the 2nd axis:

$$A^{\mu} = \left(0, A_x, 0, 0\right) \tag{C12}$$

$$k^{\mu} = \left(k^0, 0, 0, k^0\right) \tag{C13}$$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = \left(E_x, E_y, E_z\right) = \left(-k^0 \frac{dA_x}{d\phi}, 0, 0\right) \tag{C14}$$

$$\vec{B} = \mathrm{rot}\vec{A} = \left(B_x, B_y, B_z\right) = \left(0, -k^0 \frac{dA_x}{d\phi}, 0\right) \tag{C15}$$

and, second, the electron propagates along the 3rd axis asymptotically:

$$p^{\mu} = \left(p^0, 0, 0, p_z\right) \tag{C16}$$

It follows that the charge density, the spin density and zeta potential are given as

$$N = \frac{1}{cq} j^0 = 1 + \frac{1}{2p^0\left(p^0 - p_z\right)}\left(\frac{q}{c}\right)^2 \left(A_x\right)^2 \tag{C17}$$

$$\vec{s} = \pm\frac{1}{2}\hbar\left(\frac{1}{p^0}\frac{q}{c}A_x, 0, 1 - \left(N-1\right)\right) \tag{C18}$$

$$\phi_5 = \pm\frac{\hbar c}{2}\left(\frac{p_z}{p^0} - \left(N-1\right)\right) \tag{C19}$$

The spin torque and zeta force are calculated to be

$$\vec{t} = \left(t_x, t_y, t_z\right) = \pm\frac{1}{2}\hbar\left(q\frac{k^0}{p^0}\frac{dA_x}{d\phi}, 0, 0\right) \tag{C20}$$

$$\vec{\zeta} = \left(\zeta_x, \zeta_y, \zeta_z\right) = \pm\frac{1}{2}\hbar\left(0, 0, -\frac{ck^0}{p^0\left(p^0 - p_z\right)}\left(\frac{q}{c}\right)^2 A_x \frac{dA_x}{d\phi}\right) \tag{C21}$$

Consequently, we have nonnul spin dynamics, which should be so since the Volkov state is not stationary:

$$\frac{\partial}{\partial t}\vec{s} = \vec{t} + \vec{\zeta} \neq \vec{0} \tag{C22}$$

As a trivial limit of free electron in the stationary state, the torque and zeta force are calculated to be zero:

$$\vec{t} = \vec{0}, \quad \vec{\zeta} = \vec{0} \tag{C23}$$

and hence the sum:

$$\frac{\partial}{\partial t}\vec{s} = \vec{t} + \vec{\zeta} = \vec{0} \tag{C24}$$

which should be so since the state here is chosen stationary.

Appendix D

The Landau levels of the Dirac electron under a static uniform magnetic field along the 3rd axis

$$A^{\mu} = \left(0, -\frac{1}{2}Hy, \frac{1}{2}Hx, 0\right) \tag{D1}$$

is given in a textbook [16]. Using the Landau eigenfunctions $R_{n,m_\ell,k_z,\sigma}(\rho)$ with $\rho = \sqrt{x^2 + y^2}$, the torque and zeta force are calculated to be cancelled with each other, which should be so since the state is stationary:

$$\frac{\partial}{\partial t}\vec{s} = \vec{t} + \vec{\zeta} = \vec{0} \tag{D2}$$

But the vector components are nonzero in this case:

$$\vec{\zeta} = -\mathrm{grad}\phi_5 = \left(-\frac{\partial}{\partial x}\phi_5, -\frac{\partial}{\partial y}\phi_5, 0\right) \tag{D3}$$

with the zeta potential

$$\phi_5 = \frac{\hbar c}{\frac{E_{n,m_\ell,k_z,\sigma}}{c} + mc}\frac{k_z\sigma}{(2\pi)^2}\left(R_{n,m_\ell,k_z,\sigma}(\rho)\right)^2 \tag{D4}$$

where n and m_ℓ are the quantum numbers, k_z is the wave number along the 3rd axis, and where σ is the sign of the 3rd eigenvalue $\zeta = \pm\frac{1}{2}\hbar$ of spin $S^3 = \vec{S}\bullet\vec{e}_z$.

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