(2)

Dynamics of rational semigroups and Hausdorff dimension of the Julia sets[1]

Thesis

Hiroki Sumi [2]
Graduate School of Human and Environmental Studies,
Kyoto University, Kyoto 6068501, Japan
e-mail: sumi@math.h.kyoto-u.ac.jp

February, 1999

[1]1991 Mathematical Subject Classification: 30D05, 58F23
[2]Partially supported by JSPS Research Fellowships for Young Scientists.

Contents

Abstract

For a Riemann surface S, let End(S) denote the set of all holomorphic endomorphisms of S. It is a semigroup with the semigroup operation being composition of maps. A *rational semigroup* is a subsemigroup of End($\overline{\mathbb{C}}$) without any constant elements. The study of dynamics of rational semigroups can be regarded as a generalization of the study of usual one-dimensional complex dynamics, Kleinian groups and iterated function systems generated by conformal maps in $\mathbb{R}^2$. We can define Fatou sets and Julia sets for rational semigroups. In general the Julia set is not forward invariant under each element of the semigroup. This is the most difficult aspect of the study of rational semigroups. For example, the Julia sets may have non-empty interior points. But if $G = \langle f_1, \ldots , f_m\rangle$ is a finitely generated rational semigroup, then the Julia set $J(G)$ of G satisfies the *backward self-similarity* , that is, $J(G) = \cup_{j=1}^{m} f_j^{-1}(J(G))$. From this we can show that if $G = \langle f_1, \ldots , f_m\rangle$ satisfies the *open set condition*, then the interior of the Julia set of G is empty(Proposition 2.6.3). We have another sufficient condition that the Julia sets have empty interior points(Theorem 2.6.4).

In Chapter 1, we see the definition of Fatou sets and Julia sets and will see some fundamental properties.

In Chapter 2 , we define hyperbolic, sub-hyperbolic, semi-hyperbolic and expanding semigroups. These are some *standard* classes of rational semigroups. For any finitely generated rational semigroups, hyperbolicity and expandingness are equivalent under some small assumptions (Theorem 2.1.38, Theorem 2.1.39 and Theorem 2.4.17). In Section 2.1, we show that if a rational semigroup G is sub-hyperbolic or semi-hyperbolic, then there is no wadering domain in the Fatou set of G. Furthermore, if we assume G is finitely generated, then there exists an *attractor* in the Fatou set of G.

In Section 2.2, we consider the continuity of the Julia set with respect to the perturbation of the generators. If a finitely generated rational semigroup has an attractor in the Fatou set, then the Julia set moves continuously. Furthermore, if a finitely generated rational semigroup is hyperbolic and the backward images of the Julia set by the generators are mutually disjoint, then the Julia set moves by holomorphic motion.

In Section 2.3, it is shown that if G is a finitely generated rational semigroup and the post critical set is included in a component of the Fatou set of G, then the Julia set of G has a kind of self-similar-like structure.

In Section 2.4, we will consider the skew products of rational functions. The "Julia set" of any skew product is defined to be the closure of the union of the fiberwise Julia sets. We will define hyperbolicity and semi-hyperbolicity. We will show that if a skew product is semi-hyperbolic, then the Julia set is

equal to the union of the fiberwise Julia sets and the skew product has the contraction property with respect to the backward dynamics along fibers.

In Section 2.5, we will consider necessary and sufficient conditions to be semi-hyperbolic. We will show that any sub-hyperbolic semigroup without any superattracting fixed point of any element of the semigroup in the Julia set is semi-hyperbolic.

In Section 2.6, we will show that if a finitely generated rational semigroup G is semi-hyperbolic and satisfies the open set condition with the open set O such that $\sharp(\partial O \cap J(G)) < \infty$, then 2-dimensional Lebesgue measure of the Julia set is equal to 0.

We consider the Hausdorff dimension of the Julia sets of rational semigroups. To investigate that we construct the *subconformal measures*(Section 2.8). If we assume strong open set condition, we can also construct the conformal measures(Section 2.7). If a rational semigroup has at most countably many elements and the δ-Poincaré series converges, then we can construct δ-subconformal measures. We will see that if G is a finitely generated semi-hyperbolic rational semigroup, then the Hausdorff dimension of the Julia set is less than the exponent δ(Theorem 2.8.7). if G is a finitely generated semi-hyperbolic rational semigroup and the backward images of the Julia set by the generators are mutually disjoint, then the Hausdorff dimension of the Julia set is equal to the infimum of exponents whose Poincaré series converge and is equal to the unique exponent δ which allows us to construct the δ-conformal measure. Furtheremore, $\delta < 2$ and the δ-Hausdorff measure of the Julia set is strictly positive and finite. (Theorem 2.7.4, Corollary 2.7.5, Corollary 2.7.6). To show those results, the contracting property of backward dynamics will be used.

In Chapter 3, for any finitely generated rational semigroup $G = \langle f_1, \ldots , f_m\rangle$ we consider the skew product costructed by the generator system.

If $G = \langle f_1, \ldots , f_m\rangle$ is an expanding finetely generated rational semigroup, then using the *thermodynamical formalisms* on the skew product constructed by the generator system $\{f_1, \ldots , f_m\}$, we can get an upper estimate of the Julia set of G(Section 3.1). In that inequality, if we assume that $\{f_j^{-1}(J(G))\}_{j=1,\ldots,m}$ are mutually disjoint, then we see that the equality holds,

In Section 3.2, we construct the *(backward) self-similar measure*. That is, a kind of invariant measures whose projection to the base space(space of one-sided infinite words) are some Bernoulli measures. We will show the uniqueness for any weight without any assumption about hyperbolicity. Furthermore in Section 3.3, we caliculate the metric entropy of those measures. We show that the topological entropy of the skew product constructed by

the generator system $\{f_1, \dots, f_m\}$ is equal to

$$\log(\Sigma_{j=1}^m \deg(f_j))$$

and there exists a unique maximal entropy measure, which coincides with the backward self-similar measure corresponding to the weight

$$a_0 := \left(\frac{\deg(f_1)}{\sum_{j=1}^m \deg(f_j)}, \dots, \frac{\deg(f_m)}{\sum_{j=1}^m \deg(f_j)}\right).$$

Hence the projection of the maximal entropy measure of the skew product to the base space is equal to the Bernoulli measure corresponding to the above weight a_0. Applying this result if $\{f_j^{-1}(J(G))\}_{j=1,\dots,m}$ are mutually disjoint, then we get a lower estimate of Hausdorff dimension of the Julia set of G.

In the paper "Semi-hyperbolic transcendental semigroups" (a joint work with Hartje Kriete in Goettingen Univerisity, Germany), we consider some sufficient and necessary conditions to be semi-hyperbolic for semigroups generated by (transcendental) entire functions. We will consider some examples of semi-hyperbolic transcendental semigroups.

Notes

- The paper "On dynamics of hyperbolic rational semigroups " was published in Journal of Mathematics of Kyoto University, Vol.37, No.4, pp.717-733, 1997.
- The paper "On Hausdorff dimension of Julia sets of hyperbolic rational semigroups" was published in Kodai Mathematical Journal, Vol.21, No.1, pp.10-28, March 1998.
- A survey of results from Chapter 1 to Chapter 3 was published in *Dynamics of Sub-hyperbolic and semi-hyperbolic rational semigroups and conformal measures of rational semigroups*, In S.Morosawa, editor, RIMS Kokyuroku 1042: Complex Dynamics and Related Problems, Kyoto University, 1998, pp.68-78.
- Another survey of results from Chapter 1 to Chapter 3 will be published in *Dynamics of rational semigroups and skew products*, In M.Kisaka, editor, RIMS Kokyuroku: Research of Complex Dynamics, 1999.
- The paper "Semi-hyperbolic transcendental semigroups" is a joint work with Hartje Kriete in Goettingen Univerisity, Germany.

Acknowledgement

I would like to thank my advisor, Professor Shigehiro Ushiki, for many valuable advices and comments. I would also like to thank Professor Masahiko Taniguchi, Jun Kigami, Toshiuki Sugawa, Atsushi Kameyama and Masashi Kisaka for many helpful advices and discussions.

I would like to thank Professor Ren Fuyao in Fudan univerisity for many interesting discussions and comments during the international conference on complex analysis held at Beijing university in August 1997.

Chapter 3 was written during my stay in Goettingen university(April 1998-September 1998). I would like to express my gratitude to Professor Manfred Denker , Hartje Kriete and Stefan M.Heinemann for their kind hospitality in Goettingen and so many valuable advices and comments. Actually this stay was so good and fruitful for me.

I would like to thank Professor Manfred Denker, Aimo Hinkkanen , Gavin Martin and Olivier Sester for many valuable and interesting adivices and comments in some seminars and conferences in Germany.

I would like to thank David Boyd and Rich Stankewitz for so many interesting advices and discussions. It is very exciting and wanderful for me that I can discuss so much about rational semigroups with them.

I thank my family for many supports and encouragement for me.

Comments

This paper is dedicated to Masaki Hashimoto(1971-1994).

I remember he encourged me, " Sumi ha zettai suugaku dekirutte. Ore ga hoshou suru."

Chapter 1

Introduction

1.1 preliminaries

For a Riemann surface S, let $\mathrm{End}(S)$ denote the set of all holomorphic endomorphisms of S. It is a semigroup with the semigroup operation being composition of functions. A *rational semigroup* is a subsemigroup of $\mathrm{End}(\overline{\mathbb{C}})$ without any constant elements. We say that a rational semigroup G is a *polynomial semigroup* if each element of G is a polynomial.

Definition 1.1.1. Let G be a rational semigroup. We set

$$F(G) = \{z \in \overline{\mathbb{C}} \mid G \text{ is normal in a neighborhood of } z\}, \quad J(G) = \overline{\mathbb{C}} \setminus F(G).$$

$F(G)$ is called the *Fatou set* for G and $J(G)$ is called the *Julia set* for G.

Definition 1.1.2. Let G be a rational semigroup and z be a point of $\overline{\mathbb{C}}$. The backward orbit $G^{-1}(z)$ of z and the set of exceptional points $E(G)$ are defined by:

$$G^{-1}(z) \overset{\text{def}}{=} \{w \in \overline{\mathbb{C}} \mid \text{there is some } g \in G \text{ such that } g(w) = z\},$$

$$E(G) \overset{\text{def}}{=} \{z \in \overline{\mathbb{C}} \mid \sharp G^{-1}(z) \leq 2\}.$$

Definition 1.1.3. Let G be a rational semigroup and $S = \{f_\lambda \mid \lambda \in \Lambda\}$ a generator system of G. For each $g \in G$, We set

$$wl_S(g) = \min\{n \in \mathbb{N} \mid g = f_{\lambda_1} \cdots f_{\lambda_n}\}.$$

We call $wl_S(g)$ the word length of g with respect to S.

Definition 1.1.4. A subsemigroup H of a semigroup G is said to be of finite index if there is a finite collection of elements $\{g_1, g_2, \dots, g_n\}$ of G such that $G = \cup_{i=1}^n g_i H$. Similarly we say that a subsemigroup H of G has cofinite index if there is a finite collection of elements $\{g_1, g_2, \dots, g_n\}$ of G such that for every $g \in G$ there is a $j \in \{1, 2, \dots, n\}$ such that $g_j g \in H$.

Lemma 1.1.5. *Let G be a rational semigroup.*

1. *For any $f \in G$,*

$$f(F(G)) \subset F(G),\ f^{-1}(J(G)) \subset J(G),$$

$$F(G) \subset F(\langle f \rangle),\ J(\langle f \rangle) \subset J(G)$$

2. *Assume G is generated by a compact subset Λ of End($\overline{\mathbb{C}}$). Then*

$$J(G) = \bigcup_{f \in \Lambda} f^{-1}(J(G)).$$

We call this property the backward self-similarity of the Julia set.

Proof. By definition, it is easy to show 1 . Since $J(G)$ is backward invariant under G, we have

$$J(G) \supset \cup_{f \in \Lambda} f^{-1}(J(G)).$$

Suppose there exists a point $x \in J(G)$ that does not belong to $\cup_{f \in \Lambda} f^{-1}(J(G))$. There exists a neighborhood U of x in $\overline{\mathbb{C}}$ such that $f(U) \subset F(G)$ for each $f \in \Lambda$. Take any $x' \in U$. Let $\epsilon > 0$ be any small number. Since $\cup_{f \in \Lambda} f(x')$ is a compact subset of $F(G)$, there exists a number $\delta_1 > 0$ such that if $d(f(x'),\ y) < \delta_1$ for some $f \in \Lambda$, then $d(gf(x'),\ g(y)) < \epsilon$ for each $g \in G \cup \{id\}$. Take $\delta_2 > 0$ such that if $d(x', y) < \delta_2$ then $d(f(x'),\ f(y')) < \delta_1$ for each $f \in \Lambda$. Then we have that if $d(x', y) < \delta_2$, then $d(gf(x'),\ gf(y')) < \epsilon$ for each $g \in G \cup \{id\}$ and each $f \in \Lambda$. Hence we have $x \in F(G)$ and this is a contradiction. □

If a set K satisfies that $K = \cup_{i=1}^n f_i^{-1}(K)$, we say that K has backward self-similarity. Next lemma was shown in [HM1], [ZR].

Lemma 1.1.6 ([HM1], [ZR]). *Let G be a rational semigroup.*

1. *If a subsemigroup H of G is of finite or cofinite index, then*

$$J(H) = J(G).$$

In particular, when G is a rational semigroup generated by finite elements $\{f_1, f_2, \dots f_n\}$ and m is an integer, if we set

$$H_m = \{g = f_{j_1} \cdots f_{j_k} \in G \mid m \text{ devides } k\},$$

$$I_m = \{g \in G \mid g \text{ is a product of some elements of word length } m\}$$

then

$$J(G) = J(H_m) = J(I_m).$$

Here we say an element $f \in G$ is word length m if m is the minimum integer such that

$$f = f_{j_1} \cdots f_{j_m}.$$

2. *If $J(G)$ contains at least three points, then $J(G)$ is a perfect set.*

3. *If there is an element $g \in G$ such that $\deg(g) \geq 2$ or there is an element $g \in G$ such that $\deg(g) = 1$ and the order of g is infinite, then*

$$E(G) = \{z \in \overline{\mathbb{C}} \mid \sharp G^{-1}(z) < \infty\},\ \sharp E(G) \leq 2.$$

4. *If a point z is not in $E(G)$, then for every $x \in J(G)$, x belongs to $\overline{G^{-1}(z)}$. In particular if a point z belongs to $J(G) \setminus E(G)$, then*

$$\overline{G^{-1}(z)} = J(G).$$

5. *If there is an element $g \in G$ such that $\deg(g) \geq 2$ or there is an element $g \in G$ such that $\deg(g) = 1$ and the order of g is infinite and $J(G)$ contains at least three points, then $J(G)$ is the smallest closed backward invariant set containing at least three points. Here we say that a set A is backward invariant under G if for each $g \in G$, $g^{-1}(A) \subset A$.*

6. *If $J(G)$ contains at least three points, then*

$$J(G) = \overline{\{z \in \overline{\mathbb{C}} \mid z \text{ is a repelling fixed point of some } g \in G\}}$$

Proposition 1.1.7. *Let $\{Q_\lambda\}$ be a family of polynomials that are not of degree one and G be a polynomial semigroup generated by $\{Q_\lambda\}$.*

If a transformation $\sigma(z) = \mu z + \tau \in \mathrm{Aut}\mathbb{C}$, $\mu = \exp(\frac{2\pi i}{k})$, $k \in \mathbb{N}$ *satisfies for every* λ

$$\sigma(J(\langle Q_\lambda \rangle)) = J(\langle Q_\lambda \rangle),$$

then

$$\sigma(J(G)) = J(G).$$

Proof. For every polynomial Q that is not of degree one, $J(Q)$ is completely invariant under a transformation $z \mapsto (\exp(\frac{2\pi i}{k}))(z)$ if and only if $Q = az^d P(z^k)$, where P is a polynomial, a is a number, and d is an integer ([Bel]). So it is easy to see the statement using Lemma 1.1.6.6. □

Example 1.1.8. For a regular triangle $p_1p_2p_3$, we set $g_i(z) = 2(z - p_i) + p_i$, $i = 1, 2, 3$. And let G be a rational semigroup generated by $\{g_i\}$, not as a group. Then $J(G)$ is the Sierpiński Gasket.

In general, the Julia set of a rational semigroup may have non-empty interior points. For example, $J(\langle z^2, 2z \rangle) = \{|z| \leq 1\}$. In fact, in [HM2] it was shown that if G is a finitely generated rational semigroup, then any super attracting fixed point of any element of G does not belong to $\partial J(G)$. Hence we can easily get many examples that the Julia sets have non-empty interior points.

Since the Julia set of a rational semigroup may have non-empty interior points, it is significant for us to get sufficient conditions such that the Julia set has no interior points, to know when the area of the Julia set is equal to 0 or to get an upper estimate of the Hausdorff dimension of the Julia set. We will try that using various information about forward dynamics of the semigroup in the Fatou set or backward dynamics of the semigroup in the Julia set.

1.2 Limit functions

First, we will give some comments about limit functions of semigroups. The study of limit functions plays a very important role in the study of complex dynamical systems. The forward invariant domains of iteration of rational functions are classified into five types by the limit functions([Bel], [Mi]).

Let S be a hyperbolic Riemann surface, S_∞ the one point compactification of S, and H a subsemigroup of $\mathrm{End}(S)$.

Definition 1.2.1.
$\overline{\mathcal{L}}_H(S) \stackrel{\mathrm{def}}{=} \{\varphi : S \to S_\infty \mid$ there is a sequence (g_j) of mutually distinct elements of H such that $g_j \to \varphi$ locally uniformly on S as $j \to \infty\}$.

Remark 1. Every family $\mathcal{A}$ of elements of $\mathrm{End}(S)$ contains a sequence that converges to an element of $\mathrm{End}(S)$ or ∞. ([Mi]).

Lemma 1.2.2. *Let S be a hyperbolic Riemann surface and H a subsemigroup of End(S). If $g \in H$ is non-constant and φ belongs to $\overline{\mathcal{L}}_H(S)$, then $\varphi g \in \overline{\mathcal{L}}_H(S)$. Moreover if φ also belongs to End(S), then $g\varphi \in \overline{\mathcal{L}}_H(S)$.*

Proof. Let φ be an element of $\overline{\mathcal{L}}_H(S)$. There is some sequence (f_j) of mutually distinct elements of H such that $f_j \to \varphi$. Then the sequence $(f_j g)$converges to φg and $\{f_j g\}$ are mutually distinct because g is non-constant. By definition φg belongs to $\overline{\mathcal{L}}_H(S)$.

Next assume φ also belongs to $\mathrm{End}(S)$. The sequence (gf_j)converges to $g\varphi$. We will show $\{gf_j\}$ contains infinitely many elements that are mutually distinct. For each number i, j, we set

$$C_{ij} = \{z \in S \mid f_i(z) = f_j(z)\}, \ C = \cup_{i \neq j} C_{ij}.$$

C is a countable set and we can take a point x of S which does not belong to C. Then $\{f_j(x)\}$ are mutually distinct and the sequence $(f_j(x))$ converges to $\varphi(x) \in S$. Now assume that there exists a subsequence(j_k) of (j) such that $j_k \to \infty$ as $k \to \infty$ and all elements of $\{gf_{j_k}\}$ are equal to an element $h \in \mathrm{End}(S)$. Then for each k, $gf_{j_k}(x) = g\varphi(x)$ and this is a contradiction because g is non-constant. So $\{gf_j\}$ contains infinitely many elements that are mutually distinct. By definition, it follows that $g\varphi$ belongs to $\overline{\mathcal{L}}_H(S)$. □

Lemma 1.2.3. *Let S be a hyperbolic Riemann surface and H a subsemigroup of End(S) generated by a generator system $\{g_\lambda\}_{\lambda \in \Lambda}$ such that $\cup_{\lambda \in \Lambda}\{g_\lambda\}$ is a compact subset of End(S) with respect to the compact open topology. Let φ be a non-constant element of End(S) and suppose a sequence (f_j) in H converges to φ locally uniformly on S. Then the following hold.*

1. *If φ is an injective map(resp. a covering map on S, an element of Aut(S)), then there exists a $\lambda \in \Lambda$ such that g_λ is an injective map(resp. a covering map on S, an element of Aut(S)).*

2. *Suppose that there exists a word $w \in \Lambda^{\mathbb{N}}$ and a sequence (n_j) of positive integers such that $f_j = g_{w_{n_j}} \circ \cdots \circ g_{w_1}$ for each j. Then there exists a sequence (h_n) in H converging to Id_S locally uniformly on S and a $\lambda \in \Lambda$ such that $g_\lambda \in$Aut(S). Moreover, if $\sharp\Lambda < \infty$ and the word length of $g_{w_{n_j}} \circ \cdots \circ g_{w_1}$ is equal to n_j for each $j \in \mathbb{N}$, then $\sharp(H \cap$ Aut(S)$) = \infty$ and there exists a sequence (h_n) in H consisting of mutually disjoint elements of H such that $h_n \to Id_S$.*

Proof. We will show 1. For each j there exists an element $\alpha_j \in G \cup \{Id_S\}$ and a $\lambda_j \in \Lambda$ such that $f_j = \alpha_j \circ g_{\lambda_j}$. We can assume the sequence (α_j) converges to a map α locally uniformly on S and the sequence (g_{λ_j}) converges to a map g_λ locally uniformly on S for some $\lambda \in \Lambda$. Then we have that $\varphi = \alpha \circ g_\lambda$. If φ is an injective map, then g_λ is injective on S. If φ is a local isometry on S with respect to the hyperbolic metric on S, then g_λ is also a local isometry.

We will show 2. For each j there exists an element $h_j \in H$ such that $g_{w_{n_{j+1}}} \circ \cdots \circ g_{w_1} = h_j g_{w_{n_j}} \circ \cdots \circ g_{w_1}$. We can assume the sequence (h_j) converges to an element $h \in H$. Then we have $\varphi = h \circ \varphi$. Since φ is non-constant, we have that $h = Id_S$. By the statement 1, we have that there exists a $\lambda \in \Lambda$ such that g_λ belongs to Aut(S).

Now we assume $\sharp\Lambda = k < \infty$ and the word length of $g_{w_{n_j}} \circ \cdots \circ g_{w_1}$ is equal to n_j for each $j \in \mathbb{N}$. Then in the above argument we can assume that there exists a word $\tau = (\tau_1, \tau_2, \dots) \in \Lambda^{\mathbb{N}}$ such that for each $j \in \mathbb{N}$

$$h_j = \beta_j \circ g_{\tau_j} \circ \cdots \circ g_{\tau_1}$$

for some $\beta_j \in H$. Since $h_j \to Id_S$ we have that $g_{\tau_j} \circ \cdots \circ g_{\tau_1}$ is an injective map and a local isometry with respect to the hyperbolic metric on S for each j. Hence $g_{\tau_j} \circ \cdots \circ g_{\tau_1}$ belongs to Aut(S) for each j. Since the word length of $g_{w_{n_j}} \circ \cdots \circ g_{w_1}$ is equal to n_j, we have that $(g_{\tau_j} \circ \cdots \circ g_{\tau_1})$ consists of mutually disjoint elements. Hence the statement of our lemma holds. □

Lemma 1.2.4. *Let S be a hyperbolic Riemann surface and H a finitely generated subsemigroup of End(S) generated by $\{g_1, \dots, g_k\}$. If there is a non-constant element $\varphi \in \overline{\mathcal{L}}_H(S)$, then at least one of the following assertions is true:*

1. *there exists a word $w \in \{1, \dots, k\}^{\mathbb{N}}$, a sequence (n_j) of positive integers and a non-constant element $\phi \in \overline{\mathcal{L}}_H(S)$ such that the word length of $g_{w_{n_j}} \circ \cdots \circ g_{w_1}$ is equal to n_j for each j and $g_{w_{n_j}} \circ \cdots \circ g_{w_1} \to \phi$ locally uniformly on S as $j \to \infty$. Moreover we have that $Id_S \in \overline{\mathcal{L}}_H(S)$, that there exists an element $i \in \{1, \dots, k\}$ such that $g_i \in Aut(S)$ and that $\sharp(H \cap Aut(S)) = \infty$.*

2. *there exists a word $w \in \{1, \dots, k\}^{\mathbb{N}}$ and a sequence (α_n) in H such that $\liminf_{n\to\infty} \|(g_{w_n} \circ \cdots \circ g_{w_1})'(z)\| > 0$ for any $z \in S$ where we denote by $\|\cdot\|$ the norm of the derivative with respect to the hyperbolic metric on S and*

$$g_{w_n} \circ \cdots \circ g_{w_1} \to \infty, \quad \alpha_n \circ g_{w_n} \circ \cdots \circ g_{w_1} \to \varphi$$

as $n \to \infty$ locally uniformly on S.

Proof. We fix a generator system $\{g_1, \dots, g_k\}$ of H. There is a sequence (f_j) of mutually distinct elements of H such that $f_j \to \varphi$ and word length of f_j strictly increases as $j \to \infty$. We represent each f_j by its reduced word. We take a subsequence (f_{1j}) of (f_j) as follows. There is a generator g_{w_1} of H such that for each j

$$f_{1j} = \cdots \circ g_{w_1}.$$

Inductively when we get a sequence $(f_{nj})_j$, we take a subsequence $(f_{n+1,j})_j$ of it as follows. There is a generator $g_{w_{n+1}}$ of H such that for each j

$$f_{n+1,j} = \cdots \circ g_{w_{n+1}} \circ \cdots \circ g_{w_1}.$$

Now we get a sequence $(f_{nn})_n$ and a word $w \in \{1, \dots, k\}^{\mathbb{N}}$ such that for each n

$$f_{nn} = \alpha_n \circ a_n, \text{ where } \alpha_n \in H, \ a_n = g_{w_n} \circ \cdots \circ g_{w_1}.$$

Take a point $z \in S$. Suppose there exists a subsequence of $(a_n(z))$ converging to a point of S. Then there exists a subsequence (a_{n_j}) of (a_n) converging to a map $a \in$ End(S). We can assume that (α_{n_j}) converges to a map locally uniformly on S. Since φ is non-constant, we have that a is non-constant. By Lemma 1.2.3, we get the assertion 1.

Suppose there exists no subsequence of $(a_n(z))$ converging to any point in S. Then the assertion 2 holds. □

Next we define stable domains([HM1]).

Definition 1.2.5. Let G be a rational semigroup and U a connected component of $F(G)$. We say that U is a stable domain if there is an element $g \in G \setminus \mathrm{Aut}\overline{\mathbb{C}}$ such that $g(U) \subset U$. And we set

$$G_U \stackrel{\text{def}}{=} \{g \in G \mid g(U) \subset U\}.$$

Similar definitions for entire semigroup can also be given.

Definition 1.2.6. Let U be a domain of $\overline{\mathbb{C}}$ and H a subsemigroup of End(U). Then we set

$\mathcal{L}_H(U) \stackrel{\text{def}}{=} \{\varphi : U \to \overline{U} \mid$ there is a sequence (g_j) of mutually distinct elements of H such that $g_j \to \varphi$ locally uniformly on U as $j \to \infty\}$.

Remark 2. If $g \in H$ is non-constant and φ belongs to $\mathcal{L}_H(U)$, then $\varphi \circ g \in \mathcal{L}_H(U)$. Moreover if φ also belongs to End(U), then $g \circ \varphi \in \mathcal{L}_H(U)$.

Now we consider a case such that there are only finitely many constant limit functions taking its value in a domain U. In this case $\mathcal{L}_H(U)$ has only finitely many elements.

Proposition 1.2.7. *Let G be a rational semigroup and U a subdomain of $F(G)$ and we set*

$$H = \{g \in G \mid g(U) \subset U\}, \ \mathcal{A} = \{\zeta \in U \mid \exists \varphi \in \mathcal{L}_H(U), \ \varphi \equiv \zeta\}.$$

If H is finitely generated and if $1 < \sharp \mathcal{A} < \infty$, then any $\varphi \in \mathcal{L}_H(U)$ is a constant map being its value$\in U$. And $M = H \cap Aut(\overline{\mathbb{C}})$ has only finitely many elements.

Remark 3. A similar result for entire semigroup also holds. And if we set

$$G = \langle z^2, \ e^{i\theta}z \rangle, \ \frac{\theta}{2\pi} \notin \mathbb{Q}, \ U = \{\mid z \mid < 1\},$$

then

$$\sharp\{\varphi \in \mathcal{L}_H(U) \mid \exists \zeta \in U, \ \varphi \equiv \zeta\} = 1, \ Id_U \in \mathcal{L}_H(U).$$

Next we consider a case such that there are infinitely many constant limit functions taking its value in a stable domain.

Proposition 1.2.8. *Let G be a rational(entire) semigroup, U a stable domain of G. We set*

$$H = G_U,$$

$$\mathcal{A} \stackrel{\text{def}}{=} \{\zeta \in U \mid \exists \varphi \in \mathcal{L}_H(U), \ \varphi \equiv \zeta\}, \ \mathcal{B} \stackrel{\text{def}}{=} \{\zeta \in \overline{U} \mid \exists \varphi \in \mathcal{L}_H(U), \ \varphi \equiv \zeta\}.$$

If $\mathcal{A}$ has an accumulation point in U, then $\mathcal{B}$ is a perfect set, in particular an uncountable set.

Proof. First, it is easy to see that $\mathcal{B}$ is a closed subset of $\overline{U}$. Assume that $\mathcal{A}$ has an accumulation point in U and $\zeta \in \mathcal{B}$ is an isolated point. There is a sequence (g_j) of H converging ζ locally uniformly on U. By our assumpsion $\mathcal{A}$ is not empty and take a point $x \in \mathcal{A}$. Then $g_j(x) \to \zeta$ as $j \to \infty$ and $g_j(x) \in \mathcal{A}$ by the remark after Definition 1.2.6. So ζ belongs to U, for it is an isolated point. Now $g_j(\zeta) \to \zeta$ as $j \to \infty$ and $g_j(\zeta) = \zeta$ for large enough j because ζ is isolated. Also for each compact set K, g_j maps K into a small disc about ζ for large enough j. It follows that for large enough j, the point ζ is an attracting fixed point of g_j. Take a large enough number j and set $g = g_j$. For each $y \in \mathcal{A}$ the sequence $(g^n(y))$ converges to ζ as $n \to \infty$. Because ζ is an isolated point, $g^n(y) = \zeta$ for each large enough n. So $\mathcal{A} \subset \cup_n g^{-n}\{\zeta\}$, and each point of $\mathcal{A}$ is isolated in U because $\{g^n\}$ is normal in U. This is a contradiction. □

If $\mathcal{A}$ has infinitely many points and there is no accumulation point of $\mathcal{A}$ in U, then by the proof of Proposition 1.2.8, for any $\zeta \in \mathcal{A}$ there is an element g of H such that $\mathcal{A} \subset \cup_n g^{-n}\{\zeta\}$. It is a problem whether this situation can occur or not.

Conjecture 1.2.9. *If $\mathcal{A}$ has infinitely many points, then $\mathcal{A}$ has an accumulation point in U.*

If this conjecture is true, by Proposition 1.2.8, it implies the following conjecture.

Conjecture 1.2.10. *If $\mathcal{A}$ has infinitely many points, then $\mathcal{B}$ is a perfect set.*

Next we consider the nearly abelian semigroup in [HM1] and the limit functions as an example.

Definition 1.2.11. Let G be a rational semigroup containing an element g with $\deg(g) \geq 2$. We say that G is nearly abelian if there is a compact family of Möbius (or linear fractional) transformations $\Phi = \{\varphi\}$ with the following properties.

- $\varphi(F(G)) = F(G)$ for all $\varphi \in \Phi$
- for all $f, \ g \in G$ there is a $\varphi \in \Phi$ such that $fg = \varphi g f$

Then by [HM1], if $g \in G$ is of degree at least two, then $J(G) = J(g)$. And it is also shown in [HM1] that in each stable domain U, the type of each element $g \in G_U$ such that $\deg(g)$ is at least two coincides. Here we define by the type of $g \in G_U$ the type of the connected component of $F(g)$ containing U.

Let X be a subset of $\mathbb{C}$ that is not a round circle. We set

$$G = \{g \mid g \text{ is a polynomial}, \ J(g) = X\}.$$

If G contains an element g such that $\deg(g)$ is at least two, then G is nearly abelian and we can take a family Φ of Definition 1.2.11 so that it contains only finitely many elements.

Proposition 1.2.12. *Let G be a nearly abelian rational semigroup, Φ the family in Definition 1.2.11 and U a stable domain. We set $H = G_U$ and $\mathcal{B} = \{\zeta \in \overline{U} \mid \exists\varphi \in \mathcal{L}_H(U),\ \varphi \equiv \zeta\}$. If Φ has only finitely many elements, then for any element g of H,*

$$\mathcal{B} \subset \bigcup_{n,\ m, \geq 1,\ n+m \leq \sharp\Phi+1} g^{-m}\{\text{fixed point of } g^n\},$$

in particular, $\mathcal{B}$ has at most finitely many elements. Moreover if $\mathcal{B}$ is not empty, either all points of $\mathcal{B}$ belong to U or all points of $\mathcal{B}$ belong to ∂U.

Proof. Let α be an element of $\mathcal{L}_H(U)$. Then there is a sequence (g_j) of mutually distinct elements of H converging to α locally uniformly on U. Let g be any element of H. For every j there is an element $\varphi_j \in \Phi$ such that

$$gg_j = \varphi_j g_j g.$$

We can assume that (φ_j) converges to an element φ of Φ. Then

$$g\alpha = g \lim_{j\to\infty} g_j = \lim_{j\to\infty} \varphi_j g_j g = \varphi\alpha g.$$

If α is identically equal to a constant value $\zeta \in \overline{U}$, then

$$g(\zeta) = \varphi(\zeta).$$

There are some positive integers $n,\ m$ with $n + m \leq \sharp\Phi + 1$ such that $g^m(\zeta)$ is a fixed point of g^n. Now assume that $\mathcal{B} \cap U \neq \emptyset$ and $\mathcal{B} \cap \partial U \neq \emptyset$. Let $x,\ y$ be points of $\mathcal{B} \cap U,\ \mathcal{B} \cap \partial U$ respectively. Then there is a sequence (h_j) of mutually distinct elements of H converging to y locally uniformly on U. The sequence $(h_j(x))$ converges to y as $j \to \infty$ and $h_j(x)$ belongs to $\mathcal{B}$ for each j, this implies that $\mathcal{B}$ has infinitely many elements. □

Example 1.2.13. Let n be an integer such that $n \geq 2$ and we set $f(z) = z^n + c,\ \sigma(z) = \exp(\frac{2\pi i}{n})z$, and $G = \langle f,\ \sigma f,\ \cdots,\ \sigma^{n-1} f\rangle$. Then G is nearly abelian. If $|c|$ is small enough, then 0 belongs to $F(G)$. Let U be the stable domain containing 0. Then

$$\mathcal{L}_H(U) = \{\sigma^j(z_0),\ j = 0,\ \ldots,\ n-1\},$$

where z_0 is an attracting fixed point of f in U and $\sharp\mathcal{L}_H(U) = n$. Also there is a number c such that each element of $\mathcal{L}_H(U)$ is constant value of ∂U and $\sharp\mathcal{L}_H(U) = n$.

Example 1.2.14. Let $m,\ n$ be integers greater than 1. We set $f(z) = z^m(z-c),\ g(z) = z^n(z-c)+c,\ \ G = \langle f, g\rangle$. If $|c|$ is small enough, then 0 and c belong to the same connected component U of $F(G)$. Now $f(0),\ f(c) = 0$ and $g(0),\ g(c) = c$ and it implies that

$$\mathcal{L}_G(U) = \{\varphi_0,\ \varphi_c\}, \text{ where } \varphi_0 \equiv 0,\ \varphi_c \equiv c.$$

Also G is not nearly abelian, for, the type of f is super attracting and different from that of g.

Chapter 2

Hyperbolicity

2.1 No Wandering Domain

Definition 2.1.1. Let G be a rational semigroup. We set

$$P(G) = \overline{\bigcup_{g \in G} \{ \text{ critical values of } g\}}.$$

We call $P(G)$ the post critical set of G. We say that G is *hyperbolic* if $P(G) \subset F(G)$. Also we say that G is *sub-hyperbolic* if $\sharp\{P(G) \cap J(G)\} < \infty$ and $P(G) \cap F(G)$ is a compact set.

We denote by $B(x, \epsilon)$ a ball of center x and radius ϵ in the spherical metric. We denote by $D(x, \epsilon)$ a ball of center $x \in \mathbb{C}$ and radius ϵ in the Euclidian metric. Also for any hyperboplic manifold M we denote by $H(x, \epsilon)$ a ball of center $x \in M$ and radius ϵ in the hyperbolic metric. For any rational map g, we denote by $B_g(x, \epsilon)$ a connected component of $g^{-1}(B(x, \epsilon))$. For each open set U in $\overline{\mathbb{C}}$ and each rational map g, we denote by $c(U, g)$ the set of all connected components of $g^{-1}(U)$. Note that if g is a polynomial and $U = D(x, r)$ then any element of $c(U, g)$ is simply connected by the maximal principle.

For each set A in $\overline{\mathbb{C}}$, we denote by A^i the set of all interior points of A.

Definition 2.1.2. Let G be a rational semigroup and A a set in $\overline{\mathbb{C}}$. We set $G(A) = \cup_{g \in G} g(A)$ and $G^{-1}(A) = \cup_{g \in G} g^{-1}(A)$.

We can show the following Lemma immediately.

Lemma 2.1.3. *Let G be a rational semigroup. Assume that $\{f_\lambda\}_{\lambda\in\Lambda}$ is a generator system of G. Then we have*

$$\bigcup_{g\in G}\{\text{critical values of } g\} = \bigcup_{\lambda\in\Lambda}(G\cup\{Id\})(\{\text{critical values of } f_\lambda\}).$$

Definition 2.1.4. Let G be a rational semigroup and N a positive integer. We set

$$\begin{aligned} & SH_N(G) \\ = & \{x\in\overline{\mathbb{C}} \mid \exists\delta(x)>0,\ \forall g\in G,\ \forall B_g(x,\delta(x)),\ \deg(g: B_g(x,\delta)\to B(x,\delta))\le N\} \end{aligned}$$

and $UH(G)=\overline{\mathbb{C}}\setminus(\cup_{N\in\mathbb{N}}SH_N(G))$.

Remark 4. By definition, $SH_N(G)$ is an open set in $\overline{\mathbb{C}}$ and $g^{-1}(SH_N(G))\subset SH_N(G)$ for each $g\in G$. Also $UH(G)$ is a compact set and $g(UH(G))\subset UH(G)$ for each $g\in G$. For each rational map g with $\deg(g)\le 2$, any parabolic or attracting periodic point of g belongs to $UH(G)$.

Definition 2.1.5. Let G be a rational semigroup. We say that G is *semi-hyperbolic* (resp. *weakly semi-hyperbolic*) if there exists a positive integer N such that $J(G)\subset SH_N(G)$(resp.$\partial J(G)\subset SH_N(G)$).

Remark 5.

1. If G is semi-hyperbolic and $N=1$, then G is hyperbolic.
2. If G is hyperbolic, then G is semi-hyperbolic.
3. For a rational map f with the degree at least two, $\langle f\rangle$ is semi-hyperbolic if and only if f has no parabolic orbits and each critical point in the Julia set is non-recurrent([CJY], [Y]). If $\langle f\rangle$ is semi-hyperbolic, then there are neither indifferent cycles, Siegel disks nor Hermann rings.

Definition 2.1.6. Let V be a domain in $\overline{\mathbb{C}}$ and E a compact subset of V. We set

$$\operatorname{mod}(E,V)=\sup\{\operatorname{mod} A\},$$

where the supremum is taken over all annulus A such that E lies in a compact component of $V\setminus A$.

Lemma 2.1.7 ([CJY]). *For any positive integer N and real number r with $0<r<1$, there exists a constant $C=C(N,r)$ such that if $f: D(0,1)\to D(0,1)$ is a proper holomorphic map with $\deg(f)=N$, then*

$$H(f(z_0),C)\subset f(H(z_0,r))\subset H(f(z_0),r)$$

for any $z_0\in D(0,1)$. Here we can take $C=C(N,r)$ independent of f.

Corollary 2.1.8. *For any positive integer N and real number r with $0<r<1$, there exist constants r_1 and r_2 with $0<r_1\le r_2<1$ depending only on r, N such that if $f: D(0,1)\to D(0,1)$ is a proper holomorphic map with $\deg(f)=N$ and $f(0)=0$, then*

$$D(0,r_1)\subset W\subset D(0,r_2)$$

where W is the connected component of $f^{-1}(D(0,r))$ containing 0.

Corollary 2.1.9 ([Y]). *Let V be a simply connected domain in $\mathbb{C}$, $0\in V$, $f: V\to D(0,1)$ be a proper holomorphic map of degree N and $f(0)=0$, W be the component of $f^{-1}(D(0,r))$ containing 0, $0<r<1$. Then there exists a constant K depending only on r and N, not depending on V and f, so that*

$$\left|\frac{x}{y}\right|\le K$$

for all $x,y\in\partial W$.

Proof. We will follow Y.Yin's proof([Y]). Let $g: V\to D(0,1)$ be the univalent function such that $g(0)=0$. From Corollary 2.1.8,

$$r_1\le|g(x)|\le r_2$$

for all $x\in\partial W$. Applying the Koebe distortion theorem, we have that

$$|(g^{-1})'(0)|\cdot\frac{r_1}{(1+r_1)^2}\le|x|\le|(g^{-1})'(0)|\cdot\frac{r_2}{(1-r_2)^2}.$$

Then

$$\left|\frac{x}{y}\right|\le\frac{r_2(1+r_1)^2}{r_1(1-r_2)^2}=:K.$$

K is a constant depending only on N and r. □

Lemma 2.1.10. *Let V be a domain in $\overline{\mathbb{C}}$, K a continuum in $\overline{\mathbb{C}}$ with $diam_S K=a$. Assume $V\subset\overline{\mathbb{C}}\setminus K$. Let $f: V\to D(0,1)$ be a proper holomorphic map of degree N. Then there exists a constant $r(N,a)$ depending only on N and a such that for each r with $0<r\le r(N,a)$, there exists a constant $C=C(N,r)$ depending only on N and r satisfying that for each connected component U of $f^{-1}(D(0,r))$,*

$$diam_S\, U\le C,$$

where we denote by $diam_S$ the spherical diameter. Also we have $C(N,r)\to 0$ as $r\to 0$.

Proof. Let r be a number with $0 < r < 1$. Let U be a connected component of $f^{-1}(D(0,r))$ and V' be the connected component of $\overline{\mathbb{C}} \setminus \overline{V}$ containing K. Since V is connected, V' is simply connected. Let U' be the connected component of $\overline{\mathbb{C}} \setminus \overline{U}$ containing V'. Since U' is also simply connected and $\overline{V'} \subset U'$, we have that there exists a connected component of $U' \setminus \overline{V'}$ which is a ring domain.

There exists a sequence $(r_j)_{j=0}^{n}$ of real numbers with $r_0 = r < r_1 < \cdots < r_n = 1$ such that there exist no critical values of f in $D(0, r_{j+1}) \setminus \overline{D(0, r_j)}$ for $j = 0, \ldots, n-1$. For each $i = 0, \ldots, n$, let U_i'' be the connected component of $f^{-1}(D(0, r_i))$ containing U and let U_i' be the connected component of $\overline{\mathbb{C}} \setminus \overline{U_i''}$ containing V'. Then we have

$$U_0'' = U \subset U_1'' \subset \cdots \subset U_n'' = V \text{ and}$$

$$U_0' = U' \supset U_1' \supset \cdots \supset U_n' = V'.$$

By the construction, $f : U_{i+1}'' \setminus \overline{U_i''} \to D(0, r_{i+1}) \setminus \overline{D(0, r_i)}$ is a proper map for $i = 0, \ldots, n-1$. Since there exist no critical values of f in $D(0, r_{i+1}) \setminus \overline{D(0, r_i)}$, each connected component of $U_{i+1}'' \setminus \overline{U_i''}$ is a ring domain.

Now we claim that for each $i = 0, \ldots, n-1$, there exists a connencted component of $U_{i+1}'' \setminus \overline{U_i''}$ which is included in $U_i' \setminus \overline{U_{i+1}'}$. We will show that. Since $\partial U_i' \subset U_{i+1}''$, there exists a ring domain R_i in $U_{i+1}'' \setminus \overline{U_i''}$ such that $\partial U_i'$ is a connected component of ∂R_i. Let R_i' be the connected component of $U_{i+1}'' \setminus \overline{U_i''}$ containing R_i. Since

$$\partial (U_i' \setminus \overline{U_{i+1}'}) = \partial U_i' \cup \partial U_{i+1}' \subset \partial U_i'' \cup \partial U_{i+1}'',$$

we have $R_i' \cap \partial (U_i' \setminus \overline{U_{i+1}'}) = \emptyset$. Hence $R_i' \subset U_i' \setminus \overline{U_{i+1}'}$ and we have proved the above claim.

From the above claim, we get

$$\text{mod } (\overline{U_{i+1}'},\ U_i') \geq \frac{1}{2\pi N} \log \frac{r_{i+1}}{r_i}, \quad \text{for } i = 0, \ldots, n-1.$$

It follows that

$$\begin{aligned} \text{mod } (\overline{V'},\ U') &\geq \sum_{i=0}^{n-1} \text{mod } (U_i' \setminus \overline{U_{i+1}'}) \\ &\geq \frac{1}{2\pi N} \log \frac{1}{r}. \end{aligned}$$

On the other hand, by Lemma 6.1 in p34 in [LV], we have

$$\text{mod } (\overline{V'},\ U') \leq \frac{\pi^2}{2C_1^2},$$

where $C_1 = \min\{a,\ \text{diam}_S\, U\}$. Hence the statement of our lemma holds. □

Lemma 2.1.11. *Let V and W be simply connected domains in $\overline{\mathbb{C}}$. Suppose that $\overline{W} \subset V$ and mod $(\overline{W},\ V) > c > 0$. Then there exists a constant $0 < \lambda < 1$ depending only on c such that*

$$\frac{\text{diam } W}{\text{diam } V} \leq \lambda,$$

here by "diam" we mean the spherical diameter.

Proof. We can assume that $0 \in W$ and diam $V = d(0,1)$ where d is the spherical metric. Let $g : D(0,1) \to V$ be the Riemann map such that $g(0) = 0$. By Theorem 2.4 in [Mc], there exists a constant c_1 depending only on c such that

$$\text{diam}_H(g^{-1}(W)) \leq c_1,$$

where we denote by diam_H the diameter with respect to the hyperbolic metric in $D(0,1)$. Since diam $V = d(0,1)$, by the Koebe distortion theorem, we have that there exists a constant c_2 not depending on V and W such that $|g'(0)| \leq c_2$. Using the Koebe distortion theorem again, we see that there exists a constant c_3 depending only on c such that for each $z \in g^{-1}(W)$, $|g(z)| \leq c_3$. Hence there exists a constant $0 < c_4 < d(0,1)$ depending only on c such that diam $W \leq c_4$.

□

Lemma 2.1.12. *Let $G = \langle f_1, \ldots, f_m \rangle$ be a finitely generated rational semigroup. Let y be a point of $\overline{\mathbb{C}} \setminus UH(G)$. If there exists a neighborhood W of y such that $\overline{\mathbb{C}} \setminus G^{-1}(W)$ contains a continuum, then there exists a neighborhood W_1 of y such that for each simply connected open neighborhood V of y included in W_1 and for each $g \in G$, each element of $c(V, g)$ is simply connected.*

Proof. For each $j = 1, \ldots, m$, let C_j be the set of all critical points of f_j. By Lemma 2.1.10, there exists a $\delta > 0$ such that for each $g \in G$, each element of $c(B(y, \delta),\ g)$ does not contain any two different points of C_j, $j = 1, \ldots m$. Then for any simply connected open neighborhood V of y included in $B(y, \delta)$ and for any $g \in G$, each element of $c(V, g)$ is simply connected. □

Lemma 2.1.13. *Let G be a rational semigroup and N a positive integer. Then for each $g \in G$, any critical point c of g does not belong to $SH_N(G) \cap \overline{(G \cup \{id\})(g(c))}$.*

Proof. Assume that there exists a critical point c of an element $g \in G$ such that $c \in SH_N(G) \cap \overline{(G \cup \{id\})(g(c))}$. Then there exists a sequence (g_n) in G so that $g_n g(c) \to c$.

There exists a positive number ϵ such that $B(c, \epsilon) \subset SH_N(G)$. Since $g_n g(c) \to c$, we can construct a sequence (n_j) and a sequence (B_j) so that for each j, B_j is a connected component of $((g_{n_1} g)(g_{n_2} g) \cdots (g_{n_j} g))^{-1}(B(c, \epsilon))$ and $c \in B_j$, which contradicts that $c \in SH_N(G)$. □

Lemma 2.1.14. *Let g be a rational map with $\deg(g) \geq 2$ and N a positive integer. Assume that $x \in J(\langle g \rangle) \cap SH_N(\langle g \rangle)$. Then x belongs to neither boundaries of Siegel disks, boundaries of Hermann rings nor indifferent cycles.*

Proof. By Theorem 1 and Corollary in [Ma4] and Lemma 2.1.13, we can show the statement immediately. □

Definition 2.1.15. Let G be a rational semigroup and U a component of $F(G)$. For every element g of G, we denote by U_g the connected component of $F(G)$ containing $g(U)$. We say that U is a wandering domain if $\{U_g\}$ is infinite.

Remark 6. In [HM1], A.Hinkkanen and G.J.Martin showed that there exists an infinitely generated polynomial semigroup which has a wandering domain.

Lemma 2.1.16. *Let G be a rational semigroup which contains an element with the degree at least two. Let x be a point of $F(G)$ and assume that there exists a point $y \in \partial J(G)$ and a sequence (g_n) of elements of G such that $g_n(x) \to y$. Then we have $y \in P(G) \cap \partial J(G)$.*

Proof. We can assume that $\sharp P(G) \geq 3$. Suppose $y \in \overline{\mathbb{C}} \setminus P(G)$. Let δ be a number so that $\overline{B(y, \delta)} \subset \overline{\mathbb{C}} \setminus P(G)$. We can assume that for each n, $g_n(x) \in B(y, \delta)$. For each n, there exists an analytic inverse branch α_n of g_n in U such that $\alpha_n(g_n(x)) = x$. Since $\sharp P(G) \geq 3$, we have $\{\alpha_n\}$is normal in U. Hence if we take an ϵ small enough,

$$\operatorname{diam} \alpha_n(B(y, \ \epsilon\delta)) < d(x, J(G)), \text{ for each } n.$$

But $x \in \alpha_n(B(y, \ \epsilon\delta))$ for large n and $\alpha_n(B(y, \ \epsilon\delta)) \cap J(G) \neq \emptyset$ because $J(G)$ is backward invariant under G. This is a contradiction. □

Corollary 2.1.17. *Let G be a rational semigroup which contains an element with the degree at least two. If $P(G) \cap \partial J(G) = \emptyset$, then for each $x \in F(G)$, $\overline{G(x)} \setminus F(G)$ and there is no wandering domain.*

Lemma 2.1.18. *Let G be a polynomial semigroup, N a positive integer and y a point in $\partial J(G) \cap \mathbb{C}$. Assume that there exists an open neighborhood U of y such that $U \subset SH_N(G)$ and $\sharp(\overline{\mathbb{C}} \setminus G^{-1}(U)) \geq 3$. Then for each $x \in F(G)$, $\overline{G(x)} \subset \overline{\mathbb{C}} \setminus \{y\}$.*

Proof. We can assume that $\infty \in \overline{\mathbb{C}} \setminus G^{-1}(U)$. Suppose that there exists a point $x \in F(G)$ and a sequence (g_n) in G such that $g_n(x) \to y$ as $n \to \infty$. Let δ be a positive number so that for each $g \in G$,

$$\deg(g : V \to D(y, \delta)) \leq N,$$

for each $V \in c(D(y, \ \delta), \ g)$. For any r with $0 < r \leq \delta$ there exists a positive integer $n(r)$ such that for each integer n with $n \geq n(r)$, $g_n(x) \in D(y, r)$. Let $D_{g_n}(y, r)$ be the connected component of $g_n^{-1}(D(y, r))$ containing x. For each n with $n \geq n(r)$, there exists a conformal map φ_n from $D(0, 1)$ onto $D_{g_n}(y, \delta)$ such that $\varphi_n(0) = x$. From Lemma 2.1.10, there exists a constant $C(r)$ with $C(r) \to 0$ as $r \to 0$ such that for each integer n with $n \geq n(r)$,

$$\operatorname{diam} \varphi_n^{-1}(D_{g_n}(y, r)) \leq C(r).$$

Since $\sharp(\overline{\mathbb{C}} \setminus G^{-1}(U)) \geq 3$, the family $\{\varphi_n\}$ is normal in $D(0, 1)$. Hence if r is sufficiently small, then for each integer n with $n \geq n(r)$,

$$\operatorname{diam}_S D_{g_n}(y, r) < d(J(G), x),$$

where we denote by diam_S the spherical diameter and by d the spherical distance. On the other hand, since $J(G)$ is backward invariant under G and $y \in J(G)$, we have that for each n with $n \geq n(r)$, $D_{g_n}(y, r) \cap J(G) \neq \emptyset$. This is a contradiction. Therefore we have for each $x \in F(G)$, $\overline{G(x)} \subset \overline{\mathbb{C}} \setminus \{y\}$. □

Lemma 2.1.19. *Let G be a polynomial semigroup. Assume that there exists a point $x \in F(G)$, a point $y \in \partial J(G)$ and a sequence (g_n) in G such that $g_n(x) \to y$ as $n \to \infty$. Then at least one of the following holds.*

1. *$UH(G) = \emptyset$ and each element of G is a Möbius transformation. For each $z \in F(G)$, $y \in \overline{G(z)}$.*
2. *$\sharp(UH(G)) = 1$ or 2, $UH(G) \subset J(G)$ and $UH(G) \cap \partial J(G) \neq \emptyset$. For each $z \in F(G)$, $y \in \overline{G(z)}$.*

3. $y \in UH(G)$.

Proof. Suppose that $\sharp(UH(G)) \geq 3$. From Lemma 2.1.18, we have $y \in UH(G)$.

Suppose there exists a point $z \in F(G)$ such that $\overline{G(z)} \subset \overline{\mathbb{C}} \setminus \{y\}$. Then there exists a neighborhood V of z such that $G(V) \subset \overline{\mathbb{C}} \setminus \{y\}$. By Lemma 2.1.18, $y \in UH(G)$.

Now we consider the case $\sharp(UH(G)) = 1$ or 2. Then $\infty \in UH(G)$. If $\infty \in F(G)$, then since $G(\infty) = \{\infty\}$, from Lemma 2.1.18 the condition 3. holds. Now suppose $\infty \in J(G)$. There exists an element $g \in G$ with the degree at least two. From Corollary 2.1.14, g has no Siegel disks. Let z be a point in $F(G)$. Since $F(G) \subset F(\langle g \rangle)$, $z \in F(\langle g \rangle)$. From no wandering domain theorem and the fact that g has no Siegel disks, there exists an attracting or parabolic periodic point $\zeta \in \overline{F(G)}$ of g and a sequence (n_j) of positive integers such that $g^{n_j}(z) \to \zeta$. We have $\zeta \in UH(G)$. If $\zeta \in \partial J(G)$, then the condition 2. holds. If $\zeta \in F(G)$, then since G is a polynomial semigroup, we have $G(\{\zeta\}) = \{\zeta\} \subset F(G)$ and it implies $y \in UH(G)$ from Lemma 2.1.18. Hence the condition 3. holds.

Finally we consider the case $UH(G) = \emptyset$. Assume there exists an element $h \in G$ with the degree at least two. Since $F(G) \neq \emptyset$, we have $F(\langle g \rangle) \neq \emptyset$. By the no wandering domain theorem, g has (super)attracting cycles, parbolic cycles, Siegel disks or Hermann rings. Since $UH(G) = \emptyset$, this is a contradiction. □

Theorem 2.1.20. *Let G be a rational semigroup containing an element with the degree at least two and U a connected component. Assume that there exists a sequence (g_n) of elements of G such that $U_{g_n} \cap U_{g_m} = \emptyset$ if $n \neq m$ (in pariticular, U is a wandering domain). Then there exists a subsequence (g_{n_j}) of (g_n) and a point $y \in P(G) \cap \partial J(G)$ such that (g_{n_j}) converges to y locally uniformly on U.*

Proof. By the method in the proof of Theorem 2.2.3 in [S1], we can show that there exists a subsequence (g_{n_j}) of (g_n) and a point $y \in \partial J(G)$ such that (g_{n_j}) converges to y locally uniformly on U. Hence the statement of our theorem holds from Lemma 2.1.16. □

Theorem 2.1.21. *Let G be a polynomial semigroup and U a connected component of $F(G)$. Assume that there exists a sequence (g_n) of elements of G such that $U_{g_n} \cap U_{g_m} = \emptyset$ if $n \neq m$ (in pariticular, U is a wandering domain). Then at least one of the following holds.*

1. *$UH(G) = \emptyset$ and each element of G is a Möbius transformation. For each $z \in F(G)$, $\overline{G(z)} \cap \partial J(G) \neq \emptyset$.*

2. *$\sharp(UH(G)) = 1$ or 2, $UH(G) \subset J(G)$ and $UH(G) \cap \partial J(G) \neq \emptyset$. For each $z \in F(G)$, $\overline{G(z)} \cap \partial J(G) \neq \emptyset$.*

3. *There exists a subsequence (g_{n_j}) of (g_n) and a point $y \in UH(G) \cap \partial J(G)$ such that (g_{n_j}) converges to y locally uniformly on U.*

Proof. Using Lemma 2.1.19, we can show the statement in the same way as the proof of Theorem 2.1.20. □

By Lemma 2.1.10 and using the method of the proof in Lemma 2.1.18, we can show the next lemma immediately.

Lemma 2.1.22. *Let G be a rational semigroup and y a point of $\partial J(G) \setminus UH(G)$. Assume that there exists an open neighborhood U of y such that $\overline{\mathbb{C}} \setminus G^{-1}(U)$ contains a continuum K. Then for each $x \in F(G)$, $\overline{G(x)} \subset \overline{\mathbb{C}} \setminus \{y\}$.*

Lemma 2.1.23. *Let G be a rational semigroup. Assume that there exists a point $x \in F(G)$, a point $y \in \partial J(G)$ and a suquence (g_n) in G such that $g_n(x) \to y$ as $n \to \infty$. Then at least one of the following holds.*

1. *$UH(G) = \emptyset$ and each element of G is a Möbius transformation. For each $z \in F(G)$, $y \in \overline{G(z)}$.*

2. *$UH(G)$ is totally disconnected, $UH(G) \subset J(G)$ and $UH(G) \cap \partial J(G) \neq \emptyset$. For each $z \in F(G)$, $y \in \overline{G(z)}$.*

3. *$y \in UH(G)$.*

Proof. Suppose $UH(G)$ is empty. Then we can show that each element of G is a Möbius transformation in the same way as the proof of Lemma 2.1.19.

Suppose there exists a point $z \in F(G)$ such that $\overline{G(z)} \subset \overline{\mathbb{C}} \setminus \{y\}$. Then there exists a neighborhood V of z such that $G(V) \subset \overline{\mathbb{C}} \setminus \{y\}$. By Lemma 2.1.22, $y \in UH(G)$.

Suppose $UH(G) \cap F(G) \neq \emptyset$. Let $z \in UH(G) \cap F(G)$. If $\overline{G(z)} \subset \overline{\mathbb{C}} \setminus \{y\}$, then by the previous arguments, $y \in UH(G)$. If $y \in \overline{G(z)}$, we have also $y \in UH(G)$.

If $UH(G)$ contains a continuum, then from Lemma 2.1.22, we have $y \in UH(G)$.

Suppose that $\emptyset \neq UH(G) \subset J(G)$ and $UH(G)$ is totally disconnected. There exists an element $g \in G$ of degree at least two. Since $UH(G)$ is totally disconnected and $F(G) \neq \emptyset$, by no wandering domain theorem we can show that g has an (super) attracting or parabolic periodic point ζ in $\partial J(G)$. We have $\zeta \in UH(G)$. □

By Lemma 2.1.23, we can show the next result in the same way as the proof of Theorem 2.1.20.

Theorem 2.1.24. *Let G be a rational semigroup and U a connected component of $F(G)$. Assume that there exists a sequence (g_n) of elements of G such that $U_{g_n} \cap U_{g_m} = \emptyset$ if $n \neq m$ (in pariticular, U is a wandering domain). Then at least one of the following holds.*

1. *$UH(G) = \emptyset$ and each element of G is a Möbius transformation. For each $z \in F(G)$, $\overline{G(z)} \cap \partial J(G) \neq \emptyset$.*

2. *$UH(G)$ is totally disconnected, $UH(G) \subset J(G)$ and $UH(G) \cap \partial J(G) \neq \emptyset$. For each $z \in F(G)$, $\overline{G(z)} \cap \partial J(G) \neq \emptyset$.*

3. *There exists a subsequence (g_{n_j}) of (g_n) and a point $y \in UH(G) \cap \partial J(G)$ such that (g_{n_j}) converges to y locally uniformly on U.*

By Lemma 2.1.22, we can show the next result immediately.

Theorem 2.1.25. *Let G be a rational semigroup. Assume that G is weakly semi-hyperbolic and there is a point $z \in F(G)$ such that the closure of the G-orbit $\overline{G(z)}$ is included in $F(G)$. Then for each $x \in F(G)$, $\overline{G(x)} \subset F(G)$ and there is no wandering domain.*

Next theorem follows from Lemma 2.1.23.

Theorem 2.1.26. *Let G be a rational semigroup containing an element $g \in G$ with $\deg(g) \geq 2$. Assume that G is weakly semi-hyperbolic. If $F(G) \neq \emptyset$, then for each $x \in F(G)$, $\overline{G(x)} \subset F(G)$ and there is no wandering domain.*

Definition 2.1.27. Let G be a rational semigroup. We set

$$A_0(G) = \overline{G(\{z \in \overline{\mathbb{C}} \mid \exists g \in G \text{ with } \deg(g) \geq 2,\ g(x) = x \text{ and } |g'(x)| < 1.\})},$$

$$\tilde{A}_0(G) = \overline{G(\{z \in F(G) \mid \exists g \in G \text{ with } \deg(g) \geq 2,\ g(x) = x \text{ and } |g'(x)| < 1.\})},$$

$$A(G) = \overline{G(\{z \in \overline{\mathbb{C}} \mid \exists g \in G,\ g(x) = x \text{ and } |g'(x)| < 1.\})},$$

$$\tilde{A}(G) = \overline{G(\{z \in F(G) \mid \exists g \in G,\ g(x) = x \text{ and } |g'(x)| < 1.\})},$$

where the closure in the definition of $\tilde{A}_0(G)$ and $\tilde{A}(G)$ is considered in $\overline{\mathbb{C}}$.

Remark 7. By definition, $A_0(G) \subset A(G) \cap P(G)$. For each $g \in G$, $g(A_0(G)) \subset A_0(G)$ and $g(A(G)) \subset A(G)$. We have also similar statements for $\tilde{A}_0(G)$ and $\tilde{A}(G)$.

Lemma 2.1.28. *Let G be a rational semigroup. If $\tilde{A}_0(G)$ is a non-empty compact subset of $F(G)$, then*

$$\emptyset \neq \tilde{A}_0(G) = \tilde{A}(G) \subset P(G) \cap F(G).$$

Proof. Let g be any Möbius transformation in G and $x \in \overline{\mathbb{C}}$ a fixed point of g with $|g'(x)| < 1$. Since $g(\tilde{A}_0(G)) \subset \tilde{A}_0(G) \cap F(G)$ and $\tilde{A}_0(G) \neq \emptyset$, we have that $x \in \tilde{A}_0(G)$. Therefore the statement follows. □

Lemma 2.1.29. *Let G be a rational semigroup containing an element with the degree at least two. Assume that G is semi-hyperbolic and $F(G) \neq \emptyset$. Then*

$$\emptyset \neq A_0(G) = \tilde{A}_0(G) = A(G) = \tilde{A}(G) \subset F(G).$$

Proof. Let $g \in G$ be an element with the degree at least two. Since $F(G) \neq \emptyset$, the element g has a (super)attracting periodic point x in $F(G)$. By Remark 4, we have that $A_0(G) \subset F(G)$. Hence the statement follows from the proof of Lemma 2.1.28. □

Lemma 2.1.30. *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup. Assume that each element of G with the degree at least two has neither Siegel disks nor Hermann rings and each element of $\mathrm{Aut}\,\overline{\mathbb{C}} \cap G$ (if this is not empty) is loxodromic. Also assume that $\sharp J(G) \geq 3$. Let $U_1, \ldots, U_s$ be some connected components of $F(G)$ and K a non-empty compact subset of $V = \cup_{j=1}^{s} U_j$ such that $U_j \cap K \neq \emptyset$ for each $j = 1, \ldots, s$ and $g(K) \subset K$ for each $g \in G$. Then for each compact subset L of V there exists a constant c with $c > 0$ and a constant λ with $0 < \lambda < 1$ such that*

1. *$\sup\{\|(f_{i_n} \cdots f_{i_1})'(z)\| \mid z \in L,\ (i_n, \ldots, i_1) \in \{1, \ldots, m\}^n\} \leq c\lambda^n$, where we denote by $\|\cdot\|$ the norm of the derivative of with respect to the hyperbolic metric on V.*

2. *$\sup\{d(f_{i_n} \cdots f_{i_1}(z), K) \mid z \in L,\ (i_n, \ldots, i_1) \in \{1, \ldots, m\}^n\} \leq c\lambda^n$, where we denote by d the spherical metric.*

Proof. Let a be a large positive number. For each $j = 1, \ldots, s$, let K_j be the compact a-neighborhood of $K \cap U_j$ in U_j with respect to the distance induced by the hyperbolic metric in U_j. We set $K_0 = \cup_{j=1}^{s} K_j$. Then for each $g \in G$, $g(K_0) \subset K_0$. If a is large enough, we have that $L \subset K_0$.

We claim that there exists a constant $c > 0$ and a constant $\lambda < 1$ such that

$$\sup\{\|(f_{i_n}\cdots f_{i_1})'(z)\| \mid z \in L,\ (i_n,\dots,i_1) \in \{1,\dots,m\}^n\} \le c\lambda^n, \qquad (2.1)$$

where we denote by $\|\cdot\|$ the norm of the derivative of with respect to the hyperbolic metric on V. To show the claim, let z be a point of K_j and $(i_{s+1},\dots,i_1)$ an element of $\{1,\dots,m\}^{s+1}$. Then there exists an integer t with $1 \le t \le s$ such that $(f_{i_{s+1}}\cdots f_{i_{t+1}})(U_{j_t}) \subset U_{j_t}$, where U_{j_t} is the component of V containing $(f_{i_t}\cdots f_{i_1})(U_j)$. From the assumption, we have that for each $x \in K_{j_t}$, $\|(f_{i_{s+1}}\cdots f_{i_{t+1}})'(x)\| < 1$. Hence

$$\|(f_{i_{s+1}}\cdots f_{i_1})'(z)\| < 1.$$

Therefore the claim holds.

From the above claim, we can show the statement of our lemma immediately. □

Definition 2.1.31. Let G be a rational semigroup and U a open set in $\overline{\mathbb{C}}$. We say that a non-empty compact subset K of U is an *attractor* in U for G if $g(K) \subset K$ for each $g \in G$ and for any open neighborhood V of K in U and each $z \in U$, $g(z) \in U$ for all but finitely many $g \in G$.

Lemma 2.1.32. *Let $G = \langle f_1, f_2, \dots, f_m\rangle$ be a finitely generated rational semigroup and E a finite subset of $\overline{\mathbb{C}}$. Assume that each $x \in E$ is not a non-repelling fixed point of any element of G. Then for any $M > 0$, there exists a positive integer n_0 such that for any integer n with $n \ge n_0$ if z, $f_{w_1}(z)$, $f_{w_2}f_{w_1}(z)$, $\dots$, $(f_{w_{n-1}}\cdots f_{w_1})(z)$ and $(f_{w_n}\cdots f_{w_1})(z)$ belong to E and $|(f_{w_n}\cdots f_{w_1})'(z)| \neq 0$, then $|(f_{w_n}\cdots f_{w_1})'(z)| > M$.*

Proof. We will show the statement by induction on $\sharp E$. When $\sharp E = 1$, it easy to see that the statement holds. Now assume that for each finite subset E of $\overline{\mathbb{C}}$ with $\sharp E \le s$ the statement holds. Let E' be a finite subset of $\overline{\mathbb{C}}$ with $\sharp E' = s+1$ and assume that each $x \in E'$ is not a non-repelling fixed point of any element of G. Take a number M_0 so that

$$M_0(\inf\{|(f_j)'(\zeta)| \mid \zeta \in E',\ (f_j)'(\zeta) \neq 0,\ j = 1,\dots,m.\})^2 > 1.$$

From the hypothesis of the induction, there exists a positive integer n_0 such that for any subset E of E' with $E \neq E'$ and for any integer n with $n \ge n_0$, if x, $f_{w_1}(x)$, $f_{w_2}f_{w_1}(x)$, $\dots$, $(f_{w_{n-1}}\cdots f_{w_1})(x)$ and $(f_{w_n}\cdots f_{w_1})(x)$ belong to E and $|(f_{w_n}\cdots f_{w_1})'(x)| \neq 0$, then $|(f_{w_n}\cdots f_{w_1})'(x)| > M_0$. For each $y \in E$ and postive integer t with $t \le n_0 + 1$, we set

$$G_{y,t} = \{g \in G \mid g(y) = y,\ g\text{: a product of } t \text{ generators}\}.$$

Then we have that $\sharp G_{y,t} < \infty$ and for each $g \in G_{y,t}$, y is a repelling fixed point of g.

Now assume that z, $f_{w_1}(z)$, $f_{w_2}f_{w_1}(z)$, $\dots$, $(f_{w_{n-1}}\cdots f_{w_1})(z)$ and $(f_{w_n}\cdots f_{w_1})(z)$ belong to E', $(f_{w_n}\cdots f_{w_1})(z) = z$, $(f_{w_n}\cdots f_{w_1})'(z) \neq 0$ and $(f_{w_j}\cdots f_{w_1})(z) \neq z$ for each $j = 1,\dots,n-1$. If $n \le n_0 + 1$, we have

$$|(f_{w_n}\cdots f_{w_1})'(z)| > \inf\{|g'(z)| \mid g \in G_{z,t},\ 1 \le t \le n_0 + 1\} > 1.$$

If $n \ge n_0 + 2$, then we have

$$|(f_{w_n}\cdots f_{w_1})'(z)| > M_0(\inf\{|(f_j)'(\zeta)| \mid \zeta \in E',\ f_j'(\zeta) \neq 0,\ j = 1,\dots,m.\})^2 > 1.$$

From these results, we can show that for any $M > 0$, there exits a positive integer n_1 such that for any integer u with $u \ge n_1$ if z, $f_{w_1}(z)$, $f_{w_2}f_{w_1}(z)$, $\dots$, $(f_{w_{u-1}}\cdots f_{w_1})(z)$ and $(f_{w_u}\cdots f_{w_1})(z)$ belong to E' and $|(f_{w_u}\cdots f_{w_1})'(z)| \neq 0$, then

$$(f_{w_u}\cdots f_{w_1})'(z)| > M.$$

Hence we have completed the induction. □

Lemma 2.1.33. *Let $G = \langle f_1, f_2, \dots, f_m\rangle$ be a finitely generated rational semigroup and E a finite subset of $\overline{\mathbb{C}}$. Assume that each $x \in E$ is not any non-repelling fixed point of any element of G. Then there exists an open neighborhood V of E in $\overline{\mathbb{C}}$ such that for each $z \in V$, if there exists a word $w = (w_1, w_2, \dots) \in \{1,\dots,m\}^{\mathbb{N}}$ satifying that:*

1. *for each n, $(f_{w_n}\cdots f_{w_1})(z) \in V$,*
2. *$(f_{w_n}\cdots f_{w_1}(z))$ accumulates only in E and*
3. *for each n, $(f_{w_n}\cdots f_{w_1})(\zeta) \in E$ and $(f_{w_n}\cdots f_{w_1})'(\zeta) \neq 0$ where ζ is the closest point to z in E,*

then z is equal to the point $\zeta \in E$.

Proof. Let ϵ be a small number so that $B(x,\epsilon) \cap B(y,\epsilon) = \emptyset$ if $x, y \in E$ and $x \neq y$. Take an ϵ smaller, if necesarry, so that if $z_0 \in E$ and $f_j'(z_0) \neq 0$ for some j, then $f_j|_{B(z_0,\epsilon)}$ is injective. We set $V = \cup_{z\in E} B(z,\epsilon)$.

Let $z \in V$ be a point. Assume that there exists a word $w = (w_1, w_2 \dots) \in \{1,\dots,m\}^{\mathbb{N}}$ satisfying the conditions 1, 2 and 3. We set $\alpha_n = f_{w_n}f_{w_{n-1}}\cdots f_{w_1}$. From the conditions 2 and 3, there exists a point $a \in E$ and a sequence (n_j) such that $\alpha_{n_j}(z) \to a$ as $j \to \infty$ and $\alpha_{n_j}(\zeta) = a$ for each j. By lemma 2.1.32, we have $|(\alpha_n)'(\zeta)| \to \infty$ as $n \to \infty$. Hence by the Koebe distortion theorem, there exists a number $\eta > 0$ such that for each positive integer j, there exists an analytic inverse branch β_j of α_{n_j} on $B(a, \eta)$ so that $\beta_j(a) = \zeta$ and $\beta_j(B(a,\eta)) \subset V$ and diam $\beta_t(B(a,\eta)) \to 0$ as $t \to \infty$.

We set $y_j = \beta_j(\alpha_{n_j}(z))$ for each large j. We claim that for each integer l with $0 \le l \le n_j - 1$, if $(f_{w_{l+1}} f_{w_l} \cdots f_{w_1})(y_j) = (f_{w_{l+1}} f_{w_l} \cdots f_{w_1})(z)$, then $(f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(y_j) = (f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(z)$. Let us show the claim above. Assume that $(f_{w_{l+1}} f_{w_l} \cdots f_{w_1})(y_j) = (f_{w_{l+1}} f_{w_l} \cdots f_{w_1})(z)$. We have that

$$f_{w_l} f_{w_{l-1}} \cdots f_{w_1} \circ \beta_j : B(a, \eta) \to \overline{\mathbb{C}}$$

is an analytic inverse branch of $f_{w_{n_j}} f_{w_{n_j-1}} \cdots f_{w_{l+1}}$ satisfying

$$(f_{w_l} f_{w_{l-1}} \cdots f_{w_1} \beta_j)(a) = (f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(\zeta).$$

By Lemma 2.1.32 and the Koebe distortion theorem, we can assume that

$$(f_{w_l} f_{w_{l-1}} \cdots f_{w_1} \beta_j)(B(a,\eta)) \subset B((f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(\zeta),\ \epsilon).$$

Since

$$(f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(z) \in B((f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(\zeta), \epsilon), \qquad (2.2)$$

$$(f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(y_j) = (f_{w_l} f_{w_{l-1}} \cdots f_{w_1} \beta_j)(\alpha_{n_j}(z)) \in B((f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(\zeta),\ \epsilon) \qquad (2.3)$$

and $f_{w_{l+1}}|_{B(f_{w_l} f_{w_{l-1}} \cdots f_{w_1}(\zeta),\ \epsilon)}$ is injective,

$$(f_{w_{l+1}} f_{w_l} \cdots f_{w_1})(y_j) = (f_{w_{l+1}} f_{w_l} \cdots f_{w_1})(z)$$

implies that $(f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(y_j) = (f_{w_l} f_{w_{l-1}} \cdots f_{w_1})(z)$. Hence the claim above holds.

From this claim, it follows that $y_j = z$ for each large j. Since diam $\beta_j(B(a,\eta)) \to 0$ as $j \to \infty$, we have $z = \zeta$.

□

Theorem 2.1.34. *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup. Assume that $F(G) \neq \emptyset$, there is an element $g \in G$ such that $\deg(g) \ge 2$ and each element of $\mathrm{Aut}\,\overline{\mathbb{C}} \cap G$ (if this is not empty) is loxodromic. Also we assume all of the following conditions;*

1. *$\tilde{A}_0(G)$ is a compact subset of $F(G)$,*

2. *any element of G with the degree at least two has neither Siegel disks nor Hermann rings.*

3. *$\sharp(UH(G) \cap \partial J(G)) < \infty$ and each point of $UH(G) \cap \partial J(G)$ is not a non-repelling fixed point of any element of G.*

Then $\tilde{A}_0(G) = \tilde{A}(G) \neq \emptyset$ and for each compact subset L of $F(G)$,

$$\sup\{d(f_{i_n} \cdots f_{i_1}(z),\ \tilde{A}(G)) \mid z \in L,\ (i_n, \ldots, i_1) \in \{1, \ldots, m\}^n\} \to 0,$$

as $n \to \infty$, where we denote by d the spherical metric. Also $\tilde{A}(G)$ is the smallest attractor in $F(G)$ for G. Moreover we have that if (h_n) is a sequence in G consisting of mutually disjoint elements and converges to a map ϕ in a subdomain V of $F(G)$, then ϕ is constant taking its value in $\tilde{A}(G)$.

Proof. First we will show that $\tilde{A}_0(G) = \tilde{A}(G) \neq \emptyset$. By the condition 2, g has neither Siegel disks nor Hermann rings. Since $F(G) \neq \emptyset$ and by the condition 3, applying the no wandering domain theorem for $\langle g \rangle$, we see that the element g has an attracting periodic point x in $F(G)$. Hence $\tilde{A}_0(G) \neq \emptyset$. By Lemma 2.1.28, we get $\tilde{A}_0(G) = \tilde{A}(G) \neq \emptyset$.

Next we will show that for each $x \in F(G)$, $\overline{G(x)} \subset F(G)$. Assume that there exists a connected component U of $F(G)$, a sequence (g_n) of elements of G and a point $y \in \partial J(G)$ such that (g_n) converges to y locally uniformly on U. We take a subsequence $(g_{1,n})$ of (g_n) satisfying that there exists a generator f_{i_1} so that

$$g_{1,n} = \cdots f_{i_1},$$

for each n. Inductively when we get a sequence $(g_{j,n})_n$ satisfying that there exists a word $(i_1, \ldots, i_j) \in \{1, \ldots, m\}^j$ so that $g_{j,n} = \cdots f_{i_j} \cdots f_{i_1}$ for each n, we take a subsequence $(g_{j+1,n})_n$ of $(g_{j,n})_n$ satisfying that there exists a generator $f_{i_{j+1}}$ so that

$$g_{j+1,n} = \cdots f_{i_{j+1}} \cdots f_{i_1}$$

for each n. By the diagonal method, we get a subsequence $(g_{n,n})_n$ of (g_n) satisfying that there exists a word $(i_1, i_2, \ldots) \in \{1, \ldots, m\}^{\mathbb{N}}$ so that for each n,

$$g_{n,n} = \alpha_n f_{i_n} \cdots f_{i_1},$$

where α_n is an element of G. We consider the sequence (β_n) where $\beta_n = f_{i_n} \cdots f_{i_1}$. We see that $U_{\beta_n} \neq U_{\beta_m}$ if $n \neq m$. For, if there exists n and m with $n > m$ such that $U_{\beta_n} = U_{\beta_m}$, then

$$(f_{i_n} \cdots f_{i_{m+1}})(U_{\beta_m}) \subset U_{\beta_m}$$

and the element $f_{i_n}\cdots f_{i_{m+1}}$ has an (super)attracting fixed point x_0 in $\overline{U_{\beta_m}}$. By the condition 3, we have $x_0 \in \tilde{A}(G)$. From Lemma 2.1.30, it contradicts to that (g_n) converges to $y \in \partial J(G)$ in U. Hence $U_{\beta_n} \neq U_{\beta_m}$ if $n \neq m$. Now let z be a point of U. Since $U_{\beta_n} \neq U_{\beta_m}$ if $n \neq m$, we have $(\beta_n(z))$ accumulates only in $\partial J(G)$. By Theorem 2.1.24, we can show that $(\beta_n(z))$ accumulates only in $\partial J(G) \cap UH(G)$. For each large n, let ζ_n be the closest point to $\beta_{i_n}(z)$ in $\partial J(G) \cap UH(G)$. Since $\sharp(\partial J(G) \cap UH(G)) < \infty$ and there is no super attracting fixed point of any element of G in $\partial J(G)$, there exists an integer n_0 such that for each integer n with $n \geq n_0$,

$$(f_{i_n}\cdots f_{i_{n_0}+1})'(\zeta_{n_0}) \neq 0.$$

From Lemma 2.1.33, we get a contradiction. Therefore we have for each $x \in F(G)$, $\overline{G(x)} \subset F(G)$.

Now let x be a point of $F(G)$. We have $\overline{G(x)} \subset F(G)$. Let $\{U_1, \dots, U_s\}$ be the set of all connected components of $F(G)$ having non-empty intersection with $\overline{G(x)}$. We set $V = \cup_{j=1}^{s} U_j$. Suppose that $x \in U_j$. For each $(i_{s+1}, i_s, \dots, i_1) \in \{1, \dots, m\}^{s+1}$, there exists an integer t with $1 \leq t \leq s$ such that $(f_{i_{s+1}}\cdots f_{i_{t+1}})(U_{j_t}) \subset U_{j_t}$, where U_{j_t} is the component of V containing $(f_{i_t}\cdots f_{i_1})(U_j)$. From our assumption, the element $f_{i_{s+1}}\cdots f_{i_{t+1}}$ has an attracting fixed point in $U_{j_t} \cap \tilde{A}(G)$. Hence, from Lemma 2.1.30, we have

$$\sup\{d(f_{i_n}\cdots f_{i_1}(z),\ \tilde{A}(G)) \mid (i_n, \dots, i_1) \in \{1, \dots, m\}^n\} \to 0,$$

as $n \to \infty$. Therefore for each compact subset L of $F(G)$, the similar result holds.

Next we will show that $\tilde{A}(G)$ is the smallest attractor in $F(G)$ for G. From the argument above, $\tilde{A}(G)$ is an attractor in $F(G)$ for G. Let K be any attractor in $F(G)$ for G. It is easy to see that each attracting fixed point of any element of G in $F(G)$ belongs to the set K. It implies that $\tilde{A}(G) \subset K$.

Finally assume (h_n) is a sequence in G consisting mutually disjoint elements and converges to a map ϕ in a subdomain V of $F(G)$. Then by Lemma 2.1.30 we see that ϕ is constant taking its value in $\tilde{A}(G)$. □

By Theorem 2.1.34 and Lemma 2.1.29, we get the next theorem.

Theorem 2.1.35. *Let $G = \langle f_1, f_2, \dots, f_m\rangle$ be a finitely generated rational semigroup which is semi-hyperbolic. Assume that there is an element $g \in G$ such that $\deg(g) \geq 2$ and each element of Aut $\overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic. If $F(G) \neq \emptyset$, then $\emptyset \neq A(G) = A_0(G) \subset F(G)$ and for each compact subset L of $F(G)$,*

$$\sup\{d(f_{i_n}\cdots f_{i_1}(z),\ A(G)) \mid z \in L,\ (i_n, \dots, i_1) \in \{1, \dots, m\}^n\} \to 0,$$

as $n \to \infty$, where we denote by d the spherical metric. Also $A(G)$ is the smallest attractor in $F(G)$ for G. Moreover we have that if (h_n) is a sequence in G consisting mutually disjoint elements and converges to a map ϕ in a subdomain V of $F(G)$, then ϕ is constant taking its value in $\tilde{A}(G)$.

Theorem 2.1.36. *Let $G = \langle f_1, f_2, \dots, f_m\rangle$ be a finitely generated rational semigroup which is sub-hyperbolic. Assume that there is an element $g \in G$ such that $\deg(g) \geq 2$ and each element of Aut $\overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic. If $F(G) \neq \emptyset$, then $\emptyset \neq \tilde{A}(G) = \tilde{A}_0(G) \subset F(G)$ and for each compact subset L of $F(G)$,*

$$\sup\{d(f_{i_n}\cdots f_{i_1}(z),\ \tilde{A}(G)) \mid z \in L,\ (i_n, \dots, i_1) \in \{1, \dots, m\}^n\} \to 0,$$

as $n \to \infty$, where we denote by d the spherical metric. Also $\tilde{A}(G)$ is the smallest attractor in $F(G)$ for G. Moreover we have that if (h_n) is a sequence in G consisting mutually disjoint elements and converges to a map ϕ in a subdomain V of $F(G)$, then ϕ is constant taking its value in $\tilde{A}(G)$.

Proof. Since $\tilde{A}_0(G) \subset P(G)$ and G is sub-hyperbolic, we have that $\tilde{A}_0(G)$ is a compact subset of $F(G)$ and $\sharp(UH(G) \cap J(G)) < \infty$. Now let x be a point of $UH(G) \cap \partial J(G)$. Assume that there exists an element $h \in G$ such that $h(x) = x$. Since G is sub-hyperbolic, x is neither attracting nor indifferent fixed point of h. Since G is finitely generated, by [HM2], we have that there exists no superattracting fixed point of any element of G in $\partial J(G)$. Hence x is a repelling fixed point of h.

From Theorem 2.1.34, the statement of our theorem holds. □

Proposition 2.1.37. *Let G be a finitely generated rational semigroup which contains an element with the degree at least two. Assume that $\sharp P(G) < \infty$ and $P(G) \subset J(G)$. Then $J(G) = \overline{\mathbb{C}}$.*

Proof. Suppose $F(G) \neq \emptyset$. Let $g \in G$ be an element with the degree at least two. By the assumption of our Proposition, g has a super attracting periodic point in $\partial J(G)$. On the other hand, since G is finitely generated, by [HM2], there exist no super attracting fixed points of any element of G in $\partial J(G)$. This is a contradiction. □

Now we consider the expandingness of hyperbolic rational semigroups, which gives us an information about the analytic property of them.

Theorem 2.1.38. *Let $G = \langle f_1, f_2, \dots f_n\rangle$ be a finitely generated hyperbolic rational semigroup . Assume that G contains an element with the degree at least two and each Möbius transformation in G is neither the identity nor an* *elliptic element. Let K be a compact subset of $\overline{\mathbb{C}} \setminus P(G)$. Then there are a positive number c, a number $\lambda > 1$ and a Riemannian metric ρ on an open subset V of $\overline{\mathbb{C}} \setminus P(G)$ which contains $K \cup J(G)$ and is backward invariant under G such that for each k*

$$\inf\{\|(f_{i_k} \circ \cdots \circ f_{i_1})'(z)\|_\rho \mid z \in (f_{i_k} \circ \cdots \circ f_{i_1})^{-1}(K), (i_k, \dots, i_1) \in \{1, \dots, n\}^k\}$$

$\geq c\lambda^k$, here we denote by $\|\cdot\|_\rho$ the norm of the derivative measured from the metric ρ to it.

Proof. This follows from Theorem 2.4.17 which we will show later, but here we will show the statement in the way similar to that of the proof of theorem3.13 in [Mc]. We denote by B the union of all components of $F(G)$ each of which has a non-empty intersection with $P(G)$. Let $B_1, \dots, B_s$ be all the components of B. For each $j = 1, \dots, s$ we take the hyperbolic metric in B_j. Let L_j be the ϵ-neighborhood of $P(G) \cap B_j$ in B_j with respect to the distance in B_j induced by the hyperbolic metric, where ϵ is a positive number which is sufficiently small. We set $L = \cup_{j=1}^s L_j$ and $V = \overline{\mathbb{C}} \setminus L$. Then V contains $K \cup J(G)$ and for each element g of G the inverse image $g^{-1}(V)$ is included in V.

By Theorem 2.1.34, there is a positive integer m_0 such that for each number $m \geq m_0$ the closure of $g_m^{-1}(V)$ is included in V for any element g_m of G in the form $f_{i_m} \circ \cdots \circ f_{i_1}$. Now let m be any positive integer with $m \geq m_0$ and g_m any element of G in the form $f_{i_m} \circ \cdots \circ f_{i_1}$. We set $U = g_m^{-1}(V)$. We take the hyperbolic metric in each component of V and denote it by ρ. Also we take the hyperbolic metric in each component of U and denote it by τ.

We will show that the inclusion map $i : U \to V$ satisfies that $\|i'(z)\| < 1$ for each $z \in U$ where we denote by $\|\cdot\|$ the norm of the derivative measured from the Riemannian metric τ on U to the Riemannian metric ρ on V. Assume that there is a point $z_0 \in U$ such that $\|i'(z_0)\| = 1$. Let W_1 be the connected component of U containing z_0 and W_2 the connected component of V containing z_0. For each $i = 1, 2$ the universal cover of W_i is $D(0,1)$. Let $\tilde{i} : D(0,1) \to D(0,1)$ be the lift of $i : W_1 \to W_2$. Since $\|i'(z_0)\| = 1$, $\tilde{i}(D(0,1)) = D(0,1)$ from Schwartz lemma. It follows that $W_1 = W_2$ but this is a contradiction because the closure of $g_m^{-1}(V)$ is included in V. Hence $\|i'(z)\| < 1$ for each $z \in U$.

The map g_m is a covering map from U to V and is a local isometry between the Riemannian metric τ on U and ρ on V. Hence $\|g_m'(z)\|_\rho > 1$ for each $z \in U$, where we denote by $\|\cdot\|_\rho$ the norm of the derivative measured from the Riemannian metric ρ on V to it.

It is eacy to see that there exists a compact subset C of V which contains K and is backward invariant under G, hence the statement of our theorem holds. □

Now we will show the converse of Theorem 2.1.38.

Theorem 2.1.39. *Let $G = \langle f_1, f_2, \dots f_n\rangle$ be a finitely generated rational semigroup. If there are a positive number c, a number $\lambda > 1$ and a Riemannian metric ρ on an open subset U containing $J(G)$ such that for each k*

$$\inf\{\|(f_{i_k} \circ \cdots \circ f_{i_1})'(z)\|_\rho \mid z \in (f_{i_k} \circ \cdots \circ f_{i_1})^{-1}(J(G)), (i_k, \dots, i_1) \in \{1, \dots, n\}^k\}$$

$\geq c\lambda^k$, where we denote by $\|\cdot\|_\rho$ the norm of the derivative measured from the metric ρ on V to it, then G is hyperbolic and each Möbius transformation in G is loxodromic or hyperbolic.

Remark 8. Because of the compactness of $J(G)$, we can show, with an easy argument, which is familiar to us in the iteration theory of rational functions, that even if we exchange the metric ρ to another Riemannian metric ρ_1, the enequality of the assumption holds with the same number λ and a different constant c_1.

Proof. Take a positive integer k such that $c\lambda^k > 1$ and fix it. We take the compact ϵ-neighborhood K of $J(G)$ in U with respect to the distance ρ. If ϵ is sufficiently small, then

$$\inf\{\|(f_{i_k} \circ \cdots \circ f_{i_1})'(z)\|_\rho \mid z \in (f_{i_k} \circ \cdots \circ f_{i_1})^{-1}(K), (i_k, \dots, i_1) \in \{1, \dots, n\}^k\}$$

> 1 and for each $g \in G$ which is of the form $f_{i_k} \circ \cdots \circ f_{i_1}$, the set $g^{-1}(K)$ is included in K. Moreover if we take ϵ smaller, then in K there is no critical value of any element of G with the word length less than k hence there is no critical value of any element of G in K.

Now let h be any Möbius transformation in G. We will show that h is loxodromic or hyperbolic. Assume that h is parabolic. Then the fixed point x of h satisfies that $(h^n)'(x) = 1$ for each positive integer n and $x \in J(G)$ but this is a contradiction. Assume that h is elliptic. We can assume that $h(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$ and $\mathbb{C} \cap J(G) \neq \emptyset$. There is a sequence (n_j) of positive integers such that for $y \in \mathbb{C} \cap J(G)$, $h^{n_j}(y) \to y$ and $(h^{n_j})'(y) \to 1$ as $j \to \infty$ but this is a contradiction. □

Definition 2.1.40. Let $G = \langle f_1, f_2, \dots f_n\rangle$ be a finitely generated rational semigroup. We say that G is expanding if the assumption of Theorem 2.1.39 holds.

2.2 Continuity of Julia sets

Definition 2.2.1. Let E be a metric space. We denote by $\mathrm{Comp}^*(E)$ the set of non-empty compact subsets of E. For every $A, B \in \mathit{Comp}^*(E)$ we set

$$\partial(A, B) = \sup\{d(x, B) \mid x \in A\}$$

and

$$d_H(A, B) = \max\{\partial(A, B),\ \partial(B, A)\}.$$

It is well known that d_H is a distance on $\mathrm{Comp}^*(E)$. We call it the Hausdorff metric.

Definition 2.2.2. Let M be a complex manifold and Λ a set. Suppose the map

$$(z, a) \in \overline{\mathbb{C}} \times M \mapsto f_{\lambda,a}(z) \in \overline{\mathbb{C}}$$

is holomorphic for each $\lambda \in \Lambda$ and $f_{\lambda,a} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is non-constant for each $\lambda \in \Lambda$ and $a \in M$. Let G_a be the rational semigroup generated by $\{f_{\lambda,a}\}_{\lambda\in\Lambda}$ for each $a \in M$. Then we say that $\{G_a\}_{a\in M}$ is a holomorphic family of rational semigroups.

Remark 9. If a map $F : \overline{\mathbb{C}} \times M \to \overline{\mathbb{C}}$ is holomorphic, then for each $a \in M$ the map $F(\ , a)$ is a rational map and $\deg(F(\ , a))$ is a constant function on M when M is connected. For, if two maps f, g from S^2 to S^2 are continuous and homotopic, then $\deg(f) = \deg(g)$. Holomorphic families of usual iteration of rational functions have been studied in [MSS]. It is well known that the set of J-stable parameters is open and dense in the parameter space([MSS], [Mc]).

Theorem 2.2.3. *Let $\{G_a\}_{a\in M}$ be a holomorphic family of rational semigroups where $G_a = \langle f_{1,a}, \cdots, f_{n,a}\rangle$. We assume that for a point $b \in M$ there is a confinement set K of G_b. Then the map*

$$a \mapsto J(G_a) \in \mathrm{Comp}^*(\overline{\mathbb{C}})$$

is continuous at the point $a = b$ with respect to the Hausdorff metric.

Proof. By Lemma 1.1.6.6, for any $\epsilon > 0$ there is a finite set

$$X_b = \{x_{1,b}, \cdots, x_{l,b}\} \subset J(G_b)$$

of repelling fixed points of G_b such that

$$\partial(J(G_b), X_b)) \leq \epsilon/2.$$

By the implicit function theorem, there is a neighborhood W of b in M such that for every $a \in W$ and for every j=1, ... ,l there is a repelling fixed point $x_{j,a}$ of G_a such that

$$d(x_{j,b}, x_{j,a}) \leq \epsilon/2.$$

For each $a \in W$ we set $X_a = \{x_{1,a}, \cdots, x_{l,a}\}$. Then

$$\partial(X_b, J(G_a)) \leq \partial(X_b, X_a) \leq \epsilon/2.$$

So

$$\partial(J(G_b), J(G_a)) \leq \partial(J(G_b), X_b) + \partial(X_b, J(G_a)) \leq \epsilon.$$

Next, for every $a \in M$ we fix the generator system $\{f_{j,a}\}$ of G_a. We denote by A the union of all components of $F(G_b)$ that have a non empty intersection with K and we take the hyperbolic metric in each component of A. Let α be a positive number and K_2 be the compact 2α neighborhood of K in A and K_1 be the compact α neighborhood of K in A. Then if we take the neighborhood W of b smaller, there is an integer m such that for every $a \in W$ and for every integer t satisfying $m \leq t \leq 2m$ every element $g \in G_a$ of a product of t generators of G_a satisfies

$$g(K_2) \subset K_1.$$

So for every $a \in W$ and for every integer t satisfying $m \leq t$ every element $g \in G_a$ of a product of t generators of G_a satisfies the above. Now we take the ϵ neighborhood O of $J(G_b)$ with respect to the chordal metric and we denote by L the set $\overline{\mathbb{C}} \setminus O$. And if we take W smaller again there is an integer u such that for every $a \in W$ every element $g \in G_a$ of a product of u generators of G_a satisfies that $g(L) \subset K_2$ and so L is included in $F(G_a)$. So

$$\partial(J(G_a), J(G_b)) \leq \epsilon.$$

Hence $a \mapsto J(G_a)$ is continuous at the point b with respect to the Hausdorff metric. □

By Theorem 2.1.34, we get the following result.

Corollary 2.2.4. *Let M be a complex manifold. Let $\{G_a\}_{a\in M}$ be a holomorphic family of rational semigroups where $G_a = \langle f_{1,a}, \cdots, f_{n,a}\rangle$. Let b be a point of M. We assume that G_b satisfies the assumption in Theorem 2.1.34. Then the map*

$$a \mapsto J(G_a)$$

is continuous at the point $a = b$ with respect to the Hausdorff metric.

Corollary 2.2.5. *Let M be a complex manifold. Let $\{G_a\}_{a\in M}$ be a holomorphic family of rational semigroups where $G_a = \langle f_{1,a}, \cdots, f_{n,a}\rangle$. Let b be a point of M. Assume that G_b contains an element of degree at least two and that each element of $Aut\,\overline{\mathbb{C}} \cap G_b$(if this is not empty) is loxodromic. If G_b is semi-hyperbolic or sub-hyperbolic, then the map*

$$a \mapsto J(G_a)$$

is continuous at the point $a = b$ with respect to the Hausdorff metric.

Theorem 2.2.6. *Let $\{G_a\}_{a\in M}$ be a holomorphic family of rational semigroups where $G_a = \langle f_{1,a}, \cdots, f_{n,a}\rangle$. Then*

1. *Let b be a point of M. Assume that G_b is hyperbolic. And also assume that $\deg(f_{1,b})$ is at least two and each Möbius transformation in G_b is neither the identity nor an elliptic element. Then there is an open neighborhood W of b such that for every $a \in W$ the rational semigroup G_a is hyperbolic and the map $a \mapsto J(G_a)$ is continuous with respect to the Hausdorff metric.*

2. *Under the same assumption as 1, if the sets $(f_{j,b}^{-1}(J(G)))_j$ are mutually disjoint, then there is an open neighborhood V of b and a continuous map $i : \overline{\mathbb{C}} \times V \to \overline{\mathbb{C}}$ such that for every $z \in \overline{\mathbb{C}}$ the map $a \mapsto i(z,a)$ is holomorphic, and for every $a \in V$ the map $z \mapsto i(z,a)$ is a quasiconformal homeomorphism of $\overline{\mathbb{C}}$ mapping $J(G_b)$ onto $J(G_a)$.*

Proof. Proof of 1. For every $a \in M$ we fix the generator system $\{f_{j,a}\}$ of G_a. We denote by A the union of all components of $F(G_b)$ that have a non empty intersection with $K = P(G_b)$ and we take the hyperbolic metric in each component of A. Let α be a positive number and K_2 be the compact 2α neighborhood of K in A and K_1 be the compact α neighborhood of K in A. Then if we take a small neighborhood W of b there is an integer m such that for every $a \in W$ and for every integer t satisfying $m \le t \le 2m$ every element $g \in G_a$ of a product of t generators of G_a satisfies

$$g(K_2) \subset K_1.$$

So for every $a \in W$ and for every integer t satisfying $m \le t$ every element $g \in G_a$ of a product of t generators of G_a satisfies the above. Now let Q_a denote the union of all critical points of all generators of G_a. Let L be a relatively compact neighborhood of Q_b in $F(G_b)$. If we take W smaller, for every $a \in W$ the set Q_a is in L. And we can assume that there is a positive integer u such that for every $a \in W$ every element $g \in G_a$ of word length u satisfies $g(L) \subset K_2$. So for every $a \in W$ the set $P(G_a)$ is included in $F(G_a)$ and so G_a is hyperbolic. And from this fact combined with theorems 2.1.34, 2.2.3, it follows that the map $a \mapsto J(G_a)$ is continuous in W.

Proof of 2. We take a neighborhood W of b as above. We can assume that W is a polydisc and for each $a \in W$ the sets $(f_{j,a}^{-1}(J(G)))_j$ are mutually disjoint. Let c be a point of W and x be a repelling fixed point of $g_c = f_{j_1,c} \circ \cdots \circ f_{j_m,c}$ where the number m is the word length of g_c. Then there is an analytic function $x(a)$ in a small neighborhood U of c in W such that $x(a)$ is a repelling fixed point of g_a and $x(c) = x$. If a_0 is a point of $\partial U \cap W$, then $x(a_0)$ is a repelling fixed point of g_{a_0} because G_{a_0} is hyperbolic. So we can take an analytic continuation of $x(a)$ throughout W such that $x(a)$ is a repelling fixed point of g_a. Next if h_a is an element of G_a such that the word length is at most m and $x(a)$ is a fixet point of it then h_a is equal to g_a because G_a is hyperbolic and the sets $(f_{j,a}^{-1}(J(G)))_j$ are mutually disjoint. So by the λ lemma([MSS], [BR], [ST]) and Lemma 1.1.6.6 the statement follows immediately. □

Proposition 2.2.7. *Let $\{G_a\}$ be a holomorphic family of rational semigroup where G_a is a rational semigroup generated by $\{f_{\lambda,a}\}_{\lambda\in\Lambda}$ for each $a \in M$. Let $b \in M$ be a point. Assume there exists a neighborhood U of b in M such that for each $w \in \Lambda^{\mathbb{N}}$, $n \in \mathbb{N}$, the family of rational maps $\{f_{w_1,a} \circ \cdots \circ f_{w_n,a}\}_{a\in U}$ is J-stable in U. Also assume that $\sharp J(G_a) \ge 3$ for each $a \in U$ and that there exist three points x_1, x_2 and x_3 in $\overline{\mathbb{C}}$ which are mutually disjoint such that $x_i \in F(G_a)$ for each $a \in U$ and $i = 1,2,3$. Then the map $a \mapsto J(G_a)$ is continuous on U with respect to the Hausdorff metric.*

Proof. By p54 in [Mc] , we have that for each $w \in \Lambda^{\mathbb{N}}$, $n \in \mathbb{N}$, each point z_a of $J(f_{w_1,a} \circ \cdots \circ f_{w_n,a})$ moves holomorphically on U. Since $x_i \in F(G_a)$ for each $a \in U$ and $i = 1,2,3$, the family of maps $\{a \mapsto z_a\}_{w,z}$ is normal in U. Since the Julia set of a rational semigroup is equal to the closure of the union of Julia sets of all elements of the semigroup, the statement follows. □

Example 2.2.8. Let $d_1, \ldots, d_m$ be positive integers such that $d_j \ge 2$ for each $j = 1, \ldots m$. Let $G_{(c_1,\ldots,c_m)} = \langle z^{d_1}+c_1, \ldots, z^{d_m}+c_m\rangle$ for each $(c_1, \ldots, c_m) \in$

$\mathbb{C}^m$. Let

$$\mathcal{M} = \{(c_1, \ldots, c_m) \in \mathbb{C}^m \mid \overline{G_{(c_1,\ldots,c_m)}(0)} \text{ is bounded in } \mathbb{C}\}.$$

Then the map $(c_1, \ldots c_m) \mapsto J(G_{(c_1,\ldots,c_m)})$ is continuous in $\mathcal{M}^i$. This can be shown by Proposition 2.2.7 and Theorem 4.2 in [Mc](in the condition 6 in Theorem 4.2 in [Mc], we can change the "critical point " to "critical value").

2.3 Self-similarity of Julia Sets

When G is generated by a single rational function f, we know that if all the critical points are in the immediate attractive basin of a fixed point, then the Julia set is a Cantor set. Now we consider the following situation similar to that.

Theorem 2.3.1. *Let $G = \langle f_1, \ldots, f_n \rangle$ be a finitely generated rational semigroup. Assume that G contains an element with the degree at least two and each Möbius transformation in G is neither the identity nor an elliptic element. If $P(G)$ is included in a connected component U of $F(G)$, then there are simply connected domains $V_1, \ldots, V_k$ and mappings $h_1, \ldots, h_s$ from $W = \cup_j V_j$ to W such that for each j, i the map h_j is a contraction map from V_i to a domain $V_{i'}$ with respect to the hyperbolic metric with the rate of contraction bounded by a constant strictly less than one throughout V_i and*

$$J(G) \subset W, \quad \bigcup_j h_j(J(G)) = J(G).$$

Proof. There is a relatively compact subdomain V of U including $P(G)$. For each positive integer m we denote by G_m the subsemigroup of G generated by all elements $g_1, \ldots, g_l$ of word length m. If we take a number m large enough, then for each $g \in G_m$, g maps the closure of V into V. So the closure of $g^{-1}(\overline{\mathbb{C}} \setminus V)$ is included in $\overline{\mathbb{C}} \setminus V$. Each connected component of $\overline{\mathbb{C}} \setminus \overline{V}$ is simply connected because V is connected. For each component of $\overline{\mathbb{C}} \setminus \overline{V}$ we take all branches of g^{-1} on it. Then each branch is a contraction map on each component of $\overline{\mathbb{C}} \setminus \overline{V}$ with respect to the hyperbolic metric with the rate of contraction bounded by a constant strictly less than one. Now from Lemma 1.1.5.2 and Lemma 1.1.6.1,

$$J(G) = J(G_m) = \bigcup_{j=1}^{l} g_j^{-1}(J(G_m)),$$

so the statement follows. □

Remark 10. In the above proof, if we can take V as a simply connected domain, then the Julia set is a self-similar set in $\overline{\mathbb{C}} \setminus \overline{V}$ with respect to the hyperbolic metric.

By Theorem 2.3.1 and the proof, we can show the following result.

Theorem 2.3.2. *Let $G = \langle f_1, \ldots, f_n \rangle$ be a finitely generated rational semigroup. Assume that $\deg(f_1)$ is at least two. If $P(G)$ is included in a connected component U of $F(G)$ and the sets $\{f_j^{-1}(J(G))\}_{j=1,\ldots,n}$ are mutually disjoint, then the Julia set $J(G)$ is a Cantor set.*

Example 2.3.3. Let $G_c = \langle z^2 + c,\ z^2 + ci \rangle$. Then $J(G_c)$ is a Cantor set for sufficiently large positive number c.

2.4 Rational Skew Product

Definition 2.4.1 (rational skew product). Let X be a topological space. If a continuous map $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ is represented by the following form:

$$\tilde{f}((x, y)) = (p(x),\ q_x(y)),$$

where $p : X \to X$ is a continuous map and $q_x : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational map with the degree at least 1 for each $x \in X$, then we say that $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ is a rational skew product. In this paper we always assume that X is a compact metric space.

Notation: For each $n \in \mathbb{N}$ and $x \in X$, we set

$$q_x^{(n)} := q_{p^{n-1}(x)} \circ \cdots \circ q_x.$$

Definition 2.4.2. Under the above situation, we define the following sets. For each $x \in X$,

$$F_x = \{y \in \overline{\mathbb{C}} \mid \{q_x^{(n)}\}_n \text{ is normal in a neigborhood of } y\},$$

$$J_x = \overline{\mathbb{C}} \setminus F_x, \quad \tilde{J}_x = \{x\} \times J_x.$$

Further we set

$$\tilde{J}(\tilde{f}) = \overline{\bigcup_{x \in X} \tilde{J}_x}, \quad \tilde{F}(\tilde{f}) = (X \times \overline{\mathbb{C}}) \setminus \tilde{J}(\tilde{f}).$$

$$C(\tilde{f}) = \{(x, y) \in X \times \overline{\mathbb{C}} \mid q_x'(y) = 0\}, \quad P(\tilde{f}) = \overline{\bigcup_{n \in \mathbb{N}} \tilde{f}^n(C(\tilde{f}))}.$$

$C(\tilde{f})$ is called the critical set for $\tilde{f}$ and $P(\tilde{f})$ is called the post critical set for $\tilde{f}$. Moreover we set

$$(\tilde{f}^n)'((x,y)) = (q_x^{(n)})'(y).$$

If (x,y) is a period point of $\tilde{f}$ with the period n, then we say that (x,y) is repelling(resp. indifferent, attracting, etc.) if $|(\tilde{f}^n)'((x,y))| > 1$(resp.$= 1$, $<$ 1, etc.).

Lemma 2.4.3. *Let $\tilde{f} : X\times\overline{\mathbb{C}} \to X\times\overline{\mathbb{C}}$ be a rational skew product represented by $\tilde{f}((x,y)) = (p(x),\ q_x(y))$. Then the following hold.*

1. *if $x \in X$, then $q_x^{-1}(F_{p(x)}) = F_x$, $q_x^{-1}(J_{p(x)}) = J_x$, $\tilde{f}(\tilde{J}(\tilde{f})) \subset \tilde{J}(\tilde{f})$.*
2. *if $p : X \to X$ is surjective, then $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ is surjective.*
3. *if $p : X \to X$ is a surjective and open map, then $\tilde{f}^{-1}(\tilde{J}(\tilde{f})) = \tilde{f}(\tilde{J}(\tilde{f})) = \tilde{J}(\tilde{f})$.*

Proof. We will show the last statement. Suppose $\tilde{f}((x,y)) \in \tilde{J}(\tilde{f})$. Then there exists a sequence $((x_i,\ y_i))$ converging to $\tilde{f}((x,y))$ such that $y_i \in F_{x_i}$ for each i. Since p is an open map, there exists a sequence $(\tilde{x}_i)$ converging to x such that $p(\tilde{x}_i) = x_i$. Then there exists a sequence $(\tilde{y}_i)$ converging to y such that $q_{\tilde{x}_i}(\tilde{y}_i) = y_i$ for each i. Then $\tilde{y}_i \in J_{\tilde{x}_i}$. Hence $(x,y) \in \tilde{J}(\tilde{f})$. Hence $\tilde{f}^{-1}(\tilde{J}(\tilde{f})) \subset \tilde{J}(\tilde{f})$. Since $\tilde{f}(\tilde{J}(\tilde{f})) \subset \tilde{J}(\tilde{f})$ and $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ is surjective, we have $\tilde{f}^{-1}(\tilde{J}(\tilde{f})) = \tilde{f}(\tilde{J}(\tilde{f})) = \tilde{J}(\tilde{f})$. □

Definition 2.4.4. Let G be a rational semigroup generated by $\{f_\lambda\}_{\lambda\in\Lambda}$. Let $X = \Lambda^{\mathbb{N}}$. Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be the map defined by:

$$\tilde{f}((x,y)) = (p(x),\ f_{x_1}(y)),$$

where $p : X \to X$ is the shift map and $x \in X$ is represented by: $x = (x_1, x_2, \dots)$. Then we say that $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ is the rational skew product constructed by the generator system $\{f_\lambda\}_{\lambda\in\Lambda}$.

Let $G = \langle f_1, \dots, f_m\rangle$ be a finitely generated rational semigroup. Let $\tilde{f} : \Sigma_m \times\overline{\mathbb{C}} \to \Sigma_m \times\overline{\mathbb{C}}$ be the rational skew product constructed by the generator system $\{f_1, \dots, f_m\}$, where $\Sigma_m = \{1, \dots, m\}^{\mathbb{N}}$. Then $\tilde{f}$ is a finite-to-one and open map. We have that a point $(w,x) \in \Sigma_m \times \overline{\mathbb{C}}$ satisfies $f'_{w_1}(x) \neq 0$ if and only if $\tilde{f}$ is a homeomorphism in a small neighborhood of (w,x). Moreover the following proposition holds.

Proposition 2.4.5. *Let $G = \langle f_1, f_2, \dots f_m\rangle$ be a finitely generated rational semigroup. Let $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ be the rational skew product defined by $\tilde{f}((w,x)) = (\sigma(w), f_{w_1}(x))$. Then the following hold.*

1. *$\tilde{F}$ and $\tilde{J}$ are completely invariant under $\tilde{f}$. $\tilde{F}$ is open and $\tilde{J}$ is compact. $\tilde{f}(\tilde{J}_w) = \tilde{J}_{\sigma w}$. $\tilde{F}(\tilde{f})$ is equal to the set of all the points $(w,x) \in \Sigma_m \times \overline{\mathbb{C}}$ which satisfies that there exists an open neighborhood U of x and an open neighborhood V of w such that for each $a \in V$ the family of maps $\{f_{a_n} \circ \cdots \circ f_{a_1}\}$ is normal in U.*
2. *$\tilde{J} = \cap_{n=0}^{\infty}\tilde{f}^{-n}(\Sigma_m \times J(G))$. $\pi_2(\tilde{J}) = J(G)$, where we denote by $\pi_2 : \Sigma_m \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ the second projection.*
3. *$\tilde{J}$ has no interior points or is equal to $\Sigma_m \times \overline{\mathbb{C}}$.*
4. *If $\sharp(J(G)) \geq 3$, then $\tilde{J}$ is a perfect set.*
5. *If $\sharp(J(G)) \geq 3$, then $\tilde{J}$ is equal to the closure of the set of all repelling period points of $\tilde{f}$.*
6. *Assume $\sharp(J(G)) \geq 3$ and $E(G) \subset F(G)$. Let K be a compact subset of $\pi_2^{-1}(\overline{\mathbb{C}} \setminus E(G))$. If U is an open set in $\Sigma_m \times \overline{\mathbb{C}}$ satisfying $U \cap \tilde{J} \neq \emptyset$, then there exists a positive integer N such that for each integer n with $n \geq N$, we have $\tilde{f}^n(U) \supset K$.*

Proof. By definition, it is easy to see 1. and 2. Assume $\tilde{J} \neq \Sigma_m \times \overline{\mathbb{C}}$ and $\tilde{J}$ contains a non-empty open set U. Then $F(G) \neq \emptyset$ and for each positive integer n we have

$$\pi_2\tilde{f}^n(U) \subset \overline{\mathbb{C}} \setminus F(G).$$

By Montel's theorem, this is a contradiction. Hence 3. holds.

Let $z \in \tilde{J}$ be a point and assume there exists an open neighborhood U of z such that $U\setminus\{z\} \subset \tilde{F}$. There exists a positive integer n such that $\pi_1(\tilde{f}^n(U)) = \Sigma_m$. It follows that $\pi_2(\tilde{f}^n(z)) \in J(G)$ and $\pi_2(\tilde{f}^n(U \setminus \{z\})) \subset F(G)$. Since $\sharp(J(G)) \geq 3$, we have $J(G)$ is perfect and so that is a contradiction. Hence 4. holds.

Now we will show 5. Let $(w,x) \in \tilde{J}$. Let U be a neighborhood of w in Σ_m and V be a neighborhood of x in $\overline{\mathbb{C}}$. There exists a positive integer n such that if we set

$$U_n = \{\alpha \in \Sigma_m \mid \alpha_j = w_j,\ j = 1, \dots, n,\}$$

then $U_n \subset U$. We set

$$G_n = \{g \in G \mid g = \cdots f_{w_n} \circ \cdots \circ f_{w_1}\}.$$

Then this is a subsemigroup of G. We have

$$J(G_n) = (f_{w_n} \cdots f_{w_1})^{-1}J(G)$$

and since $J(G)$ has infinitely many points, $J(G_n)$ must have at least three points. By Theorem 3.1 in [HM1](Note that if we read the proof of this theorem, we can see that the statement of this theorem holds whenever the Julia set has at least three points), we get that $J(G_n)$ is the closure of the set of repelling fixed point of all elements of G_n. Since $x \in (f_{w_n} \cdots f_{w_1})^{-1}J(G) = J(G_n)$, there exists an element $g \in G_n$ and a point $y \in V$ such that y is a repelling fixed point of g. Hence 5. holds.

Now we will show 6. Let K and U be as in 6. Assume $E(G) \neq \emptyset$. If $E(G)$ has exactly two points, then it is easy to show the statement of 6. Suppose $E(G) = \{x\}$ and let V be a connencted component of $F(G)$ containing x. Let ρ be the hyperbolic metric in V. Since each f_j does not increase the metric and x is a fixed point of it, there exists an open hyperbolic ball A about x included in V such that $f_j(A) \subset A$ for each j. It implies that $\tilde{f}(\pi_2^{-1}(A)) \subset \pi_2^{-1}(A)$. Hence, from the beginning of the proof, we can assume that

$$\tilde{f}(K) \supset K. \tag{2.4}$$

We will show that for each positive integer k,

$$K \subset \cup_{j=1}^{\infty} \tilde{f}^{kj}(U). \tag{2.5}$$

There exists a positive integer j such that $\pi_1 \tilde{f}^{kj}(U) = \Sigma_m$. On the other hand, by [HM1], for any rational semigroup G_1 the closure of the backward orbit of each point $x \in \overline{\mathbb{C}} \setminus E(G)$ under G_1 contains the Julia set $J(G_1)$. By [HM1] again, the Julia set of the subsemigroup H_k of G which is generated by :

$$\{f_{\alpha_k} \circ \cdots \circ f_{\alpha_1} \mid (\alpha_k, \ldots, \alpha_1) \in \{1, \ldots, m\}^k\}$$

is equal to $J(G)$. Also we have $E(H_k) = E(G)$ by definition of the exceptional set. Hence (2.5) holds.

By 5., we obtain that there exists an open set U_0 included in U and a positive integer s such that $\tilde{f}^s(U_0) \supset U_0$. Hence by (2.5), we get that there exists a positive integer N such that $\tilde{f}^N(U) \supset K$. From (2.4), it follows that for each positive integer n with $n \geq N$, we have $\tilde{f}^n(U) \supset K$. Hence the statement of 6. holds when $E(G)$ has exactly one point. If $E(G) = \emptyset$, we can show the statement in the same way as the above. □

Definition 2.4.6 (hyperbolicity). Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product. We say that $\tilde{f}$ is hyperbolic along fibers if $P(\tilde{f}) \subset \tilde{F}(\tilde{f})$.

Definition 2.4.7. Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product. We say that $\tilde{f}$ is expanding along fibers if there exists a positive constant C and a constant λ with $\lambda > 1$ such that for each $n \in \mathbb{N}$,

$$\inf_{z \in \tilde{J}(\tilde{f})} \|(\tilde{f}^n)'(z)\| \geq C\lambda^n,$$

where we denote by $\| \cdot \|$ the norm of the derivative with respect to the spherical metric.

Definition 2.4.8 (semi-hyperbolicity). Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product. Let N be a positive integer. We say that a point $(x_0, y_0) \in X \times \overline{\mathbb{C}}$ belongs to $SH_N(\tilde{f})$ if there exists a neighborhood U of x_0 and a positive number δ satisfying that for any $x \in U$, any $n \in \mathbb{N}$, any element $x_n \in p^{-n}(x)$ and any element V of $c(B(y_0, \delta), q_{x_n}^{(n)})$,

$$\deg(q_{x_n}^{(n)} : V \to B(y_0, \delta)) \leq N.$$

We set

$$UH(\tilde{f}) = (X \times \overline{\mathbb{C}}) \setminus \cup_{N \in \mathbb{N}} SH_N(\tilde{f}).$$

We say that $\tilde{f}$ is semi-hyperbolic along fibers if for any $(x_0, y_0) \in \tilde{J}(\tilde{f})$ there exists a positive integer N such that $(x_0, y_0) \in SH_N(\tilde{f})$.

Lemma 2.4.9. *Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product. If $\tilde{f}$ is hyperbolic along fibers, then it is semi-hyperbolic along fibers.*

Lemma 2.4.10. *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup. Then G is semi-hyperbolic if and only if the rational skew product $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ constructed by the generator system $\{f_1, f_2, \ldots, f_m\}$ is semi-hyperbolic along fibers. G is hyperbolic if and only if $\tilde{f}$ is hyperbolic along fibers.*

Definition 2.4.11 (Condition(C1)). Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product. We say that $\tilde{f}$ satisfies the condition (C1) if there exists a family $\{D_x\}_{x \in X}$ of discs in $\overline{\mathbb{C}}$ such that the following three conditions are satisfied:

1. $\overline{\bigcup_{n \geq 0} \tilde{f}^n(\{x\} \times D_x)} \subset \tilde{F}(\tilde{f})$.
2. for any $x \in X$, we have that $\mathrm{diam}(q_x^{(n)}(D_x)) \to 0$, as $n \to \infty$.
3. $\inf_{x \in X}$ diam $(D_x) > 0$.

Now we will show the following lemma and theorem.

Lemma 2.4.12. *Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product satisfying the condition (C1). Assume that there exists a point $(x_0, y_0) \in X \times \overline{\mathbb{C}}$ with $y_0 \in F_{x_0}$, a connected open neighborhood U of y_0 in $\overline{\mathbb{C}}$ and a sequence (n_j) of positive integers such that $R_j := q_{x_0}^{(n_j)}$ converges to a non-constant map ϕ uniformly on U as $j \to \infty$. Let $(x_j, y_j) = \tilde{f}^{n_j}(x_0, y_0)$ and $(x_\infty, y_\infty) = \lim_{j\to\infty}(x_j, y_j)$. Let $S_{i,j} = q_{x_i}^{(n_j - n_i)}$ for $1 \le i < j$. Let*

$$V = \{y \in \overline{\mathbb{C}} \mid \exists \epsilon > 0,\ \limsup_{i\to\infty} \sup_{j>i} \sup_{d(\xi,y)\le\epsilon} d(S_{i,j}(\xi),\ \xi) = 0\}.$$

Then V is a non-empty open set and for any $y \in \partial V$, we have that

$$(x_\infty,\ y) \in \tilde{J}(\tilde{f}) \cap UH(\tilde{f}). \tag{2.6}$$

Theorem 2.4.13. *Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product. Assume $\tilde{f}$ is semi-hyperbolic along fibers and satisfies the condition (C1). Then the following hold.*

1. *Let $(x_0, y_0) \in X \times \overline{\mathbb{C}}$ be any point with $y_0 \in F_{x_0}$. Then for any open connected neighborhood U of y_0 in $\overline{\mathbb{C}}$, there exists no subsequence of $(q_{x_0}^{(n)})_n$ converging to a non-constant map locally uniformly on U.*

2. $$\tilde{J}(\tilde{f}) = \bigcup_{x\in X} \tilde{J}_x.$$

3. *If there exists a disc D in $\overline{\mathbb{C}}$ such that $D_x = D$ for all $x \in X$ in the condition (C1), then there exist positive constants δ, L and $\lambda (0 < \lambda < 1)$ such that for any $n \in \mathbb{N}$,*

$$\sup\{\mathrm{diam}\ U \mid U \in c(B(y,\delta),\ q_{x_n}^{(n)}),\ (x,y) \in \tilde{J}(\tilde{f}),\ x_n \in p^{-n}(x)\} \le L\lambda^n.$$

To show Lemma 2.4.12 and Theorem 2.4.13, we need the following lemma.

Lemma 2.4.14. *Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product satisfying the condition (C1). Assume $(x_0, y_0) \in SH_N(\tilde{f})$ for some $N \in \mathbb{N}$. Then there exists a positive number δ_0 such that for each δ with $0 < \delta < \delta_0$ there exists a neighborhood U of x_0 in $\overline{\mathbb{C}}$ satisfying that for each $n \in \mathbb{N}$, each $x \in U$ and each $x_n \in p^{-n}(x)$, we have that each element of $c(B(y_0, \delta),\ q_{x_n}^{(n)})$ is simply connected.*

Proof. Take a positive number δ_1 such that for each $x \in X$ and each $x_1 \in p^{-1}(x)$, we have that each connected component of $q_{x_1}^{-1}(D_x)$ contains a ball with the radius at least δ_1.

By the semi-hyperbolicity and Lemma 2.1.10, we can take a positive number δ_0 and a open neighborhood U of x_0 in $\overline{\mathbb{C}}$ such that for each δ with $0 < \delta < \delta_0$, each $x \in U$, each $n \in \mathbb{N}$, and each $x_n \in p^{-n}(x)$, we have that the diameter of each element of $c(B(y_0, \delta),\ q_{x_n}^{(n)})$ is less than δ_1.

Now we will show each element of $c(B(y_0, \delta),\ q_{x_n}^{(n)})$ is simply connected by induction on n. Assume an element W of $c(B(y_0, \delta),\ q_{x_n}^{(n)})$ is simply connected. Let W_1 be a connected component of $q_{x_{n+1}}^{-1}(W)$ where x_{n+1} is an element of $p^{-1}(x_n)$. Suppose W_1 is not simply connected. Each connected component of ∂W_1 is mapped onto ∂W by $q_{x_{n+1}}$. Hence the image of each connected component of $\overline{\mathbb{C}} \setminus W_1$ by $q_{x_{n+1}}$ contains D_{x_n}. Hence we have that diam $W_1 \ge \delta_1$, which contradicts to the choice of δ_0 and U. □

Now we will show the Lemma 2.4.12.

Proof. We will show the statement developing a method in M.Jonsson's Thesis([J]). By the definition, V is an open set. Since ϕ is non-constant, there exists a positive number a such that

$$R_j(U) \supset B(y_\infty,\ a)$$

for each $j \in \mathbb{N}$. We have that $B(y_\infty,\ a) \subset V$. For, if $y \in B(y_\infty,\ a)$ then $y = R_i(\xi_i)$ for some $\xi_i \in U$ and so $d(S_{i,j}(y),\ y) = d(R_j(\xi_i),\ R_i(\xi_i))$ which is small if i is large. Hence V is a non-empty open set.

Take any $y \in \partial V$. We will show

$$(x_\infty,\ y) \in \tilde{J}(\tilde{f}). \tag{2.7}$$

Assume this is false. If there exists a positive integer i_0 such that $\{S_{i,j}\}_{j\ge i\ge i_0}$ is normal in a neighborhood of y, then since $S_{i,j} \to Id$ on $V \cap W$, we have that $W \subset V$ and it is a contradiction. Hence there exist sequences (i_k), (j_k) and (ξ_k) such that $i_k \le j_k$, $j_k - i_k \to \infty$, $\xi_k \to y$ and

$$S_{i_k,\ j_k}(\xi_k) \in D_{x_{i_k}}, \tag{2.8}$$

where we denote by $(D_x)_{x\in X}$ a family of discs in $\overline{\mathbb{C}}$ in the definition of condition (C1). Since we are assuming $(x_\infty,\ y) \in \tilde{F}(\tilde{f})$, we have that there exists an disc B around y such that $B \subset F_{x_{i_k}}$ and $\xi_k \in B$ for large k. By (2.8), the condition (C1) and the definition of V, we get a contradiction. Hence (2.7) holds.

Now we will show $(x_\infty, y) \in UH(\tilde{f})$. Suppose this is false. Then there exists a positive integer N such that $(x_\infty, y) \in SH_N(\tilde{f})$. Let δ_0 be a number for (x_∞, y) in Lemma 2.4.14 and let $\delta = \delta_0/2$. We can assume that there exists a neighborhood U' of x_∞ satisfying that for any $x \in U'$, any $n \in \mathbb{N}$, any element $x_n \in p^{-n}(x)$ and any element V of $c(B(y,\ \delta_0),\ q_{x_n}^{(n)})$,

$$\deg(q_{x_n}^{(n)} : V \to B(y,\ \delta_0)) \leq N.$$

Take two domains V_1 and V_2 such that

$$y_\infty \in V_2 \subset\subset V_1 \subset\subset V,\ B(y,\ \delta) \cap V_2 \neq \emptyset. \tag{2.9}$$

If $j > i$ and i is large enough, then $S_{i,j}$ is close to id_{V_1} on V_1. Hence $S_{i,j}$ is biholomorphic on V_1 and $S_{i,j}(V_1) \supset V_2$. Let $h_{i,j} : V_2 \to V_1$ be a map such that $S_{i,j} \circ h_{i,j} = id$ on V_2 and $h_{i,j} \circ S_{i,j} = id$ on $S_{i,j}^{-1}(V_2) \cap V_1$. Then we have that

$$\lim_{i\to\infty} \sup_{j>i} \sup_{\xi \in V_2} d(h_{i,j}(\xi),\ \xi) = 0. \tag{2.10}$$

For each (i,j) such that $j > i$ and i is large enough, let $B_{i,j} \in c(B(y,\delta),\ S_{i,j})$ be an element such that $h_{i,j}(V_2 \cap B(y,\delta)) \subset B_{i,j}$. By the choice of δ_0, we have that $B_{i,j}$ is simply connected. By semi-hyperbolicity, there exists a positive integer M such that for each (i,j) with $j > i$ where i is large enough,

$$\sharp(\ \mathrm{cv}(S_{i,j}|_{B_{i,j}}) \cap B(y,\delta)) \leq M, \tag{2.11}$$

where we denote by "cv" the set of critical values. Hence there exists a positive number θ with $0 < \theta < 2\pi$ such that for each (i,j) with $i < j$ there exists a sector $U_{i,j}$ in $B(y,\delta)$ of angle θ with the center y such that $U_{i,j} \cap\ \mathrm{cv}(S_{i,j}|_{B_{i,j}}) = \emptyset$ and $U_{i,j} \cap V_2 \neq \emptyset$. Let $g_{i,j} : U_{i,j} \to B_{i,j}$ be the analytic continuation of $h_{i,j}$ on $V_2 \cap U_{i,j}$. Let $y_{i,j} \in B_{i,j} \cap \overline{g_{i,j}(U_{i,j})}$ such that $S_{i,j}(y_{i,j})) = y$. By (2.10), Corollary 2.1.9, Condition (C1) and the fact $V_2 \cap U_{i,j} \neq \emptyset$, we have that there exists a positive number δ_1 such that

$$B(y_{i,j},\ \delta_1) \subset B_{i,j}, \tag{2.12}$$

for each (i,j) such that $j > i$ and i is large enough. Now we will show the following claim:

$$\text{Claim: } \lim_{i\to\infty} \sup_{j>i} d(y_{i,j},\ y) = 0. \tag{2.13}$$

Suppose this is false. Then there exists a sequence $((i_k, j_k))$ with $j_k > i_k$ and a positive number δ_2 such that $d(y_{i_k,j_k},\ y) > \delta_2$, for each k. We can assume that (y_{i_k,j_k}) converges to a point $\tilde{y}$ as $k \to \infty$ and that there exists a sector U_0 with the center y such that $U_{i_k,j_k} = U_0$ for each k. Then we have that

$$B(\tilde{y},\ \delta_1/2) \cap V = \emptyset. \tag{2.14}$$

For, assume the left hand side is not empty. Then since $S_{i_k,j_k} \to id$ in V and the family $\{S_{i_k,j_k}|_{B(\tilde{y},\delta_1/2)}\}_k$ is normal, we have that $S_{i_k,j_k} \to id$ locally uniformly on $B(\tilde{y},\delta_1/2)$. But this is a contradiction because $\tilde{y} \neq y$. Hence we have (2.14).

By Lemma 2.1.10, there exists a positive number δ_3 with $\delta_3 < \delta_1/4$ such that for each k, the diameter of each element of $c(B(y,\delta_3),\ S_{i_k,j_k})$ is less than δ_1. Hence if we take a fixed point $z \in B(y,\delta_3) \cap U_0$, then we have that for each large k,

$$d(\tilde{y},\ g_{i_k,j_k}(z)) < \delta_1/4. \tag{2.15}$$

On the other hand, since $g_{i_k,j_k} \to id$ locally uniformly on V_2 and $(g_{i_k,j_k})_k$ is normal in U_0, we have that $g_{i_k,j_k} \to id$ locally uniformly on U_0. Hence we have that $d(g_{i_k,j_k}(z),\ y) < \delta_1/4$ for each large k. Together with (2.15) and $B(\tilde{y},\ \delta_1/2) \cap V = \emptyset$, we get a contradiction. Hence we have shown the claim (2.13).

Since $B(y_{i,j},\ \delta_1) \subset B_{i,j}$ for each (i,j) such that $j > i$ and i is large enough, by the above claim we have that there exists a positive integer i_0 such that for each (i,j) with $j > i \geq i_0$,

$$S_{i,j}(B(y,\ \delta_1/2)) \subset B(y,\ \delta).$$

Hence $(S_{i,j})_{j>i\leq i_0}$ is normal in $B(y,\ \delta_1/2)$. Since $S_{i,j} \to id$ on $B(y,\ \delta_1/2) \cap V$, we have that $y \in V$ and this is a contradiction. Hence we have shown the first statement of our lemma. □

Now we will show Theorem 2.4.13.

Proof. The statement 1 follows from Lemma 2.4.12.

Now we will show the statement 2 of our theorem. Suppose the statement is false. Then there exists a point $(x_0, y_0) \in \tilde{J}(\tilde{f})$ with $y_0 \in F_{x_0}$, a connected component U of y_0 in $\overline{\mathbb{C}}$ and a sequence (n_j) of positive integers such that $R_j := q_{x_0}^{(n_j)}$ converges to a map ϕ uniformly on U as $j \to \infty$. Let $(x_j, y_j) = \tilde{f}^{n_j}(x_0, y_0)$ and $(x_\infty, y_\infty) = \lim_{j\to\infty}(x_j, y_j)$. By Lemma 2.1.10, there exists a positive number a such that $y_j \in B(y_\infty,\ a)$ and the element $B_j \in c(B(y_\infty,\ a),\ R_j)$ containing y_j satisfies that $B_j \subset U$ for each large

j. Hence $R_j(U) \supset B(y_\infty, a)$ for each large j and it implies that ϕ is non-constant. By Lemma 2.4.12, it is a contradiction. Hence we have shown the statement 2 of our theorem.

Finally we will show the statement 3 of our theorem. By semi-hyperbolicity and Lemma 2.4.14, there exists a positive integer N and a positive real number δ_0 such that for any $(x', y') \in \tilde{J}(\tilde{f})$ there exists a neighborhood U' of x' satisfying that for any real number τ with $0 < \tau \leq \delta_0$, any $x \in U'$, any $n \in \mathbb{N}$, any element $x_n \in p^{-n}(x)$ and any element V of $c(B(y'\tau), q_{x_n}^{(n)})$, we have that V is simply connected and

$$\deg(q_{x_n}^{(n)} : V \to B(y', \tau)) \leq N.$$

Let $\delta = \delta_0/2$. We set

$$A_n = \sup\{\mathrm{diam}\, U \mid U \in c(B(y, \delta), q_{x_n}^{(n)}), (x, y) \in \tilde{J}(\tilde{f}), x_n \in p^{-n}(x)\}.$$

First we will show

$$A_n \to 0, \quad \text{as } n \to \infty. \tag{2.16}$$

Suppose this is false. Then there exists a positive constant C, a sequence $((x^k, y^k))$ of points in $\tilde{J}(\tilde{f})$, a sequence $((\tilde{x}^k, \tilde{y}^k))$ with $\tilde{f}^{n_k}((\tilde{x}^k, \tilde{y}^k)) = (x^k, y^k)$ for some $n_k \in \mathbb{N}$, $\to \infty$ and a sequence $(U_k)_k$ with $U_k \in c(B(y_k, \delta), q_{\tilde{x}^k}^{(n_k)})$ and $\tilde{y}^k \in U_k$ for each k such that

$$\mathrm{diam}\, U_k \geq C, \quad \text{for each } k.$$

We can assume that $((x^k, y^k))$ tends to a point $(x^0, y^0) \in \tilde{J}(\tilde{f})$ and that $((\tilde{x}^k, \tilde{y}^k))$ tends to a point $(\tilde{x}^0, \tilde{y}^0) \in \tilde{J}(\tilde{f})$. By Corollary 2.1.9, there exists a positive number r such that $B(\tilde{y}^0, r) \subset U_k$ for each large k. Hence

$$q_{\tilde{x}^k}^{(n_k)}(B(\tilde{y}^0, r)) \subset B(y^0, r), \tag{2.17}$$

for each large k. By the second statement of our theorem, we have that $\tilde{y}^0 \in J_{\tilde{x}^0}$. Hence there exists a positive integer j and a point $z \in B(\tilde{y}^0, r)$ such that

$$q_{\tilde{x}^0}^{(j)}(z) \in D.$$

Hence $q_{\tilde{x}^k}^{(j)}(z) \in D$ for each large k. On the other hand by the condition (C1) if we take δ_0 so small then we can assume

$$\overline{\bigcup_{n \geq 0} \tilde{f}^n(\{x\} \times D)} \cap (\{y^0\} \times B(y^0, \delta_0)) = \emptyset.$$

Since $n_k \to \infty$, by (2.17) we get a contradiction. Hence we have (2.16).

Take a positive integer n_0 such that for each $n \in \mathbb{N}$ with $n \geq n_0$,

$$A_n \leq \delta/2. \tag{2.18}$$

Fix any positive integer k. Let $((x_n, y_n))$ be a sequence such that $\tilde{f}((x_{n+1}, y_{n+1})) = (x_n, y_n)$ for each n and $(x_0, y_0) \in \tilde{J}(\tilde{f})$. For each $j = 0, \ldots, k$, let W_j be the element of $c(B(y_{(k-j)n_0}, \delta), q_{x_{kn_0}}^{(jn_0)})$ containing y_{kn_0}. By (2.18) we have that

$$W_0 \supset \cdots \supset W_k. \tag{2.19}$$

For each $j = 1, \ldots, k$,

$$q_{x_{kn_0}}^{(jn_0)} : W_j \to B(y_{(k-j)n_0}, \delta)$$

is a proper holomorphic map with the degree at most N. Since $q_{x_{kn_0}}^{(jn_0)}(W_{j+1})$ is a connected component of $(q_{x_{(k-j)n_0}}^{(n_0)})^{-1}(B(y_{(k-j-1)n_0}, \delta))$, which is included in $B(y_{(k-j)n_0}, \delta/2)$ by (2.18), we have that for each $j = 0, \ldots, k-1$,

$$\mathrm{mod}\,(\overline{W_{j+1}}, W_j) \geq c > 0, \tag{2.20}$$

where c is a constant number depending only on N. By Lemma 2.1.11, there exists a λ with $0 < \lambda < 1$ depending only on N such that

$$\mathrm{diam}\, W_{j+1}/\mathrm{diam}\, W_j \leq \lambda, \text{ for each } j = 0, \ldots, k-1.$$

Hence we get that $\mathrm{diam}\, W_k \leq \lambda^k \,\mathrm{diam}\, B(y_0, \delta)$. Therefore the statement 3 of our theorem holds.

□

Corollary 2.4.15. *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup which is semi-hyperbolic. Assume G contains an element with the degree at least two and each element of Aut $\overline{\mathbb{C}} \cap G$ (if this is not empty) is loxodromic. Also assume $F(G) \neq \emptyset$. Then there exists a $\delta > 0$, a constant L with $L > 0$ and a constant λ with $0 < \lambda < 1$ such that*

$$\sup\{\mathrm{diam}\, U \mid U \in c(B(x, \delta), f_{i_n} \cdots f_{i_1}), x \in J(G), (i_1, \ldots, i_n) \in \{1, \ldots, m\}^n\}$$

$\leq L\lambda^n$, *for each n.*

Proof. Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be the rational skew product constructed by the generator system $\{f_1, f_2, \ldots, f_m\}$. Then this is semi-hyperbolic along fibers. By the existence of an attractor in $F(G)$ for G (Theorem 2.1.35) we have that if we set $D_x = D$ for each $x \in X$ where D is a small disc around a point of the attractor, then $\tilde{f}$ satisfies the condition (C1) with that family of discs. By Theorem 2.4.13, the statement of our Corollary holds.

□

Corollary 2.4.16. *Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product such that q_x is a polynomial of degree at least two for each $x \in X$. Assume $\tilde{f}$ is semi-hyperbolic along fibers. Then the following hold.*

1.
$$\tilde{J}(\tilde{f}) = \bigcup_{x \in X} \tilde{J}_x.$$

2. *There exist positive constants δ, L and $\lambda (0 < \lambda < 1)$ such that for any $n \in \mathbb{N}$,*

$$\sup\{diam\ U \mid U \in c(B(y,\delta),\ q_{x_n}^{(n)}),\ (x,y) \in \tilde{J}(\tilde{f}),\ x_n \in p^{-n}(x)\} \le L\lambda^n.$$

3. *For any compact subset K of $\tilde{F}(\tilde{f})$, we have that $\overline{\cup_{n \ge 0} \tilde{f}^n(K)} \subset \tilde{F}(\tilde{f})$ and there exist constants $C > 0$ and $\tau < 1$ such that for each n, $\sup_{z \in K} \|(\tilde{f}^n)'(z)\| \le C\tau^n$.*

Proof. Take a small disc D around ∞ and set $D_x = D$ for each $x \in X$. Then $\tilde{f}$ safisfies the condition C1 with the family of discs $\{D_x\}$. Hence the statement 1 and 2 follow from Theorem 2.4.13. Let K be a compact subset of $\tilde{F}(\tilde{f})$. Suppose there exists a sequence (x_k, y_k) of points in K and a sequence (n_k) of positive integers such that $(\tilde{f}^{n_k}((x_k, y_k)))$ converges to a point $(x', y') \in \tilde{J}(\tilde{f})$. As in Chapter 1 of [Se] or [J], we have that the family $\{J_x\}_{x \in X}$ is lower semi-continuous. Hence we have that for each k there exists a point $z_k \in J_{p^{n_k}(x_k)}$ and (z_k) converges to y'. Since $K \cap \tilde{J}(\tilde{f}) = \emptyset$, by semihyperbolicity and Lemma 2.1.10 we get a contradiction. Hence we get that $\overline{\cup_{n \ge 0} \tilde{f}^n(K)} \subset \tilde{F}(\tilde{f})$. Let $K' = \overline{\cup_{n \ge 0} \tilde{f}^n(K)}$. By the statement 1 of Theorem 2.4.13, we have that for each $z = (x,y) \in K'$, there exists a neighoborhoood $U(z)$ of x in X, a neighborhood $V(z)$ of y in $\overline{\mathbb{C}}$ and a positive integer $n(z)$ such that $\|(\tilde{f}^{(n(z))})'(z')\| < 1/2$ for each $z' \in U(z) \times V(z)$. Since K' is a compact set, we get the statement 3. □

Theorem 2.4.17. *Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product. Assume $\tilde{f}$ is hyperbolic along fibers and satisfies the condition (C1) with a family of discs $(D_x)_{x \in X}$ such that there exists a disc D satisfying $D_x = D$ for all $x \in X$. Then $\tilde{f}$ is expanding along fibers.*

Proof. We have only to show that there exists a positive integer n_0 such that for each $n \in \mathbb{N}$ with $n \ge n_0$ and $z \in \tilde{J}(\tilde{f})$,

$$\|(\tilde{f}^n)'(z)\| \ge 2.$$

Suppose this is false. Then there exists a sequence (n_j) of positive integers and a sequence $(z_j) = ((x_j, y_j))$ in $\tilde{J}(\tilde{f})$ such that

$$\|(\tilde{f}^{n_j})'(z_j)\| \le 2. \tag{2.21}$$

We can assume that $\tilde{f}^{n_j}(z_j)$ converges to a point $(x,y) \in \tilde{J}(\tilde{f})$. Let δ be a small positive number. For each j let $B_j \in c(B(y,\delta),\ q_{x_j}^{(n_j)})$ be the element containing y_j. By (2.21) and Koebe distortion theorem, there exists a positive constant c such that for each j, $\operatorname{diam} B_j \ge c$. But this contradicts to the statement 2.4.13 of Theorem 2.4.13. □

2.5 Conditions to be semi-hyperbolic

Theorem 2.5.1. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. Let $z_0 \in J(G)$ be a point. Assume all of the following conditions:*

1. *there exists a neighborhood U_1 of z_0 in $\overline{\mathbb{C}}$ such that for any sequence $(g_n) \subset G$, any domain V in $\overline{\mathbb{C}}$ and any point $\zeta \in U_1$, we have that the sequence (g_n) does NOT converge to ζ locally uniformly on V.*

2. *there exists a neighborhood U_2 of z_0 in $\overline{\mathbb{C}}$ and a positive real number $\tilde{\epsilon}$ such that if we set*

$$T = \{c \in \overline{\mathbb{C}} \mid \exists j,\ f_j'(c) = 0,\ (G \cup \{id\})(f_j(c)) \cap U_2 \ne \emptyset\}$$

then for each $c \in T \cap C(f_j)$, we have $d(c,\ (G \cup \{id\})(f_j(c))) > \tilde{\epsilon}$.

3. *$F(G) \ne \emptyset$.*

Then $z_0 \in SH_N(G)$ for some $N \in \mathbb{N}$.

Notation: For any family $\{g_\lambda\}_{\lambda \in \Lambda}$ of rational functions, we denote by $F(\{g_\lambda\})$ the set of all points $z \in \overline{\mathbb{C}}$ such that z has a neighborhood where the family $\{g_\lambda\}$ is normal. We set $J(\{g_\lambda\}) = \overline{\mathbb{C}} \setminus F(\{g_\lambda\})$. $F(\{g_\lambda\})$ is called the Fatou set and $J(\{g_\lambda\})$ is called the Julia set for the family.

Corollary 2.5.2. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. Let $z_0 \in J(G)$ be a point. Assume all of the following conditions:*

1. *there exists a neighborhood U_1 of z_0 in $\overline{\mathbb{C}}$ such that for any sequence $(g_n) \subset G$ consisting of mutually distinct elements and any domain V in $F((g_n))$, there exists a point $x \in V$ such that the sequence $\overline{\cup_n \{g_n(x)\}} \cap \overline{\mathbb{C}} \setminus U_1 \ne \emptyset$.*

2. *there exists a neighborhood U_2 of z_0 in $\overline{\mathbb{C}}$ and a positive real number $\tilde{\epsilon}$ such that if we set*

$$T = \{c \in \overline{\mathbb{C}} \mid \exists j,\ f_j'(c) = 0,\ (G \cup \{id\})(f_j(c)) \cap U_2 \neq \emptyset\}$$

then for each $c \in T \cap C(f_j)$, we have $d(c,\ (G \cup \{id\})(f_j(c))) > \tilde{\epsilon}$.

3. *$F(G) \neq \emptyset$.*

Then $z_0 \in SH_N(G)$ for some $N \in \mathbb{N}$ and there exists a neighborhood W of z_0 in $\overline{\mathbb{C}}$ such that for any sequence $(g_n) \subset G$ consisting of mutually distinct elements, we have

$$\sup\{\ diam\ S \mid S \in c(W, g_n)\} \to 0,\ as\ n \to \infty.$$

We will consider the proof of Theorem 2.5.1. We may assume $U_1 = U_2 = U$ for some small disc U . By condition 1 and 3, we may assume $\infty \in F(G)$ and $g^{-1}(U) \subset \mathbb{C}$ for each $g \in G$. Now we will show the above theorem by developing a lemma in [Ma4] and using the methods in [KS]. The stories are almost same as those in [KS], except some modifications.

First we need some new notations. An "square" is a set S of the form

$$S = \{z \in \mathbb{C} \mid |\Re(z-p)| < \delta,\ |\Im(z-p)| < \delta\}.$$

The point p is called the center of S and δ its radius . For each $k > 0$, given a square S with center p and radius δ, we denote by S^k the square with the center p and radius $k\delta$. Take a $\sigma > 0$ such that U contains a closed square Q' with the center a point in U and its radius 2σ. Let $Q'' = (Q')^{1/2}$. Q'' is called "admissable square at level 1." We will define asmissable squares at level n for each $n \in \mathbb{N}$. Let Q be an admissable square at level n with the radius a. Then Q is covered by 16 squares with the radius $a/8$. We have 20 squares with the radius $a/8$ adjacent to Q. We call all these 36 squares admissable at level $n+1$. These squares are denoted by $\{Q_{\mu,n+1}\}$. The union of these 36 squares is denoted by $\tilde{Q}$, which is called the "square attached to Q." Each admissable and each attached square is a relative compact subset of U.

Notation: For any open set V_1 and for any rational map g, if $V_2 \in c(V_1,\ g)$ then we set $\Delta(V_1,\ g) = \sharp\{x \in V_1 \mid g'(x) = 0\}$, counting the multiplicity.

We need some lemmas to show Theorem 2.5.1.

Lemma 2.5.3. *For given $\epsilon > 0$ and $N \in \mathbb{N}$, there exists some $n_0 \in \mathbb{N}$ such that the following holds: If Q is an admissable square at some level $n \geq n_0$, $\tilde{Q}$ the corresponding attached square, V an element of $c(\tilde{Q},\ f)$ for some $f \in G$, and $\Delta(V, f) \leq N$, then diam $(K) \leq \epsilon$ for each element $K \in c(Q, f)$ contained in V.*

Proof. Fix $\epsilon > 0$ and $N \in \mathbb{N}$. If the lemma is false then there exists a sequence $(n_k)_{k \in \mathbb{N}}$ converging to ∞, admissable squares Q_{μ_k,n_k} and functions $g_k \in G$ such that diam $(K_k) \geq \epsilon > 0$ and $\Delta(V_k,\ g_k) \leq N$ for some element $V_k \in c(\tilde{Q}_{\mu_k,n_k},\ g_k)$ and some element $K_k \in c(Q_{\mu_k,n_k},\ g_k)$ contained in V_k. Take $\gamma > 1$ such that $\gamma^{N+1} = \frac{3}{2}$. Then there exists $0 < j \leq N+1$ such that, denoting by $\hat{R}_k = Q^{\gamma^j}_{\mu_k,n_k}$, the set $\hat{R}_k^\gamma - \hat{R}_k$ does not contain any critical values of $g_k|_{V_k}$. We have $\hat{R}_k \supset Q_{\mu_k,n_k}$. Take $\hat{K}_k \in c(\hat{R}_k,\ g_k)$ such that $K_k \subset \hat{K}_k \subset V_k$. Then diam $\hat{K}_k > \epsilon$. The element g_k is represented by the following form: $g_k = f_{s_l} \circ \cdots \circ f_{s_1}$. Then there exists a positive integer i such that

$$\text{diam } f_{s_i} \circ \cdots \circ f_{s_1}(\hat{K}_k) > \epsilon,\ \text{diam } f_{s_{i+1}} \circ \cdots \circ f_{s_1}(\hat{K}_k) \leq \epsilon.$$

We may assume i is the largest one satisfying the above. Then taking $\epsilon > 0$ small enough, since the cardinality of generator system of G is finite we see that $f_{s_i} \circ \cdots \circ f_{s_1}(\hat{K}_k)$ is simply connected. Set $\tilde{K}_k = f_{s_i} \circ \cdots \circ f_{s_1}(\hat{K}_k)$ and $\tilde{g}_k = f_{s_l} \circ \cdots \circ f_{s_{i+1}}$. Since $\hat{R}_k^\gamma - \hat{R}_k$ does not contain any critical values of $g_k|_{V_k}$, the element $\tilde{K}_k^+ \in c(\hat{R}_k^\gamma,\ \tilde{g}_k)$ containing $\tilde{K}_k$ is also simply connected. Since $\deg(\tilde{g}_k|_{\tilde{K}_k^+}) \leq 1 + N$ and diam $\tilde{K}_k > \epsilon$, by Corollary 2.1.9 we see that there exists a positive real number r such that for each k,

$$B(z_k,\ r) \subset \tilde{K}_k,$$

for some $z_k \in \overline{\mathbb{C}}$. We can assume that (z_k) converges to a point $z \in \overline{\mathbb{C}}$ and $(\hat{R}_k)$ converges to a point $y \in U$. Then $(\tilde{g}_k)$ is normal in $B(z,\ r/2)$ and we can assume that $(\tilde{g}_k)$ converges to y locally uniformly on $B(z,\ r/2)$. But this contradicts to the assumption 1. Hence the lemma holds. □

Now, let $t = \sharp T$, $N = (\max_{j=1,\dots,m} \deg(f_j))^t$ and $\epsilon < \frac{\tilde{\epsilon}}{36N}$. We can assume that $\tilde{\epsilon}$ is sufficiently small and diam $U \leq \tilde{\epsilon}$. Let $n_0 \in \mathbb{N}$ be an integer in Lemma 2.5.3 for these ϵ and N.

Lemma 2.5.4. *Let G be an element of the form $f = f_{w_1} \circ \cdots \circ f_{w_k}$. Let B be a simply connected subdomain of U, $B' \in c(B, f)$ an element such that $\Delta(B', f) > N$. Then there exists some $\nu \in \{0, \dots, k-1\}$ such that if we set $B_\nu = f_{w_{k-\nu}} \circ \cdots \circ f_{w_k}(B')$, then B_ν is simply connected, diam $(B_\nu) \geq \tilde{\epsilon}$, and*

$$\deg(f_{w_1} \circ \cdots f_{w_{k-\nu-1}}|B_\nu : B_\nu \to B) \leq N.$$

Proof. Suppose diam $B_\nu < \tilde{\epsilon}$ for each $\nu = 1, \dots k-1$, then B' is simply connected(Note that we can assume $\tilde{\epsilon}$ is sufficiently small) and $\deg(f|_{B'}$:

$B' \to B) \leq N$. Hence $\Delta(B', f) \leq N$ and it is a contradiction. Hence there exists a $\nu \in \{1, \ldots, k-1\}$ such that

$$\text{diam } B_\nu \geq \tilde{\epsilon}.$$

Take the maximal $\nu(1 \leq \nu \leq k-1)$ satisfying the above. Then B_ν is simply connected and

$$\deg(f_{w_1} \circ \cdots \circ f_{w_{k-\nu-1}}|_{B_\nu} : B_\nu \to B) \leq N.$$

□

Now we will show the Theorem 2.5.1.

Proof. Take $\tilde{\epsilon}$, ϵ and N as before. Take n_0 in Lemma 2.5.3 for ϵ and N. Let k be the smallest integer such that there exists some admissable square $Q = Q_{\mu,n}$ at level $n \geq n_0$ with diam $(K) > \epsilon$ for some element K of $c(Q,\ f_{w_1} \circ \cdots \circ f_{w_k})$ where $(w_1, \ldots, w_k)$ is some word of length k. We have $k \geq 1$. Let $\tilde{Q}$ be the square attached to Q. By lemma 2.5.3, there exists an element $S \in c(\tilde{Q},\ f_{w_1} \circ \cdots \circ f_{w_k})$ such that $\Delta(S,\ f_{w_1} \circ \cdots \circ f_{w_k}) > N$. Take an integer ν with $1 \leq \nu < k$ in Lemma 2.5.4. Then we have

$$\text{diam } (f_{w_{k-\nu}} \circ \cdots \circ f_{w_k}(S)) > \tilde{\epsilon}.$$

If we set $\tilde{S} = f_{w_{k-\nu}} \circ \cdots \circ f_{w_k}(S)$ then

$$\deg(f_{w_1} \circ \cdots \circ f_{w_{k-\nu-1}}|_{\tilde{S}}) \leq N$$

and

$$\tilde{S} \subset \bigcup_{\mu} (f_{w_1} \circ \cdots \circ f_{w_{k-\nu-1}})^{-1}(Q_{\mu,\ n+1}).$$

By the minimality of k, we have that the diameter of each element of $c(Q_{\mu,\ n+1},\ f_{w_1} \circ \cdots \circ f_{w_{k-\nu-1}})$ is less than ϵ. Since $\deg(f_{w_1} \circ \cdots \circ f_{w_{k-\nu-1}}|_{\tilde{S}}) \leq N$, we have that

$$\tilde{\epsilon} < \text{ diam } \tilde{S} \leq 36N\epsilon.$$

This contradicts to $\epsilon < \frac{\tilde{\epsilon}}{36N}$. Hence we have proved that for each admissable square $Q_{\mu,n}$ with $n \geq n_0$ and each $g \in G$, each element $K \in c(Q_{\mu,n},\ g)$ satisfies that diam $(K) < \epsilon$. Since ϵ is sufficiently small, K is simply connected. By Lemma 2.5.4, we have that

$$\deg(f|_K : K \to Q_{\mu,n}) \leq N + 1.$$

Hence $z_0 \in SH_{N+1}$. □

Now we will show the Corollary 2.5.2.

Proof. If we assume the conditions in the assumption of Corollary 2.5.2, then clearly the conditions in the assumption of Theorem 2.5.1 are satisfied. Hence we have $z_0 \in SH_N(G)$ for some $N \in \mathbb{N}$. Now take a small disc W around z_0 contained in $SH_N(G) \cap U_2$. If there exists a constant $C > 0$, a sequence $(g_n) \subset G$ consisting of mutually distinct elements and a sequence (W_n) with $W_n \in c(W, g_n)$ such that diam $W_n > C$ for each n, then by Corollary 2.1.9, there exists a positive real number r such that for each n, we have $B(z_n,\ r) \subset W_n$ for some $z_n \in \overline{\mathbb{C}}$. We can assume (z_n) converges to a point $y \in \overline{\mathbb{C}}$. Then (g_n) is normal in $B(y,\ r/2)$. Since $g_n(B(y,\ r/2)) \subset W \subset U_1$ for each large $n \in \mathbb{N}$, we get a contradiction. Hence the statement of the Corollary holds. □

Theorem 2.5.5. *Let $G = \langle f_1, f_2, \ldots f_n \rangle$ be a finitely generated rational semigroup. Assume that there exists an element of G with the degree at least two, that each element of Aut $\overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic and that $F(G) \neq \emptyset$. Then G is semi-hyperbolic if and only if all of the following conditions are satisfied.*

1. *for each $z \in J(G)$ there exists a neighborhood U of z in $\overline{\mathbb{C}}$ such that for any sequence $(g_n) \subset G$, any domain V in $\overline{\mathbb{C}}$ and any point $\zeta \in U$, we have that the sequence (g_n) does NOT converge to ζ locally uniformly on V*

2. *for each $j = 1, \ldots, m$ each $c \in C(f_j) \cap J(G)$ satisfies*

$$d(c,\ (G \cup \{id\})(f_j(c))) > 0$$

Proof. First assume the conditions 1 and 2. Then by Theorem 2.5.1, we have that G is semi-hyperbolic.

Conversely, suppose G is semi-hyperbolic. By Lemma 2.1.13, the condition 2. holds. Now we will show the condition 1.holds. By Theorem 2.1.35, there exists an attractor K in $F(G)$ for G. Let z_0 be any point and U a neighborhood of z_0 such that $\overline{U} \cap K = \emptyset$. Suppose that there exists a sequence $(g_n) \subset G$, a domain V in $\overline{\mathbb{C}}$ and a point $\zeta \in U$ such that $g_n \to \zeta$ as $n \to \infty$ locally uniformly on V. We will deduce a contradiction. We can assume that there exists a word $w \in \{1, \ldots, m\}^{\mathbb{N}}$ such that for each n,

$$g_n = \alpha_n \circ f_{w_n} \circ \cdots \circ f_{w_1},$$

where $\alpha_n \in G$ is an element. Then from Theorem 2.1.35 and that $\overline{U} \cap K = \emptyset$, we have that

$$f_{w_n} \circ \cdots \circ f_{w_1}(V) \subset J(G), \qquad (2.22)$$

for each n. Hence $(f_{w_n} \circ \cdots \circ f_{w_1})_n$ is normal in V. Now let us consider the rational skew product $\tilde{f}$ constructed by the generator system $\{f_1, \ldots, f_m\}$. By the second statement of Theorem 2.4.13, we have that $\{w\} \times V \subset \tilde{F}(\tilde{f})$. Hence there exists a positive integer n such that $f_{w_n} \circ \cdots \circ f_{w_1}(V) \subset F(G)$, if we take V sufficiently small. But this contradicts to (2.22). Hence we have shown that the condition 1. holds. □

Theorem 2.5.6. *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated sub-hyperbolic rational semigroup. Assume that there exists an element of G with the degree at least two, that each element of $\mathrm{Aut}\,\overline{\mathbb{C}} \cap G$ (if this is not empty) is loxodromic and that there is no super attracting fixed point of any element of G in $J(G)$. Then there exists a Riemannian metric ρ on a neighborhood V of $J(G) \setminus P(G)$ such that for each $z_0 \in J(G) \setminus G^{-1}(P(G) \cap J(G))$, if there exists a word $w = (w_1, w_2, \ldots,) \in \{1, \ldots, m\}^{\mathbb{N}}$ satisfying $(f_{w_n} \cdots f_{w_1})(z_0) \in J(G)$ for each n, then*

$$\|(f_{w_n} \cdots f_{w_1})'(z_0)\| \to \infty, \text{ as } n \to \infty,$$

where $\|\cdot\|$ is the norm of the derivative measured from ρ on V to it.

Proof. By Theorem 2.1.36, there exists an attractor K in $F(G)$ for G such that $K^i \supset P(G) \cap F(G)$. Let $\{V_1, \ldots, V_t\}$ be the set of all connected components of $\overline{\mathbb{C}} \setminus K$ having non-empty intersection with $J(G)$. We take the hyperbolic metric in $V_i \setminus P(G)$ for each $i = 1, \ldots, t$. We denote by ρ the Riemannian metric in $V = \cup_{i=1}^{t} V_i \setminus P(G)$. First we will show the following.

- Claim 1. there exists a $k \in \mathbb{N}$ such that for each n,

$$\|(f_{w_{n+k}} \cdots f_{w_n})'(f_{w_n} \cdots f_{w_1}(z_0))\| > 1,$$

where $\|\cdot\|$ is the norm of the derivative measured from ρ to it. For each $i = 1, \ldots, t$, let x_i be a point of $V_i \cap F(G)$. Since K is an attractor in $F(G)$ for G, there exists a $k \in \mathbb{N}$ such that for each $(i_1, \ldots, i_k) \in \{1, \ldots, m\}^k$,

$$(f_{i_k} \cdots f_{i_1})(x_i) \in K, \quad \text{for } i = 1, \ldots, t. \qquad (2.23)$$

Let x be a point of $J(G) \cap V_i \setminus P(G)$. Suppose $(f_{i_k} \cdots f_{i_1})(x) \in V_j \setminus P(G)$ for some $(i_1, \ldots, i_k) \in \{1, \ldots, m\}^k$ and j. Let U be the connected component of $(f_{i_k} \cdots f_{i_1})^{-1}(V_j \setminus P(G)) \cap (V_i \setminus P(G))$ containing x. Then

$$(f_{i_k} \cdots f_{i_1}) : U \to V_j \setminus P(G)$$

is a covering map. Hence we have

$$\|(f_{i_k} \cdots f_{i_1})'(z)\|_{U,\, V_j \setminus P(G)} = 1, \quad \text{for each } z \in U, \qquad (2.24)$$

where we denote by $\|\cdot\|_{U,\, V_j \setminus P(G)}$ the norm of the derivative measured from the hyperbolic metric on U to that on $V_j \setminus P(G)$. On the other hand, by (2.23), $U \neq V_i \setminus P(G)$. Therefore the inclusion map $i : U \to V_i \setminus P(G)$ satisfies that

$$\|i'(z)\|_{U,\, V_i \setminus P(G)} < 1, \text{ for each } z \in U, \qquad (2.25)$$

where we denote by $\|\cdot\|_{U,\, V_i \setminus P(G)}$ the norm of the derivative measured from the hyperbolic metric on U to that on $V_i \setminus P(G)$. By (2.24) and (2.25), we get

$$\|(f_{i_k} \cdots f_{i_1})'(z)\|_{V_i \setminus P(G),\, V_j \setminus P(G)} > 1, \text{ for each } z \in U, \qquad (2.26)$$

where we denote by $\|\cdot\|_{V_i \setminus P(G),\, V_j \setminus P(G)}$ the norm of the derivative measured from the hyperbolic metric on $V_i \setminus P(G)$ to that on $V_j \setminus P(G)$. Hence the Claim 1. holds.

By Claim 1., we get that if the sequence $(f_{w_n} \cdots f_{w_1})(z_0))_{n=1}^{\infty}$ does not accumulate to any point of $P(G) \cap J(G)$, then $\|(f_{w_n} \cdots f_{w_1})'(z_0)\| \to \infty$ as $n \to \infty$. Hence we can assume that the sequence accumulates to a point of $P(G) \cap J(G)$. We set

$$g_n = f_{w_{nk}} \cdots f_{w_1}, \text{ for each } n.$$

We will show the following.

- Claim 2. $\|(g_n)'(z_0)\| \to \infty$ as $n \to \infty$.

Since $z_0 \in J(G) \setminus G^{-1}(P(G) \cap J(G))$, by the same arguments as that in the proof of Theorem 2.1.34, we can show that there exists an $\epsilon_1 > 0$ and a sequence (n_j) of integers such that

$$g_{n_j}(z_0) \in J(G) \setminus B(P(G), \epsilon_1), \; g_{n_j+1}(z_0) \in J(G) \cap B(P(G), \epsilon_1).$$

Suppose the case there exists a constant ϵ_2 such that for each j,

$$d(g_{n_j+1}(z_0),\; P(G)) \geq \epsilon_2.$$

Then from Claim 1, there exists a constant $c > 1$ such that for each j,

$$\|(f_{w_{(n_j+1)k}} \cdots f_{w_{n_jk+1}})'((f_{w_{n_jk}} \cdots f_{w_1})(z_0))\| > c.$$

Using the Claim 1 again, we can show that $\|(g_n)'(z_0)\| \to \infty$ as $n \to \infty$.

Next suppose the case there exists a subsequence $(h_l)_{l=1}^{\infty}$ of $(g_{n_j+1})_{j=1}^{\infty}$ such that $d(h_l(z_0),\ P(G)) \to 0$ as $l \to \infty$. There exists a subsequence $(\beta_l)_{l=1}^{\infty}$ of $(g_{n_j})_{j=1}^{\infty}$ such that for each l $h_l = \alpha_l \circ \beta_l$ where α_l is an element of G. Then there exists a constant $c_1 \in \mathbb{N}$ such that for each l, $wl_S(\alpha_l) \le c_1$ where $S = \{f_1, \dots, f_m\}$. Hence there exists a sequence (x_l) such that $d(x_l,\ \beta_l(z_0)) \to 0$ as $l \to \infty$ and $\alpha_l(x_l) \in P(G)$ for each $l \in \mathbb{N}$. We can assume that $x_l \in B(\beta_l(z_0),\ \epsilon_1)$ for each $l \in \mathbb{N}$. Let γ_l be the analytic inverse branch of β_l in $B(\beta_l(z_0),\ \epsilon_1)$ such that

$$\gamma_l(\beta_l(z_0)) = z_0, \text{ for each } l \in \mathbb{N}.$$

Since $\cup_{l=1}^{\infty}\gamma_l(B(\beta_l(z_0),\ \epsilon_1)) \subset \overline{\mathbb{C}} \setminus K$ and $d(x_l, \beta_l(z_0)) \to 0$. We get $\gamma_l(x_l) \to z_0$ as $l \to \infty$. Hence we have

$$d(z_0,\ h_l^{-1}(P(G))) \to 0, \text{ as } l \to \infty. \tag{2.27}$$

There exists an i such that $z_0 \in V_i \setminus P(G)$. For each l let V_{j_l} be the element of $\{V_1, \dots, V_t\}$ such that $h_l(z_0) \in V_{j_l} \setminus P(G)$. Let W_l be the connected component of $h_l^{-1}(V_{j_l} \setminus P(G)) \cap V_i \setminus P(G)$ containing z_0. Then $h_l : W_l \to V_{j_l}$ is a covering map. Hence we have

$$\|(h_l)'(z)\|_{W_l,\ V_{j_l}\setminus P(G)} = 1, \text{ for } z \in W_l,$$

where $\|\cdot\|_{W_l,\ V_{j_l}\setminus P(G)}$ is the norm of the derivative measured from the hyperbolic metric on W_l to that on V_{j_l}. By Theorem 2.25 in [Mc], (2.27) implies that

$$\|(i_l)'(z)\|_{W_l,\ V_i\setminus P(G)} \to 0 \text{ as } l \to \infty,$$

where we denote by i_l the inclusion map from W_l into $V_i \setminus P(G)$ for each $l \in \mathbb{N}$. It follows that

$$\|h_l'(z)\|_{V_i\setminus P(G),\ V_{j_l}\setminus P(G)} \to \infty \text{ as } l \to \infty, \tag{2.28}$$

where $\|\cdot\|_{V_i\setminus P(G),\ V_{j_l}\setminus P(G)}$ is the norm of the derivative measured from the hyperbolic metric on $V_i \setminus P(G)$ to that on $V_{j_l} \setminus P(G)$. By (2.28) and Claim 1, we get $\|(g_n)'(z_0)\| \to \infty$ as $n \to \infty$. Hence the Claim 2 holds.

In the same way we can show that for each $i = 1, \dots, k-1$,

$$\|(f_{w_{nk+i}} \cdots f_{w_1})(z_0)\| \to \infty \text{ as } n \to \infty.$$

We have thus proved the Theorem. □

Theorem 2.5.7. *Let $G = \langle f_1, f_2, \dots, f_m\rangle$ be a finitely generated sub-hyperbolic rational semigroup. Assume that there exists an element of G with the degree at least two, that each element of Aut $\overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic and that there is no super attracting fixed point of any element of G in $J(G)$. Then G is semi-hyperbolic.*

Proof. We will appeal to Theorem 2.5.5. Since there is no super attracting fixed point of any element of G in $J(G)$, the condition 2. in Theorem 2.5.5 is satisfied. By Theorem 2.1.36, there exists an attractor K in $F(G)$ for G. Let z_0 be any point and U a neighborhood of z_0 such that $\overline{U} \cap K = \emptyset$. Suppose that there exists a sequence $(g_n) \subset G$, a domain V in $\overline{\mathbb{C}}$ and a point $\zeta \in U$ such that $g_n \to \zeta$ as $n \to \infty$ locally uniformly on V. We will deduce a contradiction. We can assume that there exists a word $w \in \{1, \dots, m\}^{\mathbb{N}}$ such that for each n,

$$g_n = \alpha_n f_{w_n} \circ \cdots \circ f_{w_1},$$

where $\alpha \in G$ is an element. Then from Theorem 2.1.35 and that $\overline{U} \cap K = \emptyset$, we have that

$$f_{w_n} \circ \cdots \circ f_{w_1}(V) \subset J(G), \tag{2.29}$$

for each n. Hence $(f_{w_n} \circ \cdots \circ f_{w_1})_n$ is normal in V. Let $z_1 \in V \cap G^{-1}(P(G) \cap J(G))$ be a point. By the backward self-similarity of $J(G)$ and Lemma 2.1.33, there exists a sequence (n_j) of positive integers and a neighborhood W of $P(G) \cap J(G)$ in $\overline{\mathbb{C}}$ such that for each j,

$$f_{w_{n_j}} \circ \cdots \circ f_{w_1}(z_1) \in \overline{\mathbb{C}} \setminus W.$$

By Theorem 2.5.6, we have that

$$\|(f_{w_{n_j}} \circ \cdots \circ f_{w_1})'(z_1)\| \to \infty, \quad \text{as } j \to \infty, \tag{2.30}$$

where $\|\cdot\|$ denotes the norm of the derivative with respect to the spherical metric. Since $(f_{w_n} \circ \cdots \circ f_{w_1})_n$ is normal in V, this is a contradiction. Hence the condition 1 in Theorem 2.5.5 is satisfied. By Theorem 2.5.5, we get that G is semi-hyperbolic. □

2.6 Open Set Condition and Area 0

Definition 2.6.1. Let $G = \langle f_1, f_2, \dots, f_m\rangle$ be a finitely generated rational semigroup. We say that G satisfies the *open set condition* with respect to the generators $f_1, f_2, \dots, f_m$ if there exists an open set O such that for each $j = 1, \dots, m$, $f_j^{-1}(O) \subset O$ and $\{f_j^{-1}(O)\}_{j=1,\dots,m}$ are mutually disjoint.

Definition 2.6.2. Let $G = \langle f_1, f_2, \ldots f_n\rangle$ be a finitely generated rational semigroup. We say that G satisfies the strong open set condition if there is an open neighborhood O of $J(G)$ such that each set $f_j^{-1}(O)$ is included in O and is mutually disjoint.

Proposition 2.6.3. *Let $G = \langle f_1, f_2, \ldots, f_m\rangle$ be a finitely generated rational semigroup. Assume that G satisfies the open set condition with respect to the generators $f_1, f_2, \ldots, f_m$ and $O \setminus J(G) \neq \emptyset$ where O is an open set in the definition of the open set condition. Then $J(G)^i = \emptyset$ where we denote by $J(G)^i$ the interior of $J(G)$.*

Proof. Let $S = \{f_1, \ldots, f_m\}$. Assume that $J(G)^i \neq \emptyset$.

Then we claim that for each element $g \in G$ and each point $x \in J(G)^i$,

$$g(x) \in \overline{\mathbb{C}} \setminus (O \setminus J(G)).$$

Suppose that there exists a point $y \in J(G)^i$ and an element $g_1 \in G$ such that $g_1(y) \in O \setminus J(G)$. Since $J(G) = \cup_{i=1}^n f_i^{-1}(J(G))$, there exists an element $h \in G$ with $wl_S(h) = wl_S(g_1)$ such that $h(y) \in J(G)$. Since $f_j^{-1}(O) \subset O$ for each $j = 1, \ldots, m$, we have $J(G) \subset \overline{O}$ and $J(G)^i \subset O$. Hence $g_1^{-1}(O) \cap h^{-1}(O) \neq \emptyset$. But $g_1 \neq h$ and that is a contradiction because of the open set condition. Therefore the above claim holds.

Now the claim implies that G is normal in $J(G)^i$ but this is a contradiction and so we have $J(G)^i = \emptyset$. □

Theorem 2.6.4. *Let $G = \langle f_1, f_2, \ldots f_n\rangle$ be a finitely generated rational semigroup. We assume that the set $\cup_{(i,j):i\neq j} f_i^{-1}(J(G)) \cap f_j^{-1}(J(G))$ does not contain any continuum. Then the Julia set $J(G)$ has no interior points.*

Proof. Assume $J(G)$ has non-empty interior points and let U be a component of $\mathrm{int}(J(G))$. Let x be a point of U. From Lemma 1.1.5.2, there exists a positive integer i_1 with $i_1 \leq n$ such that $x \in f_{i_1}^{-1}(J(G))$. From Lemma 1.1.6.4, we have $U \cap f_{i_1}^{-1}(\mathrm{int}J(G)) \neq \emptyset$. Let V be the connected component of $U \cap f_{i_1}^{-1}(\mathrm{int}J(G))$.

We will show that V is dense in U. To show that, we can assume that $V \neq U$. Then $\partial V \cap U \neq \emptyset$. If $\partial V \cap U$ contains a continuum K, then from the assumption of our theorem, there exists a point $z \in K$ such that for each j with $j \neq i_1$, $z \notin f_j^{-1}(J(G))$. We denote the open disk centered at z of radius ϵ by $D(z, \epsilon)$. Hence there is a small positive number ϵ such that $D(z, \epsilon)$ is included in U and disjoint from $\cup_{j\neq i_1} f_j^{-1}(J(G))$. From Lemma 1.1.5.2, $D(z, \epsilon) \subset f_{i_1}^{-1}(\mathrm{int}J(G))$ and this is a contradiction because V is a connected component of $U \cap f_{i_1}^{-1}(\mathrm{int}J(G))$. Therefore $\partial V \cap U$ does not contain any continuum and V is dense in U.

It follows that $f_{i_1}(U)$ is included in a component U_1 of $\mathrm{int}(J(G))$. In this way, we can take a sequence $(i_k)_k$ such that for each k the number i_k is in $\{1, \ldots, n\}$ and

$$f_{i_k} \circ \cdots \circ f_{i_1}(U) \subset U_k,$$

where U_k is a component of $\mathrm{int}(J(G))$. Now let (g_j) be a sequence of elements of G. If the sequence contains infinite elements of $(f_{i_k} \circ \cdots \circ f_{i_1})$, then (g_j) is a normal family on U. Unless (g_j) contains any element of the form $f_{i_k} \circ \cdots \circ f_{i_1}$, then for each l the set $g_l(U)$ is included in $F(G)$ because of the assumption of our theorem and so (g_j) is a normal family on U. It follows that U is included in $F(G)$ and this is a contradiction. □

Remark 11. If $\cup_{(i,j):i\neq j} f_i^{-1}(J(G)) \cap f_j^{-1}(J(G))$ contains a continuum, then the Julia set may have non-empty interior points. For example, let $p_1 = 0$, $p_2 = 1$, $p_3 = 1 + i$ and $p_4 = i$. For each $j = 1, \ldots, 4$, we set $f_j(z) = 2(z - p_j) + p_j$. Then $J(\langle f_1, \ldots, f_4\rangle)$ is equal to the closed rectangle $p_1p_2p_3p_4$.

Definition 2.6.5. Let G be a polynomial semigroup. We denote by $K(G)$ the closure of the set $K_1(G)$ consisting of the points, for each z of which, there is a sequence $(g_m)_m$ consisting of mutually distinct elements of G such that the sequence $(g_m(z))_m$ is bounded. $K(G)$ is called the filled-in Julia set of G.

Remark 12. For each $g \in G$ the inverse image $g^{-1}(K(G))$ is included in $K(G)$ and $J(G) \subset K(G)$. If $G = \langle f_1, f_2, \ldots f_n\rangle$ is a finitely generated polynomial semigroup, then

$$K(G) = \bigcup_{j=1}^{n} f_j^{-1}(K(G)).$$

Theorem 2.6.6. *Let $G = \langle f_1, f_2, \ldots f_n\rangle$ be a finitely generated polynomial semigroup. Assume that the set $\cup_{(i,j):i\neq j} f_i^{-1}(K(G)) \cap f_j^{-1}(K(G))$ does not contain any continuum. Then*

$$\partial(K(G)) = J(G).$$

Proof. Let z be a point of $\partial(K(G))$ and let U be an open neighborhood of z. For each $x \in K_1(G) \cap U$ there is a sequence $(g_m)_m$ of elements of G such that $(g_m(x))_m$ is bounded. But for each $y \in U \setminus K(G)$ the sequence $(g_m(y))_m$ tends to infinity so G is not normal in U and $z \in J(G)$. So $\partial(K(G)) \subset J(G)$. Next let U be a component of $\mathrm{int}(K(G))$. From the fact $K(G) = \cup_{j=1}^n f_j^{-1}(K(G))$ and our assumption we can show that G is normal in U in the same way as the proof of Theorem 2.6.4. □

Theorem 2.6.7. *Let $G = \langle f_1, f_2, \dots, f_m\rangle$ be a finitely generated rational semigroup which is semi-hyperbolic, contains an element with the degree at least two and satisfies the open set condition with respect to the generators $f_1, f_2, \dots, f_m$. Let O be an open set in Definition 2.6.1. Assume that $\sharp(\partial O \cap J(G)) < \infty$. Then the 2-dimensional Lebesgue measure of $J(G)$ is equal to 0.*

Proof. We will show the statement using the method of Theorem 1.3 in [Y]. We fix a gemerator system $S = \{f_1, \dots, f_m\}$. By the assumption of our Theorem, we have each element of Aut $\overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic. By Theorem 2.1.35, $A(G)$ is an attractor in $F(G)$ for G. We can assume $\infty \in A(G)$. Suppose that the 2-dimensional Lebesgue measure of $J(G)$ is positive.

Since $\sharp(\partial O \cap J(G)) < \infty$, $G^{-1}(G(\partial O \cap J(G)))$ is a countable set. Hence there exists a Lebesgue density point x of $J(G)$ such that $x \in J(G) \setminus (G^{-1}(G(\partial O \cap J(G)))$. Since we have $J(G) = \cup_{j=1}^m f_j^{-1}(J(G))$, there exists a word $w = (w_1, w_2, \dots) \in \{1, \dots, m\}^{\mathbb{N}}$ such that for each positive integer u, $f_{w_u} \cdots f_{w_1}(x) \in J(G)$.

We will show that the sequence $(f_{w_u} \cdots f_{w_1}(x))_u$ has an accumulation point in $J(G) \setminus \partial O$. Assume that is false. For each large u, let ζ_u be the closest point to $f_{w_u} \cdots f_{w_1}(x)$ in $\partial O \cap J(G)$. Since there exists no super attracting fixed point of any point of any element of G in $J(G)$, there exists a positive integer s such that for each integer t with $t \geq s$, $(f_{w_t} \cdots f_{w_s})'(\zeta_{s-1}) \neq 0$. Since G is semi-hyperbolic, we have that for each $x \in \partial O \cap J(G)$, if there exists an element $g \in G$ such that $g(x) = x$, then x is a repelling fixed point of g. Applying Lemma 2.1.33, we get a contradiction. Hence the sequence $(f_{w_u} \cdots f_{w_1}(x))_u$ has an accumulation point in $J(G) \setminus \partial O$.

By the argument above, we have that there exists an $\epsilon > 0$ and a sequence (g_n) of elements of G such that for each n, $g_{n+1} = h_n g_n$ for some $h_n \in G$ and $g_n(x) \in J(G) \setminus D(\partial O, \epsilon)$. Let δ be a small number so that $\delta < \epsilon$ and for each $g \in G$ and each $x \in J(G)$,

$$\deg(g : U \to D(x, \delta)) \leq N$$

for each $U \in c(D(x, \delta), g)$, where N is a positive integer independent of x, g and U. By Lemma 2.1.12, we can assume that for each $g \in G$ and each $x \in J(G)$, if V is a simply connected open neighborhood of x contained in $D(x, \delta)$, then each element of $c(D(x, \delta), g)$ is simply connected.

For each n, we set $x_n = g_n(x)$. Let U_n be the connected component of $g^{-1}(D(x_n, \frac{1}{2}\delta))$ containing x. Now we will claim that

$$\lim_{n\to\infty} \frac{m_2(U_n \cap J(G))}{m_2(U_n)} = 1, \tag{2.31}$$

where we denote by m_2 the 2-dimensional Lebesgue measure. By Corollary 2.1.9, Proposition 2.6.3 and Corollary 2.4.15, there exist a constant $K > 0$, two sequences (r_n) and (R_n) such that $\frac{1}{K} \leq \frac{r_n}{R_n} < 1$, $R_n \to 0$ and

$$D(x, r_n) \subset U_n \subset D(x, R_n).$$

Since x is a Lebesgue density point of $J(G)$, the claim holds. Now we get

$$\lim_{n\to\infty} \frac{m_2(U_n \cap F(G))}{m_2(U_n)} = 0. \tag{2.32}$$

For each n, Let $\phi_n : D(0,1) \to D_{g_n}(x_n, \delta)$ be the Riemann map such that $\phi_n(0) = x$, where $D_{g_n}(x_n, \delta)$ is the element of $c(D(x_n, \delta), g_n)$ containing U_n. By (2.32) and the Koebe distortion theorem, we get

$$\lim_{n\to\infty} \frac{m_2(\phi_n^{-1}(U_n \cap F(G)))}{m_2(\phi_n^{-1}(U_n))} = 0. \tag{2.33}$$

By Corollary 2.1.8, there exists a constant $0 < c_1 < 1$ such that for each n, the Euclidian diameter of $\phi_n^{-1}(U_n)$ is less than c_1. Since we can assume that $D_{g_n}(x_n, \delta) \subset \mathbb{C}$ for each n and uniformly bounded in $\mathbb{C}$, by Cauchy's formula, we get that there exists a constant c_2 such that

$$|(g_n\phi_n)'(z)| \leq c_2 \text{ on } \phi_n^{-1}(U_n),\ n = 1, 2, \dots \tag{2.34}$$

Now we will show

$$D(x_n, \frac{1}{2}\delta) \cap F(G) = g_n(U_n \cap F(G)), \text{ for each } n. \tag{2.35}$$

It is easy to see that $D(x_n, \frac{1}{2}\delta) \cap F(G) \supset g_n(U_n \cap F(G))$. Now let z be a point of $D(x_n, \frac{1}{2}\delta) \cap F(G)$ and assume that there exists a point $w \in U_n \cap J(G)$ such that $g_n(w) = z$. Since $J(G) = \cup_{j=1}^m f_j^{-1}(J(G))$ and $g_n(w) \in F(G)$, there exists an element $g \in G$ with $wl_S(g) = wl_S(g_n)$ such that $g(w) \in J(G) \subset \overline{O}$. Hence we have $g \neq g_n$ and $g^{-1}(O) \cap g_n^{-1}(O) \neq \emptyset$. But this contradicts to the open set condition. Therefore (2.35) holds.

By (2.33) , (2.34) and (2.35), we have

$$\begin{aligned}
\frac{m_2(D(x_n, \frac{1}{2}\delta) \cap F(G))}{m_2(D(x_n, \frac{1}{2}\delta))} &= \frac{m_2((g_n \circ \phi_n)(\phi_n^{-1}(U_n \cap F(G)))}{m_2(D(x_n, \frac{1}{2}\delta))} \\
&\leq \frac{\int_{\phi_n^{-1}(U_n \cap F(G))} |(g_n \circ \phi_n)'(z)|^2 dm_2(z)}{m_2(\phi_n^{-1}(U_n))} \frac{m_2(\phi_n^{-1}(U_n))}{m_2(D(x_n, \frac{1}{2}))} \\
&\to 0,
\end{aligned}$$

as $n \to \infty$. Hence we have

$$\lim_{n\to\infty} \frac{m_2(D(x_n, \frac{1}{2}\delta) \cap J(G))}{m_2(D(x_n, \frac{1}{2}\delta))} = 1.$$

We can assume that there exists a point $x_\infty \in J(G)$ such that $x_n \to x_\infty$. Then

$$\frac{m_2(D(x_\infty, \frac{1}{2}\delta) \cap J(G))}{m_2(D(x_\infty, \frac{1}{2}\delta))} = 1.$$

This implies that $D(x_\infty, \frac{1}{2}\delta) \subset J(G)$ but this is a contradiction because we have $J(G)^i = \emptyset$ by Proposition 2.6.3. □

Corollary 2.6.8. *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup which is sub-hyperbolic, contains an element with the degree at least two and satisfies the open set condition with respect to the generators $f_1, f_2, \ldots, f_m$. Let O be an open set in Definition 2.6.1. Assume that $\sharp(\partial O \cap J(G)) < \infty$. Then the 2-dimensional Lebesgue measure of $J(G)$ is equal to 0.*

Proof. By Proposition 2.6.3, $J(G)^i = \emptyset$. Since G is finitely generated, by [HM2], there is no super attracting fixed point of any element of G in $\partial J(G) = J(G)$. Therefore by Theorem 2.5.7, we have that G is semi-hyperbolic. By Theorem 2.6.7, the statement holds. □

2.7 δ-conformal measures and Hausdorff dimension of the Julia sets

We construct δ-conformal measures on Julia sets of rational semigroups and consider Hausdorff dimension of the Julia sets. δ-conformal measures on Julia sets of rational functions were introduced in [Sul1]. See also [MTU].

Definition 2.7.1. Let $G = \langle f_1, f_2, \ldots f_n \rangle$ be a finitely generated rational semigroup satisfying the strong open set condition and let δ be a non-negative number. We say that a probability measure μ on $\overline{\mathbb{C}}$ is δ-conformal if for each $j = 1, \ldots, n$ and for each measurable set A included in $f_j^{-1}(J(G))$ where f_j is injective on A,

$$\mu(f_j(A)) = \int_A \|f_j'(z)\|^\delta d\mu,$$

where $\|\cdot\|$ denotes the norm of the derivative with respect to the spherical metric. And we set

$$\delta(G) = \inf\{\delta \mid \text{there is a } \delta\text{-conformal measure on } J(G)\}.$$

Theorem 2.7.2. *Let $G = \langle f_1, f_2, \ldots f_n \rangle$ be a finitely generated rational semigroup satisfying the strong open set condition. We assume that when n is equal to one the degree of f_1 is at least two. Then there are a number $0 < \delta \le 2$ and a probability measure μ whose support is equal to $J(G)$ such that μ is δ-conformal. Also $\delta(G) > 0$.*

We will show the statement in the same way as [Sul1] or [MTU]. We need the following lemma.

Lemma 2.7.3. *Under the same assumption in Theorem 2.7.2, let O be an open set in Definition 2.6.2. Then there exists an open set U whose closure is included in $O \cap F(G)$ such that for each open neighborhood W of $J(G)$ there is a positive integer m satisfying*

$$\bigcup_{g \in G:\ \text{word length} \ge m} g^{-1}(U) \subset W. \tag{2.36}$$

Proof. Let V be an open set whose closure is included in $O \cap F(G)$. First we consider the case such that for each $z \in F(G)$, the G-orbit of z does not accumulate at any point of $\overline{V}$. If there are a sequence (x_k) converging to a point $y \in F(G)$ and a sequence (g_k) of G such that for each k, $g_k(x_k) \in V$ and the word length of g_m tends to infinity as $m \to \infty$, then the sequence $(g_k(y))$ accumulates in $\overline{V}$ because (g_k) is normal in a neighborhood of y. This is a contradiction. Putting $U = V$, we have (2.36).

Now let $V_1, \ldots, V_m$ be all connected components of $F(G)$ each of which has a non-empty intersection with O^c. For each j we take the hyperbolic metric in V_j. We set

$$H = \{g \in G \mid g(V_1 \cap O) \cap (V_1 \cap O) \neq \emptyset\}.$$

If H is empty, take U in $V_1 \cap O$ so small that whose closure is included in $V_1 \cap O$. For each $z \in F(G)$, the G-orbit of z does not accumulate at any point of $\overline{U}$, so by the previous argument, (2.36) holds. Hence we can assume that H is non-empty. We have for each j, $f_j(O^c) \subset O^c$. Therefore for each $h \in H$, $h(V_1 \cap O^c) \subset V_1 \cap O^c$. If V_1 is included in a Siegel disc or a Hermann ring of an element of $g \in G$, then $g(V_1 \cap O) = V_1 \cap O$. We can assume the word length of g is less than that of any other element of G which has a Siegel disc or a Hermann ring containing V_1. We represent g as

$$g = f_{i_k} \cdots f_{i_1}.$$

Take small open set U in $B = g^{-1}(V_1 \cap O) \setminus (V_1 \cap O)$. Note that because of the backward self-similarity of $J(G)$ and the strong open set condition, for each open set D in $O \cap F(G)$ and each element $h \in G$, $h^{-1}(D) \subset O \cap F(G)$ and so $B \subset O \cap F(G)$. We have for each $z \in B$,

$$G(z) \cap O = \{f_{i_s} \cdots f_{i_1} g^t(z) \mid 0 \leq s \leq k,\ t \geq 0\} \setminus \{z\}.$$

Hence $G(z) \cap O$ does not accumulate at any point of $\overline{U}$ and it follows that for each $y \in F(G)$, $G(y)$ does not accumulate at any point of $\overline{U}$. Therefore (2.36) holds.

So we can assume that each $h \in H$ has a (super)attracting basin containing V_1, here note that $h(V_1 \cap O^c) \subset V_1 \cap O^c$. Let K be the compact ϵ-neighborhood of O^c in $\cup_{j=1}^m V_j$ with respect to the hyperbolic metric. For each $j = 1, \ldots, n$ and $i = 1, \ldots, m$, we set

$$a_{ji} = \sup\{\|f_j'(z)\| \mid z \in K \cap V_i\},$$

where $\|\cdot\|$ is the norm of the derivative measured from the hyperbolic metric on V_i to that on some V_u which contains $f_j(V_i)$. We denote by $d_H(\cdot,\cdot)$ the distance on V_1 induced by the hyperbolic metric. Then for each $h \in H$ and for each $z \in K \cap V_1 \cap O$, $d_H(h(z),\ V_1 \cap O^c)/d_H(z,\ V_1 \cap O^c)$ is less than

$$\sup\{a_{ji} \mid j = 1, \ldots, n,\ i = 1, \ldots, m,\ a_{ji} \neq 1\} < 1.$$

Since $h(O^c) \subset O^c$ for each $h \in H$, if we take U small enough in $K \cap V_1 \cap O$, then for each $y \in F(G)$, $G(y)$ does not accumulate at any point of $\overline{U}$ and so (2.36) holds. □

Proof. of Theorem 2.7.2. Let O be the open set in Definition 2.6.2. Let U be the open set in Lemma 2.7.3. we can assume that U is a simply connected domain in $O \setminus (P(G) \cup J(G))$. Now we have

$$\sum_S \int_U \|S'(z)\|^2 dm < \infty, \tag{2.37}$$

where S is taken all holomorphic inverse branches of all elements of G defined on U, $\|\cdot\|$ denotes the norm of the derivative with respect to the spherical metric and m is the Lebesgue measure. For, assume that there are sequences $(m_k)_k$ and $(l_k)_k$ of integers with $m_k \to \infty$ such that for each k there is an element $g_{m_k+l_k} \in G$ of word length $m_k + l_k$ and $\overline{g}_{l_k} \in G$ of word length l_k so that

$$g_{m_k+l_k}^{-1}(U) \cap \overline{g}_{l_k}^{-1}(U) \neq \emptyset.$$

Then because of the strong open set condition for each k there is an element $h_{m_k} \in G$ of word length m_k such that

$$U \cap h_{m_k}^{-1}(U) \neq \emptyset.$$

But this is a contradiction by (2.36) and so (2.37) holds.

Now for each $x \in U$ we set

$$I(x) = \bigcup_m \bigcup_{g \in G:\ \text{word length } m} g^{-1}(x)$$

and

$$d(y) = \|g'(y)\|^{-1},$$

for $y \in g^{-1}(x)$. By (2.37) for almost everywhere $x \in U$

$$\sum_{y \in I(x)} d(y)^2 < \infty. \tag{2.38}$$

We fix a point $x \in U$ such that (2.38) holds. And we set

$$\delta = \inf\{s | \sum_{y \in I(x)} d(y)^s < \infty\}.$$

For each j there is a positive number C_j such that $\|f_j'(z)\| \leq C_j$ in a neighborhood of $f_j^{-1}(J(G))$ and the set

$$\bigcup_{g \in G:\ \text{word length } m} g^{-1}(x)$$

has $(\sum_{j=1}^n \deg(f_j))^m$ points so $\delta > 0$.

Now we consider the case $\sum_{y \in I(x)} d(y)^\delta = \infty$. For each number $s > \delta$ we denote by μ_s the probability measure on $\overline{\mathbb{C}}$ such that for each $y \in I(x)$

$$\mu_s(\{y\}) = \frac{d(y)^s}{\sum_{w \in I(x)} d(w)^s}.$$

Let μ be a weak limit of μ_s when $s \searrow \delta$. Then the support of μ is included in $J(G)$ because $\sum_{y \in I(x)} d(y)^\delta = \infty$. Let ζ be a point of $f_j^{-1}(J(G))$. Also let V be a neighborhood of ζ in $f_j^{-1}(O)$ and assume that f_j is injective on V. Then f_j is a bijection from $I(x) \cap V$ to $I(x) \cap f_j(V)$. We set $\lambda = \|f_j'(\zeta)\|$. Let $\epsilon > 0$ be a small number. We take V smaller such that for each $z \in V$

$$\lambda(1-\epsilon) < \|f_j'(z)\| < \lambda(1+\epsilon).$$

Then

$$\lambda^s \mu_s(V)(1-\epsilon)^s \leq \mu_s(f_j(V)) \leq \lambda^s \mu_s(V)(1+\epsilon)^s.$$

Let $s \searrow \delta$ and we get

$$\lambda^\delta \mu(V)(1-\epsilon)^\delta \leq \mu(f_j(V)) \leq \lambda^\delta \mu(V)(1+\epsilon)^\delta.$$

If $f_j'(\zeta) = 0$, we can show that $\mu(f_j(\zeta)) = 0$. It follows that μ is a δ-conformal measure on $J(G)$.

Next we consider the case $\sum_{y \in I(x)} d(y)^\delta < \infty$. We take Patterson's function h i.e. h is a continuous and non-decreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and satisfies that

1. $\sum_{y \in I(x)} h(d(y)^{-1}) d(y)^s$ converges for each $s > \delta$ and does not converge for each $s \leq \delta$.

2. for each ϵ there is a number r_0 such that $h(rt) \leq t^\epsilon h(r)$ for each $r > r_0$ and $t > 1$.

For more detail about Patterson's function, see [Pat] . We set

$$\mu_s = \frac{1}{\sum_{y \in I(x)} h(d(y)^{-1}) d(y)^s} \sum_{y \in I(x)} h(d(y)^{-1}) d(y)^s \delta_y,$$

where we denote by δ_y the dirac measure which is concentrated on $\{y\}$. Letting $s \searrow \delta$ we get a δ-conformal measure on $J(G)$ in the same way as the case $\sum_{y \in I(x)} d(y)^\delta = \infty$.

We will show that support of μ is equal to $J(G)$. By the construction, the support of μ is included in $J(G)$. Now assume that there are a point $\zeta \in J(G)$ and a positive number a such that $\mu(D(\zeta, a)) = 0$. By Lemma 1.1.6, there exists an element $g \in G$ such that $g(D(\zeta, a)) \supset J(G)$. Since μ is a conformal measure, it follows that $\mu(J(G)) = 0$ and this is a contradiction. Therefore the support of μ is equal to $J(G)$.

We now consider $\delta(G)$. There is a $\delta(G)$ conformal measure μ on $J(G)$. Assume that $\delta(G)$ is equal to zero. If there exists a point $x \in f^{-1}(J(G))$ such that $\mu(\{x\}) > 0$, then $\mu(\{f_j(x)\}) = \mu(\{x\})$. Since backward orbit of any point of $J(G)$ has infinitely many points and μ is a probability measure, it is a contradiction. Hence μ is non-atomic. For each measurable set A included in $J(G)$ we set

$$\tau(A) = \mu(\cup_{j=1}^n f_j^{-1}(A)).$$

Then τ is a probability measure on $J(G)$. But if A is a measurable set in $J(G)$ such that for each j all branches of f_j^{-1} are well defined on A then

$$\tau(A) = (\sum_{j=1}^n \deg(f_j))\mu(A),$$

and this is a contradiction, since $J(G)$ is a disjoint union of some finintely many points and some sets on each of which for each j all branches of f_j^{-1} are well defined. □

Theorem 2.7.4. *Let $G = \langle f_1, f_2, \dots f_n \rangle$ be a finitely generated hyperbolic rational semigroup satisfying the strong open set condition. We assume that when n is equal to one the degree of f_1 is at least two. Let δ be a number satisfying that $0 < \delta \leq 2$ and assume that there is a δ-conformal measure μ on $J(G)$. Then $\delta = \delta(G)$ and*

$$\dim_H(J(G)) = \delta(G), \ 0 < H_{\delta(G)}(J(G)) < \infty,$$

where $\dim_H$ is the Hausdorff dimension and H_α is the $\alpha-$Hausdorff measure.

By Theorem 2.6.7 , Theorem 2.7.2 and Theorem 2.7.4, we get the next result.

Corollary 2.7.5. *Let $G = \langle f_1, f_2, \dots f_n \rangle$ be a finitely generated hyperbolic rational semigroup satisfying the strong open set condition. We assume that when n is equal to one the degree of f_1 is at least two. Then*

$$0 < \dim_H(J(G)) < 2.$$

And if we set $\alpha = \dim(J(G))$, then

$$0 < H_\alpha(J(G)) < \infty.$$

Corollary 2.7.6. *Let $G = \langle f_1, f_2, \dots f_n \rangle$ be a finitely generated hyperbolic rational semigroup. We assume that when n is equal to one the degree of f_1 is at least two and the sets $\{f_j^{-1}(J(G))\}_{j=1,\dots,n}$ are mutually disjoint. Then*

$$0 < \dim_H(J(G)) < 2.$$

And if we set $\alpha = \dim(J(G))$, then

$$0 < H_\alpha(J(G)) < \infty.$$

Proof. of Corollary 2.7.6. By Lemma 1.1.6.1 and Theorem 2.1.38, we can assume that

$$\inf_j \inf_{z \in f_j^{-1}(J(G))} \|f_j'(z)\| > 1,$$

where we denote by $\|\cdot\|$ the norm of the derivative with respect to the spherical metric. Then it is easy to see that G satisfies the strong open set condition. Now the statement follows from Corollary 2.7.5. □

Proof. of Theorem 2.7.4. To prove our theorem it is sufficient to show that if for a number δ satisfying $0 < \delta \leq 2$ there is a δ-conformal measure μ on $J(G)$, then $0 < H_\delta(J(G)) < \infty$. We set

$$\lambda = \inf_j \inf_{z \in f_j^{-1}(J(G))} \|f_j'(z)\|.$$

By Lemma 1.1.6.1 and Theorem 2.1.38 we can assume that $\lambda > 1$ by replacing G by a subsemigroup I_m of G. As G is hyperbolic, there is a number $r > 0$ such that for each $\zeta \in J(G)$ and for each $g \in G$ we can take well defined branches of g^{-1} on $D(\zeta, r)$ where $D(\zeta, r)$ is the r disc about ζ. Also we can assume that for each j and for each $\zeta \in J(G)$ the map f_j is injective on $D(\zeta, r)$. We set

$$S_\zeta = \{S \mid \text{ a branch of } g^{-1} \text{ on } D(\zeta, r),\ g \in G\}.$$

By the Koebe theorem, there is a positive number c_0 such that for each $\zeta \in J(G)$ and for each $S \in S_\zeta$

$$\sup\{\|S'(z)\| \mid z \in D(\zeta, \frac{r}{2})\} \leq c_0 \cdot \inf\{\|S'(z)\| \mid z \in D(\zeta, \frac{r}{2})\}.$$

We fix a point $z_0 \in J(G)$. For each positive integer n there is a unique element $g_n \in G$ of word length n such that $g_n(z_0) \in J(G)$ because of Lemma 1.1.5.2 and the strong open set condition. We take a branch S_n of g_n^{-1} such that $S_n(g_n(z_0)) = z_0$. By the Koebe theorem there are a positive constant α, β such that if we set

$$r_n = \frac{r \cdot \|(S_n)'(z_n)\|}{\alpha},$$

for each n where $z_n = g_n(z_0)$, then

$$D(z_0, r_n) \subset S_n(D(z_n, r)),$$

$$D(z_0, r_n) \supset S_n(D(z_n, \frac{r}{\beta})).$$

Also we have $r_n \to 0$ as $n \to \infty$. Since the support of μ is equal to $J(G)$, for each small number $a > 0$ there is a number $M(a) > 0$ such that for each $\zeta \in J(G)$

$$\mu(D(\zeta, a)) > M(a).$$

From above and since μ is δ conformal we get

$$\mu(D(z_0, r_n)) \geq c_0^{-\delta} \|(S_n)'(z_n)\|^\delta \mu(D(z_n, \frac{r}{\beta}))$$

$$\geq (\frac{rc_0}{\alpha})^{-\delta} r_n^\delta M(\frac{r}{\beta}),$$

$$\mu(D(z_0, r_n)) \leq c_0^\delta |(S_n)'(z_n)|^\delta \mu(D(z_n, r))$$

$$\leq c_0^\delta (\frac{r}{\alpha})^{-\delta} r_n^\delta.$$

So there is a number $c_1 > 1$ such that for each n

$$c_1^{-1} \leq \frac{\mu(D(z_0, r_n))}{r_n^\delta} \leq c_1.$$

We can take c_1 independent of $z_0 \in J(G)$. We set

$$c_2 = \frac{\max_j \max_{z \in f_j^{-1}(J(G))} \|f_j'(z)\|}{\lambda}.$$

There is a number n such that

$$c_2^{-1} r_n \leq r' \leq c_2 r_n,$$

for all r' with $r' < r_1$. Then for each r' such that $r' < r_1$

$$\mu(D(z_0, r_n)) \geq c_1^{-1} r_n^\delta \geq (c_1 c_2^\delta)^{-1} (r')^\delta,$$

$$\mu(D(z_0, r_n)) \leq c_1 r_n^\delta \leq c_1 c_2^\delta (r')^\delta.$$

So if we set $c = c_1 c_2^\delta$, for any small r'

$$c^{-1}(r')^\delta \leq \mu(D(z_0, r')) \leq c(r')^\delta.$$

Now the statement of our theorem follows immediately. □

In [DU1], M.Denker and M.Urbański gave a conjecture that for any rational map f, $\dim_H(J(\langle f\rangle)) = \delta(\langle f\rangle)$. Similary we give the following conjecture.

Conjecture 2.7.7. *Let $G = \langle f_1, f_2, \dots f_n\rangle$ be a finitely generated rational semigroup satisfying the strong open set condition. We assume that when n is equal to one the degree of f_1 is at least two. Then*

$$\dim_H(J(G)) = \delta(G).$$

2.8 δ-subconformal measures and Hausdorff dimension of the Julia sets

Definition 2.8.1. Let G be a rational semigroup and δ a non-negative number. We say that a Borel probability measure μ on $\overline{\mathbb{C}}$ is δ-subconformal if for each $g \in G$ and for each Borel measurable set A

$$\mu(g(A)) \leq \int_A \|g'(z)\|^\delta d\mu,$$

where we denote by $\|\cdot\|$ the norm of the derivative with respect to the spherical metric. For each $x \in \overline{\mathbb{C}}$ and each real number s we set

$$S(s,\ x) = \sum_{g\in G}\sum_{g(y)=x} \|g'(y)\|^{-s}$$

counting multiplicities and

$$S(x) = \inf\{s \mid S(s,\ x) < \infty\}.$$

If there is not s such that $S(s,\ x) < \infty$, then we set $S(x) = \infty$.Also we set

$$s_0(G) = \inf\{S(x)\},\ s(G) = \inf\{\delta \mid \exists\mu : \delta\text{-subconformal measure}\}$$

It is not difficult for us to prove the next result using the same method as that in the section of δ-conformal measure.

Theorem 2.8.2. *Let G be a rational semigroup which has at most countably many elements. If there exists a point $x \in \overline{\mathbb{C}}$ such that $S(x) < \infty$ then there is a $S(x)$-subconformal measure. In particular, we have $s(G) \leq s_0(G)$.*

Proposition 2.8.3. *Let G be a rational semigroup and τ a δ-subconformal measure for G where δ is a real number Assume that $\sharp J(G) \geq 3$ and for each $x \in E(G)$ there exists an element $g \in G$ such that $g(x) = x$ and $|g'(x)| < 1$. Then the support of τ contains $J(G)$.*

Proposition 2.8.4. *Let $G = \langle f_1, f_2, \dots, f_m\rangle$ be a finitely generated rational semigroup. Assume that G satisfies the open set condition with respect to the generators $f_1, f_2, \dots, f_m$ and $O \setminus J(G) \neq \emptyset$ where O is an open set in the definition of the open set condition. If there exists an attractor in $F(G)$ for G, then*

$$s_0(G) \leq 2.$$

Proof. We can assume $m \geq 2$. Let K be an attractor in $F(G)$ for G. There exists a simply connected domain U in $(O \cap F(G)) \setminus (K \cup P(G))$ such that $g(U) \cap U = \emptyset$ for each $g \in G$. By the open set condition, it is easy to see that if $g \neq h$, then $g^{-1}(U) \cap h^{-1}(U) = \emptyset$. Hence we have

$$\sum_S \int_U \|S'(z)\|^2 dm_2(z) < \infty,$$

where S is taken over all holomorphic inverse branches of all elements of G defined on U, $\|\cdot\|$ denotes the norm of the derivative with respect to the spherical metric and m_2 is the 2-dimensional Lebesgue measure on $\overline{\mathbb{C}}$. It follows that for almost every where $x \in U$, $S(2, x) < \infty$. □

Lemma 2.8.5. *Let G be a rational semigroup. Assume that $\infty \in F(G)$, $\sharp J(G) \geq 3$ and for each $x \in E(G)$ there exists an element $g \in G$ such that $g(x) = x$ and $|g'(x)| < 1$. We also assume that there exist a countable set E in $\overline{\mathbb{C}}$, positive numbers a_1 and a_2 and a constant c with $0 < c < 1$ such that for each $x \in J(G)\setminus E$, there exist two sequences (r_n) and (R_n) of positive real numbers and a sequence (g_n) of elements of G satisfying all of the following conditions:*

1. *$r_n \to 0$ and for each n, $0 < \frac{r_n}{R_n} < c$ and $g_n(x) \in J(G)$.*
2. *for each n, $g_n(D(x, R_n)) \subset D(g_n(x), a_1)$.*
3. *for each n $g_n(D(x, r_n)) \supset D(g_n(x), a_2)$.*

Let δ be a real number with $\delta \geq s(G)$ and μ a δ- subconformal measure. Then δ-Hausdorff measure on $J(G)$ is absolutely continuous with respect to μ such that the Radon-Nikodim derivative is bounded from above. In particular, we have

$$dim_H(J(G)) \leq s(G).$$

Proof. By Proposition 2.8.3, the support of μ contains $J(G)$. Hence there exists a constant $c_1 > 0$ such that for each $x \in J(G)$, $\mu(D(x, a_2)) > c_1$.

Fix any $x \in J(G)\setminus E$. For each n we set $\tilde{R_n}(z) = R_n z + x$. By the condition 1 and 2, the family $\{g_n \circ \tilde{R_n}\}$ is normal in $D(0,1)$. By Marty's theorem, there exists a constant c_2 such that for each n and each $w \in D(0,c)$,

$$\|(g_n \circ \tilde{R_n})'(w)\| \leq c_2.$$

Note that we can take the constant c_2 independent of $x \in J(G) \setminus E$. Hence we have for each n,

$$\begin{aligned} c_1 &\leq \mu(D(g_n(x), a_2)) \\ &\leq \mu(g_n(D(x, r_n))) \\ &\leq \int_{D(x,r_n)} \|g_n'(z)\|^\delta d\mu(z) \\ &= \int_{D(x,r_n)} \|(g_n \circ \tilde{R_n} \circ \tilde{R_n}^{-1})'(z)\|^\delta d\mu(z) \\ &\leq c_3 \frac{1}{R_n^\delta} \mu(D(x, r_n)) \\ &\leq c_3 \frac{1}{r_n^\delta} \mu(D(x, r_n)), \end{aligned}$$

where c_3 is a constant not depending on n and $x \in J(G) \setminus E$. Therefore we get that there exists a constant c_4 not depending on n and $x \in J(G) \setminus E$ such that

$$\frac{\mu(D(x, r_n))}{r_n^\delta} \geq c_4. \tag{2.39}$$

Now we can show the statement of our lemma in the same way as the proof of Theorem 14 in [DU2]. We will follow it. Let A be any Borel set in $J(G)$. We set $A_1 = A \setminus E$. We denote by H_δ the δ-Hausdorff measure. Since E is a countable set, we have $H_\delta(A) = H_\delta(A_1)$. Fix γ, ϵ. For every $x \in A_1$, denote by $\{r_n(x)\}_{j=1}^\infty$ the sequence constructed in the above paragraph. Since μ is regular, for every $x \in A_1$ there exists a radius $r(x)$ being of the form $r_n(x)$ such that

$$\mu(\bigcup_{x \in A_1} D(x, r(x)) \setminus A_1) < \epsilon.$$

By the Besicovič theorem we can choose a countable subcover $\{D(x_i, r_{x_i})\}_{i=1}^\infty$ from the cover $\{D(x, r(x))\}_{x \in A_1}$ of A_1, of multiplicity bounded by some constant $C \geq 1$, independent of the cover. By (2.39), we obtain

$$\begin{aligned} \sum_{i=1}^\infty r(x_i)^\delta &\leq c_4^{-1} \sum_{i=1}^\infty \mu(D(x_i, r(x_i))) \\ &\leq c_4^{-1} C \mu(\bigcup_{i=1}^\infty D(x_i, r(x_i))) \\ &\leq c_4^{-1} C(\epsilon + \mu(A_1)). \end{aligned}$$

Letting $\epsilon \to 0$ and then $\gamma \to 0$ we get

$$H_\delta(A) = H_\delta(A_1) \leq c_4^{-1} C \mu(A_1) \leq c_4^{-1} C \mu(A).$$

□

Theorem 2.8.6. *Let G be a rational semigroup generated by a generator system $\{f_\lambda\}_{\lambda \in \Lambda}$ such that $\cup_{\lambda \in \Lambda}\{f_\lambda\}$ is a compact subset of $End(\overline{\mathbb{C}})$. Let $\tilde{f}$ be a rational skew product constructed by the generator system. Assume $\tilde{f}$ is semi-hyperbolic along fibers and satisfies the condition C1 with a family of discs $\{D_x\}_{x \in X}$ such that $D_x = D$, $\forall x \in X$ with some D. Then we have*

$$\dim_H(J(G)) \leq s(G).$$

Proof. We can assume $\infty \in F(G)$. Let x be any point of $J(G)$. Since we have $J(G) = \cup_{\lambda \Lambda} f_\lambda^{-1}(J(G))$, for each $n \in \mathbb{N}$ there exists an element $g_n \in G$ which is a product of n generators such that $g_n(x) \in J(G)$. Let δ be a small positive number. For each n, we denote by $D_{g_n}(g_n(x), \delta)$ the element of $c(D(g_n(x), \delta), \ \delta)$ containing x. By Theorem 2.4.13, if we take a δ smaller, then

$$\text{diam } (D_{g_n}(g_n(x), \delta)) \to 0, \text{ as } n \to \infty. \tag{2.40}$$

By Lemma 2.4.14, we can assume that $D_{g_n}(g_n(x), \delta)$ is simply connected for each n. Let $\phi_n : D(0,1) \to D_{g_n}(g_n(x), \delta)$ be the Riemann map such that $\phi_n(0) = x$. By the Koebe distortion theorem, we have that for each n,

$$D_{g_n}(g_n(x), \delta) \supset D(x, \ \frac{1}{4}|\phi_n'(0)|).$$

Since G is semi-hyperbolic, we can assume that $D(J(G), \ \delta) \subset SH_N(G)$ where N is a positive integer. By Corollary 2.1.10, we get

$$\sup_{n \in \mathbb{N}}\{\text{diam } (\phi_n^{-1}(D_{g_n}(g_n(x), \epsilon\delta)))\} \to 0, \text{ as } \epsilon \to 0.$$

Therefore by the Koebe distortion theorem, there exists an ϵ such that

$$\begin{aligned} D_{g_n}(g_n(x),\ \epsilon\delta) &= \phi_n(\phi_n^{-1}(D_{g_n}(g_n(x),\ \epsilon\delta))) \\ &\subset D(x,\ \frac{1}{8}|\phi_n'(0)|),\ \text{for each } n. \end{aligned}$$

By (2.40), we have $|\phi_n'(0)| \to 0$ as $n \to \infty$. Applying Lemma 2.8.5, we get

$$\dim_H(J(G)) \leq s(G).$$

□

Theorem 2.8.7. *Let $G = \langle f_1, f_2, \dots, f_m\rangle$ be a finitely generated rational semigroup which is semi-hyperbolic. Assume that G contains an element with the degree at least two, each element of $Aut\,\overline{\mathbb{C}} \cap G$ (if this is not empty) is loxodromic and $F(G) \neq \emptyset$. Then we have*

$$\dim_H(J(G)) \leq s(G) \leq s_0(G).$$

Proof. By Theorem 2.8.6 and Theorem 2.8.2. □

Remark 13. Let $G = \langle f_1, f_2, \dots, f_m\rangle$ be a finitely generated hyperbolic rational semigroup which satisfies the strong open set condition. We assume that when $n = 1$ the degree of f_1 is at least two. By the results in Theorem 2.7.2 and the proof, Theorem 2.7.4 and Corollary 2.7.5, we have

$$0 < \dim_H J(G) = s(G) = s_0(G) = \delta(G) < 2.$$

Theorem 2.8.8. *Let $G = \langle f_1, f_2, \dots f_n\rangle$ be a finitely generated rational semigroup which is expanding. Let λ be the number in the assumption of Theorem 2.1.39. Then*

$$\dim_H(J(G)) \leq \frac{\log(\sum_j \deg(f_j))}{\log\lambda}. \tag{2.41}$$

Proof. This theorem follows from a Corollary which we will show later using thermodynamical formalisms, but here we will show the statement directly. By replacing G by a subsemigroup I_m of G, we can assume that for each j and $z \in f_j^{-1}(J(G))$

$$\|f_j'(z)\| \geq \lambda,$$

where $\|\cdot\|$ denotes the norm of the derivative with respect to the spherical metric. We take a point $x \in J(G)$. Now for each m the set

$$\bigcup_{g\in G:\ \text{word length } m} g^{-1}\{x\}$$

has at most $(\sum_j(\deg(f_j)))^m$ points. Also for each $z \in J(G)$ and for each m, if $g \in G$ is word length m and $g(z) \in J(G)$, then

$$\|g'(z)\| \geq \lambda^m.$$

So for each number s such that $s > \frac{\log(\sum_j \deg(f_j))}{\log\lambda}$ we have

$$\sum_{g\in G}\sum_{g(y)=x} \|g'(y)\|^{-s} \leq \sum_{m=0}^{\infty}(\sum_j \deg(f_j))^m\lambda^{-ms}$$

$$= \sum_{m=0}^{\infty} \exp\{m\log(\sum_j \deg(f_j))(1 - s\frac{\log\lambda}{\log(\sum_j \deg(f_j))})\} < \infty.$$

From the way of construction of δ-subconformal measure and Theorem 2.8.7, the statement of Theorem 2.8.8 follows. □

Example 2.8.9. Let n be a positive integer such that $n \geq 4$. We set $G = \langle z^n,\ n(z-4)+4\rangle$. Then G is a finitely generated hyperbolic rational semigroup satisfying the strong open set condition. For, let $f(z) = z^n$, $g(z) = n(z-4)+4$ and $U = \{|z| < 5\}$. Then the closures of $f^{-1}(U)$ and $g^{-1}(U)$ are included in U and mutually disjoint. Hence $J(G) \subset U$ and G satisfies the strong open set condition. Since $|g(0)| > 5$, G is hyperbolic. By Theorem 2.8.8, we get

$$1 \leq \dim_H J(G) \leq \frac{\log(n+1)}{\log(n)}.$$

Example 2.8.10. Let $G = \langle f_1, f_2\rangle$ where $f_1(z) = z^2 + 2$, $f_2(z) = z^2 - 2$. Since $P(G) \cap J(G) = \{2, -2\}$ and $P(G) \cap F(G)$ is compact, we have G is sub-hyperbolic. By Theorem 2.5.7, G is also semi-hyperbolic. Since $f_j^{-1}(D(0,2)) \subset D(0,2)$ for $j = 1,2$ and $f_1^{-1}(D(0,2)) \cap f_2^{-1}(D(0,2)) = \emptyset$, G satisfies the open set condition. Also $J(G)$ is included in $B = \cup_{j=1}^2 f_j^{-1}(\overline{D(0,2)})$. Since $B \cap \partial D(0,2) = \{2, -2, 2i, -2i\}$, we get $\sharp(J(G) \cap \partial D(0,2)) < \infty$. By Corollary 2.6.7, we have $m_2(J(G)) = 0$, where we denote by m_2 the 2-dimensional Lebesgue measure. By Theorem 2.8.7 and Proposition 2.8.4, we have also

$$dim_H(J(G)) \leq s(G) \leq s_0(G) \leq 2.$$

Chapter 3

Skew products related to finitely generated rational semigroups

In this chapter we assume the following situation. Let m be a positive integer and $\Sigma_m = \{1, \dots, m\}^{\mathbb{N}}$. We denote by $\sigma : \Sigma_m \to \Sigma_m$ the shift map, that is

$$(w_1, \dots) \mapsto (w_2, \dots).$$

Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. Let $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ be the rational skew product constructed by the generator system $\{f_1, \dots, f_m\}$. Hence for each $(w, x) \in \Sigma_m \times \overline{\mathbb{C}}$,

$$\tilde{f}((w, x)) = (\sigma w, f_{w_1} x).$$

3.1 thermodynamical formalisms

We consider to use thermodynamical formalisms to get some estimates of Hausdorff dimension of Julia sets of expanding rational semigroups.

For each $j = 1, \dots, m$, let φ_j be a Hölder continuous function on $f_j^{-1}(J(G))$. We set for each $(w, x) \in \tilde{J}$, $\varphi((w, x)) = \varphi_{w_1}(x)$. Then φ is a Hölder continuous function on $\tilde{J}$. We define an operater L on $C(\tilde{J}) = \{\psi : \tilde{J} \to \mathbb{C} \mid$ continuous $\}$ by

$$L\psi((w, x)) = \sum_{\tilde{f}((w', y)) = (w, x)} \frac{\exp(\varphi((w', y)))}{\exp(P)} \psi((w', y)),$$

counting multiplicities, where we denote by $P = P(\tilde{f}|_{\tilde{J}}, \varphi)$ the pressure of $(\tilde{f}|_{\tilde{J}}, \varphi)$.

Lemma 3.1.1. *With the same notations as the above, let $G = \langle f_1, f_2, \dots f_m\rangle$ be a finitely generated expanding rational semigroup. Then for each set of Hölder continuous functions $\{\varphi_j\}_{j=1,\dots,m}$, there exists a unique probability measure τ on $\tilde{J}$ such that*

- $L^*\tau = \tau$.
- *for each $v \in C(\tilde{J})$, $\|L^n v - \tau(v)\alpha\|_{\tilde{J}} \to 0, n \to \infty$, where we set $\alpha = \lim_{l\to\infty} L^l(1) \in C(\tilde{J})$ and we denote by $\|\cdot\|_{\tilde{J}}$ the supremum norm on $\tilde{J}$.*
- *$\alpha\tau$ is an equilibrium state for $(\tilde{f}|_{\tilde{J}}, \varphi)$.*

Lemma 3.1.2. *Let $G = \langle f_1, f_2, \dots f_m\rangle$ be a finitely generated expanding rational semigroup. Then there exists a unique number $\delta > 0$ such that if we set $\varphi_j(x) = -\delta\log(\|f_j'(x)\|), j = 1, \dots, m,$ then $P = 0$.*

From Lemma 3.1.1, for this δ there exists a unique probability measure τ on $\tilde{J}$ such that $L_\delta^*\tau = \tau$ where L_δ is an operator on $C(\tilde{J})$ defined by

$$L_\delta v((w,x)) = \sum_{\tilde{f}((w',y))=(w,x)} \frac{v((w',y))}{\|(f_{w_1'})'(y)\|^\delta}.$$

Also δ satisfies that

$$\delta = \frac{h_{\alpha\tau}(\tilde{f})}{\int_{\tilde{J}} \tilde{\varphi}\alpha d\tau} \le \frac{\log(\sum_{j=1}^m \deg(f_j))}{\int_{\tilde{J}} \tilde{\varphi}\alpha d\tau},$$

where $\alpha = \lim_{l\to\infty} L_\delta^l(1)$, we denote by $h_{\alpha\tau}(\tilde{f})$ the metric entropy of $(\tilde{f}, \alpha\tau)$ and $\tilde{\varphi}$ is a function on $\tilde{J}$ defined by $\tilde{\varphi}((w,x)) = \log(\|f_{w_1}'(x)\|)$.

By these arguments and results in section of δ-(sub) conformal measures , we get the following result.

Theorem 3.1.3. *Let $G = \langle f_1, f_2, \dots f_m\rangle$ be a finitely generated expanding rational semigroup and δ the number in the above argument. Then*

$$\dim_H(J(G)) \le s(G) \le \delta.$$

Moreover, if the sets $\{f_j^{-1}(J(G))\}$ are mutually disjoint, then $\dim_H(J(G)) = \delta < 2$ and $0 < H_\delta(J(G)) < \infty$, where we denote by H_δ the δ-Hausdorff measure.

Corollary 3.1.4. *Let $G = \langle f_1, f_2, \dots f_m\rangle$ be a finitely generated expanding rational semigroup. Then*

$$\dim_H(J(G)) \le \frac{\log(\sum_{j=1}^m \deg(f_j))}{\log\lambda},$$

where λ denotes the number in Definition 2.1.40.

Example 3.1.5.

1. Let $G = \langle f_1, f_2\rangle$ where $f_1(z) = z^2$ and $f_2(z) = 2.3(z-3)+3$. Then we can see easily that $\{|z| < 0.9\} \subset F(G)$ and G is hyperbolic. By the corollary 3.1.4, we get

$$\dim_H(J(G)) \le \frac{\log 3}{\log 1.8} < 2.$$

In particular, $J(G)$ has no interior points. In Proposition 2.6.3, it was shown that if a finitely generated rational semigroup satisfies the open set condition with an open set O, then the Julia set is equal to the closure of the open set O or has no interior points. Note that the fact that the Julia set of the above semigroup G has no interior points was shown by only using analytic quantity. It seems to be true that G does not satisfy the open set condition.

2. Let $G = \langle \frac{z^3}{4}, z^2+8\rangle$. Then we can see easily that $\{|z| < 2\} \subset F(G)$ and G is hyperbolic. Hence we have

$$\dim_H J(G) \le \frac{\log 5}{\log 3} < 2.$$

In particular, $J(G)$ has no interior points.

3.2 backward self-similar measure

We now consider about invariant measures and self-similar measures on Julia sets. In the cases of iterations of rational functions, Brolin's and Lyubich's studies are well known([Br], [L]). Recently, D.Boyd investigated "invariant measure" (that is, the measure $(\pi_2)_*\tilde{\mu}$ in the notation in Theorem 3.2.3) in the case that each f_j is of degree at least two and have shown the uniqueness in [Bo]. We introduce some notations and results from [L]. Let A be a bounded operator in the complex Banach space $\mathcal{B}$. The operator A is called *almost periodic* if the orbit $\{A^m\varphi\}_{m=1}^\infty$ of any vector $\varphi \in \mathcal{B}$ is strongly conditionally compact. The eigenvalue λ and related eigenvector are called unitary if $|\lambda| = 1$. The set of unitary eigenvectors of the operator A will be denoted by $\mathrm{spec}_u A$. We denote by $\mathcal{B}_u$ the closure of the linear span of the unitary eigenvectors of the operator A. And we set

$$\mathcal{B}_0 = \{\varphi \mid A^m\varphi \to 0 \ (m \to \infty)\},$$

here the convergence is assumed to be strong.

Theorem 3.2.1. *([L]) If $A : \mathcal{B} \to \mathcal{B}$ is an almost periodic operator in the complex Banach space $\mathcal{B}$, then*

$$\mathcal{B} = \mathcal{B}_u \oplus \mathcal{B}_0.$$

Corollary 3.2.2. *([L]) Let $A : \mathcal{B} \to \mathcal{B}$ be an almost periodic operator in the complex Banach space $\mathcal{B}$. Assume that $\mathrm{spec}_u A = \{1\}$ and the point $\lambda = 1$ is a simple eigenvalue. Let $h \neq 0$ be an invariant vector of the operator A. Then there exists an A^* invariant functional $\mu \in \mathcal{B}^*$, $\mu(h) = 1$, such that*

$$A^m\varphi \to \mu(\varphi)h \quad m \to \infty.$$

Let $G = \langle f_1, f_2, \dots f_m\rangle$ be a finitely generated rational semigroup. We set $d_j = \deg(f_j)$ for each $j = 1, \dots, m$ and $d = \sum_{j=1}^m d_j$. For each compact set K of $\overline{\mathbb{C}}$ we denote by $C(K)$ all continuous complex valued functions on K. It is a Banach space with supremum norm on K. Assume that K is backward invariant under G. For each j and for each element φ we set

$$(A_j\varphi)(z) = \frac{1}{d_j}\sum_{\zeta\in f_j^{-1}(z)}\varphi(\zeta),$$

where z is any point of K. Then $A_j\varphi$ is an element of $C(K)$ and A_j is a bounded operator on $C(K)$. We set

$$\mathcal{W} = \{(a_1, \dots, a_n) \in \mathbb{R}^n \mid \sum_j a_j = 1,\ a_j \geq 0\}.$$

And for each $a \in \mathcal{W}$ we set

$$(B_a\varphi)(z) = \sum_{j=1}^{n} a_j(A_j\varphi)(z).$$

Then B_a is a bounded operator on $C(K)$.

Similarly, let $\tilde{K}$ be a compact subset of $\Sigma_m \times \overline{\mathbb{C}}$ which is backward invariant under $\tilde{f}$. We define an operator $\tilde{B}_a$ on $C(\tilde{K})$ as follows. For each element $\tilde{\varphi} \in C(\tilde{K})$ we set

$$(\tilde{B}_a\tilde{\varphi})(z) = \Sigma_{\zeta\in\tilde{f}^{-1}(z)}\tilde{\varphi}(\zeta)\tilde{\psi}_a(\zeta)$$

where $\tilde{\psi}_a(\zeta) = \frac{a_{w_1}}{d_{w_1}}$ if $\pi_1(\zeta) = (w_1, w_2, \dots)$.

$\tilde{B}_a$ is a bounded operator on $C(\tilde{K})$. Furtheremore, if $\pi_2(\tilde{K}) = K$, then we get

$$\pi_2^* B_a = \tilde{B}_a \pi_2^*$$

and $\pi_2^*; C(K) \to C(\tilde{K})$ is an isometry.

Theorem 3.2.3. *Let $G = \langle f_1, \dots, f_m\rangle$ be a finitely generated rational semigroup. Assume that there exists an element $g_0 \in G$ of degree at least two, the exceptional set $E(G)$ for G is included in $F(G)$ and $F(H) \supset J(G)$ where H is a rational semigroup defined by $H = \{h^{-1} \mid h \in Aut(\overline{\mathbb{C}}) \cap G\}$.(if H is empty, put $F(H) = \overline{\mathbb{C}}$.) Then for each $a \in \mathcal{W}$ with $a \neq 0$ there exists a unique regular Borel probability measure $\tilde{\mu}_a$ on $\Sigma_m \times \overline{\mathbb{C}}$ such that for each compact set $\tilde{K}$ which is included in $\pi_2^{-1}(\overline{\mathbb{C}} \setminus E(G))$ and backward invariant under $\tilde{f}$,*

$$\|\tilde{B}_a^n(\tilde{\varphi}) - \tilde{\mu}_a(\varphi)\mathbf{1}\|_{\tilde{K}} \to 0,$$

as $n \to \infty$, for each $\tilde{\varphi} \in C(\tilde{K})$, where we denote by $\mathbf{1}$ the constant function taking its value 1. Similarly, there exists a unique regular Borel probability measure μ_a on $\overline{\mathbb{C}}$ such that for each compact set K which is included in $\overline{\mathbb{C}} \setminus E(G)$ and backward invariant under G,

$$\|B_a^n(\varphi) - \mu_a(\varphi)\mathbf{1}\|_K \to 0,$$

as $n \to \infty$, for each $\varphi \in C(K)$.

Moreover, $(\pi_2)_*(\tilde{\mu}_a) = \mu_a$. The support of $\tilde{\mu}_a$ is equal to $\tilde{J}$ and the support of μ_a is equal to $J(G)$.

Definition 3.2.4. We call $\tilde{\mu}_a$ or μ_a the *self-similar measure* with respect to the weight a.

We need some lemmas to prove Theorem 3.2.3.

Lemma 3.2.5. *Let $G = \langle f_1, \dots, f_m\rangle$ be a finitely generated rational semigroup. Assume that there exists an element $g_0 \in G$ of degree at least two and $F(H) \supset J(G)$ where H is a rational semigroup defined by $H = \{h^{-1} \mid h \in Aut(\overline{\mathbb{C}}) \cap G\}$.(if H is empty, put $F(H) = \overline{\mathbb{C}}$.) Then there exists a $\delta > 0$ such that for each $x \in J(G)$, if we denote by $\mathcal{F}_{x,\delta}$ the family of maps satisfying that each element of it is a well-defined inverse branch of some element of G on $B(x, \delta)$ where $B(x, \delta)$ is a ball about x with the radius δ with respect to the spherical metric, then $\mathcal{F}_{x,\delta}$ is a normal family on $B(x, \delta)$.*

Proof. Take $x \in J(G)$. Let $D = B(x, \epsilon)$ where ϵ is a small positive number. Let G_1 be the subsemigroup of G such that for each element $g \in G_1$ there exists an element $g_1 \in G$ and integer j with $1 \le j \le m$ and $d_j \ge 2$ satisfying $g = g_1 f_j$. Let $\mathcal{F}_1$ be the family of maps from D to $\overline{\mathbb{C}}$ such that each element of $\mathcal{F}_1$ is an inverse branch of some element of G_1. Then from Theorem 2.1 in [HM3], $\mathcal{F}_1$ is normal in D. Taking ϵ smaller, we can assume that

$$\overline{\cup_{\beta \in \mathcal{F}_1} \beta(D)} \subset F(H). \tag{3.1}$$

Now let $(g_j)_j$ be any sequence of elements of $\mathcal{F}_{x,\epsilon}$ where g_j is an element of G. For each j there exists an element $\psi_j \in G_1 \cup \{Id\}$ and $h_j \in H \cup \{Id\}$ such that $g_j = \psi_j h_j$. Then for each j, $\psi_j(h_j g_j^{-1}) = g_j g_j^{-1} = Id_D$. Hence we have $h_j g_j^{-1} \in \mathcal{F}_1$. Since $\mathcal{F}_1$ is normal in D, there exists a map g from D to $\overline{\mathbb{C}}$ and a sequence of positive integers (j_k) such that $h_{j_k} g_{j_k}^{-1} \to g$ locally uniformly on D as $k \to \infty$. By (3.1), we can assume that there exists a map h from a neighborhood V of $\overline{g(D)}$ to $\overline{\mathbb{C}}$ such that $h_{j_k}^{-1} \to h$ locally uniformly on V as $k \to \infty$. It follows that $g_{j_k}^{-1} = h_{j_k}^{-1} h_{j_k} g_{j_k} \to hg$ locally uniformly on D as $k \to \infty$. Hence we have $\mathcal{F}_{x,\epsilon}$ is normal on D. Since $J(G)$ is compact, the statement of our lemma holds.

□

Lemma 3.2.6. *Under the same assumption as Theorem 3.2.3, let $\tilde{K}$ be a compact subset of $\pi_2^{-1}(\overline{\mathbb{C}} \setminus E(G))$ which is backward invariant under $\tilde{f}$. If $\tilde{B}_a \varphi = \lambda \varphi$, $|\lambda| = 1$, then $\lambda = 1$ and φ is constant. That is, $(C(\tilde{K}))_u = \mathbb{C} \cdot 1$.*

Proof. Let z be a point of $\tilde{K}$ such that

$$|\varphi(z)| = \sup_{w \in \tilde{K}} |\varphi(w)|.$$

Then

$$\begin{aligned} |\varphi(z)| &= |(\tilde{B}_a \varphi)(z)| \\ &\le \sum_{\zeta \in \tilde{f}^{-1}(z)} |\psi_a(\zeta)||\varphi(\zeta)| \\ &\le \sum_j a_j |\varphi(z)| = |\varphi(z)|. \end{aligned}$$

Hence if ζ is a point of $\tilde{f}^{-1}(z)$, then $|\varphi(\zeta)| = |\varphi(z)|$ and it implies $\varphi(\zeta) = \lambda \varphi(z)$. Fix any point $\zeta_0 \in \tilde{J}$. By Proposition 2.4.5.6, there exists a sequence (ζ_n) such that $\zeta_n \in \tilde{f}^{-n}(z)$ for each positive integer n and $\zeta_n \to \zeta_0$ as $n \to \infty$. Hence we have $\lambda^n \varphi(z) = \varphi(\zeta_n) \to \varphi(\zeta)$ as $n \to \infty$. It implies $\lambda = 1$.

Now we will show that φ is constant. We put $\varphi = \Re\varphi + i\Im\varphi$. Then

$$\tilde{B}_a(\Re\varphi) = \Re\varphi, \ \tilde{B}_a(\Im\varphi) = \Im\varphi.$$

Let z be a point of $\tilde{K}$ such that

$$\Re\varphi(z) = \sup_{w \in \tilde{K}} \Re\varphi(w).$$

By a similar argument we can show that $\Re\varphi(\zeta) = \Re\varphi(z)$ for each $\zeta \in \tilde{f}^{-1}(z)$. Let ζ be any point of $\tilde{J}$. Let $(\zeta_n)_n$ be a sequence such that for each n the point ζ_n belongs to $\tilde{f}^{-n}(z)$ and $\zeta_n \to \zeta$. Then $\Re\varphi(\zeta_n) \to \Re\varphi(\zeta)$ so $\Re\varphi(z) = \Re\varphi(\zeta)$. In the same way we can show that if x is the minimum point of the function $\Re\varphi$, then $\varphi(x) = \varphi(\zeta)$, where ζ is any point of $\tilde{J}$. Hence $\Re\varphi$ is constant and by the same argument $\Im\varphi$ is also constant. Whence φ is constant. □

Lemma 3.2.7. *Under the same assumption as Theorem 3.2.3, if $\tilde{K}$ is a compact subset of $\pi_2^{-1}(\overline{\mathbb{C}} \setminus E(G))$ which is backward invariant under $\tilde{f}$, then $\tilde{B}_a$ is an almost periodic operator on $C(\tilde{K})$.*

Proof. We will develop the methods of key lemma about equicontinuity of $\{B_{a_0}^n \phi\}_n$ where $a_0 = (\frac{d_1}{d}, \dots, \frac{d_m}{d})$, $\phi \in C(K)$ in [Bo]. Let $\varphi \in C(\tilde{K})$ be any element. We have $\|\tilde{B}_a{}^n \varphi\|_{\tilde{K}} \le \|\varphi\|_{\tilde{K}}$ for each positive integer n. By the Ascoli-Arzela Theorem, we have only to show that the family $\{\tilde{B}_a{}^n \varphi\}_n$ is equicontinuous on $\tilde{K}$.

For each $t = (t_1, \dots, t_m) \in \mathbb{N}^m$, we set

$$a_{r,t} = \frac{t_r d_r}{\sum_{k=1}^n t_k d_k}, \ r = 1, \dots, m,$$

and $a(t) = (a_{1,t}, \dots, a_{m,t}) \in \mathcal{W}$. Then there exists a sequence $(t^l)_l$ of elements of $\mathbb{N}^m$ such that $a(t^l) \to a$, as $l \to \infty$.

For each $p \in \mathbb{N}$, we set

$$Z_p = \cup_{(i_p, \dots, i_1) \in \{1, \dots, m\}^p} \{\mathrm{cv}(f_{i_p} \cdots f_{i_1})\} = \pi_2(\mathrm{cv}(\tilde{f}^p)),$$

where cv means the critical values.

Let U be any simply connected domain such that $U \subset \overline{\mathbb{C}} \setminus Z_p$. For each $i = 1, \dots, m$ and $l \in \mathbb{N}$, we set

$$g_{i,j}^l = f_i, \ j = 1, \dots, t_i^l.$$

For each $l \in \mathbb{N}$ we consider $\{g_{i,j}^l\}_{i,j}$ as a generator system and let $\tilde{f}_l : \Sigma_{m(l)} \times \overline{\mathbb{C}} \to \Sigma_{m(l)} \times \overline{\mathbb{C}}$ be the skew product map constructed by that generator system in the same way as the beginning of this section where $m(l) = \sum_{i=1}^{m} t_i^l$. For each $s \in \mathbb{N}$ and $l \in \mathbb{N}$, we denote by $\sigma_{s,l} = \sigma_{s,l}(U)$ the cardinality of the family consisting of well-defined inverse branches of $\tilde{f_l}^s$ on $\Sigma_{m(l)} \times U$. For each finite word $\{1, 2, \dots, m(l)\}^s$, let $\sigma_{s,k,\alpha}$ be the cardinality of the family consisting of well-defined inverse branches of the element in $\langle g_{i,j}^l \rangle_{i,j}$ corresponding to the word α on U. Then by definition, we have

$$\sigma_{s,l} = \sum_{\alpha \in \{1,2,\dots,m(l)\}^s} \sigma_{s,l,\alpha}.$$

For each $k = 1, 2, \dots, m(l)$, let e_k be the degree of k-th element of $\{g_{i,j}^l\}_{i,j}$. Then we have

$$\sigma_{s+1,l,\alpha k} \geq e_k(\sigma_{s,l,\alpha} - (2e_k - 2)).$$

Hence we get

$$\begin{aligned}
\sigma_{s+1,l} &= \sum_{\alpha \in \{1,2,\dots,m(l)\}^s} \sum_{k \in \{1,2,\dots,m(l)\}} \sigma_{s+1,l,\alpha k} \\
&\geq \sum_{\alpha \in \{1,2,\dots,m(l)\}^s} \sum_{k \in \{1,2,\dots,m(l)\}} e_k(\sigma_{s,l,\alpha} - (2e_k - 2)) \\
&= \Big(\sum_{k \in \{1,2,\dots,m(l)\}} e_k \Big) \sigma_{s,l} - m(l)^s \sum_{k \in \{1,2,\dots,m(l)\}} e_k(2e_k - 2) \\
&= \Big(\sum_{j=1}^{m} t_j^l d_j\Big)\sigma_{s,l} - m(l)^s \Big(\sum_{j=1}^{m} 2t_j^l(d_j^2 - d_j)\Big) \\
&= d(l)\sigma_{s,l} - m(l)^s e(l),
\end{aligned}$$

where $d(l) = \sum_{j=1}^{m} t_j^l d_j$ and $e(l) = \sum_{j=1}^{m} 2t_j^l(d_j^2 - d_j)$. It follows that $\sigma_{p,l} = d(l)^p$ and for each positive integer n,

$$\sigma_{p+n,l} \geq d(l)^{p+n} - e(l)m(l)^p \sum_{i=0}^{n-1} m(l)^{n-1-i} d(l)^i.$$

Hence we get for each $l \in \mathbb{N}$ and $n \in \mathbb{N}$,

$$\frac{d(l)^{p+l} - \sigma_{p+n,l}}{d(l)^{p+n}} \leq e(l)\Big(\frac{m(l)}{d(l)}\Big)^p \frac{1}{d(l)} \sum_{i=1}^{n-1} \Big(\frac{m(l)}{d(l)}\Big)^i. \tag{3.2}$$

We have

$$\frac{m(l)}{d(l)} = \sum_{j=1}^{m} a_{j,l'} \frac{1}{d_j} \to \sum_{j=1}^{m} a_j \frac{1}{d_j} < 1, \tag{3.3}$$

$$\frac{e(l)}{d(l)} = \sum_{j=1}^{m} a_{j,l'} 2(d_j^2 - d_j) \to \sum_{j=1}^{m} a_j 2(d_j^2 - d_j), \tag{3.4}$$

as $l \to \infty$. By (3.3), we can assume that there exists a number δ with $0 < \lambda < 1$ such that for each $l \in \mathbb{N}$,

$$\frac{m(l)}{d(l)} < \lambda. \tag{3.5}$$

Now let $\epsilon > 0$ be arbitrary small positive number. From (3.2), (3.4) and (3.5), we get that there exists a positive integer p such that for each simply connected domain U satisfying $U \subset \overline{\mathbb{C}} \setminus Z_p$, the number $\sigma_{p+n,l} = \sigma_{p+n,l}(U)$ satisfies that

$$\frac{d(l)^{p+n} - \sigma_{p+n,l}}{d(l)^{p+n}} \leq \epsilon, \tag{3.6}$$

for each $l \in \mathbb{N}$ and $n \in \mathbb{N}$.

Let d_{Σ_m} be a fixed metric in Σ_m and $d_{\overline{\mathbb{C}}}(\,,\,)$ the spherical metric on $\overline{\mathbb{C}}$. Let $\tilde{d}(\,,\,)$ be the metric on $\Sigma_m \times \overline{\mathbb{C}}$ defined by: $\tilde{d}((w', y'), (w, y)) = \max\{d_{\Sigma_m}(w', w),\ d_{\overline{\mathbb{C}}}(y', y)\}$.

Let δ be a number in Lemma 3.2.5. Let $K' = \overline{B(J(G), \frac{1}{2}\delta)}$. Let $(w, x) \in \Sigma_m \times \overline{\mathbb{C}}$ be a point such that $x \in \pi_2(K) \cap K' \setminus Z_p$. We can easily see that there exists a positive number δ_1 such that if $\tilde{d}(z, z') < \delta_1$, $z, z' \in K$ and $\pi_2(z) = \pi_2(z')$, then

$$|\tilde{B}_{a(l)}^n \varphi(z) - \tilde{B}_{a(l)}^n \varphi(z')| < \epsilon, \tag{3.7}$$

for each $l \in \mathbb{N}$ and $n \in \mathbb{N}$. Hence by Lemma 3.2.5, (3.6) and (3.7), we get that if we take δ_2 so small then for each $(w', x') \in K$ with $\tilde{d}((w, x), (w', x')) < \delta_2$, we have

$$\begin{aligned}
&|\tilde{B}_{a(l)}^n \varphi((w, x)) - \tilde{B}_{a(l)}^n \varphi((w', x'))| \\
\leq\ &|\tilde{B}_{a(l)}^n \varphi((w, x)) - \tilde{B}_{a(l)}^n \varphi((w, x'))| + |\tilde{B}_{a(l)}^n \varphi((w, x')) - \tilde{B}_{a(l)}^n \varphi((w', x'))| \\
\leq\ &\epsilon + 2M\epsilon + \epsilon = \epsilon(2 + 2M),
\end{aligned} \tag{3.8}$$

where $M = \sup_{z \in K} |\varphi(z)|$, for each $l \in \mathbb{N}$ and $n \in \mathbb{N}$.

Now, let $z \in K$ be any point. By Proposition 2.4.5.6, there exists a positive integer τ such that for each $y \in K$, we have

$$\pi_2(\tilde{f}^{-\tau}(y)) \cap (\pi_2(K) \cap K' \setminus Z_p) \neq \emptyset. \tag{3.9}$$

For each $l \in \mathbb{N}$, we set

$$\beta(l) = \min_{(w_1,\dots,w_\tau)\in\{1,\dots,m\}^\tau} t^l_{w_1}\cdots t^l_{w_\tau}. \tag{3.10}$$

Then we have

$$0 < (\min_{j=1,\dots,m} \frac{a_{w_j}}{d_{w_j}})^\tau \le \liminf_{l\to\infty} \frac{\beta(l)}{d(l)^\tau}, \tag{3.11}$$

$$\limsup_{l\to\infty} \frac{\beta(l)}{d(l)^\tau} \le (\max_{j=1,\dots,m} \frac{a_{w_j}}{d_{w_j}})^\tau < 1. \tag{3.12}$$

Hence we can assume that there exist constants c_1 and c_2 such that for each $l \in \mathbb{N}$,

$$0 < c_1 \le \frac{\beta(l)}{d(l)^\tau} \le c_2 < 1. \tag{3.13}$$

For each $l \in \mathbb{N}$, let $\iota_l : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_{m(l)} \times \overline{\mathbb{C}}$ be an natural embedding and $\pi^l : \Sigma_{m(l)} \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ the natural projection. For each $l \in \mathbb{N}$ and $n \in \mathbb{N}$, let $S_{n,l}$ be the set of solution of $\tilde{f}_l^{\tau n}(z') = \iota_l(z)$ and $\sharp S_{n,l}$ the cardinality counting multiplicity. Let $S_{1,l,1}$ be a subset of $S_{1,l}$ such that the second projection of each point of the set belongs to $\pi_2(K) \cap K' \setminus Z_p$ and $\sharp S_{1,l,1} = \beta(l)$. And let $S_{1,l,2} = S_{1,l} \setminus S_{1,l,1}$. Inductively, for each $n \ge 1$, let $S_{n+1,l,1}$ be a set of backward images of $S_{n,l,2}$ by $\tilde{f}_l^\tau$ such that the second projection of each point of the set belongs to $\pi_2(K) \cap K' \setminus Z_p$ and $\sharp S_{n+1,l,1} = \beta(l) \sharp S_{n,l,2}$ where the cardinalities are counted considering multiplicity. And let $S_{n+1,l,2} = \tilde{f}^{-\tau}(S_{n,l}) \setminus S_{n+1,l,1}$. Then inductively we can see that for each $n \in \mathbb{N}$,

$$\sharp S_{n,l,2} = (d(l)^\tau - \beta(l))^{n-1} \tag{3.14}$$

and

$$\sharp S_{n,l,1} = (d(l)^\tau - \beta(l))^{n-2} \beta(l). \tag{3.15}$$

By (3.13), there exists a positive integer N such that for each $l \in \mathbb{N}$,

$$(\frac{d(l)^\tau - \beta(l)}{d(l)^\tau})^N < \epsilon. \tag{3.16}$$

By (3.7), there exists a number $\eta > 0$ such that for each $n \in \mathbb{N}, l \in \mathbb{N}$ and $j = 1, \dots N$, if $z' \in \pi^l(S_{j,l,1})$ and $\tilde{d}(z', x)$, $\tilde{d}(z', y) < \eta$,

$$|\tilde{B}^n_{a(l)}(\varphi)(x) - \tilde{B}^n_{a(l)}(\varphi)(y)| < 2\epsilon(2 + 2M). \tag{3.17}$$

By (3.14), (3.15), (3.16) and (3.17), we can see that if we take $\delta_2 > 0$ small enough then $\tilde{d}(z, z') < \delta_2$, $z' \in K$ implies that for each $n \in \mathbb{N}$ and $l \in \mathbb{N}$,

$$\begin{aligned} &|\tilde{B}^{n+\tau N}_{a(l)}(\varphi)(z) - \tilde{B}^{n+\tau N}_{a(l)}(\varphi)(z')| \\ \le\ & d(l)^{-N\tau} \sum_{i=1}^{d(l)^{\tau N}} |\tilde{B}^n_{a(l)}(\varphi)(\pi^l(z_i)) - \tilde{B}^n_{a(l)}(\varphi)(\pi^l(z'_i))| \\ \le\ & \frac{1}{d(l)^{N\tau}} (\sum_{j=1}^{N} \frac{\sharp S_{j,l,1}}{d(l)^{N\tau - j\tau}} 2\epsilon(2 + 2M) + \sharp S_{N,l,2} 2M) \\ =\ & (\sum_{j=1}^{N} (\frac{d(l)^\tau - \beta(l)}{d(l)^\tau})^{j-1} \frac{\beta(l)}{d(l)^\tau} \cdot 2\epsilon(2 + 2M)) + 2M\epsilon, \end{aligned}$$

where on the above we set $\{z_1, \dots z_b\} = S_{\tau N,l}$, $b = d(l)^{\tau N}$ and we denoted by z'_i the point of $\tilde{f}_l^{-\tau N}(\iota_l(z'))$ corresponding to z_i. By (3.13), there exists a constant $C > 0$, not depending on N, such that for each $l \in \mathbb{N}$,

$$\sum_{j=1}^{N} (\frac{d(l)^\tau - \beta(l)}{d(l)^\tau})^{j-1} \frac{\beta(l)}{d(l)^\tau} \le C. \tag{3.18}$$

Hence we get

$$|\tilde{B}^{n+\tau N}_{a(l)}(\varphi)(z) - \tilde{B}^{n+\tau N}_{a(l)}(\varphi)(z')| \le \epsilon(4C + 4MC + 2M). \tag{3.19}$$

Letting $l \to \infty$, we get that for each $n \in \mathbb{N}$,

$$|\tilde{B}^{n+\tau N}_a(\varphi)(z) - \tilde{B}^{n+\tau N}_a(\varphi)(z')| \le \epsilon(4C + 4MC + 2C). \tag{3.20}$$

Thus we have proved the lemma. □

Proof. of Theorem 3.2.3. By Corollary 3.2.2, Lemma 3.2.6 and Lemma 3.2.7 we can show the statement about convergence of the operator and that the support of $\tilde{\mu}_a$ is included in $\tilde{J}$ in the same way as that in [L]. Since $\tilde{\mu}_a$ is $\tilde{B}^*_a$-invariant and $\inf_{z\in\tilde{J}} \tilde{\psi}_a(z) > 0$, by Proposition 2.4.5.6, we can show that the support of $\tilde{\mu}_a$ is equal to $\tilde{J}$ immediately. It implies that the support of μ_a is equal to $J(G)$. □

Lemma 3.2.8. *Under the same assumption as Theorem 3.2.3, for any $a \in \mathcal{W}$ with $a \neq 0$, we have μ_a is non-atomic.*

Proof. We set for each $n \in \mathbb{N}, l \in \mathbb{N}$ and $z \in J(G)$,

$$c(n,l)(z) = \sum_{\alpha\in\{1,\dots,m(l)\}^n,\ g_{\alpha_1}\circ\cdots\circ g_{\alpha_n}(z)\in J(G)} (\text{mul } (g_{\alpha_1} \circ \cdots \circ g_{\alpha_n}) \text{at } z), \tag{3.21}$$

where we denote by g_{α_j} any element of $\{g^l_{i,j}\}$ and mul denotes the multiplicity. We will show the following claim.

Claim 1. for any $z \in J(G)$, there exists an open neighborhood $U(z)$ of z and a word $(w_1(z), \dots, w_2(z)) \in \{1, \dots, m\}^2$ such that for each $y \in U(z)$,

$$(\text{mul } (f_{w_2(z)} \circ f_{w_1(z)}) \text{at } y) < d_{w_2(z)} d_{w_1(z)}.$$

Suppose there exists a point $z \in J(G)$ such that for each $(w_2, w_1) \in \{1, \dots, m\}^2$,

$$\text{mul } (f_{w_2} \circ f_{w_1}) \text{at } z = d_{w_2} d_{w_1}.$$

For each $j = 1, \dots m$, we set $z_j = f_j(z)$. We can assume that there exists a positive integer t with $1 \le t \le m$ such that $d_1, \dots, d_t \ge 2$ and $d_{t+1} = \dots = d_m = 1$.

If there exists an integer i such that $z \ne z_i$ then for each integer s with $1 \le s \le t$, mul f_s at z and at z_i are equal to d_s. Hence, conjugating G by some Möbius transformation, we can assume that $z = 0$, $z_s = \infty$, $f_s(z) = \frac{1}{z^{d_s}}$ for each s with $1 \le s \le t$ and $z_{t+1}, \dots, z_m \in \{0, \infty\}$. It implies $z \in E(G)$ but this contradicts to the assumption $E(G) \subset F(G)$.

If $z = z_i$ for each $i = 1, \dots, m$, then conjugating G by some Möbius transformation, we can assume that $z = \infty$ and $f_1, \dots, f_m$ are polynomials. It contradicts to $E(G) \subset F(G)$. Hence the claim 1. holds.

From claim 1, there exists a finite collection $U(x_1), \dots, U(x_k)$ with $\cup_{j=1}^k U(x_j) \supset J(G)$ where $x_1, \dots x_k \in J(G)$ such that for each $j = 1, \dots, k$, there exists a word $(w_2(x_j), w_1(x_j)) \in \{1, \dots m\}^2$ satisfying that for each $y \in U(x_j)$,

$$(\text{mul } (f_{w_2(x_j)} \circ f_{w_1(x_j)}) \text{at } y) < d_{w_2(x_j)} d_{w_1(x_j)}.$$

We set

$$c = \min_{j=1,\dots,k} \min_{y \in U(x_k)} (d_{w_2(x_j)} d_{w_1(x_j)} - (\text{mul } f_{w_2(x_j)} \circ f_{w_1(x_j)} \text{at } y)) > 0.$$

We get for each $z \in J(G)$ and $l \in \mathbb{N}$,

$$c(2, l)(z) \le d(l)^2 - (\min_{j=1,\dots,m} t^l_j)^2 c.$$

Hence for each $n \in \mathbb{N}, l \in \mathbb{N}$ and $z \in J(G)$,

$$\frac{c(2n, l)(z)}{d(l)^{2n}} \le (\frac{d(l)^2 - (\min_{j=1,\dots,m} t^l_j)^2 c}{d(l)^2})^n. \tag{3.22}$$

Let $\epsilon > 0$ be any small number. And fix $z \in J(G)$. By (3.22), there exists a positive integer n_0 such that for each $l \in \mathbb{N}$,

$$\frac{c(2n_0, l)(z)}{d(l)^{2n_0}} \le \epsilon. \tag{3.23}$$

Take $\zeta \in J(G)$. For each $l \in \mathbb{N}$ and $n \in \mathbb{N}$, we set

$$\mu^\zeta_{l,n} = \frac{1}{d(l)^n} \sum_{\alpha \in \{1,\dots,m(l)\}^n} \sum_{y \in (g_{\alpha_1} \circ \dots \circ g_{\alpha_n})^{-1}(\zeta)} \delta_y,$$

where δ_y denotes the dirac measure concentrated at y and g_k denotes the k-th element of $\{g^l_{i,j}\}_{i,j}$. Note that by Theorem 3.2.3, $\mu^\zeta_{l,n} \to \mu_{a(t^l)}$ weakly as $n \to \infty$. There exists an open neighborhood U of z such that if we set

$$c'(2n_0, l)(U) = \sum_{\alpha \in \{1,\dots,m(l)\}^{n_0},\ g_{\alpha_1} \circ \dots \circ g_{\alpha_{n_0}}(z) \in J(G)} \deg(g_{\alpha_1} \circ \dots \circ g_{\alpha_{n_0}}|_U),$$

then we have $c'(2n_0, l)(U) = c(2n_0, l)(z)$. Hence by (3.23), we get that for each $n \in \mathbb{N}$ and $l \in \mathbb{N}$,

$$\mu^\zeta_{l,2n_0+n}(U) \le \frac{d(l)^n c'(2n_0, l)(U)}{d(l)^{2n_0+n}} \le \epsilon,$$

Letting $n \to \infty$, since we can assume that $\mu_{a(t^l)}(\partial U) = 0$ for each $l \in \mathbb{N}$, we get for each $l \in \mathbb{N}$,

$$\mu_{a(t^l)}(U) \le \epsilon. \tag{3.24}$$

By the uniqueness of the self-similar measure with respect to the weight a, we have $\mu_{a(t^l)} \to \mu_a$ weakly as $l \to \infty$. Since we can assume $\mu_a(\partial U) = 0$, by (3.24), we get

$$\mu_a(U) \le \epsilon.$$

Since ϵ can be taken arbitrary small, we get $\mu_a(\{z\}) = 0$. Hence μ_a is non-atomic. □

3.3 entropy of $\tilde{f}$

Lemma 3.3.1. *Under the same assumption as Theorem 3.2.3, let $\tilde{\mu}_a$ be the self-similar measure with respect to the weight $a \in \mathcal{W}$. Then $\tilde{\mu}_a$ is $\tilde{f}$-invariant and*

1. $(\tilde{f}, \tilde{\mu}_a)$ *is exact.*

2. $h_{\tilde{\mu}_a}(\tilde{f}) \geq H(\epsilon|(\tilde{f})^{-1}\epsilon) = -\sum_{j=1}^m a_j \log a_j + \sum_{j=1}^m a_j \log d_j$, *where we denote by* ϵ *the partition of* $\Sigma_m \times \overline{\mathbb{C}}$ *into one point subsets.*

Proof. By Theorem 3.2.3, the measure $\tilde{\mu}_a$ is $\tilde{B}_a^*$-invariant. Hence for each $\varphi \in C(\Sigma_m \times \overline{\mathbb{C}})$,

$$\int \varphi \circ \tilde{f} d\tilde{\mu} = \int \tilde{B}_a(\varphi \circ \tilde{f}) d\tilde{\mu} = \int \varphi d\tilde{\mu}.$$

Hence $\tilde{\mu}_a$ is $\tilde{f}$-invariant.

Let ν_z denote the conditional measure on the element of partition $\tilde{f}^{-1}\epsilon$ containing $z \in \Sigma_m \times \overline{\mathbb{C}}$with respect to the measure $\tilde{\mu}_a$. Then by Theorem 3.2.3 and using the same argument as that in p366-367 in [L], we can show that

$$\nu_z = \sum_{j=1}^m \frac{a_j}{d_j} \sum_{\zeta \in \tilde{f}^{-1}\tilde{f}(z) \cap \Sigma_{m,j}} \delta_\zeta, \tag{3.25}$$

where $\Sigma_{m,j} = \{w \in \Sigma_m \mid w_1 = j\}$. By Theorem 3.2.3 and (3.25), using the same argument as that in P367 in [L] again, we can show that $(\tilde{f}, \tilde{\mu}_a)$ is exact.

By Lemma 3.2.8, we have $\pi_{2*}\tilde{\mu}_a$ is non-atomic. In particular,

$$\tilde{\mu}_a(\text{cv}(\tilde{f})) = 0. \tag{3.26}$$

By (3.25) and (3.26), we get that

$$I(\epsilon|\tilde{f}^{-1}\epsilon)(z) = -\sum_{j=1}^m d_j \cdot \frac{a_j}{d_j} \log \frac{a_j}{d_j} = -\sum_{j=1}^m a_j \log \frac{a_j}{d_j}, \tag{3.27}$$

for $\tilde{\mu}$-almost all $z \in \Sigma_m \times \overline{\mathbb{C}}$. Hence

$$H(\epsilon|\tilde{f}^{-1}\epsilon) = \int I(\epsilon|\tilde{f}^{-1}\epsilon)(z) d\tilde{\mu}(z) = -\sum_{j=1}^m a_j \log \frac{a_j}{d_j}.$$

□

Now we will estimate the topological entropy of $\tilde{f}$ from above.

Theorem 3.3.2. *Let* $G = \langle f_1, \dots, f_m \rangle$ *be a rational semigroup and* $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ *the skew product map constructed by the generator system* $\{f_1, \dots, f_m\}$. *Then the toplogical entropy* $h(\tilde{f})$ *on* $\Sigma_m \times \overline{\mathbb{C}}$ *satifies that*

$$h(\tilde{f}) \leq \log(\sum_{j=1}^m \deg f_j).$$

To prove this theorem, we need several lemmas.

The first one is the *Ruelle's inequality* for skew product maps. Let X be a compact metric space and M a compact C^∞ manifold. Let $f : X \times M \to X \times M$ be a continuous map such that $f(x,y) = (\sigma(x), g_x(y))$ where $\sigma : X \to X$ is a continuous map, $g_x : M \to M$ is a differential map for each $x \in X$. Let $D_y g_x : T_y M \to T_{g_x(y)} M$ be the linear map induced by g_x. Assume that $(x, y) \mapsto D_y g_x$ is continuous. For each positive integer n and $(x, y) \in X \times M$, we define $D_{(x,y)} f^n : T_y M \to T_{\pi_2(f^n(x,y))} M$ as $v \mapsto D(g_{\sigma^n(x)} \circ \dots \circ g_x)(v)$. Then we get the following result by a slight modification of Theorem 2. in [Ru].

Lemma 3.3.3. *Under the above, let* ρ *be an* f*-invariant probability measure on* $X \times M$. *Then,*

1. *there exists a Borel set* Ω *in* $X \times M$ *such that* $\rho(\Omega) = 1$ *and for each* $(x, y) \in \Omega$ *the following holds. There is a strictly increasing sequence of subspaces:*

$$0 = V_{x,y}^{(0)} \subset V_{x,y}(1) \subset \dots \subset V_{x,y}^{(s(x,y))} = T_y M$$

such that, for $r = 1, \dots, s(x,y)$,

$$\lim_{n \to \infty} \frac{1}{n} \log \|D_{(x,y)} f^n u\| = \lambda_{x,y}^{(r)} \text{ if } u \in V_{x,y}^{(r)} \setminus V_{x,y}^{(r-1)}$$

and $\lambda_{x,y}^{(1)} < \lambda_{x,y}^{(2)} < \dots < \lambda_{x,y}^{(s(x,y))}$ *: here we may have* $\lambda_{x,y}^{(1)} = -\infty$. *The* $V_{x,y}^{(r)}$ *and* $\lambda_{x,y}^{(r)}$ *are uniquely defined with these properties and independent of the choice of the Riemannian metric on* M. *The maps* $(x, y) \mapsto s(x, y)$, $(V_{x,y}^{(1)}, \dots, V_{x,y}^{(s(x,y))})$, $(\lambda_{x,y}^{(1)}, \dots, \lambda_{x,y}^{(s(x,y))})$ *are Borel.*

2. *Let* $m_{x,y}^{(r)} = \dim V_{x,y}^{(r)} - \dim V_{x,y}^{(r-1)}$ *for* $r = 1, \dots, s(x,y)$ *and define*

$$\lambda_+(x,y) = \sum_{r : \lambda_{x,y}^{(r)} > 0} m_{x,y}^{(r)} \lambda_{x,y}^{(r)}.$$

Then, the metric entropy $h_\rho(f)$ *of* (f, ρ) *satisfies that*

$$h_\rho(f) \leq \chi_\rho(f) + h_{(\pi_1)_*\rho}(\sigma),$$

where $\chi_\rho(f) = \int \lambda_+(x,y) d\rho(x,y)$.

Corollary 3.3.4. *Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated rational semigroup and $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ the skew product map constructed by the generator system $\{f_1, \dots, f_m.\}$ Let ρ be an $\tilde{f}$-invariant probability measure on $\Sigma_m \times \overline{\mathbb{C}}$. Then we have*

$$h_\rho(\tilde{f}) \leq 2\max\{0, \int_{\Sigma_m \times \overline{\mathbb{C}}} \lim_{n\to\infty} \frac{1}{n} \log \|(\tilde{f}^n)'(z)\| d\rho(z)\} + h_{(\pi_1)_*\rho}(\sigma).$$

Let ρ be an $\tilde{f}$-invariant probability measure on $\Sigma_m \times \overline{\mathbb{C}}$. As in p108 in [Par], there exists a ρ-integrable function $J_\rho : \Sigma_m \times \overline{\mathbb{C}} \to [1, \infty)$ such that

$$\rho(\tilde{f}(A)) = \int_A J_\rho(z) d\rho(z),$$

for any Borel set A in $\Sigma_m \times \overline{\mathbb{C}}$ such that $\tilde{f}_{|A}$ is injective. Now we will generalize some Mañé's results([Ma1]), using the methods in [Ma1] and Corollary 3.3.4.

Lemma 3.3.5. *Let ρ be an $\tilde{f}$-invariant ergodic probability measure on $\Sigma_m \times \overline{\mathbb{C}}$ with $h_\rho(\tilde{f}) > h_{(\pi_1)_*\rho}(\sigma)$. Then the function $z \mapsto \log \|\tilde{f}'(z)\|$ is ρ-integrable and*

$$\int_{\Sigma_m \times \overline{\mathbb{C}}} \log \|\tilde{f}'(z)\| d\rho(z) \geq \frac{1}{2}(h_\rho(\tilde{f}) - h_{(\pi_1)_*\rho}(\sigma)). \tag{3.28}$$

Moreover,

$$\lim_{n\to\infty} \frac{1}{n} \log \|(\tilde{f}^n)'(z)\| = \int_{\Sigma_m \times \overline{\mathbb{C}}} \log \|\tilde{f}'(z)\| d\rho(z), \tag{3.29}$$

for ρ-almost all $z \in \Sigma_m \times \overline{\mathbb{C}}$.

Proof. $\log \|\tilde{f}'(z)\|$ is upper bounded. Since ρ is ergodic, we have either $\log \|\tilde{f}'(z)\|$ is not ρ-integrable and then

$$\lim_{n\to\infty} \frac{1}{n} \log \|(\tilde{f}^n)'(z)\| = -\infty \tag{3.30}$$

for ρ-a.e. $z \in \Sigma_m \times \overline{\mathbb{C}}$, or $\log \|\tilde{f}'\|$ is μ-integrable and :

$$\lim_{n\to\infty} \frac{1}{n} \log \|(\tilde{f}^n)'(z)\| = \int_{\Sigma_m \times \overline{\mathbb{C}}} \log \|\tilde{f}'(z)\| d\rho(z) \tag{3.31}$$

for ρ-a.e. $z \in \Sigma_m \times \overline{\mathbb{C}}$. By Corollary 3.3.4, we have (3.30) contradicts to our assumption. Hence (3.31) holds. Using again Corollary 3.3.4, we get that $\int_{\Sigma_m \times \overline{\mathbb{C}}} \log \|\tilde{f}'(z)\| d\rho(z) > 0$ and

$$h_\rho(\tilde{f}) \leq 2 \int_{\Sigma_m \times \overline{\mathbb{C}}} \log \|\tilde{f}'(z)\| d\rho(z) + h_{(\pi_1)_*\rho}(\sigma).$$

□

Corollary 3.3.6. *Let $x \in \overline{\mathbb{C}}$ be a critical point of some f_j, $j = 1, \dots, m$. We set $A = \{(w, x) \in \Sigma_m \times \overline{\mathbb{C}} \mid w_1 = j\}$. Then the function $z \mapsto \tilde{d}(z, A)$ is ρ-integrable for each ergodic $\tilde{f}$-invariant probability measure ρ with $h_\rho(\tilde{f}) > h_{(\pi_1)_*\rho}(\sigma)$.*

We set

$$\{x_1, \dots, x_b\} = \cup_{j=1}^m \mathrm{cp}(f_j),$$

where cp means the critical points. For each $j = 1, \dots, m$, we set

$$X_j = \{(w, x_j) \in \Sigma_m \times \overline{\mathbb{C}} \mid f'_{w_1}(x_j) = 0\}.$$

Then the following lemma holds.

Lemma 3.3.7. *For each k with $0 < k < 1$, there exists a continuous function τ on $\Sigma_m \times \overline{\mathbb{C}}$, a constant $C > 0$ and a constant $\alpha > 0$ such that*

1. *$\tau(z) \geq C \prod_{j=1}^b \tilde{d}(z, X_j)^\alpha$, (if $d_j = 1$ for each $j = 1, \dots, m$, then $\tau(z) \geq C$)*
2. *if $z \in (\Sigma_m \times \overline{\mathbb{C}}) \setminus \cup_{j=1}^b X_j$ and $\tilde{d}(z_1, z), \tilde{d}(z_2, z) < \tau(z)$, then*

$$d_{\overline{\mathbb{C}}}(\pi_2(\tilde{f}(z_1)), \pi_2(\tilde{f}(z_2))) \geq k\|\tilde{f}'(z)\| d_{\overline{\mathbb{C}}}(\pi_2(z_1), \pi_2(z_2)).$$

Proof. By Lemma II.5 in [Ma1] and the proof of it, for each $i = 1, \dots, m$, there exists a continuous function τ_i, a constant $C_i > 0$ and a constant $\alpha_i > 0$ such that

1. $\tau_i(x) \geq C_i \prod_{k=1}^{b_i} d_{\overline{\mathbb{C}}}(x, y_k)^{\alpha_i}$, where $y_1, \dots, y_{b_i}$ are critical points of f_i. (if $d_i = 1$, then $\tau_i(x) \geq C_i$.)
2. if $x \in \overline{\mathbb{C}}$ is not a critical point of f_i and $d_{\overline{\mathbb{C}}}(a_1, x), d_{\overline{\mathbb{C}}}(a_2, x) < \tau(x)$, then

$$d_{\overline{\mathbb{C}}}(f_i(a_1), f_i(a_2)) \geq k\|f'_i(x)\| d_{\overline{\mathbb{C}}}(a_1, a_2).$$

We set $\tau(w, x) = \tau_{w_1}(x)$ for each $(w, x) \in \Sigma_m \times \overline{\mathbb{C}}$. Then there exists a constant $C > 0$ such that 1. of our lemma holds. We can assume $\sup_{z\in\Sigma_m\times\overline{\mathbb{C}}} \tau(z) < 1$. Then we can assume that if $\tilde{d}(z_1, z_2) < \sup_{z\in\Sigma_m\times\overline{\mathbb{C}}} \tau(z)$, then $\pi_1(z_1) = \pi_1(z_2)$. By the property of τ_i, $i = 1, \dots m$, we have 2. of our lemma holds. □

We can show the following lemma using the same proof as that of Lemma 13.3 in [Ma2](with a slight modification).

Lemma 3.3.8. *Let ρ be an $\tilde{f}$-invariant probability measure on $\Sigma_m \times \overline{\mathbb{C}}$ and $\tau : \Sigma_m \times \overline{\mathbb{C}} \to [0,1)$ a function such that $\log \tau$ is a ρ-integrable function. Then there exists a measurable partition $\mathcal{P}$ of $\Sigma_m \times \overline{\mathbb{C}}$ such that $h_\rho(\tilde{f}, \mathcal{P}) < \infty$ and $\mathrm{diam}\mathcal{P}(z) \le \tau(z)$ for ρ-almost all $z \in \Sigma_m \times \overline{\mathbb{C}}$, where $\mathcal{P}(z)$ denotes the atom of $\mathcal{P}$ containing z.*

Lemma 3.3.9. *Let ρ be an $\tilde{f}$-invariant ergodic probability measure on $\Sigma_m \times \overline{\mathbb{C}}$ with $h_\rho(\tilde{f}) > h_{(\pi_1)_*\rho}(\sigma)$. Then there exists a measurable partition $\mathcal{P}$ of $\Sigma_m \times \overline{\mathbb{C}}$ such that $h_\rho(\tilde{f}, \mathcal{P}) < \infty$ and $\mathcal{P}$ is a generator for $(\tilde{f}, \rho)$ i.e. $\vee_{i=1}^{\infty} \tilde{f}^{-n}(\mathcal{P}) = \epsilon$ (mod 0) where ϵ denotes the partition of $\Sigma_m \times \overline{\mathbb{C}}$ into one point subsets.*

Proof. By Lemma 3.3.5, there exists a constant k with $0 < k < 1$ such that for ρ-almost all $z \in \Sigma_m \times \overline{\mathbb{C}}$,

$$\lim_{n\to\infty} \frac{1}{k^n} \|(\tilde{f}^n)'(z)\|^{-1} = 0. \tag{3.32}$$

For this k, take $\tau : \Sigma_m \times \overline{\mathbb{C}} \to [0,1)$ in Lemma 3.3.7. By Lemma 3.3.6 and Lemma 3.3.7, we have $\log \tau$ is ρ-integrable. By Lemma 3.3.8, we get that there exists a measurable partition $\mathcal{P}$ on $\Sigma_m \times \overline{\mathbb{C}}$ such that $h_\rho(\tilde{f}, \mathcal{P}) < \infty$ and $\mathrm{diam}\mathcal{P}(z) \le \tau(z)$ for ρ-almost all $z \in \Sigma_m \times \overline{\mathbb{C}}$. We will show that $\mathcal{P}$ is a generator for $(\tilde{f}, \rho)$. For each $n \in \mathbb{N}$, let $\mathcal{P}_n = \vee_{i=0}^{n} \tilde{f}^{-n}(\mathcal{P})$. It is sufficient to show that

$$\lim_{n\to\infty} \mathrm{diam}\ \mathcal{P}_n(z) = 0 \tag{3.33}$$

for ρ-almost all $z \in \Sigma_m \times \overline{\mathbb{C}}$. Let $z_i \in \mathcal{P}_n(z)$, $i = 1, 2$. Then $\tilde{f}^j(z_i) \in \mathcal{P}(\tilde{f}^j(z))$, $i = 1, 2$, for all $j = 1, \ldots n$. Since $\mathrm{diam}\ \mathcal{P}(\tilde{f}^j(z)) \le \tau(\tilde{f}^j(z))$, $j = 1, \ldots n$, we have

$$d_{\overline{\mathbb{C}}}(\pi_2 \tilde{f}^j(z_1),\ \pi_2 \tilde{f}^j(z_2)) \ge k \|\tilde{f}'(\tilde{f}^{j-1}(z))\| d_{\overline{\mathbb{C}}}(\pi_2 \tilde{f}^{j-1}(z_1),\ \pi_2 \tilde{f}^{j-1}(z_2)),$$

for each $j = 1, \ldots, n$. Hence we get

$$d_{\overline{\mathbb{C}}}(\pi_2 \tilde{f}^n(z_1),\ \pi_2 \tilde{f}^n(z_2)) \ge k^n \|(\tilde{f}^n)'(z)\| d_{\overline{\mathbb{C}}}(\pi_2(z_1),\ \pi_2(z_2)).$$

Let C be the diameter of $\overline{\mathbb{C}}$. We get

$$d_{\overline{\mathbb{C}}}(\pi_2(z_1),\ \pi_2(z_2)) \le C \cdot \frac{1}{k^n} \|(\tilde{f}^n)'(z)\|^{-1}. \tag{3.34}$$

Hence

$$\mathrm{diam}\ \pi_2(\mathcal{P}_n(z)) \le C \cdot \frac{1}{k^n} \|(\tilde{f}^n)'(z)\|^{-1}. \tag{3.35}$$

We can assume that for each $i = 1, \ldots m$, the set $Y_i = \{(w, x) \in \Sigma_m \times \overline{\mathbb{C}} \mid w_1 = i\}$ is a union of atoms of $\mathcal{P}$. Hence by (3.32) and (3.35), we get that (3.33) holds. Thus we have proved the lemma. □

Lemma 3.3.10. *Let ρ be an $\tilde{f}$-invariant ergodic probability measure on $\Sigma_m \times \overline{\mathbb{C}}$ with $h_\rho(\tilde{f}) > h_{(\pi_1)_*\rho}(\sigma)$. Then*

$$h_\rho(\tilde{f}) = \int_{\Sigma_m \times \overline{\mathbb{C}}} \log J_\rho(z) d\rho(z) = \int_{\Sigma_m \times \overline{\mathbb{C}}} I(\epsilon | \tilde{f}^{-1}(\epsilon))(z) d\rho(z).$$

Proof. By Lemma 3.3.9, there exists a generator $\mathcal{P}$ with $h_\rho(\tilde{f}, \mathcal{P}) < \infty$. By Remark 8.10 and Lemma 10.5 in [Par], we get $h_\rho(\tilde{f}) = \int_{\Sigma_m \times \overline{\mathbb{C}}} \log J_\rho(z) d\rho(z)$. □

Proof. of Theorem 3.3.2 Suppose $h(\tilde{f}) \le \log m$. Then we have nothing to do. Suppose $h(\tilde{f}) > \log m$. Let ρ be any $\tilde{f}$-invariant ergodic probability measure on $\Sigma_m \times \overline{\mathbb{C}}$ with $h_\rho(\tilde{f}) > \log m$. Then since $h(\sigma) = \log m$, by variational principle we get

$$h_\rho(\tilde{f}) > h_{(\pi_1)_*\rho}(\sigma).$$

By Lemma 10.5 in [Par] and Lemma 3.3.10, we have $I(\epsilon | \tilde{f}^{-1}\epsilon)(z) = \log J_\rho(z)$ and $h_\rho(\tilde{f}) = \int_{\Sigma_m \times \overline{\mathbb{C}}} \log J_\rho(z) d\rho(z)$. Since $\tilde{f}$ is a $d:1$ map where $d = \sum_{j=1}^m \deg(f_j)$, we have $I(\epsilon | \tilde{f}^{-1}\epsilon)(z) \le \log(\sum_{j=1}^m \deg(f_j))$. Hence we get

$$h_\rho(\tilde{f}) \le \log(\sum_{j=1}^m \deg(f_j)).$$

By the variational principle, we get

$$h(\tilde{f}) \le \log(\sum_{j=1}^m \deg(f_j)).$$

□

Theorem 3.3.11. *Let $G = \langle f_1, \ldots, f_m \rangle$ be a finitely generated rational semigroup. Assume that there exists an element $g_0 \in G$ of degree at least two, the exceptional set $E(G)$ for G is included in $F(G)$ and $F(H) \supset J(G)$ where H is a rational semigroup defined by $H = \{h^{-1} \mid h \in Aut\ (\overline{\mathbb{C}}) \cap G\}$.(if H is empty, put $F(H) = \overline{\mathbb{C}}$.) Let $\tilde{\mu}_a$ be the self-similar measure with respect to the weight $a \in \mathcal{W}$(See Theorem 3.2.3). Then it is $\tilde{f}$-invariant and*

$$h_{\tilde{\mu}_a}(\tilde{f}) = -\sum_{j=1}^m a_j \log a_j + \sum_{j=1}^m a_j \log d_j.$$

Also we have that $(\pi_1)_*\tilde{\mu}_a$ is the Bernoulli measure on Σ_m corresponding to the weight a. Moreover, let $\tilde{\mu}$ be the self-similar measure with respect to the *weight* $(\frac{d_1}{d}, \ldots, \frac{d_m}{d})$. *Then* $\tilde{\mu}$ *is the unique maximizing measure for* $\tilde{f}$ *and we have*

$$h(\tilde{f}) = h_{\tilde{\mu}}(\tilde{f}) = \log(\sum_{j=1}^{m} \deg(f_j)).$$

Also we have $(\tilde{f}, \tilde{\mu}_a)$ *is exact.*

Proof. By Lemma 3.3.1 and Theorem 3.3.2, we have

$$h(\tilde{f}) = h_{\tilde{\mu}}(\tilde{f}) = \log(\sum_{j=1}^{m} \deg(f_j)).$$

Now assume there exists an $\tilde{f}$-invariant probability measure ρ on $\Sigma_m \times \overline{\mathbb{C}}$ with $\tilde{\mu} \neq \rho$ and $h_\rho(\tilde{f}) = \log d$ where $d = \sum_{j=1}^{m} \deg(f_j)$. We will show it causes a contradiction. We can assume ρ is ergodic. Since there exists an element $g \in G$ with the degree at least two, we have $\log d > \log m$. Hence $h_\rho(\tilde{f}) > h_{(\pi_1)_*\rho}(\sigma)$. By Lemma 3.3.10, we have

$$h_\rho(\tilde{f}) = \int_{\Sigma_m \times \overline{\mathbb{C}}} \log J_\rho(z) d\rho(z).$$

By Lemma 10.5 in [Par], we have $I(\epsilon|\tilde{f}^{-1}\epsilon)(z) = \log J_\rho(z)$. Since $\tilde{f}$ is a $d:1$ map, we have $\log J_\rho(z) \leq \log d$ for ρ almost all $z \in \Sigma_m \times \overline{\mathbb{C}}$. Hence we get $\log J_\rho(z) = \log d$ for ρ almost all $z \in \Sigma_m \times \overline{\mathbb{C}}$. By Proposition 2.2 in [DU], we get that $\tilde{B}_a^*(\rho) = \rho$ where $a = (\frac{d_1}{d}, \ldots, \frac{d_m}{d})$ and $\tilde{B}_a$ denotes the operator on $C(\Sigma_m \times \overline{\mathbb{C}})$ defined in Section 3.2. If $E(G) = \emptyset$, then by Theorem 3.2.3, we get $\rho = \tilde{\mu}$ and this is a contradiction. Assume $E(G) \neq \emptyset$. Let V be the union of connected components of $F(G)$ having non-empty intersection with $E(G)$. Let $\varphi \in C(\Sigma_m \times \overline{\mathbb{C}})$ be any element with $\varphi(z) \geq 0$ for all $z \in \Sigma_m \times \overline{\mathbb{C}}$. Let $\epsilon > 0$ be any number. Let A_ϵ be the ϵ-open hyperbolic neighborhood in V. Then $K_\epsilon = \pi_2^{-1}(\overline{\mathbb{C}} \setminus A_\epsilon)$ is compact and backward invariant under $\tilde{f}$. Then by Theorem 3.2.3,

$$\begin{aligned}\int_{\Sigma_m \times \overline{\mathbb{C}}} \varphi(z) d\rho(z) &= \int_{\Sigma_m \times \overline{\mathbb{C}}} (\tilde{B}_a^n \varphi)(z) d\rho(z) \\ &\geq \int_{K_\epsilon} (\tilde{B}_a^n \varphi)(z) d\rho(z) \\ &\to \rho(K_\epsilon) \cdot \int_{K_\epsilon} \varphi(z) d\tilde{\mu}(z),\end{aligned}$$

as $n \to \infty$. Hence we have for each $\epsilon > 0$,

$$\int_{\Sigma_m \times \overline{\mathbb{C}}} \varphi(z) d\rho(z) \geq \rho(K_\epsilon) \cdot \int_{K_\epsilon} \varphi(z) d\tilde{\mu}(z).$$

Since $h_\rho(\tilde{f}) > h_{(\pi_1)_*\rho}(\sigma)$ and ρ is ergodic, we have $\rho(\pi_2^{-1}(E(G))) = 0$. Letting $\epsilon \to 0$, we get

$$\int_{\Sigma_m \times \overline{\mathbb{C}}} \varphi(z) d\rho(z) \geq \int_{\Sigma_m \times \overline{\mathbb{C}}} \varphi(z) d\tilde{\mu}(z).$$

It implies that $\rho \geq \tilde{\mu}$. Since ρ and $\tilde{\mu}$ are probability measures, it follows that $\rho = \tilde{\mu}$ but it is a contradiction. □

3.4 lower estimate of Hausdorff dimension of Julia sets

Now we consider a generalization of Mañé's result([Ma3]).

Theorem 3.4.1. *Let* $G = \langle f_1, f_2, \ldots f_m \rangle$ *be a finitely generated rational semigroup. Assume that the sets* $\{f_i^{-1}(J(G))\}_{j=1,\ldots,m}$ *are mutually disjoint. We define a map* $f : J(G) \to J(G)$ *by* $f(x) = f_i(x)$ *if* $x \in f_i^{-1}(J(G))$. *If* μ *is an ergodic invariant probability measure for* $f : J(G) \to J(G)$ *with* $h_\mu(f) > 0$, *then*

$$\int_{J(G)} \log(\|f'\|) \, d\mu > 0$$

and

$$HD(\mu) = \frac{h_\mu(f)}{\int_{J(G)} \log(\|f'\|) \, d\mu},$$

where we set

$$HD(\mu) = \inf\{\dim_H(Y) \mid Y \subset J(G), \ \mu(Y) = 1\}.$$

Proof. We can show the statement in the same way as [Ma3]. Note that the Ruelle's inequality([Ru]) also holds for the map $f : J(G) \to J(G)$. □

By Theorem 3.3.11 and Theorem 3.4.1, we get the following result.

Theorem 3.4.2. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. Assume that $F(H) \supset J(G)$ where $H = \{h^{-1} \mid h \in Aut(\overline{\mathbb{C}}) \cap G\}$ (if $H = \emptyset$, put $F(H) = \overline{\mathbb{C}}$.) Also assume that the sets $\{f_i^{-1}(J(G))\}_{j=1,\dots,m}$ are mutually disjoint. Then*

$$\dim_H(J(G)) \geq \frac{\log(\sum_{j=1}^m \deg(f_j))}{\int_{J(G)} \log(\|f'\|)\, d\mu},$$

where $\mu = (\pi_2)_* \tilde{\mu}_a$, $a = (\frac{d_1}{d}, \dots, \frac{d_m}{d})$ and $f(x) = f_i(x)$ if $x \in f_i^{-1}(J(G))$.

Bibliography

[Be1] A.F.Beardon, *Symmetries of Julia sets*, Bull.London.Math.Soc.**22** (1990), 576-582.

[Be2] A.F.Beardon, *Iteration of Rational Functions,* Springer-Verlag GTM132.

[Bo] D.Boyd, *Invariant measure for finitely generated rational semigroups,* to appear in Complex Variables.

[Br] H.Brolin, *Invariant sets under iteration of rational functions,* Ark.Mat.**6**(1965), 103-144.

[BR] L.Bers and H.L.Royden, *Holomorphic families of injections,* Acta.Math.**157**(1986), 259-286.

[CJY] L.Carleson, P.W.Jones and J.-C.Yoccoz, *Julia and John,* Bol.Soc.Bras.Mat.**25, N.1** 1994, 1-30.

[D] A. Douady, *Does a Julia set depend continuously on the Polynomial?,* Complex Dynamical System, The Mathematics behind the Mandelbrot and Julia Sets, (editor R. Devaney), p91-p138, Proceedings of Symposia in Applied Mathematics, vol.49, AMS.

[DU] M.Denker, M.Urbański, *On the existence of conformal measure,* Trans.Ame.Math.Soc. Vol.328, No.2, 1991.

[DU1] M.Denker, M.Urbański, *The dichotomy of Hausdorff measure and equilibrium states for parabolic rational maps.* Ergodic Theory and Related Topics III, Proceedings Güstow 1990, eds. U.Krengel, K.Richter and V.Warstat. Lecture Notes in Mathematics **1514**. Springer, Berlin, 1992, pp. 90-113.

[DU2] M.Denker, M.Urbański, *Haudorff and Conformal Measures on Julia Sets with a Rationally Indifferent Periodic Point,* J.London.Math.Soc. (2)43(1991)107-118.

[HM1] A.Hinkkanen, G.J.Martin, *The Dynamics of Semigroups of Rational Functions I*, Proc.London Math.Soc. (3)**73**(1996), 358-384.

[HM2] A.Hinkkanen, G.J.Martin, *Julia Sets of Rational Semigroups* , Math.Z. **222**, 1996, no.2, 161-169.

[HM3] A.Hinkkanen, G.J.Martin, *Some Properties of Semigroups of Rational Functions,* XVIth Rolf Nevanlinna Colloquium(Joensuu,1995), 53-58, de Gruyter, Berlin, 1996.

[J] M.Jonsson, *Dynamics of polynomial skew products on* $\mathbb{C}^2$*: exponents, connectedness and expansion* , in Thesis: *Dynamical studies in several complex variables,* Department of Mathematics Royal Institute of Technology, Stockholm, Sweden, 1997.

[KS] H.Kriete and H.Sumi, *Semihyperbolic transcendental semigroups,* preprint.

[L] M.J.Lyubich, *Entropy Properties of Rational Endomorphisms of the Riemann Sphere,* Ergod.th.&Dynam.Sys. (1983), **3**, 351-385

[LV] O.Lehto, K.I.Virtanen, *Quasiconformal Mappings in the plane,* Springer-Verlag, 1973.

[Ma1] R.Mañé, *On the uniqueness of the maximizing measure for rational maps,* Bol.Soc.Bras.Mat.Vol.14 No.1 (1983), 27-43.

[Ma2] R.Mañé, *Ergodic Theory and Differentiable Dynamics*, Springer-Verlag, 1987.

[Ma3] R.Mañé, *The Hausdorff dimension of invariant probabilities of rational maps,* Dynamical systems, Valparaiso 1986, Lecture Notes in Mathematics 1331(Springer, Berlin, 1988)86-117.

[Ma4] R.Mañé, *On a Theorem of Fatou,* Bol.Soc.Bras.Mat., Vol.24, N. 1, 1-11.1992.

[Mi] J.Milnor, *Dynamics in One Complex Variable : Introductory Lectures,*preprint, SUNY StonyBrook Institute for Mathematical Sciences, (1990).

[Mc] C.McMullen, *Complex Dynamics and Renormalization,* Princeton University Press, Princeton, New Jersey.

[MSS] R.Mañé, P.Sad, D.Sullivan. *On the dynamics of rational maps,* Ann.Sci.Ec.Norm.Sup.**16**(1983),193-217.

[MTU] S.Morosawa, M. Taniguchi and T. Ueda *A Primer on Complex Dynamics,* (Japanese version; Baihuukan, 1995) English version, in preparation.

[Par] W.Parry, *Entropy and generators in ergodic theory,* W.A.Benjamin Inc.1969.

[Pat] S.J.Patterson, *The Limit Set of a Fuchsian Group,* Acta Math. **136**(1976), 241-273.

[Re] F.Ren, Advances and problems in random dynamical systems, preprint.

[ZR] W.Zhou, F.Ren, The Julia sets of the random iteration of rational functions, Chinese Bulletin, **37**(12), 1992, 969-971.

[GR] Z.Gong, F.Ren, A random dynamical system formed by infinitely many functions, Journal of Fudan University, **35**, 1996, 387-392.

[Ru] D.Ruelle, *An inequality for the entropy,* Bol. Soc.Bras.Mat.vol.9, No.1(1978), 83-87.

[Sc] J.L.Schiff, *Normal Families,*Springer, New York, (1993).

[Se] O.Sester, Thesis: *Etude Dynamique Des Polynomes Fibres,* Universite De Paris-Sud, Centre d'Orsey, 1997.

[ST] D.Sullivan and W.P.Thurston, *Extending holomorphic motions,* Acta Math. **157**(1986), 243-258.

[St1] R.Stankewitz, *Completely Invariant Julia Sets For Polynomial Semigroups* (to appear in Proceedings of the AMS) http://www.math.tamu.edu/~richs/

[St2] R.Stankewitz, *Completely Invariant Sets of Normality For Rational Semigroups* (to appear in Complex Variables) http://www.math.tamu.edu/~richs/

[St3] R.Stankewitz, *Uniformly perfect set, rational semigroups, Kleinian groups and IFS's* preprint. http://www.math.tamu.edu/~richs/

[Sul1] D.Sullivan, *Conformal Dynamical System, in Geometric Dynamics,* Springer Lecture Notes **1007**(1983), 725-752.

[Sul2] D.Sullivan, *Quasiconformal homeomorphisms and dynamics I:Solution of the Fatou-Julia problem on wanderling domains,* Ann.of Math. **122**(1985), 401-418.

[S1] H.Sumi, *On Dynamics of Hyperbolic Rational Semigroups,* Journal of Mathematics of Kyoto University, Vol.37, No.4, 1997, 717-733.

[S2] H.Sumi, *On Hausdorff Dimension of Julia Sets of Hyperbolic Rational Semigroups,* Vol.21, No.1, pp10-28, 1998.

[S3] H.Sumi, *Dynamics of sub-hyperbolic and semi-hyperbolic rational semigroups and skew products*, preprint.

[S4] H.Sumi, *Skew product maps related to finitely generated rational semigroups,* preprint.

[Y] Y.Yin, *On the Julia Set of Semi-hyperbolic Rational Maps,* preprint.