An explicit solution for a tandem queue with retrials and losses

Tuan Phung-Duc

Abstract We consider a retrial tandem queueing system with two servers whose service times follow two exponential distributions. There are two types of customers: type one and type two. Customers of type one arrive at the first server according to a Poisson process. An arriving customer of type one that finds the first server busy joins an orbit and retries to enter the server after some time. We assume that the arrival rate of customers from the orbit is a linear function of the number of retrial customers. After being served at the first server, a customer of type one moves to the second server. Customers of type two directly arrive at the second server according to another Poisson process. Customers of both types one and two are lost if the second server is busy upon arrival. For this model, we derive explicit expressions of the joint stationary distribution between the number of customers in the orbit and the states of the servers. We prove that the stationary distribution is computed by a numerically stable algorithm. Numerical examples are provided to show the influence of parameters on the performance of the system.

Keywords Tandem queue · retrial queue · hypergeometric function · linear retrial rate · call centers

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1 Introduction

Tandem queues have been extensively studied because they have applications on many systems, such as, telecommunications, computer networks and production systems [9,

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10, 12, 13, 17]. Latouche and Neuts [17] present algorithmic solutions to two-stage exponential tandem queues with blocking and feedback, in which there are multiple servers at each stage and customers that complete services from both stages may return to the first stage again for an additional service. Klimenok et al. [12] consider a BMAP/G/1/N $\to \cdot$ /PH/1/M tandem queue with losses due to the finite capacity of the second server. Kim et al. [13] further consider a tandem queue with feedback. Gomez-Corral [9] studies a tandem queue with two servers and a Markovian Arrival Process (MAP), where the service times of the first and the second server follow a PH (phase type) and a general distribution, respectively. Recently, Lian and Zhao [18] consider the departure processes of a tandem network with an infinite dimensional MAP input.

On the other hand, retrial queues have attracted much attention in recent years because they are widely used in modelling and performance analysis of telecommunication networks and call centers [1, 4, 5]. Retrial queues are characterized by the phenomenon that an arriving customer that sees all the servers busy, joins an orbit and retries to enter a server after some random time. Due to the lack of the homogeneity of the underlying Markov chain, analytical solutions for retrial queues are obtained in a few special cases [3, 8, 11].

In comparison with tandem queues or retrial queues, there is a lack of extensive research concerning tandem queues with retrials. Furthermore, explicit results for tandem queues with retrials are even more rarely obtained. Moutzoukis and Langaris [19] derive an analytical solution for the M/G/1/1 $\to \cdot$ /G/1/1 retrial tandem queue with a constant retrial rate. In the tandem queue in [19], if a customer that finishes the service at the first server and sees the second server busy, then the customer is kept in the first server and the first server is blocked until the second server is available again. If the second server is idle, the blocked customer in the first server immediately occupies the second server and the first server is released. While the first server is blocked, customers that arrive at this server enter the orbit. Gomez-Corral [10] proposes a matrix-geometric approximation for retrial tandem queues with blocking, in which there is a finite waiting room in the second server.

Avrachenkov and Yechiali [4] derive an analytical solution for two servers in tandem with retrials and a common orbit, in which customers that depart from the first server and then find the second server busy, also join the orbit to retry to enter the first server again after some time. The same authors in [5] further develop a fixed point approximation to analyze a more general model with multiserver in a tandem queue and a common orbit for retrial customers. It should be noted that in [4, 5], the retrial rate is assumed to be constant.

As for retrial tandem queues with losses at the second server, Kim et al. [14, 15] analyze BMAP/G/1 $\to \cdot$ /PH/1/M retrial queues. Taramin [20] investigates a tandem queue with two Markovian inputs, in which there is a single server in the first stage

and there are multiple servers in the second stage. The authors in [14,15,20] use the so-called quasi-Toeplitz Markov chains in order to analyze the models. Although the quasi-Toeplitz approach is an approximation method, it allows the authors to analyze more general models. However, to the best of our knowledge, an explicit solution for a tandem queue with retrials and losses has not been obtained in the literature yet.

In this paper, we consider a simple retrial tandem queue with losses. Customers of *type one* and *type two* arrive at the first server and the second server according to two independent Poisson processes, respectively. An arriving customer of type one either occupies the server immediately if the server is idle or if the server is busy, it enters the orbit to retry again after some time. Customers of both types are lost if the second server is busy upon the arrival epoch. For this model, we derive an explicit expression for the joint stationary distribution of the number of customers in the orbit and the states of the servers.

The rest of the paper is organized as follows. A detailed description of the model and some preliminary results are presented in Section 2. Section 3 is devoted to the presentation of the main results of this paper, in which explicit expressions for the partial generating functions and the joint stationary distribution are derived. Section 4 presents some performance measures and a computational algorithm for the stationary distribution. Section 5 provides some numerical examples.

2 Mathematical Model and Preliminaries

2.1 Model description and research background

2.1.1 Model description

We consider two servers in a tandem, whose service times are exponentially distributed with means $1/\nu_1$ and $1/\nu_2$, respectively. Type one customers arrive at the first server according to a Poisson process with rate λ. Type two customers directly arrive at the second server according to a Poisson process with rate λ^*. If the first server is busy, an arriving customer joins the orbit and retries to enter the server again at a later time. Provided that the number of retrial customers is n, the retrial rate of the customers is given by $\gamma_n = \nu(1 - \delta_{0,n}) + n\mu$, where $\delta_{0,n}$ denotes the Kronecker delta. When a customer finishes receiving the service at the first server, the customer moves to the second server. Customers of both types either occupy the second server if the server is idle or are lost if the server is busy upon arrivals. See Fig. 1 for details. For some applications, in which the retrial rate is linear, the readers are referred to the paper by Artalejo and Gomez-Corral [2].
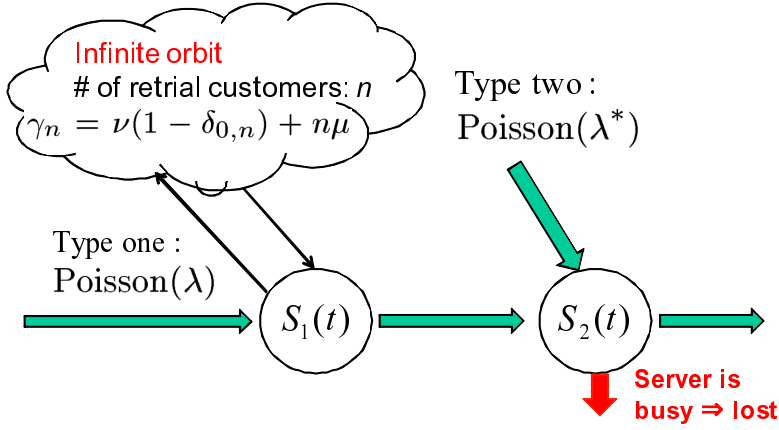
Fig. 1 Retrial tandem queue with losses.

2.1.2 Practical applications

In our everyday life, there are many situations where a service is provided in two stages for which the model of this paper can be applied. First, we consider two applications in telecommunication systems. Second, we present an application in service systems.

We consider a local area network (LAN) which is connected to a global network. In a local area network, a random access protocol such as ALOHA or CSMA is implemented. In a LAN, multiple nodes share a channel for data transmission. The channel corresponds to the first server in our model. In ALOHA and CSMA protocol, messages that are blocked at the channel are retransmitted at a later time which justifies the retrial phenomenon at the first server. The second server represents an edge node in an optical network at which the LAN connects to a global network. At the edge node, not only messages from the channel but also those from other networks arrive and a message is lost if the node is fully occupied upon arrival due to the lack of optical buffer.

Another application can be found in the performance analysis of IP telephony systems presented by Aida et al. [1], for which a control plane and a data plane can be modeled by the first and the second server in a tandem queue, respectively. The model of this paper simplifies that presented in [1], where there is a single server at the first stage and there are multiple servers in the second stage.

One more application is found in two-stage call centers [16]. A regular customer who calls to a call center is asked to input some information according to some automatic guidance. This procedure can be considered as a service at the first server. A regular customer either is served if the first server is idle or retries again after some time if the server is busy. After finishing the service at the first server, the regular customer is

forwarded to an operator that is modeled by the second server. Customers with high priority are directly forwarded to the operator. Customers of both types are lost if the operator is busy upon arrivals.

2.2 Preliminary results

Let $X(t) = (S_1(t), S_2(t), N(t))$, where $S_1(t)$, $S_2(t)$ and $N(t)$ denote the numbers of customers in the first and the second server, and in the orbit at time $t \geq 0$, respectively. It is easy to confirm that $\{X(t); t \geq 0\}$ forms a continuous time Markov chain in the state space $\{0, 1\} \times \{0, 1\} \times \mathbb{Z}_+$, where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.

Lemma 1 *The ergodic condition for $\{X(t)\}$ is given as follows.*

(i) If $\mu = 0$, then $\{X(t)\}$ is ergodic if and only if

$$\frac{\lambda(\lambda + \nu)}{\nu \nu_1} < 1.$$

(ii) If $\lim_{n \to \infty} \gamma_n = \infty$, $\{X(t)\}$ is ergodic if and only if

$$\frac{\lambda}{\nu_1} < 1.$$

Proof Because customers are lost at the second server, the stability of $\{X(t)\}$ is the same as that of $\{(S_1(t), N(t)); t \geq 0\}$, which describes the behavior of an M/M/1/1 retrial queue with linear retrial rate. Therefore, (i) and (ii) are obtained by similar methods as presented in [2] and in [8], respectively.

Remark 1 The probabilistic interpretation for (i) is as follows. We see that λ/ν_1 is the average number of customers that arrive at the system during a service period of the first server. This is also the average number of customer that are forced to enter to the orbit in between two consecutive service ending times. On the other hand, $\nu/(\lambda + \nu)$ represents the average number of retrial customer that successfully enter the server during two consecutive service ending times. Condition (i) is equivalent to say that the average number of customers coming into the orbit must be smaller than that going out the orbit between two consecutive service ending times.

It is well known that the stability condition does not depend on μ and ν for the case where $\lim_{n \to \infty} \gamma_n = \infty$. An intuitive interpretation of (ii) is that the arrival rate must be smaller than the service rate.

In what follows, we consider the queueing system under the ergodic condition. Let $\pi_{i,j,n} = \lim_{t \to \infty} \Pr\{S_1(t) = i, S_2(t) = j, N(t) = n\}$ $((i, j, n) \in \{0, 1\} \times \{0, 1\} \times \mathbb{Z}_+)$ denote the joint stationary probability of $\{X(t)\}$. The system of balance equations for

the stationary distribution $\{\pi_{i,j,n}; (i,j,n) \in \{0,1\} \times \{0,1\} \times \mathbb{Z}_+\}$ is given as follows. For $n \in \mathbb{Z}_+$,

$$(\lambda + \gamma_n + \lambda^*)\pi_{0,0,n} = \nu_2 \pi_{0,1,n}, \tag{1}$$

$$(\lambda + \gamma_n + \nu_2)\pi_{0,1,n} = \nu_1 \pi_{1,0,n} + \nu_1 \pi_{1,1,n} + \lambda^* \pi_{0,0,n}, \tag{2}$$

$$(\lambda + \nu_1 + \lambda^*)\pi_{1,0,n} = \lambda \pi_{0,0,n} + \lambda \pi_{1,0,n-1} + \nu_2 \pi_{1,1,n} + \gamma_{n+1} \pi_{0,0,n+1}, \tag{3}$$

$$(\lambda + \nu_1 + \nu_2)\pi_{1,1,n} = \lambda \pi_{0,1,n} + \lambda \pi_{1,1,n-1} + \gamma_{n+1} \pi_{0,1,n+1} + \lambda^* \pi_{1,0,n}, \tag{4}$$

where $\pi_{i,j,-1} = 0 \ (i,j = 0,1)$. Let $\pi_{i,j}(z)$ denote the partial generating function of $\{\pi_{i,j,n}\}$ with respect to n,

$$\pi_{i,j}(z) = \sum_{n=0}^{\infty} \pi_{i,j,n} z^n, \qquad i,j = 0,1, \qquad |z| \le 1. \tag{5}$$

From equations (1) to (4) and then using (5), we obtain,

$$(\lambda + \lambda^*)\pi_{0,0}(z) + \nu(\pi_{0,0}(z) - \pi_{0,0,0}) + \mu z \pi'_{0,0}(z) = \nu_2 \pi_{0,1}(z), \tag{6}$$

$$(\lambda + \nu_2)\pi_{0,1}(z) + \nu(\pi_{0,1}(z) - \pi_{0,1,0}) + \mu z \pi'_{0,1}(z) = \nu_1 \pi_{1,0}(z) + \nu_1 \pi_{1,1}(z)$$
$$+ \lambda^* \pi_{0,0}(z), \tag{7}$$

and

$$(\lambda + \nu_1 + \lambda^*)\pi_{1,0}(z) = \lambda \pi_{0,0}(z) + \lambda z \pi_{1,0}(z) + \nu_2 \pi_{1,1}(z)$$
$$+ \mu \pi'_{0,0}(z) + \frac{\nu}{z}(\pi_{0,0}(z) - \pi_{0,0,0}), \tag{8}$$

$$(\lambda + \nu_1 + \nu_2)\pi_{1,1}(z) = \lambda \pi_{0,1}(z) + \lambda z \pi_{1,1}(z) + \mu \pi'_{0,1}(z)$$
$$+ \frac{\nu}{z}(\pi_{0,1}(z) - \pi_{0,1,0}) + \lambda^* \pi_{1,0}(z). \tag{9}$$

Summing up equations (6) to (9) and rearranging the result yields

$$(z-1)\lambda(\pi_{1,0}(z) + \pi_{1,1}(z))$$
$$= (z-1)\left(\mu(\pi'_{0,0}(z) + \pi'_{0,1}(z)) + \frac{\nu}{z}(\pi_{0,0}(z) - \pi_{0,0,0}) + \frac{\nu}{z}(\pi_{0,1}(z) - \pi_{0,1,0})\right). \tag{10}$$

Dividing both sides of (10) by $(z-1)$, we obtain

$$\lambda(\pi_{1,0}(z) + \pi_{1,1}(z))$$
$$= \mu(\pi'_{0,0}(z) + \pi'_{0,1}(z)) + \frac{\nu}{z}(\pi_{0,0}(z) - \pi_{0,0,0}) + \frac{\nu}{z}(\pi_{0,1}(z) - \pi_{0,1,0}). \tag{11}$$

It should be noted that the left and right hand sides of (11) correspond to the flows coming into and out from the orbit, respectively. Thus, (11) represents the balance equation between these flows. Using (7), the left hand side of (11) is transformed as

follows:

$$\lambda(\pi_{1,0}(z) + \pi_{1,1}(z))$$
$$= \frac{\lambda(\lambda + \nu + \nu_2)}{\nu_1}\pi_{0,1}(z) + \frac{\lambda\mu z}{\nu_1}\pi'_{0,1}(z) - \frac{\lambda\nu}{\nu_1}\pi_{0,1,0} - \frac{\lambda\lambda^*}{\nu_1}\pi_{0,0}(z)$$
$$= \frac{\lambda\mu^2 z^2}{\nu_1\nu_2}\pi''_{0,0}(z) + \frac{\lambda\mu z(2\lambda + 2\nu + \mu + \nu_2 + \lambda^*)}{\nu_1\nu_2}\pi'_{0,0}(z)$$
$$+ \frac{\lambda(\lambda + \nu + \nu_2 + \lambda^*)(\lambda + \nu)}{\nu_1\nu_2}\pi_{0,0}(z) - \frac{\lambda\nu(\lambda + \nu + \nu_2)}{\nu_1\nu_2}\pi_{0,0,0} - \frac{\lambda\nu}{\nu_1}\pi_{0,1,0}. \quad (12)$$

On the other hand, the right hand side of (11) is expressed in terms of $\pi_{0,0}(z)$ as

$$\mu(\pi'_{0,0}(z) + \pi'_{0,1}(z)) + \frac{\nu}{z}(\pi_{0,0}(z) - \pi_{0,0,0}) + \frac{\nu}{z}(\pi_{0,1}(z) - \pi_{0,1,0})$$
$$= \frac{\mu^2 z}{\nu_2}\pi''_{0,0}(z) + \frac{\mu(\lambda + \mu + 2\nu + \nu_2 + \lambda^*)}{\nu_2}\pi'_{0,0}(z)$$
$$+ \frac{\nu(\lambda + \lambda^* + \nu + \nu_2)}{z\nu_2}\pi_{0,0}(z) - \frac{\nu(\nu + \nu_2)}{z\nu_2}\pi_{0,0,0} - \frac{\nu}{z}\pi_{0,1,0}. \quad (13)$$

Equations (11) to (13) are the keys for the derivation of the analytical solutions in this paper.

3 Main results

In this section, we present the main results of the paper. First, we consider a classical retrial policy, where $\nu = 0$. Second, we consider a constant retrial rate policy, namely, $\mu = 0$. Third, we deal with the linear retrial rate policy, where $\nu, \mu > 0$. We can see that the classical retrial policy and the constant retrial policy are special cases of the linear retrial policy. However, because the analytical results for the two special cases can be presented in simple and elegant forms, we analyze them separately.

3.1 Classical retrial policy

In this section, we consider the classical retrial policy, i.e. $\nu = 0$.

Definition 1 For any complex number ϕ and $n \in \mathbb{Z}_+$, let $(\phi)_n$ denote the Pochhammer symbol (see e.g. page 222 in [7]), whose definition is given by

$$(\phi)_n = \begin{cases} 1, & n = 0, \\ \phi(\phi + 1)\cdots(\phi + n - 1), & n \in \mathbb{N}, \end{cases}$$

where $\mathbb{N} = \{1, 2, \ldots\}$.

Theorem 1 *The stationary distribution $\{\pi_{i,j,n}\}$ is given as follows. For $n \in \mathbb{Z}_+$,*

$$\pi_{0,0,n} = \pi_{0,0,0} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{1}{n!} \left(\frac{\lambda}{\nu_1}\right)^n, \tag{14}$$

$$\pi_{0,1,n} = \frac{(\lambda + \lambda^* + n\mu)\pi_{0,0,n}}{\nu_2}, \tag{15}$$

$$\pi_{1,1,n} = \sum_{k=0}^{n} \left(\frac{\lambda^* \left(\lambda \pi_{0,1,k} + (k+1)\mu \pi_{0,1,k+1}\right) p^{n-k}}{(\lambda^* + \nu_2)(\lambda + \nu_1)} \right.$$
$$+ \frac{\nu_2 \left(\lambda \pi_{0,1,k} + (k+1)\mu \pi_{0,1,k+1}\right) r^{n-k}}{(\lambda^* + \nu_2)(\lambda + \lambda^* + \nu_1 + \nu_2)}$$
$$\left. + \frac{\lambda^* \left(\lambda \pi_{0,0,k} + (k+1)\mu \pi_{0,0,k+1}\right) \sum_{m=0}^{n-k} p^m r^{n-k-m}}{(\lambda + \nu_1)(\lambda + \lambda^* + \nu_1 + \nu_2)} \right), \tag{16}$$

$$\pi_{1,0,n} = \frac{\sum_{k=0}^{n} \left(\lambda \pi_{0,0,k} + \nu_2 \pi_{1,1,k} + \mu(k+1)\pi_{0,0,k+1}\right) q^{n-k}}{\lambda + \lambda^* + \nu_1}, \tag{17}$$

where

$$\gamma = \frac{\lambda + \lambda^* + \mu + \nu_2}{\mu}, \qquad \alpha = \frac{\lambda}{\mu}, \qquad \beta = \frac{\lambda + \lambda^* + \nu_2}{\mu},$$

and

$$p = \frac{\lambda}{\lambda + \nu_1}, \qquad q = \frac{\lambda}{\lambda + \lambda^* + \nu_1}, \qquad r = \frac{\lambda}{\lambda + \lambda^* + \nu_1 + \nu_2}.$$

The unknown probability $\pi_{0,0,0}$ is given by

$$\pi_{0,0,0} = \frac{\nu_1 \nu_2}{(\lambda + \nu_1)(\lambda + \lambda^* + \nu_2)a + \mu(2\lambda + \lambda^* + \mu + \nu_1 + \nu_2)b + \mu^2 c} \tag{18}$$

where a, b and c are expressed in terms of given parameters.

Proof From equations (11) to (13) with $\nu = 0$, we obtain a differential equation for $\pi_{0,0}(z)$ as follows:

$$\mu^2 z(\nu_1 - \lambda z)\pi''_{0,0}(z) + \mu(\nu_1(\lambda + \lambda^* + \mu + \nu_2)$$
$$- \lambda z(2\lambda + \lambda^* + \mu + \nu_2))\pi'_{0,0}(z) - \lambda^2(\lambda + \lambda^* + \nu_2)\pi_{0,0}(z) = 0. \tag{19}$$

Let $x = \lambda z/\nu_1$ and $q(x) = \pi_{0,0}(\nu_1 x/\lambda)$. We then have

$$q'(x) = \pi'_{0,0}(z)\frac{\nu_1}{\lambda}, \qquad q''(x) = \pi''_{0,0}(z)\frac{\nu_1^2}{\lambda^2}. \tag{20}$$

Substituting (20) into (19) and rearranging the result yields

$$x(1-x)q''(x)$$
$$+ \left(\frac{\lambda + \lambda^* + \mu + \nu_2}{\mu} - x\frac{2\lambda + \lambda^* + \mu + \nu_2}{\mu}\right) q'(x) - \frac{\lambda(\lambda + \lambda^* + \nu_2)}{\mu^2}q(x) = 0,$$

which is rewritten as

$$x(1-x)q''(x) + (\gamma - x(1+\alpha+\beta))q'(x) - \alpha\beta q(x) = 0. \qquad (21)$$

Let $F(\alpha, \beta, \gamma; x)$ denote the hypergeometric function [7], i.e.,

$$F(\alpha, \beta, \gamma; x) = \sum_{j=0}^{\infty} \frac{(\alpha)_j (\beta)_j}{(\gamma)_j} \frac{1}{j!} x^j, \qquad |x| \le 1,$$

whose radius of convergence is 1, and nth derivative is given by

$$\frac{d^n}{dx^n} F(\alpha, \beta, \gamma; x) = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} F(\alpha + n, \beta + n, \gamma + n; x).$$

Note that (21) is the hypergeometric differential equation, whose solution is given by

$$q(x) = q(0)F(\alpha, \beta, \gamma; x) = \pi_{0,0,0}F(\alpha, \beta, \gamma; x).$$

Thus we have

$$\pi_{0,0}(z) = \pi_{0,0,0}F(\alpha, \beta, \gamma; \frac{\lambda z}{\nu_1}), \qquad (22)$$

which yields (14). From (6), we have

$$\pi_{0,1}(z) = \frac{(\lambda + \lambda^*)\pi_{0,0}(z) + \mu z \pi_{0,0}'(z)}{\nu_2}, \qquad (23)$$

and thus from this result, we obtain (15). Eliminating $\pi_{1,0}(z)$ in (8) and (9) yields an expression for $\pi_{1,1}(z)$ in terms of $\pi_{0,0}(z)$ and $\pi_{0,1}(z)$ as follows.

$$\pi_{1,1}(z) = \frac{(X + \lambda^*)(\lambda \pi_{0,1}(z) + \mu \pi_{0,1}'(z))}{X(X + \lambda^* + \nu_2)} + \frac{\lambda^*(\lambda \pi_{0,0}(z) + \mu \pi_{0,0}'(z))}{X(X + \lambda^* + \nu_2)}, \qquad (24)$$

where

$$X = \lambda + \nu_1 - \lambda z. \qquad (25)$$

Remark 2 There are several methods in order to obtain an explicit expression for $\{\pi_{1,1,n}; n \in \mathbb{Z}_+\}$ from (24). However, because the coefficient of z in (25) is negative, we might obtain an expression in which both positive and negative terms are mixed, for which a procedure in a digital computer is numerically unstable. Therefore, in order to derive an expression with all positive terms, we transform some components of (24) as follows.

We have

$$\frac{\lambda^* + X}{(\lambda^* + \nu_2 + X)X} = \frac{\lambda^*}{\lambda^* + \nu_2}\frac{1}{X} + \frac{\nu_2}{\lambda^* + \nu_2}\frac{1}{X + \lambda^* + \nu_2}. \qquad (26)$$

On the other hand, because $p, q, r < 1$ and $|z| \leq 1$, we have

$$\frac{1}{X} = \frac{1}{\lambda + \nu_1} \sum_{n=0}^{\infty} (pz)^n, \qquad \frac{1}{X + \lambda^* + \nu_2} = \frac{1}{\lambda + \lambda^* + \nu_1 + \nu_2} \sum_{n=0}^{\infty} (rz)^n, \quad (27)$$

which yields

$$\frac{\lambda^*}{(\lambda^* + \nu_2 + X)X} = \frac{\lambda^*}{(\lambda + \nu_1)(\lambda + \lambda^* + \nu_1 + \nu_2)} \sum_{n=0}^{\infty} (pz)^n \sum_{n=0}^{\infty} (rz)^n,$$

$$= \frac{\lambda^*}{(\lambda + \nu_1)(\lambda + \lambda^* + \nu_1 + \nu_2)} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} p^k r^{n-k} \right) z^n. \quad (28)$$

From equations (24) to (28), we obtain (16). It follows from (7) that

$$\pi_{1,0}(z) = \frac{\lambda \pi_{0,0}(z) + \nu_2 \pi_{1,1}(z) + \mu \pi_{0,0}'(z)}{\lambda^* + X},$$

$$= \frac{\lambda \pi_{0,0}(z) + \nu_2 \pi_{1,1}(z) + \mu \pi_{0,0}'(z)}{\lambda + \lambda^* + \nu_1} \sum_{n=0}^{\infty} (qz)^n, \quad (29)$$

where we use

$$\frac{1}{\lambda^* + X} = \frac{1}{\lambda + \lambda^* + \nu_1} \sum_{n=0}^{\infty} (qz)^n, \quad (30)$$

in the second equality of (29). Thus, from (29), we obtain (17). It should be noted that the expressions of the joint stationary distribution in equations (14) to (17) include only positive terms.

From (23), (24) and the first equality of (29), the partial generating functions $\pi_{0,1}(z), \pi_{1,1}(z)$ and $\pi_{1,0}(z)$ are expressed in terms of $\pi_{0,0}(z)$, which includes $\pi_{0,0,0}$. The unknown $\pi_{0,0,0}$ is uniquely determined by the normalization condition:

$$\pi_{0,0}(1) + \pi_{0,1}(1) + \pi_{1,1}(1) + \pi_{1,0}(1) = 1. \quad (31)$$

Let

$$a = F(\alpha, \beta, \gamma, \frac{\lambda}{\nu_1}), \qquad b = \pi_{0,0}'(1) = \frac{\alpha\beta}{\gamma} \frac{\lambda}{\nu_1} F(\alpha + 1, \beta + 1, \gamma + 1, \frac{\lambda}{\nu_1}),$$

and

$$c = \pi_{0,0}''(1) = \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{\gamma(\gamma + 1)} \frac{\lambda^2}{\nu_1^2} F(\alpha + 2, \beta + 2, \gamma + 2, \frac{\lambda}{\nu_1}).$$

Therefore, according to (23), (24) and (12), we obtain

$$\pi_{0,0}(1) = \pi_{0,0,0} a, \qquad \pi_{0,1}(1) = \pi_{0,0,0} \frac{(\lambda + \lambda^*)a + \mu b}{\nu_2},$$

and

$$\pi_{1,0}(1) + \pi_{1,1}(1) = \pi_{0,0,0} \frac{\lambda(\lambda + \lambda^* + \nu_2)a + \mu(2\lambda + \lambda^* + \mu + \nu_2)b + \mu^2 c}{\nu_1 \nu_2}.$$

Thus, we have

$$\pi_{0,0}(1) + \pi_{0,1}(1) + \pi_{1,1}(1) + \pi_{1,0}(1)$$
$$= \pi_{0,0,0} \frac{(\lambda + \nu_1)(\lambda + \lambda^* + \nu_2)a + \mu(2\lambda + \lambda^* + \mu + \nu_1 + \nu_2)b + \mu^2 c}{\nu_1 \nu_2}. \tag{32}$$

Thus, (18) follows from (31) and (32).

Remark 3 From equations (22) to (24) and also the first equality of (29), we obtain explicit expressions for the generating functions $\pi_{i,j}(z)$ $(i, j = 0, 1)$, from which we can obtain explicit formulae for several performance measures such as the averages and moments. Furthermore, these expressions allow us to derive explicit formulae for the joint stationary distribution.

Remark 4 It should be noted that hypergeometric functions also have been used in the analysis of some retrial queueing models, such as M/M/2/2 and M/M/1/1+1 retrial queues [8,11]. However, the parameters of the hypergeometric functions in this paper are much simpler than those in the literature [8,11].

Corollary 1 *For the case of $\lambda^* = 0$, the joint stationary distribution is significantly simplified as follows. For $n \in \mathbb{Z}_+$, we have*

$$\pi_{0,0,n} = \pi_{0,0,0} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{1}{n!} \left(\frac{\lambda}{\nu_1} \right)^n,$$

$$\pi_{0,1,n} = \frac{(\lambda + n\mu\pi_{0,0,n})\pi_{0,0,n}}{\nu_2},$$

$$\pi_{1,1,n} = \frac{\sum_{k=0}^{n} \left(\lambda\pi_{0,1,k} + (k+1)\mu\pi_{0,1,k+1} \right) r^{n-k}}{\lambda + \nu_1 + \nu_2},$$

$$\pi_{1,0,n} = \frac{\sum_{k=0}^{n} \left(\lambda\pi_{0,0,k} + \nu_2\pi_{1,1,k} + \mu(k+1)\pi_{0,0,k+1} \right) q^{n-k}}{\lambda + \nu_1}.$$

3.2 Constant retrial rate policy

We also consider the case of constant retrial rate, where $\mu = 0$.

Theorem 2 *The stationary distribution $\{\pi_{i,j,n}\}$ is given as follows. For $n \in \mathbb{Z}_+$,*

$$\pi_{0,0,n} = \pi_{0,0,0} \frac{\lambda(\lambda + \lambda^* + \nu_2)}{(\lambda + \nu)(\lambda + \lambda^* + \nu + \nu_2)} \left(\frac{\lambda(\lambda + \nu)}{\nu\nu_1} \right)^n, \qquad n \in \mathbb{N},$$

$$\pi_{0,1,n} = \frac{(\lambda + \nu(1 - \delta_{n,0}))\pi_{0,0,n}}{\nu_2},$$

$$\pi_{1,1,n} = \sum_{k=0}^{n} \left(\frac{\lambda^* \left(\lambda \pi_{0,1,k} + \nu \pi_{0,1,k+1} \right) p^{n-k}}{(\lambda^* + \nu_2)(\lambda + \nu_1)} \right.$$

$$+ \frac{\nu_2 (\lambda \pi_{0,1,k} + \nu \pi_{0,1,k+1}) r^{n-k}}{(\lambda^* + \nu_2)(\lambda + \lambda^* + \nu_1 + \nu_2)}$$

$$\left. + \frac{\lambda^* \left(\lambda \pi_{0,0,k} + \nu \pi_{0,0,k+1} \right) \sum_{m=0}^{n-k} p^m r^{n-k-m}}{(\lambda + \nu_1)(\lambda + \lambda^* + \nu_1 + \nu_2)} \right),$$

$$\pi_{1,0,n} = \frac{\sum_{k=0}^{n} (\lambda \pi_{0,0,k} + \nu_2 \pi_{1,1,k} + \nu \pi_{0,0,k+1}) q^{n-k}}{\lambda + \lambda^* + \nu_1},$$

where the unknown probability $\pi_{0,0,0}$ is determined by the normalization condition.

Proof In this case, the equation for $\pi_{0,0}(z)$ is given by

$$\frac{\lambda(\lambda + \lambda^* + \nu + \nu_2)(\lambda + \nu)}{\nu_1 \nu_2} \pi_{0,0}(z) - \frac{\lambda \nu(\lambda + \nu + \nu_2)}{\nu_1 \nu_2} \pi_{0,0,0} - \frac{\lambda \nu}{\nu_1} \pi_{0,1,0}$$

$$= \frac{\nu(\lambda + \lambda^* + \nu + \nu_2)}{z \nu_2} \pi_{0,0}(z) - \frac{\nu(\nu + \nu_2)}{z \nu_2} \pi_{0,0,0} - \frac{\nu}{z} \pi_{0,1,0}.$$

Rearranging this yields

$$\pi_{0,0}(z) = \frac{Cz - D}{Az - B} = \frac{C}{A} + \frac{AD - CB}{AB} \frac{1}{1 - \frac{Az}{B}}, \tag{33}$$

where

$$A = \frac{\lambda(\lambda + \lambda^* + \nu + \nu_2)(\lambda + \nu)}{\nu_1 \nu_2}, \qquad B = \frac{\nu(\lambda + \lambda^* + \nu + \nu_2)}{\nu_2},$$

and

$$C = \frac{\lambda \nu(\lambda + \nu + \nu_2)}{\nu_1 \nu_2} \pi_{0,0,0} + \frac{\lambda \nu}{\nu_1} \pi_{0,1,0}, \qquad D = \nu \left(\frac{\nu + \nu_2}{\nu_2} \pi_{0,0,0} + \pi_{0,1,0} \right).$$

It follows from (33) that in order for $\pi_{0,0}(z)$ to converge in $|z| < 1$, the following inequality must be satisfied.

$$A < B \quad \Longleftrightarrow \quad \frac{\lambda(\lambda + \nu)}{\nu \nu_1} < 1.$$

This result also conforms with the ergodic condition presented in Lemma 1. Under the ergodic condition, we further transform (33) as follows,

$$\pi_{0,0}(z) = \frac{C}{A} + \frac{AD - CB}{AB} \frac{1}{1 - \frac{Az}{B}} = \frac{C}{A} + \frac{AD - CB}{AB} \sum_{n=0}^{\infty} \left(\frac{Az}{B} \right)^n. \tag{34}$$

It follows from (1) that

$$\pi_{0,1,0} = \frac{\lambda + \lambda^*}{\nu_2} \pi_{0,0,0}.$$

Therefore,

$$C = \frac{\lambda\nu(2\lambda + \lambda^* + \nu + \nu_2)}{\nu_1\nu_2}\pi_{0,0,0}, \qquad D = \frac{\nu(\lambda + \lambda^* + \nu + \nu_2)}{\nu_2}\pi_{0,0,0}. \qquad (35)$$

Furthermore, we have

$$\frac{AD - CB}{AB} = \frac{\lambda(\lambda + \lambda^* + \nu_2)}{(\lambda + \nu)(\lambda + \lambda^* + \nu + \nu_2)}\pi_{0,0,0}.$$

This result and (34) yield

$$\pi_{0,0,n} = \pi_{0,0,0}\frac{\lambda(\lambda + \lambda^* + \nu_2)}{(\lambda + \nu)(\lambda + \lambda^* + \nu + \nu_2)}\left(\frac{\lambda(\lambda + \nu)}{\nu\nu_1}\right)^n, \qquad n \in \mathbb{N}.$$

From (6), (8) and (9), we obtain

$$\pi_{0,1}(z) = \frac{(\lambda + \lambda^*)\pi_{0,0}(z) + \nu(\pi_{0,0}(z) - \pi_{0,0,0})}{\nu_2}, \qquad (36)$$

$$\pi_{1,1}(z) = \frac{(\lambda^* + X)\left(\lambda\pi_{0,1}(z) + \frac{\nu}{z}(\pi_{0,1}(z) - \pi_{0,1,0})\right)}{(\lambda^* + \nu_2 + X)X}$$
$$+ \frac{\lambda^*\left(\lambda\pi_{0,0}(z) + \frac{\nu}{z}(\pi_{0,0}(z) - \pi_{0,0,0})\right)}{(\lambda^* + \nu_2 + X)X}, \qquad (37)$$

$$\pi_{1,0}(z) = \frac{\lambda\pi_{0,0}(z) + \nu_2\pi_{1,1}(z) + \frac{\nu}{z}(\pi_{0,0}(z) - \pi_{0,0,0})}{\lambda^* + X}. \qquad (38)$$

Finally, from equations (36) to (38) and also using (27), (28) and (30), we obtain the announced result, where the unknown probability $\pi_{0,0,0}$ is uniquely determined by the normalization condition as shown in (32). Indeed, we can obtain simple expression for $\pi_{0,0,0}$ and $\pi_{i,j}$ $(i, j = 0, 1)$ as follows. Let

$$\pi_{i,j}(z) = \pi_{0,0,0}\widetilde{\pi}_{i,j}(z), \qquad i, j = 0, 1.$$

We then have

$$\widetilde{\pi}_{0,0}(z) = \frac{\widetilde{C}z - \widetilde{D}}{Az - B},$$

where

$$\widetilde{C} = \frac{\lambda\nu(2\lambda + \lambda^* + \nu + \nu_2)}{\nu_1\nu_2}, \qquad \widetilde{D} = \frac{\nu(\lambda + \lambda^* + \nu + \nu_2)}{\nu_2}.$$

From (36) to (38), we obtain simple expressions for $\widetilde{\pi}_{0,1}(z)$, $\widetilde{\pi}_{1,1}(z)$ and $\widetilde{\pi}_{1,0}(z)$ which are independent of $\pi_{0,0,0}$. It follows from the normalization condition that

$$\pi_{0,0,0} = \frac{1}{\widetilde{\pi}_{0,0}(1) + \widetilde{\pi}_{0,1}(1) + \widetilde{\pi}_{1,1}(1) + \widetilde{\pi}_{1,0}(1)}.$$

Corollary 2 *The joint stationary distribution for the case of $\lambda^* = 0$ is given by*

$$\pi_{0,0,n} = \pi_{0,0,0} \frac{\lambda(\lambda + \nu_2)}{(\lambda + \nu)(\lambda + \nu + \nu_2)} \left(\frac{\lambda(\lambda + \nu)}{\nu\nu_1}\right)^n, \qquad n \in \mathbb{N},$$

$$\pi_{0,1,n} = \frac{(\lambda + \nu(1 - \delta_{n,0}))\pi_{0,0,n}}{\nu_2}, \qquad n \in \mathbb{Z}_+,$$

$$\pi_{1,1,n} = \frac{\sum_{k=0}^{n} \left(\lambda\pi_{0,1,k} + \nu\pi_{0,1,k+1}\right) r^{n-k}}{\lambda + \nu_1 + \nu_2}, \qquad n \in \mathbb{Z}_+,$$

$$\pi_{1,0,n} = \frac{\sum_{k=0}^{n} \left(\lambda\pi_{0,0,k} + \nu_2\pi_{1,1,k} + \nu\pi_{0,0,k+1}\right) q^{n-k}}{\lambda + \nu_1}, \qquad n \in \mathbb{Z}_+.$$

3.3 Full linear retrial rate policy

Now we consider the case where both μ and ν are positive.

Theorem 3 *The stationary distribution $\{\pi_{i,j,n}\}$ is given as follows. For $n \in \mathbb{Z}_+$,*

$$\pi_{0,0,1} = \frac{\lambda^2(\lambda + \lambda^* + \nu_2)}{\nu_1\left(\mu(\lambda + \lambda^* + \mu + 2\nu + \nu_2) + \nu(\lambda + \lambda^* + \nu + \nu_2)\right)}\pi_{0,0,0}, \qquad (39)$$

$$\pi_{0,0,n+1} = \frac{w_{2,3}n(n-1) + w_{1,2}n + w_{0,1}}{w_{2,2}(n+1)n + w_{1,1}(n+1) + w_{0,0}}\pi_{0,0,n}, \qquad n \in \mathbb{N}, \qquad (40)$$

$$\pi_{0,1,n} = \frac{(\lambda + \lambda^* + \gamma_n)\pi_{0,0,n}}{\nu_2}, \qquad (41)$$

$$\begin{aligned}
\pi_{1,1,n} = \sum_{k=0}^{n} \Bigg(&\frac{\lambda^* \left(\lambda\pi_{0,1,k} + \gamma_{k+1}\pi_{0,1,k+1}\right) p^{n-k}}{(\lambda^* + \nu_2)(\lambda + \nu_1)} \\
&+ \frac{\nu_2 \left(\lambda\pi_{0,1,k} + \gamma_{k+1}\pi_{0,1,k+1}\right) r^{n-k}}{(\lambda^* + \nu_2)(\lambda + \lambda^* + \nu_1 + \nu_2)} \\
&+ \frac{\lambda^* \left(\lambda\pi_{0,0,k} + \gamma_{k+1}\pi_{0,0,k+1}\right) \sum_{m=0}^{n-k} p^m r^{n-k-m}}{(\lambda + \nu_1)(\lambda + \lambda^* + \nu_1 + \nu_2)} \Bigg), \qquad (42)
\end{aligned}$$

$$\pi_{1,0,n} = \frac{\sum_{k=0}^{n} \left(\lambda\pi_{0,0,k} + \gamma_{k+1}\pi_{0,0,k+1} + \nu_2\pi_{1,1,k}\right) q^{n-k}}{\lambda + \lambda^* + \nu_1}, \qquad (43)$$

where

$$w_{2,3} = \lambda\mu^2, \qquad w_{2,2} = \mu^2\nu_1, \qquad w_{1,2} = \lambda\mu(2\lambda + \lambda^* + 2\nu + \mu + \nu_2),$$

$$w_{1,1} = \mu\nu_1(\lambda + \lambda^* + \mu + 2\nu + \nu_2), \qquad w_{0,1} = \lambda(\lambda + \lambda^* + \nu + \nu_2)(\lambda + \nu),$$

$$w_{0,0} = \nu\nu_1(\lambda + \lambda^* + \nu + \nu_2),$$

and the unknown probability $\pi_{0,0,0}$ is determined by the normalization condition as shown in (31).

Proof From equations (11) to (13), we obtain a differential equation for $\pi_{0,0}(z)$ as

$$(w_{2,3}z^3 - w_{2,2}z^2)\pi_{0,0}''(z) + (w_{1,2}z^2 - w_{1,1}z)\pi_{0,0}'(z)$$
$$+ (w_{0,1}z - w_{0,0})\pi_{0,0}(z) - gz + h = 0, \tag{44}$$

where

$$g = \lambda\nu(\lambda + \nu + \nu_2)\pi_{0,0,0} + \lambda\nu\nu_2\pi_{0,1,0}, \qquad h = \nu\nu_1(\nu + \nu_2)\pi_{0,0,0} + \nu\nu_1\nu_2\pi_{0,1,0}.$$

Recall that

$$\pi_{0,0}(z) = \sum_{n=0}^{\infty} \pi_{0,0,n} z^n.$$

Substituting this into (44) yields

$$-w_{0,0}\pi_{0,0,0} + h + (w_{0,1}\pi_{0,0,0} - (w_{1,1} + w_{0,0})\pi_{0,0,1} - g)z + \sum_{n=2}^{\infty} a_n z^n = 0, \tag{45}$$

where

$$a_n = (w_{2,3}(n-1)(n-2) + w_{1,2}(n-1) + w_{0,1})\pi_{0,0,n-1}$$
$$- (w_{2,2}n(n-1) + w_{1,1}n + w_{0,0})\pi_{0,0,n}, \qquad n \geq 2.$$

In order for (45) to be true for all $|z| \leq 1$, the following equations must be satisfied.

$$-w_{0,0}\pi_{0,0,0} + h = 0, \qquad w_{0,1}\pi_{0,0,0} - (w_{1,1} + w_{0,0})\pi_{0,0,1} - g = 0, \tag{46}$$

and

$$a_n = 0, \qquad n \geq 2. \tag{47}$$

Equations (39) and (40) are obtained from (46) and (47), respectively. Since

$$\lim_{n\to\infty} \frac{w_{2,3}n(n-1) + w_{1,2}n + w_{0,1}}{w_{2,2}(n+1)n + w_{1,1}(n+1) + w_{0,0}} = \frac{\lambda}{\nu_1} < 1,$$

the radius of convergence for $\pi_{0,0}(z)$ is $\nu_1/\lambda > 1$. From equations (6) to (9), we have

$$\pi_{0,1}(z) = \frac{(\lambda + \lambda^*)\pi_{0,0}(z) + \mu z \pi_{0,0}'(z) + \nu(\pi_{0,0}(z) - \pi_{0,0,0})}{\nu_2}, \tag{48}$$

$$\pi_{1,1}(z) = \frac{(\lambda^* + X)(\lambda\pi_{0,1}(z) + \mu\pi_{0,1}'(z) + \frac{\nu}{z}(\pi_{0,1}(z) - \pi_{0,1,0}))}{(\lambda^* + \nu_2 + X)X}$$
$$+ \frac{\lambda^*(\lambda\pi_{0,0}(z) + \mu\pi_{0,0}'(z) + \frac{\nu}{z}(\pi_{0,0}(z) - \pi_{0,0,0}))}{(\lambda^* + \nu_2 + X)X}, \tag{49}$$

$$\pi_{1,0}(z) = \frac{\lambda\pi_{0,0}(z) + \mu\pi_{0,0}'(z) + \nu_2\pi_{1,1}(z) + \frac{\nu}{z}(\pi_{0,0}(z) - \pi_{0,0,0})}{\lambda^* + X}. \tag{50}$$

Therefore, from equations (48) to (50) and also using (26) to (28) and (30), we obtain equations (41) to (43). The unknown probability $\pi_{0,0,0}$ is determined by the normalization condition as shown in (32).

Corollary 3 *The joint stationary distribution for the case of $\lambda^* = 0$ is given in a simple form as follows. For $n \in \mathbb{Z}_+$,*

$$\pi_{0,0,1} = \frac{\lambda^2(\lambda + \nu_2)}{\nu_1\left(\mu(\lambda + \mu + 2\nu + \nu_2) + \nu(\lambda + \nu + \nu_2)\right)}\pi_{0,0,0},$$

$$\pi_{0,0,n+1} = \frac{w_{2,3}n(n-1) + w_{1,2}n + w_{0,1}}{w_{2,2}(n+1)n + w_{1,1}(n+1) + w_{0,0}}\pi_{0,0,n}, \qquad n \in \mathbb{N},$$

$$\pi_{0,1,n} = \frac{(\lambda + \gamma_n)\pi_{0,0,n}}{\nu_2},$$

$$\pi_{1,1,n} = \frac{\sum_{k=0}^{n}\left(\lambda\pi_{0,1,k} + \gamma_{k+1}\pi_{0,1,k+1}\right)r^{n-k}}{\lambda + \nu_1 + \nu_2},$$

$$\pi_{1,0,n} = \frac{\sum_{k=0}^{n}\left(\lambda\pi_{0,0,k} + \gamma_{k+1}\pi_{0,0,k+1} + \nu_2\pi_{1,1,k}\right)q^{n-k}}{\lambda + \nu_1}.$$

4 Performance Measures and Computational Algorithm

4.1 Performance measures

Let $\pi_{i,j}$ $(i, j = 0, 1)$ denote the joint stationary distribution of the states of the servers. We have

$$\pi_{i,j} = \sum_{n=0}^{\infty}\pi_{i,j,n}, \qquad i, j = 0, 1.$$

Let ρ_1 and ρ_2 denote the utilizations of the first and the second server, i. e.

$$\rho_1 = \pi_{1,0} + \pi_{1,1}, \qquad \rho_2 = \pi_{0,1} + \pi_{1,1}.$$

According to the Little's law, we have

$$\rho_1 = \frac{\lambda}{\nu_1}, \tag{51}$$

provided that the system is stable.

4.2 Computational algorithm

4.2.1 The case: $\mu > 0$

In the case where $\mu > 0$, we observe that the expression of $\pi_{0,0,0}$ includes an infinite sum for which a simple explicit expression does not exist. Therefore, we need to truncate the

finite sum at some level N_0. It is desired that N_0 is the level where the tail probability is small enough to be neglected. In other words, we need an N_0 such that

$$\sum_{n=N_0+1}^{\infty} (\pi_{0,0,n} + \pi_{0,1,n} + \pi_{1,1,n} + \pi_{1,0,n}) < \epsilon, \tag{52}$$

where ϵ is a small enough positive number. Recall that an explicit solution for the joint stationary distribution of an M/M/1/1 retrial queue with classical retrial policy is given in [8]. We consider an M/M/1/1 retrial queue with an arrival rate λ, a service rate ν_1 and a retrial rate μ. This retrial queue is also stable because $\rho_1 = \lambda/\nu_1 < 1$. Let $p_{i,n}$ $(i = 0, 1, n \in \mathbb{Z}_+)$ denote the joint stationary probability that there are i busy server and n customers in the orbit. According to [8], we have the following result.

$$p_{0,n} = \frac{\rho_1^n}{n!} (1 - \rho_1)^{\frac{\lambda}{\mu}+1} \left(\frac{\lambda}{\mu}\right)_n, \qquad p_{1,n} = \frac{\rho_1^{n+1}}{n!} (1 - \rho_1)^{1+\frac{\lambda}{\mu}} \left(1 + \frac{\lambda}{\mu}\right)_n.$$

We choose the truncation point N_0 as follows.

$$N_0 = \inf\{n \mid \sum_{i=0}^{n}(p_{0,i} + p_{1,i}) > 1 - \epsilon\}. \tag{53}$$

Because the retrial rate of our tandem queue is $\gamma_n > n\mu$ $(n \in \mathbb{N})$, thus we also expect that (52) is satisfied. Using the truncation point determined by (53), we compute approximations $\{\widehat{\pi}_{i,j,n}; (i, j, n) \in \{0, 1\} \times \{0, 1\} \times \{0, 1, \ldots, N_0\}\}$, $\widehat{\pi}_{i,j}$ $(i, j = 0, 1)$, $\widehat{\rho}_1$ and $\widehat{\rho}_2$ to $\{\pi_{i,j,n}\}$, $\pi_{i,j}$, ρ_1 and ρ_2, respectively, by the algorithm presented in Table 1. It should be noted that the algorithm is numerically stable since it manipulates only positive numbers.

Table 1 Computational algorithm.

Begin Algorithm
Input: $\lambda, \mu, \nu, \nu_1, \nu_2, \epsilon$.
Output: $\{\widehat{\pi}_{i,j,n}; i, j = 0, 1, n = 0, 1, \ldots, N_0\}, \widehat{\pi}_{i,j}(i, j = 0, 1), \widehat{\rho}_1, \widehat{\rho}_2$.
Set $\widehat{\pi}_{0,0,0} = 1$.
Compute $\widehat{\pi}_{0,0,n}$ $(n = 0, 1, \ldots, N_0 + 2)$ using (39) and (40).
Compute $\widehat{\pi}_{0,1,n}$ $(n = 0, 1, \ldots, N_0 + 1)$ using (41).
Compute $\widehat{\pi}_{1,1,n}$ $(n = 0, 1, \ldots, N_0)$ using (42).
Compute $\widehat{\pi}_{1,0,n}$ $(n = 0, 1, \ldots, N_0)$ using (43).
Set $sum = \sum_{n=0}^{N_0}(\widehat{\pi}_{0,0,n} + \widehat{\pi}_{0,1,n} + \widehat{\pi}_{1,1,n} + \widehat{\pi}_{1,0,n})$.
Set $\widehat{\pi}_{i,j,n} = \widehat{\pi}_{i,j,n}/sum$ $(i, j = 0, 1, n = 0, 1, \ldots, N_0)$.
Set $\widehat{\pi}_{i,j} = \sum_{n=0}^{N_0} \widehat{\pi}_{i,j,n}$ $(i, j = 0, 1)$.
Set $\widehat{\rho}_1 = \widehat{\pi}_{1,0} + \widehat{\pi}_{1,1}, \qquad \widehat{\rho}_2 = \widehat{\pi}_{0,1} + \widehat{\pi}_{1,1}$.
End Algorithm

4.2.2 The case: $\mu = 0$

As for the constant retrial rate policy, i.e. $\mu = 0$, from equations (33), (35), (36) to (38) and Theorem 2, we can obtain $\{\pi_{i,j,n}\}$ and $\pi_{i,j}$ $(i, j = 0, 1)$ in a simple explicit form without an infinite sum. Therefore, it is not necessary to use the algorithm in Table 1.

5 Numerical Examples

In this section, we present some numerical examples in order to explore the influence of the parameters on the performance of the system. In all the figures, we set $\nu_1 = 1$. We use $\epsilon = 10^{-7}$ in the algorithm presented in Table 1. Because the utilization of the first server is given in a simple explicit form as shown in (51), we do not plot a graph for this performance measure.
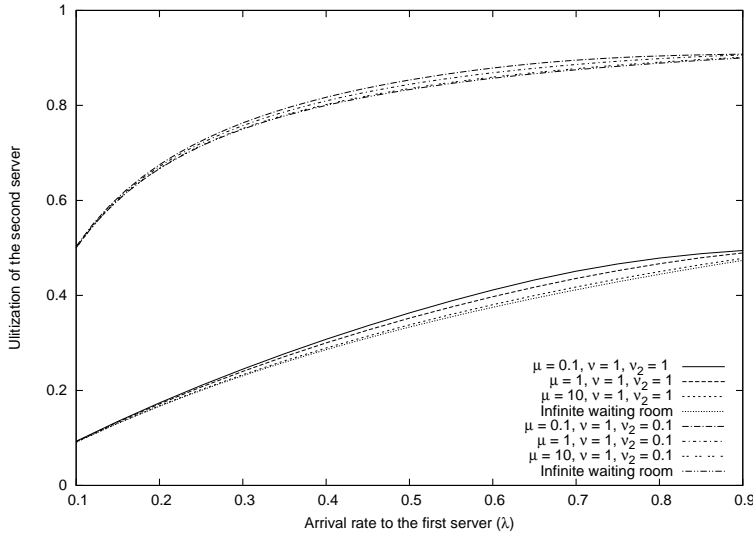

Fig. 2 Utilization of the second server vs. λ.

Fig. 2 shows the influence of λ on the utilization of the second server where $\nu = 1$ and $\lambda^* = 0$. We also consider a tandem queue with losses and without retrials where the capacity of the buffer at the first server is infinite and the arrival processes and service time distributions are the same as our retrial tandem queue. For this model the output process from the first server is also a Poisson process with rate λ. Thus, the utilization of the second server is equal to $(\lambda + \lambda^*)/(\lambda + \lambda^* + \nu_2)$.

We observe that the utilization of the second server increases with λ as expected. A significant observation is that the utilization of the second server decreases with
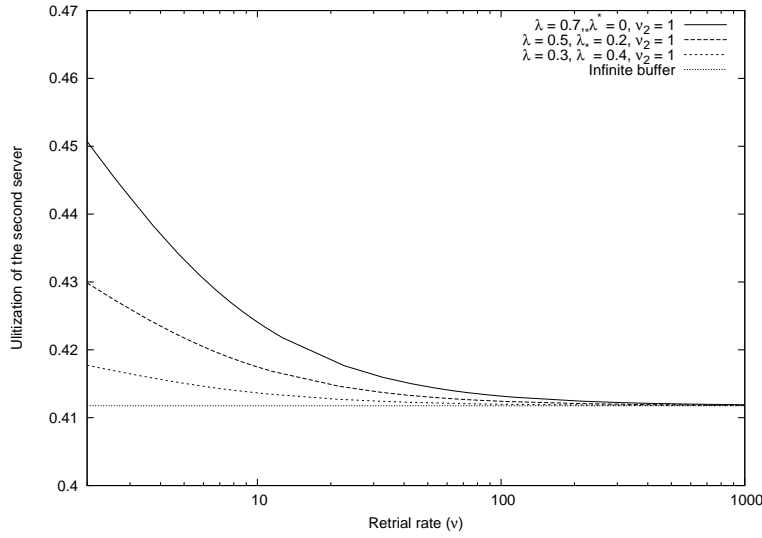
Fig. 3 Utilization of the second server vs. ν.

μ and approaches to that of the infinite buffer model. It suggests that we should keep the retrial interval as long as possible in order to achieve high utilization of the second server. The reason is that the orbit of the first server can be considered as a waiting room for the second server. Therefore, if the retrial interval is long, a large number of customers can wait in the orbit instead of being blocked at the second server. The observation also implies that the output process of an M/M/1/1 retrial queue is different from a Poisson process.

In Fig. 3, we investigate the impact of the constant retrial rate ν on the utilization of the second server for the cases: $\lambda = 0.7, 0.5$ and 0.3, while $\lambda + \lambda^* = 0.7$ and $\mu = 0$. We observe that the utilization of the second server decreases with ν and is asymptotic to that of the tandem queue with and infinite buffer at the first server. We observe that the utilization of the second server is sensitive to the retrial rate. Since the constant retrial rate policy could be used in modelling of LANs [6], this observation is important in setting optimal parameters for these systems.

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