Graph Orientations
with Set Connectivity Requirements

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Abstract

In an undirected or a directed graph, the edge-connectivity between two disjoint vertex sets X and Y is defined as the minimum number of edges or arcs that should be removed for disconnecting all vertices in Y from those in X. This paper discusses how to construct a directed graph from a given undirected graph by orienting edges so as to preserve the edge-connectivity on pairs of vertex sets as much as possible. We present several bounds on the gap between the edge-connectivities in the undirected graph and in the obtained directed graphs, which extends the Nash-Williams' strong orientation theorem.
Keywords: edge-connectivity, graph orientation, group Steiner tree packing

1   Introduction

An *orientation* $D = (V, A)$ of an undirected graph $G = (V, E)$ is a digraph obtained by replacing each undirected edge $e \in E$ with an arc from one end vertex of e to the other. In an orientation problem, we are asked whether G has an orientation satisfying given connectivity demands. This is a basic problem in combinatorial optimization, and many beautiful results have been produced so far (e.g., [1, 2, 3, 4, 5]). The main purpose of this paper is to discuss possibility to extend those results by introducing a general concept of the edge-connectivity.

Usually the edge-connectivity is defined on pairs of vertices. On the other hand, this paper deals with the edge-connectivity defined on pairs of vertex sets. Let X and Y be non-empty disjoint subsets of V, i.e., $X, Y \in 2^V$ and $X \cap Y = \emptyset$. We define the edge-connectivity $\lambda_G(X, Y)$ between X and Y in an undirected graph $G = (V, E)$ as $\min\{d_G(Z) \mid Z \in 2^V, X \subseteq Z \subseteq V - Y\}$ where $d_G(Z)$ stands for the number of edges joining vertices in Z to those in $V - Z$. Equivalently $\lambda_G(X, Y)$ is the edge-connectivity $\lambda_{G'}(x, y)$ between two vertices x and y in the graph G' obtained from G by shrinking X and Y into single vertices x and y, respectively. For a digraph D, the arc-connectivity $\lambda_D(X, Y)$ from X to Y is defined as $\min\{\delta_D(Z) \mid Z \in 2^V, X \subseteq Z \subseteq V - Y\}$ where $\delta_D(Z)$ stands for the number of arcs from vertices in Z to those in $V - Z$. $\lambda_D(X, Y)$ is also defined as the arc-connectivity $\lambda_{D'}(x, y)$ from x to y in the digraph D' obtained from D by shrinking X and Y into single vertices x and y, respectively.

The connectivity between vertex sets is a useful notion in practice. For example, let X be a set of servers providing the same service in a communication network represented by an undirected graph $G = (V, E)$, and suppose that a vertex $v \in V - X$ represents a client of the
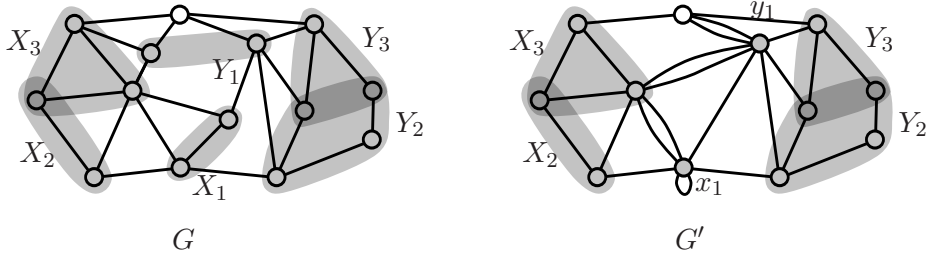
Figure 1: An undirected graph and pairs of vertex subsets

service. Then $\lambda_G(\{v\}, X)$ stands for the minimum number of links which should be broken for disconnecting the client from all servers. By such motivation, several optimization problems defined by the edge-connectivity between vertex sets are considered (e.g., graph augmentation problem [6, 7, 8], source location problem [9], and minimum cost subgraph problem [10]).

In this paper, we discuss the existence of orientations that satisfy demands defined on given pairs of vertex sets. We ask how large connectivity is necessary for guaranteeing that undirected graphs have such orientations. This is formulated as the following question. Note that $\mathbb{Z}$ denotes the set of integers, and $\mathbb{R}$ denotes the set of reals.

Question 1. *Let $G = (V, E)$ be an undirected graph, and $\{X_i, Y_i\}$ be pairs of disjoint subsets of V with connectivity demands $f_i \in \mathbb{Z}$ for $i \in \{1, 2, \ldots, \ell\}$. What is the smallest $C \in \mathbb{R}$ such that each undirected graph G with $\lambda_G(X_i, Y_i) \geq C f_i$, $i \in \{1, 2, \ldots, \ell\}$ has an orientation D with $\min\{\lambda_D(X_i, Y_i), \lambda_D(Y_i, X_i)\} \geq f_i$, $i \in \{1, 2, \ldots, \ell\}$?*

As mentioned in [5], $C \geq 2$ is necessary for satisfying the statement in Question 1 even if the connectivity demands are defined on vertex pairs. With respect to the edge-connectivity between two vertices, Nash-Williams gave the following best possible result.

Theorem 1 (Nash-Williams [5]). *Let $f : \binom{V}{2} \to \mathbb{Z}$ be a demand function, where $\binom{V}{2}$ denotes the set of unordered pairs of vertices. Every undirected graph G has an orientation D such that $\lambda_D(u, v) \geq f(u, v)$ for each $u, v \in V$ if $\lambda_G(u, v) \geq 2f(u, v)$ for each $u, v \in V$.*

Question 1 is a natural extension of Theorem 1.

One may consider that an answer to Question 1 can be derived by applying Theorem 1 to the graph obtained by shrinking vertex sets in $\{X_i, Y_i \mid i = 1, 2, \ldots, \ell\}$ into single vertices. We notice that this is not true because of the following two reasons. While we are assuming $X_i \cap Y_i = \emptyset$ for $i = 1, 2, \ldots, \ell$ in Question 1, there are possibly intersecting sets belonging to different pairs, i.e., $Z \cap Z' \neq \emptyset$ may hold for some $Z \in \{X_i, Y_i\}$ and $Z' \in \{X_j, Y_j\}$ with $i \neq j$. In addition, even if all sets in the given pairs are disjoint, shrinking a set may change the edge-connectivity of other pairs. For example, see Figure 1 illustrating a graph G with pairs $\{X_1, Y_1\}$, $\{X_2, Y_2\}$ and $\{X_3, Y_3\}$ of subsets of V, and G' obtained by shrinking X_1 and Y_1 into single vertices x_1 and y_1. Although the edge-connectivity of $\{X_2, Y_2\}$ is not changed by the shrinking ($\lambda_G(X_2, Y_2) = \lambda_{G'}(X_2, Y_2) = 4$), the edge-connectivity of $\{X_3, Y_3\}$ is ($\lambda_G(X_3, Y_3) = 4$ and $\lambda_{G'}(X_3, Y_3) = 5$).

In this paper, we prove the next theorem, which consists of two upper-bounds on C in Question 1.

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Theorem 2. *Let $G = (V, E)$ be an undirected graph, and $\{X_i, Y_i\}$ be pairs of disjoint subsets of V with connectivity demands $f_i \in \mathbb{Z}$ for $i \in \{1, 2, \ldots, \ell\}$. Let $C_1 = 2 \max\{|X_i| + |Y_i| - 1 \mid i = 1, 2, \ldots, \ell\}$ and $C_2 = 4\ell - 2$. If an undirected graph G satisfies $\lambda_G(X_i, Y_i) \geq \min\{C_1, C_2\} f_i$ for $i \in \{1, 2, \ldots, \ell\}$, then G has an orientation D with $\min\{\lambda_D(X_i, Y_i), \lambda_D(Y_i, X_i)\} \geq f_i$ for $i \in \{1, 2, \ldots, \ell\}$.*

We present the proofs for C_1 in Section 3 and for C_2 in Section 4. The former bound is derived from an observation presented in [10], and the latter one is proven by applying the theorem due to Frank [1] iteratively. In Section 4, we also show that C_2 can be improved slightly when an orientation D needs to satisfy $\lambda_D(X_i, Y_i) \geq f_i$ instead of $\min\{\lambda_D(X_i, Y_i), \lambda_D(Y_i, X_i)\} \geq f_i$ for $i \in \{1, 2, \ldots, \ell\}$. Moreover we give upper- and lower-bounds on the difference between $\min\{\lambda_D(X_i, Y_i), \lambda_D(Y_i, X_i)\}$ and $\lambda_G(X_i, Y_i)$ in Section 2.

Furthermore, we consider the case where the demand is rooted. Namely G contains a vertex $r \in V$, called root, such that $X_i = \{r\}$ for all $i \in \{1, 2, \ldots, \ell\}$, and the demand is defined on only the arc-connectivity from the root to Y_i. In this case, we have the next question similar to Question 1.

Question 2. *Let $G = (V, E)$ be an undirected graph, $r \in V$, $Y_i \subseteq V - r$ for $i \in \{1, 2, \ldots, \ell\}$, and $f_i \in \mathbb{Z}$ for $i \in \{1, 2, \ldots, \ell\}$. What is the smallest $C \in \mathbb{R}$ such that each undirected graph G with $\lambda_G(r, Y_i) \geq C f_i$, $i \in \{1, 2, \ldots, \ell\}$ has an orientation D with $\lambda_D(r, Y_i) \geq f_i$, $i \in \{1, 2, \ldots, \ell\}$?*

If some C satisfies the statement in Question 1, then it also satisfies the statement in Question 2. Moreover $C = 2$ remains best possible for Question 2 even if each Y_i, $i \in \{1, 2, \ldots, \ell\}$ is singleton. This is implied by an example where G is the union of k copies of a cycle and $|V| \geq 3$. This graph satisfies $\lambda_G(r, v) = 2k$ for each $v \in V - r$, but it has no orientation D such that $\lambda_D(r, v) \geq k + 1$ for all $v \in V - r$. In Section 5, we prove the next theorem, which answers to Question 2.

Theorem 3. *Let $G = (V, E)$ be an undirected graph, $r \in V$, $Y_i \subseteq V - r$ for $i \in \{1, 2, \ldots, \ell\}$, and $f_i \in \mathbb{Z}$ for $i \in \{1, 2, \ldots, \ell\}$. If an undirected graph G satisfies $\lambda_G(r, Y_i) \geq \ell f_i$ for $i \in \{1, 2, \ldots, \ell\}$, then G has an orientation $D = (V, A)$ with $\lambda_D(r, Y_i) \geq f_i$ for $i \in \{1, 2, \ldots, \ell\}$.*

In the rest of this introduction, let us review the difficulty of our problems from the view point of demand functions. We say that a digraph $D = (V, A)$ *covers* a demand function $h : 2^V \to \mathbb{Z}$ if $\rho_D(X) \geq h(X)$ for all non-empty $X \in 2^V$ where $\rho_D(X)$ denotes the number of arcs from vertices in $V - X$ to those in X.

Subsets X and Y of V are called *intersecting* if all of $X - Y, Y - X, X \cap Y$ are non-empty. A set function $h : 2^V \to \mathbb{Z}$ is called *intersecting G-supermodular* if $h(X) + h(Y) \leq h(X \cup Y) + h(X \cap Y) + d_G(X, Y)$ holds for each intersecting $X, Y \in 2^V$ where $d_G(X, Y)$ denotes the number of edges in G joining vertices in $X - Y$ and those in $Y - X$. If h satisfies $h(X) + h(Y) \leq h(X \cup Y) + h(X \cap Y)$ for each intersecting $X, Y \in 2^V$, then h is called *intersecting supermodular*. C8 For example, $-\rho_D$ is intersecting supermodular (see e.g., [2]).

The following theorem is due to Frank [1].

Theorem 4 (Frank [1])**.** *Let G be an undirected graph and h be an intersecting G-supermodular function (with possible negative values). There is an orientation of G covering h if and only if $d_G(\mathcal{P}) \geq \sum_{i=1}^t h(V_i)$ holds for every subpartition $\mathcal{P} = \{V_1, V_2, \ldots, V_t\}$ of V where $d_G(\mathcal{P})$ denotes the number of edges in G entering at least one member of $\mathcal{P}$.*

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This theorem is so general that it includes several known orientation theorems. However our setting is not included by this because demands on the edge-connectivity between vertex sets is not captured by intersecting G-supermodular functions. In Section 5, we observe that if Theorem 4 can be extended to skew-supermodular demand functions (defined in Section 5), then $C = 2$ satisfies the statement in Question 2.

2 Upper- and lower-bounds based on the number of odd-degree vertices

If G is Eulerian, we can obtain an orientation satisfying the demand in Question 1 with $C = 2$ by orienting edges along an Eulerian walk of G. By this fact, we can give an upper-bound following the approach taken by Nash-Williams [5] for proving Theorem 1. We let $\tau(G)$ stand for the number of odd-degree vertices in G.

Theorem 5. *Let $G = (V, E)$ be an undirected graph, and let $\{X_1, Y_1\}, \{X_2, Y_2\}, \ldots, \{X_\ell, Y_\ell\}$ be pairs of disjoint subsets of V. Then G has an orientation D such that*

$$\min\{\lambda_D(X_i, Y_i), \lambda_D(Y_i, X_i)\} \geq \lceil \lambda_G(X_i, Y_i)/2 \rceil - \tau(G)/2$$

for each $i \in \{1, 2, \ldots, \ell\}$.

Proof. Suppose that $\tau(G) = 0$. Then $d_G(X)$ is even for all $X \in 2^V$. We define D as the digraph obtained by orienting edges in G along an Eulerian walk. Since D satisfies $\rho_D(X) = d_G(X)/2$ for all $X \in 2^V$, $\min\{\lambda_D(X_i, Y_i), \lambda_D(Y_i, X_i)\} \geq \lambda_G(X_i, Y_i)/2$ for each $i \in \{1, 2, \ldots, \ell\}$.

Let us consider the case where $\tau(G) > 0$. Augment G by adding a perfect matching M on the odd-degree vertices in G. Then the obtained undirected graph $G + M$ is Eulerian (i.e., $\tau(G+M) = 0$). Hence $G+M$ has an orientation D' such that $\min\{\lambda_{D'}(X_i, Y_i), \lambda_{D'}(Y_i, X_i)\} \geq \lambda_{G+M}(X_i, Y_i)/2$ for all $i \in \{1, 2, \ldots, \ell\}$ as mentioned above. Define D as the digraph obtained by removing arcs corresponding to M from D'. Then D is an orientation of G. Since $|M| = \tau(G)/2$, $\lambda_D(X_i, Y_i) \geq \lambda_{D'}(X_i, Y_i) - \tau(G)/2$ and $\lambda_D(Y_i, X_i) \geq \lambda_{D'}(X_i, Y_i) - \tau(G)/2$ for all $i \in \{1, 2, \ldots, \ell\}$. Since $G + M$ is Eulerian, every cut has even capacity. It means that $\lambda_{G+M}(X_i, Y_i)$ is even, and thus $\lambda_{G+M}(X_i, Y_i) \geq 2\lceil \lambda_G(X_i, Y_i)/2 \rceil$ holds for all $i \in \{1, 2, \ldots, \ell\}$.

From these facts, we can derive

$$\lambda_D(X_i, Y_i) \geq \lambda_{D'}(X_i, Y_i) - \tau(G)/2$$
$$\geq \lambda_{G+M}(X_i, Y_i)/2 - \tau(G)/2 \geq \lceil \lambda_G(X_i, Y_i)/2 \rceil - \tau(G)/2$$

and

$$\lambda_D(Y_i, X_i) \geq \lambda_{D'}(Y_i, X_i) - \tau(G)/2$$
$$\geq \lambda_{G+M}(X_i, Y_i)/2 - \tau(G)/2 \geq \lceil \lambda_G(X_i, Y_i)/2 \rceil - \tau(G)/2$$

for all $i \in \{1, 2, \ldots, \ell\}$. $\qquad\square$

We also have a negative result for Question 1.

Theorem 6. *Define* $\{X_1, Y_1\}, \{X_2, Y_2\}, \ldots, \{X_\ell, Y_\ell\}$ *as all partitions of* V *into two non-empty subsets (i.e.,* $\{X_i \mid i = 1, 2, \ldots, \ell\} = \{X \in 2^V \mid 0 < |X| < |V|/2$ *and* $Y_i = V - X_i)$. *Then* G *has no orientation* D *such that*

$$\min\{\lambda_D(X_i, Y_i), \lambda_D(Y_i, X_i)\} > \lambda_G(X_i, Y_i)/2 - \tau(G)/4 \tag{1}$$

for all $i \in \{1, 2, \ldots, \ell\}$.

Proof. Suppose that G has an orientation D that satisfies (1) for all $i \in \{1, 2, \ldots, \ell\}$. Let us consider the case where at least $\tau(G)/2$ vertices in D have the in-degrees larger than the out-degrees. Let X denote the set of those vertices in G, and $E(X)$ denote the set of edges in G whose both end vertices are in X. Then X satisfies

$$\rho_D(X) = \sum_{v \in X} \rho_D(v) - E(X) \geq \sum_{v \in X} (\delta_D(v) + 1) - E(X) \geq \delta_D(X) + \tau(G)/2.$$

On the other hand, $\rho_D(X) + \delta_D(X) = d_G(X)$. By these facts, $\delta_D(X) \leq d_G(X)/2 - \tau(G)/4$ holds. Hence we have $\lambda_D(X, V - X) = \delta_D(X) \leq d_G(X)/2 - \tau(G)/4 = \lambda_G(X, V - X)/2 - \tau(G)/4$, a contradiction.

If D has at least $\tau(G)/2$ vertices having the out-degrees larger than the in-degrees, then consider the digraph D' obtained by reversing all arcs in D. By applying the above argument to D', we have a contradiction also in this case. □

From Theorem 6, we can observe that the bound in Theorem 5 is higher than the best possible by at most $\tau(G)/4$. Theorem 6 also implies that $C < \min\{4/(2 - \tau(G)/\lambda_G(X_i, Y_i)) \mid 1 \leq i \leq \ell\}$ does not satisfy the statement in Question 1 in general.

3 Proof of the bound C_1 in Theorem 2

Fukunaga and Nagamochi [10] gave the following useful relationship between the edge-connectivity between vertices and that between vertex sets.

Lemma 1 (Fukunaga, Nagamochi [10]). *Let* $\{X, Y\}$ *be a pair of disjoint subsets of* V. *If* $\lambda_G(X, Y) \geq k(|X| + |Y| - 1)$, *then there exists a pair of vertices* $x \in X$ *and* $y \in Y$ *such that* $\lambda_G(x, y) \geq k$.

From this fact, we can derive the bound C_1 in Theorem 2.

Theorem 7. *Let* $G = (V, E)$ *be an undirected graph, and let* $\{X_1, Y_1\}, \{X_2, Y_2\}, \ldots, \{X_\ell, Y_\ell\}$ *be pairs of disjoint subsets of* V *associated with connectivity demands* $f_1, f_2, \ldots, f_\ell \in \mathbb{Z}$. *If* G *satisfies* $\lambda_G(X_i, Y_i) \geq 2(|X_i| + |Y_i| - 1)f_i$ *for each* $i \in \{1, 2, \ldots, \ell\}$, *then it has an orientation* D *such that*

$$\min\{\lambda_D(X_i, Y_i), \lambda_D(Y_i, X_i)\} \geq f_i$$

for each $i \in \{1, 2, \ldots, \ell\}$.

Proof. Since $\lambda_G(X_i, Y_i) \geq 2(|X_i| + |Y_i| - 1)f_i$ holds, there exists a pair of vertices $x_i \in X_i$ and $y_i \in Y_i$ such that $\lambda_G(x_i, y_i) \geq 2f_i$ by Lemma 1. The orientation D of G given by Theorem 1 satisfies $\min\{\lambda_D(x_i, y_i), \lambda_D(y_i, x_i)\} \geq f_i$ for each $i \in \{1, 2, \ldots, \ell\}$. Since $\lambda_D(X_i, Y_i) \geq \lambda_D(x_i, y_i)$ and $\lambda_D(Y_i, X_i) \geq \lambda_D(y_i, x_i)$, D is a required orientation. □

Theorem 7 coincides with Theorem 1 when all demands are defined for vertex pairs. The bound in Theorem 7 is tight in this sense, but this does not deny possibility to improve the bound for the other case.

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4 Proof of the bound C_2 in Theorem 2

In this section, we provide the proof for bound C_2 in Theorem 2. For this, we need the next lemma.

Lemma 2. *Let $G = (V, E)$ be an undirected graph. Moreover let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_\ell, Y_\ell)$ be ordered pairs of disjoint subsets of V associated with connectivity demands $f_1, f_2, \ldots, f_\ell \in \mathbb{Z}$ such that $f_1 \leq f_2 \leq \cdots \leq f_\ell$. If G satisfies $\lambda_G(X_i, Y_i) \geq (2i - 1)f_i$ for all $i \in \{1, 2, \ldots, \ell\}$, then G has an orientation D such that $\lambda_D(X_i, Y_i) \geq f_i$ for all $i \in \{1, 2, \ldots, \ell\}$.*

Proof. Let $M_i = (V, E_i, A_i)$, $i \in \{1, 2, \ldots, \ell\}$ denote mixed graphs obtained from G by orienting some edges where E_i is the set of undirected edges and A_i is the set of arcs in M_i. We let G_i denote the undirected graph (V, E_i), and D_i denote the digraph (V, A_i). For proving Lemma 2, we show that it is possible to construct $M_1, M_2, \ldots, M_\ell$ inductively so that M_i satisfies

$$\lambda_{D_i}(X_j, Y_j) \geq f_j \text{ for } j \in \{1, 2, \ldots, i\}, \tag{2}$$

and

$$\delta_{D_i}(Z) \leq i(\rho_{D_i}(Z) + f_i) \text{ for } Y_j \subseteq Z \subseteq V - X_j \text{ with } j \in \{i + 1, i + 2, \ldots, \ell\}. \tag{3}$$

First, let us show how to construct M_1. Since $\lambda_G(X_1, Y_1) \geq f_1$, G contains at least f_1 edge-disjoint paths between X_1 and Y_1. Orient edges in the paths from X_1 to Y_1. Then the obtained mixed graph satisfies the conditions (2) and (3) for $i = 1$. In fact, the first condition $\lambda_{D_1}(X_1, Y_1) \geq f_1$ holds by the existence of arc-disjoint f_1 directed paths from X_1 to Y_1. The second condition $\delta_{D_1}(Z) \leq \rho_{D_1}(Z) + f_1$ holds because a directed path containing k arcs entering Z can have at most k arcs leaving Z if the path starts at a vertex in $V - Z$, and at most $k + 1$ arcs leaving Z otherwise.

Now suppose that we have M_i for some $1 \leq i < \ell$. We show how to construct M_{i+1} from M_i. Let $M' = (V', E', A')$ be the mixed graph obtained from M_i by shrinking X_{i+1} into a single vertex x, shrinking Y_{i+1} into a single vertex y, and deleting generated loops. Define a set function $h : 2^{V'} \to \mathbb{Z}$ so that

$$h(Z) = \begin{cases} f_{i+1} - \rho_{D'}(Z) & \text{if } y \in Z \subseteq V' - x, \\ -f_{i+1} - \rho_{D'}(Z) & \text{if } x \in Z \subseteq V' - y, \\ -\rho_{D'}(Z) & \text{otherwise.} \end{cases}$$

Then we have a helpful property of h as follows.

Claim 1. *Function h is intersecting supermodular.*

Proof. As mentioned in Section 1, $-\rho_{D'}$ is intersecting supermodular. Since intersecting supermodularity is closed under addition, it suffices to show that $h' := h + \rho_{D'}$ is intersecting supermodular. That is to say, the claim is proven if we show that for intersecting $Z, W \in 2^{V'}$ (i.e., all of $Z \cap W$, $Z - W$ and $W - Z$ are non-empty),

$$h'(Z) + h'(W) \leq h'(Z \cap W) + h'(Z \cup W) \tag{4}$$

holds. Notice that h' returns only three different values. We prove (4) by investigating cases defined according to the values of $h'(Z)$ and $h'(W)$.

Let $h'(Z) = h'(W) = f_{i+1}$. In this case, $y \in Z \cap W, Z \cup W$ and $x \notin Z \cap W, Z \cup W$ hold, which implies $h'(Z \cap W) = h'(Z \cup W) = f_{i+1}$. Hence (4) holds.

Let $h'(Z) = -f_{i+1}$ and $h'(W) = f_{i+1}$. In this case, $h'(Z \cap W) = 0$ because $x, y \notin Y \cap Z$, and $h'(Z \cup W) = 0$ because $x, y \in Y \cup Z$. Hence (4) holds.

Let $h'(Z) = 0$ and $h'(W) = f_{i+1}$. If $x, y \in Z$, then $x \notin Z \cap W$, $y \in Z \cap W$ and $x, y \in Z \cup W$ hold, which implies that $h'(Z \cap W) = f_{i+1}$ and $h'(Z \cup W) = 0$. If $x, y \notin Z$, then $x, y \notin Z \cap W$, $x \notin Z \cup W$ and $y \in Z \cup W$ hold, which implies that $h'(Z \cap W) = 0$ and $h'(Z \cup W) = f_{i+1}$. In both cases, (4) holds.

Let $h'(Z) = 0$ and $h'(W) = -f_{i+1}$. If $x, y \in Z$, then $x \in Z \cap W$, $y \notin Z \cap W$ and $x, y \in Z \cup W$ hold, which implies that $h'(Z \cap W) = -f_{i+1}$ and $h'(Z \cup W) = 0$. If $x, y \notin Z$, then $x, y \notin Z \cap W$, $x \in Z \cup W$ and $y \notin Z \cup W$ hold, which implies that $h'(Z \cap W) = 0$ and $h'(Z \cup W) = -f_{i+1}$. In both cases, (4) holds.

Let $h'(Z) = h'(W) = 0$. If $x, y \in Z$ and $x, y \in W$, then $x, y \in Z \cap W$ and $x, y \in Z \cup W$, and hence $h'(Z \cap W) = h'(Z \cup W) = 0$ holds. If $x, y \in Z$ and $x, y \notin W$, then $x, y \notin Z \cap W$ and $x, y \in Z \cup W$, and hence $h'(Z \cap W) = h'(Z \cup W) = 0$ holds. If $x, y \notin Z$ and $x, y \notin W$, then $x, y \notin Z \cap W$ and $x, y \notin Z \cup W$, and hence $h'(Z \cap W) = h'(Z \cup W) = 0$ holds. In any cases, (4) holds.

(4) holds when $h'(Z) = h'(W) = -f_{i+1}$ because $h'(Z \cap W) \geq -f_{i+1}$ and $h'(Z \cup W) \geq -f_{i+1}$. Therefore the claim is proven. $\qquad\square$

Recall that intersecting supermodular set functions on V' are intersecting G'-supermodular. Hence we can apply Theorem 4 for obtaining the following fact.

Claim 2. G' *has an orientation covering* h.

Proof. We first see that $d_{G'}(Z) \geq 2h(Z)$ holds for any $Z \subseteq V'$. It suffices to consider the case where $y \in Z$ and $x \notin Z$ because $h(Z) \leq 0$ in the other case. Notice that $d_G(Z) = d_{G'}(Z) + \rho_{D'}(Z) + \delta_{D'}(Z)$. Since $\lambda_G(X_{i+1}, Y_{i+1}) \geq (2i + 1)f_{i+1}$, $d_G(Z) \geq (2i + 1)f_{i+1}$ holds by Menger's theorem. Recall the assumption that $f_i \leq f_{i+1}$. By these and condition (3), we have

$$
\begin{aligned}
d_{G'}(Z) + (i + 1)\rho_{D'}(Z) &\geq d_{G'}(Z) + \rho_{D'}(Z) + \delta_{D'}(Z) - if_i \\
&\geq d_G(Z) - if_{i+1} \\
&\geq (i + 1)f_{i+1}.
\end{aligned}
$$

Hence $d_{G'}(Z) \geq (i + 1)(f_{i+1} - \rho_{D'}(Z)) = (i + 1)h(Z) \geq 2h(Z)$.

Let $\mathcal{P} = \{V_1, V_2, \ldots, V_t\}$ be a subpartition of V'. It then satisfies $d_{G'}(\mathcal{P}) \geq \sum_{j=1}^{t} d_{G'}(V_j)/2 \geq \sum_{j=1}^{t} h(V_j)$. This means that G' satisfies the condition presented in Theorem 4. Therefore G' has an orientation covering h. $\qquad\square$

Let D'' denote the orientation of G' covering h, and $D' + D''$ denote the digraph whose arc set consists of the arcs in D' and D''. Then $\lambda_{D'+D''}(x, y) \geq f_{i+1}$ since each $Z \subseteq V'$ with $y \in Z$ and $x \notin Z$ satisfies $\rho_{D'+D''}(Z) = \rho_{D'}(Z) + \rho_{D''}(Z) \geq \rho_{D'}(Z) + h(Z) \geq f_{i+1}$. Choose f_{i+1} arc-disjoint directed paths from x to y in $D' + D''$, and call them by $P_1, P_2, \ldots, P_{f_{i+1}}$. We denote the set of edges both in E_i and in P_j by $E(P_j)$. Define M_{i+1} as the mixed graph obtained by orienting the edges in $E(P_1) \cup E(P_2) \cup \cdots \cup E(P_{f_{i+1}})$ from x to y. In the following, we see that the constructed M_{i+1} satisfies conditions (2) and (3).

For each $j \in \{1, 2, \ldots, i\}$, $\lambda_{D_{i+1}}(X_j, Y_j) \geq f_j$ holds because $A_i \subseteq A_{i+1}$ and M_i satisfies (2). Moreover, $\lambda_{D_{i+1}}(X_{i+1}, Y_{i+1}) \geq f_{i+1}$ holds since D_{i+1} contains f_{i+1} arc-disjoint directed paths from X_{i+1} to Y_{i+1}. Hence M_{i+1} satisfies (2).

Let $Z \in 2^V$ such that $Y_j \subseteq Z \subseteq V - X_j$ for some $j \in \{i+2, i+3, \ldots, \ell\}$. Notice that each arc entering or leaving Z in $A_{i+1} - A_i$ is part of a directed path from x to y in D''. Hence $\delta_{D_{i+1}}(Z) - \delta_{D_i}(Z) \leq \rho_{D_{i+1}}(Z) + f_{i+1}$ holds. By this fact and the assumption that M_i satisfies (3), it holds that

$$
\begin{aligned}
\delta_{D_{i+1}}(Z) &= \delta_{D_{i+1}}(Z) - \delta_{D_i}(Z) + \delta_{D_i}(Z) \\
&\leq \rho_{D_{i+1}}(Z) + f_{i+1} + i(\rho_{D_i}(Z) + f_i) \\
&\leq (i+1)(\rho_{D_{i+1}}(Z) + f_{i+1}),
\end{aligned}
$$

where the last inequality follows from the properties $\rho_{D_i}(Z) \leq \rho_{D_{i+1}}(Z)$ and $f_i \leq f_{i+1}$. Therefore M_{i+1} satisfies (3). This completes the proof of Lemma 2. $\square$

The result on C_2 in Theorem 2 is obtained as follows.

Theorem 8. *Let $G = (V, E)$ be an undirected graph, and $\{X_i, Y_i\}$ be pairs of disjoint subsets of V with connectivity demands $f_i \in \mathbb{Z}$ for $i \in \{1, 2, \ldots, \ell\}$. If an undirected graph G satisfies $\lambda_G(X_i, Y_i) \geq (4\ell - 2)f_i$ for $i \in \{1, 2, \ldots, \ell\}$, then G has an orientation D with $\min\{\lambda_D(X_i, Y_i), \lambda_D(Y_i, X_i)\} \geq f_i$ for $i \in \{1, 2, \ldots, \ell\}$.*

Proof. The theorem can be proven by applying Lemma 2 for the set of ordered pairs (X_i, Y_i) and (Y_i, X_i), $i = 1, 2, \ldots, \ell$. $\square$

In addition, we would like to note that Lemma 2 gives a theorem on a slightly weaker connectivity demand.

Theorem 9. *Let $G = (V, E)$ be an undirected graph, and (X_i, Y_i) be ordered pairs of disjoint subsets of V with connectivity demands $f_i \in \mathbb{Z}$ for $i \in \{1, 2, \ldots, \ell\}$. If an undirected graph G satisfies $\lambda_G(X_i, Y_i) \geq (2\ell - 1)f_i$ for $i \in \{1, 2, \ldots, \ell\}$, then G has an orientation D with $\lambda_D(X_i, Y_i) \geq f_i$ for $i \in \{1, 2, \ldots, \ell\}$.*

5 Proof of Theorem 3

In this section, we discuss the case where the connectivity demands are defined on pairs containing the root. Theorem 3 is an immediate consequence of the following lemma.

Lemma 3. *Let $r \in V$ and $Y_1, Y_2, \ldots, Y_\ell \subseteq V - r$. An undirected graph $G = (V, E)$ has an orientation D such that $\lambda_D(r, Y_i) \geq f_i$ for all $i \in \{1, 2, \ldots, \ell\}$ if G satisfies $\lambda_G(r, Y_i) \geq if_i$ for all $i \in \{1, 2, \ldots, \ell\}$.*

Proof. Most part of the proof of Lemma 3 is same with the proof of Lemma 2. In Lemma 3, we can show that it is possible to construct mixed graphs $M_1, M_2, \ldots, M_\ell$ such that M_i satisfies (2) and

$$\delta_{D_i}(Z) \leq i\rho_{D_i}(Z) \quad \text{for } Y_j \subseteq Z \subseteq V - r \text{ with } j \in \{i+1, i+2, \ldots, \ell\}, \tag{5}$$

which improves (3). Here we mention only the different part from the proof of Lemma 2.

We consider constructing M_{i+1} from M_i satisfying (2) and (5). Define $M' = (V', E', A')$ as the mixed graph obtained from M_i by shrinking Y_{i+1} into a single vertex y. Define a set function $h : 2^{V'} \to \mathbb{Z}$ as in the proof of Lemma 2 by replacing x with r.

Let $G' = (V', E')$ and $D' = (V', A')$. If $y \in Z \subseteq V' - r$, then $d_{G'}(Z) + \rho_{D'}(Z) + \delta_{D'}(Z) = d_G(Z) \geq \lambda_G(r, Y_{i+1}) \geq (i+1)f_{i+1}$ holds. This and (5) imply

$$d_{G'}(Z) + (i+1)\rho_{D'}(Z) \geq d_{G'}(Z) + \rho_{D'}(Z) + \delta_{D'}(Z) \geq (i+1)f_{i+1},$$

and hence $G' = (V', E')$ satisfies $d_{G'}(Z) \geq (i+1)h(Z) \geq 2h(Z)$ for any Z such that $y \in Z \subseteq V' - r$. For the other $Z \in 2^{V'}$, $d_{G'}(Z) \geq 2h(Z)$ holds because $h(Z) \leq 0$. Consequently we can see that G' has an orientation covering h as in Claim 2.

From the orientation of G' covering h, define M_{i+1} in the same way with the proof of Lemma 2. The existence of f_{i+1} arc-disjoint directed paths from r to Y_{i+1} means that M_{i+1} satisfies (2). (5) is also satisfied by M_{i+1} because each of the paths starts at $r \notin Z$. □
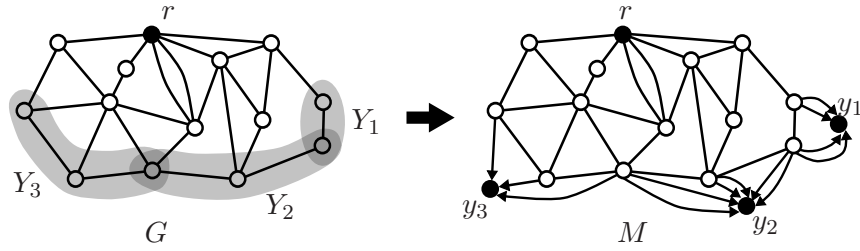

Figure 2: Construction of M from G

In the remainder of this section, we show that the orientation problem with arc-connectivity demands from r to Y_i, $i \in \{1, 2, \ldots, \ell\}$ can be reduced to the orientation problem in mixed graphs with connectivity demands from r to single vertices. Given graph $G = (V, E)$ with $r \in V$ and $Y_1, Y_2, \ldots, Y_\ell \subseteq V - r$, augment G with a new vertex y_i for each $i \in \{1, 2, \ldots, \ell\}$ and f_i parallel arcs from each $v \in Y_i$ to y_i. Let M denote the obtained mixed graph (see Figure 2). If we can orient the undirected edges in M so that the resultant digraph D satisfies $\lambda_D(r, y_i) \geq f_i$ for $i \in \{1, 2, \ldots, \ell\}$, then it gives an orientation D' of G such that $\lambda_{D'}(r, Y_i) \geq f_i$ for $i \in \{1, 2, \ldots, \ell\}$.

Unfortunately we do not know how to solve this orientation problem. If the connectivity demand is defined from r to v for all $v \in V - r$, then Theorem 4 gives a necessary and sufficient condition. However, in the above reduced problem, the connectivity demand is defined only from r to y_i, $i \in \{1, 2, \ldots, \ell\}$. This demand can not be formulated by G-supermodular functions, but by skew supermodular functions. If Theorem 4 can be extended to skew supermodular functions, then it implies that $C = 2$ is an answer to Question 2.

6   Concluding Remarks

As a concluding remark, let us mention a relationship between rooted k-arc-connectivity and tree packings.

For an undirected graph $G = (V, E)$ with a root $r \in V$ and subsets Y_i, $i \in \{1, 2, \ldots, \ell\}$ of $V - \{r\}$, a *group Steiner tree* is defined as a tree T in G spanning r and at least one vertex in Y_i for each $i \in \{1, 2, \ldots, \ell\}$. The *packing number* of group Steiner trees is defined as the maximum number of edge-disjoint group Steiner trees contained by G. Notice that if the packing number is at least k, then G has obviously an orientation D that satisfies $\lambda(r, Y_i) \geq k$ for all $i \in \{1, 2, \ldots, \ell\}$. We do not know whether its converse holds or not.

In [10], Fukunaga and Nagamochi have shown that the packing number is at least k if G satisfies $\lambda_G(r, Y_i) \geq 2k|Y_i|$ for all $i \in \{1, 2, \ldots, \ell\}$. Based on this observation, they have presented an approximation algorithm for the minimum group Steiner tree problem, which is the problem of finding a minimum cost group Steiner tree[1]. The approximation factor of their algorithm is the gap between the edge-connectivity in G and the packing number. Hence it is important to improve this gap.

Notice that the gap presented by Fukunaga and Nagamochi [10] coincides with Theorem 7. A natural question is whether Theorem 3 can be strengthened to obtain another gap between the edge-connectivity in G and the packing number. This question is formulated as the following conjecture.

Conjecture 1. *Let $G = (V, E)$ be an undirected graph with a root $r \in V$ and subsets $Y_1, Y_2, \ldots, Y_\ell \subseteq V - r$. If $\lambda_G(r, Y_i) \geq \ell k$ for all $i \in \{1, 2, \ldots, \ell\}$, then G contains k edge-disjoint group Steiner trees.*

This conjecture implies an ℓ-approximation algorithm for the group Steiner tree problem. Since the union of minimum cost paths from r to Y_i for $i \in \{1, 2, \ldots, \ell\}$ is an ℓ-approximate solution, this implication is not interesting. Nevertheless we believe that the packing of group Steiner trees itself deserves attention.

Acknowledgments

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