International Journal of Control
Vol. 00, No. 00, Month 200x, 1–19

RESEARCH ARTICLE

Properties of Discrete-Time Noncausal Linear Periodically Time-Varying Scaling and Their Relationship with Shift-Invariance in Lifting-Timing

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(*Received 00 Month 200x; final version received 00 Month 200x*)

This paper is concerned with the technique called discrete-time noncausal linear periodically time-varying (LPTV) scaling for robust stability analysis. Noncausal LPTV scaling has already been shown to be effective for reducing the conservativeness of robustness analysis in theoretical and numerical ways. However, there still remain some issues to be resolved for further understanding and exploiting noncausal LPTV scaling, e.g., its relationship with the conventional analysis approach of causal linear time-invariant scaling. In this paper, by introducing the key idea of shift-invariance in lifting-timing, we discuss the difference and corresponding relationship between the conventional approach and noncausal LPTV scaling.

Keywords: robustness analysis; discrete-time systems; lifting; linear periodic systems; dynamic scaling

1 Introduction

In this paper, we discuss the properties of discrete-time noncausal linear periodically time-varying (LPTV) scaling (Hagiwara and Ohara 2007, 2010), which is an approach to the robustness analysis of discrete-time linear time-invariant (LTI) and LPTV systems. The famous lifting technique (Bittanti and Colaneri 2000, 2009) enables us to treat discrete-time LPTV systems as if they were LTI. Hence, given an LPTV system (or an LTI system as a special case), we can analyze its robust stability by applying the separator-type robust stability theorem (Iwasaki and Hara 1998) to the lifted LTI system. Noncausal LPTV scaling is an idea that can be introduced quite naturally in such an analysis by allowing some noncausal operations of signals through the lifted treatment. Noncausality thus introduced in the scaling approach has been demonstrated to be effective for reducing the conservativeness in the robustness analysis of LTI and LPTV systems, both theoretically and numerically (Hagiwara and Ohara 2007, 2010). In particular, as far as LTI systems are concerned, it has been proved that even if we confine ourselves to *static* noncausal LPTV scaling, it induces some dynamic causal LTI scaling when it is interpreted in the lifting-free (i.e., conventional) treatment. This property endows (even static) noncausal LPTV scaling with a promising ability in achieving less conservative analysis, in spite of its simple treatment. Such a feature of noncausal LPTV scaling has already been exploited also in the development of robust controller synthesis methods (Hosoe and Hagiwara 2010a,b), and their effectiveness in comparison with the μ-synthesis (Zhou and Doyle 1998) has also been confirmed.

Despite the promising properties on the practical side of noncausal LPTV scaling described above as a new approach to robust control, however, its comprehensive properties have not necessarily been revealed entirely. The missing arguments include, e.g., the characterization of the class of dynamic causal LTI scaling in the lifting-free treatment that can equivalently be dealt with by working instead on *static* noncausal LPTV scaling in the lifted treatment; or what theoretical differences there are between noncausal LPTV scaling and the conventional causal

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ISSN: 0020-7179 print/ISSN 1366-5820 online

DOI: 10.1080/0020717YYxxxxxxxx
http://www.informaworld.com

LTI scaling. These issues must be resolved for further clarifying the advantages (or drawbacks) of noncausal LPTV scaling compared with the conventional method and thus establishing a further solid theoretical base for noncausal LPTV scaling. This paper aims at making a step forward to such issues by clarifying further properties of noncausal LPTV scaling. In particular, as a key idea, we introduce the notion of the timing of lifting into the framework of noncausal LPTV scaling. The effect of shifting the lifting-timing can be studied easily by using what we call the timing-shift matrix, and thus this matrix plays an important role throughout the paper. More precisely, this paper introduces through this matrix the notion called shift-invariance (with respect to lifting-timing) of the separator in the robust stability theorem, as well as that notion of a class of separators. It is then shown that this notion plays a crucial role in revealing the properties of noncausal LPTV scaling through its theoretical comparisons with *causal* LPTV scaling and causal LTI scaling.

This paper is organized as follows. Section 2 states the robust stability analysis problem studied in this paper, presents the basic idea of robust stability analysis in the lifted framework, and reviews the definitions of causal/noncausal LPTV scaling and the associated causal/noncausal separators. Section 3 confines itself to the robust stability analysis of LTI systems, and revisits and slightly extends the existing results about the relationship between causal/noncausal LPTV scaling and the conventional causal LTI scaling. Section 4 introduces the timing shift about lifting and the timing-shift matrix, as well as the shift-invariance notion of a separator and a class of separators, and then discusses the implication of the presence (or lack) of shift-invariance on the properties of noncausal LPTV scaling. Section 5 introduces shift-invariant reconstruction of a given class of noncausal LPTV separators that is not necessarily shift-invariant, and shows an important equivalence relationship between the two approaches: one is noncausal LPTV scaling with the reconstructed class of separators, while the other is the dynamic causal LTI scaling (in the lifting-free framework) with the associated separator class induced by the given class of noncausal LPTV separators. The implication of such a relationship is further discussed, and important observations are given on the properties of noncausal LPTV scaling.

2 Robust stability analysis based on noncausal LPTV scaling

This section states the robust stability analysis problem studied in this paper, and reviews an approach to the problem called discrete-time noncausal LPTV scaling (Hagiwara and Ohara 2007, 2010), naturally introduced through the discrete-time lifting treatment (Bittanti and Colaneri 2000, 2009).

2.1 *Robust stability analysis problem*

This paper studies the robust stability problem of the discrete-time closed-loop system Σ shown in Fig. 1 consisting of the nominal system G and the uncertainty Δ. The nominal system G is assumed to be internally stable, finite-dimensional, linear N-periodic, and represented by

$$x_{k+1} = A_k x_k + B_k u_k, \quad y_k = C_k x_k + D_k u_k \tag{1}$$

where A_k, B_k, C_k and D_k are N-periodic matrices, $x_k \in \mathbf{R}^n$, $u_k \in \mathbf{R}^p$, and $y_k \in \mathbf{R}^p$. The uncertainty Δ is assumed to belong to some given set $\boldsymbol{\Delta}$ satisfying the following assumption.

Assumption 1 *Every $\Delta \in \boldsymbol{\Delta}$ is stable, finite-dimensional and linear N-periodic, and $\boldsymbol{\Delta}$ is star-convex with a center at the origin (i.e., $k\Delta \in \boldsymbol{\Delta}$ whenever $\Delta \in \boldsymbol{\Delta}$ and $0 \le k \le 1$).*

The above problem reduces to the robust stability problem of LTI systems when $N = 1$, and this paper is largely interested in applying the discrete-time noncausal LPTV scaling technique to such a case. This means that if we are given an LTI system Σ, then we view it as a special case of an N-periodic system with a prescribed N, unless stated otherwise.

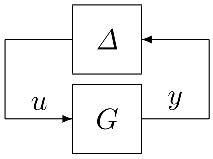

Figure 1. Closed-loop system Σ.

2.2 *Separator-type robust stability theorem via lifting*

This subsection gives a brief review of the lifting technique (Bittanti and Colaneri 2000, 2009) and a separator-type robust stability theorem via lifting.

The operation of getting new signal representations

$$\widehat{u}_\kappa := \left[u_{\kappa N}^T, u_{\kappa N+1}^T, \cdots, u_{\kappa N+N-1}^T\right]^T, \quad \widehat{y}_\kappa := \left[y_{\kappa N}^T, y_{\kappa N+1}^T, \cdots, y_{\kappa N+N-1}^T\right]^T \tag{2}$$

from the discrete-time signals u and y is called the lifting of signals. This converts the treatment of systems with input u and output y into that of systems with lifted input $\widehat{u}$ and lifted output $\widehat{y}$, and such treatment is called the lifting of systems. The resulting lifted representations of systems are called N-lifted systems. By defining $\widehat{x}_\kappa := x_{\kappa N}$, we can describe the N-lifted nominal system $\widehat{G}$ by

$$\widehat{x}_{\kappa+1} = \widehat{A}\widehat{x}_\kappa + \widehat{B}\widehat{u}_\kappa, \quad \widehat{y}_\kappa = \widehat{C}\widehat{x}_\kappa + \widehat{D}\widehat{u}_\kappa \tag{3}$$

All the coefficient matrices of $\widehat{G}$ can be constructed with the coefficient matrices in (1). We denote the transfer matrix of $\widehat{G}$ by $\widehat{G}(z)$; it is called the N-lifted transfer matrix of G. We can also get the N-lifted representation $\widehat{\Delta}$ and the N-lifted transfer matrix $\widehat{\Delta}(z)$ from Δ. Through these ideas, we can get the lifted representation $\widehat{\Sigma}$ (Fig. 2) from the closed-loop system Σ.

It follows that Σ is robustly stable with respect to $\boldsymbol{\Delta}$ if and only if $\widehat{\Sigma}$ is with respect to $\widehat{\boldsymbol{\Delta}} := \{\widehat{\Delta} \mid \Delta \in \boldsymbol{\Delta}\}$. Here, we have the following robust stability theorem via lifting.

Theorem 1 *Suppose that G is internally stable and N-periodic, and $\boldsymbol{\Delta}$ satisfies Assumption 1. If Σ is well-posed, $\forall \Delta \in \boldsymbol{\Delta}$, then Σ is robustly stable with respect to $\boldsymbol{\Delta}$ if and only if there exists $\widehat{\Theta}(z) = \widehat{\Theta}(z)^*$ $(\forall z \in \partial\mathbf{D})$ such that*

$$\begin{bmatrix} I \\ \widehat{G}(z) \end{bmatrix}^* \widehat{\Theta}(z) \begin{bmatrix} I \\ \widehat{G}(z) \end{bmatrix} \leq 0 \quad (\forall z \in \partial\mathbf{D}) \tag{4}$$

$$\begin{bmatrix} \widehat{\Delta}(z) \\ I \end{bmatrix}^* \widehat{\Theta}(z) \begin{bmatrix} \widehat{\Delta}(z) \\ I \end{bmatrix} > 0 \quad \begin{pmatrix} \forall \Delta \in \boldsymbol{\Delta}, \\ \forall z \in \partial\mathbf{D} \end{pmatrix} \tag{5}$$

where $\partial\mathbf{D} := \{z \in \mathbf{C} : |z| = 1\}$ denotes the unit circle.

Remark 1 *If we assume Σ is LTI, then by letting $N = 1$, Assumption 1 and Theorem 1 reduce, respectively, to the assumption for LTI uncertainties and the usual separator-type robust stability theorem (Iwasaki and Hara 1998) from which the above theorem immediately follows. We refer to them as Assumption 1_{LTI} and Theorem 1_{LTI}, respectively. Similarly, we use the labels $(4)_{\mathrm{LTI}}$ and $(5)_{\mathrm{LTI}}$ to refer to the inequalities corresponding to (4) and (5) with $N = 1$, respectively.*

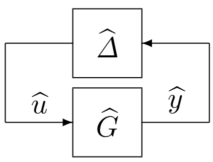

Figure 2. Lifted closed-loop system $\widehat{\Sigma}$.

By Theorem $1_{\rm LTI}$ mentioned in the above remark, the robust stability of LTI systems can be analyzed without the lifting technique, and this is nothing but the conventional causal LTI scaling (or frequency-dependent scaling). We call such a framework for robust stability analysis the *lifting-free framework*, but this is in sharp contrast with the fundamental standpoint of the present paper; as mentioned before, this paper is rather interested in viewing LTI systems as N-periodic systems and then applying Theorem 1. We call such a framework the *lifted framework*. To clearly discriminate these two frameworks, the (lifting-free) transfer matrices of G and $\varDelta$ will be denoted by $G(\zeta)$ and $\varDelta(\zeta)$, respectively, and the corresponding separator will be denoted by $\varTheta(\zeta)$, where ζ is used to denote the z-variable in the lifting-free framework.

In the following arguments, if the separator $\widehat{\varTheta}(z)$ satisfies (4) and (5), then we say that it is eligible with respect to (4) and (5) (or simply in the lifted framework). Similarly, if $\varTheta(\zeta)$ satisfies $(4)_{\rm LTI}$ and $(5)_{\rm LTI}$, then we say that it is eligible with respect to $(4)_{\rm LTI}$ and $(5)_{\rm LTI}$ (or in the lifting-free framework). In addition, if there exists an eligible $\varTheta(\zeta) \in \boldsymbol{\varTheta}(\zeta)$ (or $\widehat{\varTheta}(z) \in \widehat{\boldsymbol{\varTheta}}(z)$), then we say that the separator class $\boldsymbol{\varTheta}(\zeta)$ (or $\widehat{\boldsymbol{\varTheta}}(z)$) is eligible.

2.3 *Definition of noncausal LPTV scaling*

By Theorem 1, the robust stability problem of the closed-loop system $\widehat{\varSigma}$ (i.e., $\varSigma$) reduces to searching for separators $\widehat{\varTheta}(z)$ satisfying (4) and (5) against the given $\boldsymbol{\varDelta}$. This naturally leads to the idea of noncausal LPTV scaling reviewed in the following. For facilitating extensive discussions in the following, however, it is important to introduce noncausal LPTV scaling in contrast with causal LPTV scaling. This is carried out by classifying the separators $\widehat{\varTheta}(z)$ in Theorem 1 into two types (Hagiwara and Ohara 2007, 2010).

First, causal LPTV separators are defined as follows.

Definition 1 *We call a separator given by* $\widehat{\varTheta}(z) = \left[\widehat{V}_1(z)\ \widehat{V}_2(z)\right]^* \widehat{\varLambda} \left[\widehat{V}_1(z)\ \widehat{V}_2(z)\right]$ *a causal LPTV separator, where* $\widehat{V}_1(z)$ *and* $\widehat{V}_2(z)$ *are the* N*-lifted transfer matrices of causal* N*-periodic systems* V_1 *and* V_2 *with* p *inputs, respectively, and* $\widehat{\varLambda} = \widehat{\varLambda}^*$ *is a constant matrix of the form* $\widehat{\varLambda} = {\rm diag}[\varLambda_1, \cdots, \varLambda_N]$ *with the size of* $\varLambda_i$ *being the same for all* $i = 1, \cdots, N$*. In particular, if we take static* V_1 *and* V_2*, then we call the corresponding separator a static causal LPTV separator.*

The approach to robust stability analysis based on causal LPTV separators is called causal LPTV scaling.

On the other hand, more general noncausal LPTV separators have been defined as follows.

Definition 2 *We call a separator given by* $\widehat{\varTheta}(z) = \widehat{V}(z)^* \varGamma \widehat{V}(z)$ *a noncausal LPTV separator, where* $\widehat{V}(z)$ *is the transfer matrix of a causal LTI system* $\widehat{V}$ *with* $2Np$ *inputs defined directly on the lifted time axis* κ *in (3)* [1] *and* $\varGamma = \varGamma^*$ *is a constant matrix. In particular, if we take a static* $\widehat{V}$*, then we call the corresponding separator a static noncausal LPTV separator.*

The approach to robust stability analysis based on noncausal LPTV separators is called noncausal LPTV scaling. Even though Definition 2 is more general than Definition 1, in general, they degenerate into an identical definition when $N = 1$. We refer to the degenerated separators as causal LTI separators (in the lifting-free framework).

3 Noncausal LPTV scaling applied to LTI system

This paper discusses the properties of noncausal LPTV scaling that follows naturally from Theorem 1 as a method for robust stability analysis, where we place particular emphasis on

[1]This means that $\widehat{V}$ is not required to be an N-lifted representation of a system in the original time axis k in (1) before the application of lifting.

(but do not limit our attention exclusively to) the case when Σ is LTI. In that case, we have two alternatives for robust stability analysis: lifted framework (i.e., noncausal LPTV scaling) and lifting-free framework (i.e., the conventional causal LTI scaling). Whichever framework one may take, however, it is generally difficult to search for eligible separators, and thus one often introduces some tractable class of separators within which the search of eligible separators is to be carried out. It should be remarked that, under such a restrictive search, the inequalities (4) and (5) as well as $(4)_{\mathrm{LTI}}$ and $(5)_{\mathrm{LTI}}$ in these theorems become a conservative sufficient condition for robust stability. With this in mind, this paper aims at studying the properties of noncausal LPTV scaling that are expected to be useful in clarifying its ability in reducing the aforementioned conservativeness in the analysis, particularly in comparison with the conventional causal LTI scaling.

To facilitate the arguments that motivate the study in the remainder of this paper, this section first introduces some important results suggesting possible advantages of noncausal LPTV scaling over causal LTI scaling. Some of these results have in fact been reported in our preceding studies (with or without proof), but such remarks will be deferred to the end of this section to avoid distracting the attention of the reader. Instead, we opt to suggest immediately after these results some open problems that are not covered by these results. These problems will motivate further discussions about the properties of noncausal LPTV scaling studied in the remainder of this paper.

For readability, we explicitly state Assumption 1_{LTI} introduced in Remark 1.

Assumption 1_{LTI} Every $\Delta \in \boldsymbol{\Delta}$ is stable, finite-dimensional and LTI, and $\boldsymbol{\Delta}$ is star-convex with a center at the origin.

Then the first result is as follows.

Theorem 2 *Suppose that G is LTI, and $\boldsymbol{\Delta}$ satisfies Assumption 1_{LTI}. If there exists an eligible causal LTI separator $\Theta(\zeta)$ in the lifting-free framework, there exists an eligible causal LPTV separator $\widehat{\Theta}(z)$ in the lifted framework. In particular, if a causal LTI separator given by*

$$\Theta(\zeta) = \left[V_1(\zeta)\ V_2(\zeta)\right]^* \Lambda \left[V_1(\zeta)\ V_2(\zeta)\right] \tag{6}$$

in the lifting-free framework is eligible, the separator

$$\widehat{\Theta}(z) = \left(\frac{1}{N}\widehat{V}_i(z)^*\widehat{\Lambda}\widehat{V}_j(z)\right)_{i,j=1,2} \tag{7}$$

is eligible in the lifted framework.

An important implication of the above theorem is that if we apply causal/noncausal LPTV scaling to LTI systems, we can perform at least as good robust stability analysis as causal LTI scaling. The separator in the lifted framework given in this theorem, i.e., (7), satisfies the requirement in Definition 1 in a particular way, that is, with LTI systems V_1 and V_2, and with the constraint $\Lambda_i = \Lambda_j$ $(i, j = 1, \cdots, N)$. Hence, we refer to the separator of the form (7) constructed from the causal LTI separator (6) (in the lifting-free framework) as an *equivalent* causal LTI separator *in the lifted framework*. We denote such an embedding mapping from (6) to (7) by $\widehat{\Theta}(z) = \widehat{\mathrm{E}}[\Theta(\zeta)]$. Similarly, we call the treatment with such separators causal LTI scaling *in the lifted framework*. The validity of introducing such terms can be verified in a strong sense since not only Theorem 2 but also a sort of its converse holds as follows.

Theorem 3 *Suppose that G is LTI, $\boldsymbol{\Delta}$ satisfies Assumption 1_{LTI}, and a causal LTI separator $\Theta(\zeta)$ described by (6) is given. If the embedded separator $\widehat{\Theta}(z) = \widehat{E}[\Theta(\zeta)]$ equivalent to $\Theta(\zeta)$ is eligible in the lifted framework, $\Theta(\zeta)$ is eligible in the lifting-free framework.*

Remark 2 *Even though this section is confined to the case when Σ is LTI, we can similarly define causal LTI scaling for LPTV systems; such scaling refers to the approach in the lifted* *framework that uses only equivalent causal LTI separators constructed from causal LTI separators in the lifting-free framework. The properties of causal LTI scaling for LPTV systems will be discussed in Section 4.*

The following is another important result closely related to the advantage of noncausal LPTV scaling over causal LTI scaling.

Theorem 4 *Suppose that G is LTI, and $\boldsymbol{\Delta}$ satisfies Assumption $1_{\rm LTI}$. If a noncausal LPTV separator $\widehat{\Theta}(z)$ is eligible in the lifted framework, the causal LTI separator*

$$\Theta(\zeta) = \mathcal{T}(\zeta)^* \widehat{\Theta}(\zeta^N)\mathcal{T}(\zeta) \tag{8}$$

is eligible in the lifting-free framework, where

$$\mathcal{T}(\zeta) := {\rm diag}[T_p(\zeta), T_p(\zeta)],\ T_p(\zeta) := \begin{bmatrix} \zeta^{-(N-1)} I_p \\ \vdots \\ \zeta^{-1} I_p \\ I_p \end{bmatrix}. \tag{9}$$

This theorem implies that if we find an eligible separator $\widehat{\Theta}(z)$ in the lifted framework, it immediately means that we have also found an eligible separator $\Theta(\zeta)$ in the lifting-free framework. In particular, even if we were to confine ourselves to the search of *static* noncausal LPTV separators $\widehat{\Theta}$ (which is nothing but a constant matrix) in the lifted framework, it would induce some frequency-dependent scaling (i.e., dynamic causal LTI scaling) in the lifting-free framework by (8). Furthermore, it follows from Theorem 2 that the induced scaling in the lifting-free framework is ensured to be, at least, as effective as the static causal LTI scaling in that framework. This might suggest that static noncausal LPTV scaling could possibly replace dynamic causal LTI scaling, which sounds attractive because static separators are much more tractable than general dynamic separators.

However, Theorems 2 and 4 alone are deficient in the theoretical depth for affirming the above prospect. In other words, the properties of noncausal LPTV scaling have not been revealed completely, and there still remain important issues that should be investigated much further. For example, let us take a class $\widehat{\boldsymbol{\Theta}}_0^{\rm noncausal}(z)$ of noncausal LPTV separators, and denote by

$$\boldsymbol{\Theta}(\zeta) := \left\{ \mathcal{T}(\zeta)^* \widehat{\Theta}(\zeta^N)\mathcal{T}(\zeta) \;\middle|\; \widehat{\Theta}(z) \in \widehat{\boldsymbol{\Theta}}_0^{\rm noncausal}(z) \right\} \tag{10}$$

the class of separators $\Theta(\zeta)$ in the lifting-free framework given by (8) with $\widehat{\Theta}(z) \in \widehat{\boldsymbol{\Theta}}_0^{\rm noncausal}(z)$. An important unresolved issue is whether the eligibility of the class $\boldsymbol{\Theta}(\zeta)$ always implies that of the original class $\widehat{\boldsymbol{\Theta}}_0^{\rm noncausal}(z)$, or to put it another way, whether it is ensured that we can convert the problem of searching for an eligible $\Theta(\zeta) \in \boldsymbol{\Theta}(\zeta)$ equivalently into that of searching for an eligible noncausal LPTV separator $\widehat{\Theta}(z) \in \widehat{\boldsymbol{\Theta}}_0^{\rm noncausal}(z)$. If this question has an affirmative answer, then the prospect mentioned above is also resolved affirmatively by taking $\widehat{\boldsymbol{\Theta}}_0^{\rm noncausal}(z)$ to be the set of static separators.

This paper aims at making a step forward to answering the question raised above by revealing further properties of noncausal LPTV scaling. To proceed in that direction, the idea of shifting the timing of lifting (timing-shift of lifting) plays a crucial role. Hence, we first study in Section 4 some fundamental properties of noncausal LPTV scaling with respect to the timing-shift of lifting, where we deal with N-periodic systems as well as LTI systems. We then proceed the arguments about the timing-shift in Section 5 for the special case when Σ is LTI. In particular, we discuss further relationship and difference between noncausal LPTV scaling and the conventional causal LTI scaling, and provide a partial answer to the question raised above.

Before closing this section, we give some remarks about the proof of the above theorems. Regarding Theorem 2, its special case confining only to *static* separators has been given in our previous paper (Hagiwara and Ohara 2010) in a less explicit form; see Theorem 2 and its proof therein. A similar but again less explicit assertion has been given about a general case in the earlier conference previous paper (Hagiwara and Ohara 2007), in which the proof was omitted because of limited space. Hence, the proof of this theorem is given in Appendix A, which is indeed important in the following arguments because the ideas therein are closely related with the discussions of this paper (in particular, Theorem 6 to be derived later). On the other hand, Theorem 3 is asserted for the first time in this paper (except for the case of static separators, which is again asserted by Theorem 2 in Hagiwara and Ohara (2010) in an implicit way), whose proof is given in Appendix B. Finally, Theorem 4 is nothing but Theorem 1 in Hagiwara and Ohara (2010), which, together with the other two theorems, strongly motivates the further studies in the remainder of this paper.

4 Timing-shift in noncausal LPTV scaling

The previous section confined itself to the case when Σ is LTI. This section returns to the treatment of N-periodic systems, introduces the idea of timing-shift about the lifting treatment in noncausal LPTV scaling, and discusses the properties of noncausal LPTV scaling in connection with timing-shift.

4.1 *Timing-shift matrix and its properties*

To begin with, the timing of lifting (or lifting-timing for short) means the basic time instant that we take in the lifting treatment of signals and systems. For example, for a signal f_k related with N-periodic system H, the lifted representation of f_k is usually given by $\widehat{f}_\kappa = [f_{\kappa N}^T, f_{\kappa N+1}^T, \cdots, f_{\kappa N+N-1}^T]^T$. However, when we consider the lifting-timing denoted by l, then by definition, the lifted representation is given by $\widehat{f}_\kappa^{(l)} = [f_{\kappa N+l}^T, f_{\kappa N+1+l}^T, \cdots, f_{\kappa N+N-1+l}^T]^T$. Under the lifting-timing l, we denote the resulting lifted system by $\widehat{H}^{(l)}$, and its associated transfer matrix by $\widehat{H}^{(l)}(z)$. Obviously, it is enough to consider the lifting-timing l only in $\{0, 1, \cdots, N-1\}$, and if H is LTI, its lifted representation $\widehat{H}^{(l)}$ obtained by regarding H as N-periodic is independent of the lifting-timing l. However, this is not the case if H is not LTI. Hence, we can easily see that lifting-timing could be an important factor to study especially when the system Σ is LPTV. Nevertheless, we will eventually see that it is equally important even when Σ is LTI.

Even though shifting the lifting-timing is equivalent to shifting signals before applying the standard lifting with $l = 0$, its effect can easily be treated in the lifted framework (i.e., after lifting has been applied) by introducing the (backward) timing-shift matrix

$$S_p(z) := \begin{bmatrix} 0 & z^{-1} I_p \\ I_{(N-1)p} & 0 \end{bmatrix}. \tag{11}$$

Let us denote the z-transform of the lifted signal $\widehat{f}_\kappa^{(l)}$ by $\widehat{F}^{(l)}(z)$. Then, ignoring the influence of the "initial value f_l," we readily have $\widehat{F}^{(l)}(z) = S_p(z)\widehat{F}^{(l+1)}(z)$. This immediately leads to

$$\widehat{H}^{(l+1)}(z) = S_p(z)^{-1}\widehat{H}^{(l)}(z)S_p(z). \tag{12}$$

It is immediate from the definition that the timing-shift matrix $S_p(z)$ has the properties

$$S_p(z)S_p(z)^* = S_p(z)^*S_p(z) = I \quad (z \in \partial\mathbf{D}), \quad S_p(z)^N = z^{-1}I. \tag{13}$$

4.2 *Effect of timing-shift in noncausal LPTV scaling*

By applying the congruence transformation by the matrix $S_p(z)$ on the conditions in Theorem 1 and noting (12), we are led to the following theorem.

Theorem 5 *Suppose that G is N-periodic, and $\boldsymbol{\Delta}$ satisfies Assumption 1. Let us define*

$$\mathcal{S}(z) := \mathrm{diag}[S_p(z), S_p(z)]. \tag{14}$$

Then, $\widehat{\Theta}(z) = \widehat{\Theta}(z)^*$ $(\forall z \in \partial\mathbf{D})$ satisfies

$$\begin{bmatrix} I \\ \widehat{G}^{(0)}(z) \end{bmatrix}^* \widehat{\Theta}(z) \begin{bmatrix} I \\ \widehat{G}^{(0)}(z) \end{bmatrix} \le 0 \quad (\forall z \in \partial\mathbf{D}) \tag{15}$$

$$\begin{bmatrix} \widehat{\Delta}^{(0)}(z) \\ I \end{bmatrix}^* \widehat{\Theta}(z) \begin{bmatrix} \widehat{\Delta}^{(0)}(z) \\ I \end{bmatrix} > 0 \quad \begin{pmatrix} \forall \Delta \in \boldsymbol{\Delta}, \\ \forall z \in \partial\mathbf{D} \end{pmatrix} \tag{16}$$

under the standard lifting-timing $l = 0$ if and only if it satisfies

$$\begin{bmatrix} I \\ \widehat{G}^{(l)}(z) \end{bmatrix}^* (\mathcal{S}(z)^l)^* \widehat{\Theta}(z) \mathcal{S}(z)^l \begin{bmatrix} I \\ \widehat{G}^{(l)}(z) \end{bmatrix} \le 0 \quad (\forall z \in \partial\mathbf{D}) \tag{17}$$

$$\begin{bmatrix} \widehat{\Delta}^{(l)}(z) \\ I \end{bmatrix}^* (\mathcal{S}(z)^l)^* \widehat{\Theta}(z) \mathcal{S}(z)^l \begin{bmatrix} \widehat{\Delta}^{(l)}(z) \\ I \end{bmatrix} > 0 \quad \begin{pmatrix} \forall \Delta \in \boldsymbol{\Delta}, \\ \forall z \in \partial\mathbf{D} \end{pmatrix} \tag{18}$$

under at least one lifting-timing $l = 0, \cdots, N-1$, and also if and only if it satisfies (17) and (18) under all $l = 0, 1, \cdots, N-1$.

According to this theorem, if we take a set of some tractable (e.g., static) separators denoted by $\widehat{\boldsymbol{\Theta}}_0(z)$, the approach under the lifting-timing $l = 0$ that searches for eligible separators $\widehat{\Theta}(z) \in \widehat{\boldsymbol{\Theta}}_0(z)$ is, if it is interpreted under another lifting-timing l, equivalent to the approach of searching for eligible separators $(\mathcal{S}(z)^l)^* \widehat{\Theta}(z) \mathcal{S}(z)^l$ such that $\widehat{\Theta}(z) \in \widehat{\boldsymbol{\Theta}}_0(z)$. Hence, it is not obvious, in general, whether the approach of searching for eligible separators $\widehat{\Theta}(z) \in \widehat{\boldsymbol{\Theta}}_0(z)$ under $l = 0$ is equivalent to that under another l that searches for eligible separators $\widehat{\Theta}(z)$ within the same class $\widehat{\boldsymbol{\Theta}}_0(z)$. That is, even if we were to search for eligible separators within the common tractable class $\widehat{\boldsymbol{\Theta}}_0(z)$ regardless of l, the effects obtained by noncausal LPTV scaling might vary in general, depending on the underlying lifting-timing l. If this is indeed the case, it would be related with the fact that an eligible separator $\widehat{\Theta}(z) \in \widehat{\boldsymbol{\Theta}}_0(z)$ under some lifting-timing l may not satisfy

$$\widehat{\Theta}(z) = \mathcal{S}(z)^* \widehat{\Theta}(z) \mathcal{S}(z) \ (z \in \partial\mathbf{D}). \tag{19}$$

Hence, the remainder of this section is devoted to discussing the properties of causal LTI, causal LPTV and noncausal LPTV scaling approaches, all in connection with the condition (19). In particular, we suggest that noncausal LPTV scaling has different properties in this respect from the other two approaches.

In the rest of this paper, we regard that separators are defined only on the unit circle $\partial\mathbf{D}$, and identify an operator with another if they take the same value for every z (or ζ) on the unit circle. For example, I and $z^* z I$ are regarded as the same separator.

(a) **Causal LTI scaling** We first consider causal LTI scaling (in the lifted framework, i.e., in the sense of Remark 2), assuming that Σ is N-periodic, in general. A causal LTI separator (7) in the lifted framework is described by $\widehat{V}_1$, $\widehat{V}_2$ and $\widehat{\Lambda}$, which are the lifted representations of the LTI systems V_1, V_2 and Λ, respectively (hence, $\widehat{V}_i = \widehat{V}_i^{(0)} = \widehat{V}_i^{(l)}$, $l = 0, 1, \cdots, N-1$). This, together with (12), leads to

$$\begin{aligned}(S_p(z)^l)^*\widehat{\Theta}_{ij}(z)S_p(z)^l =&(1/N)(S_p(z)^l)^*\widehat{V}_i(z)^*\widehat{\Lambda}\widehat{V}_j(z)S_p(z)^l\\ =&(1/N)\widehat{V}_i(z)^*(S_p(z)^l)^*\widehat{\Lambda}S_p(z)^l\widehat{V}_j(z)\\ =&(1/N)\widehat{V}_i(z)^*\widehat{\Lambda}\widehat{V}_j(z) = \widehat{\Theta}_{ij}(z) \quad (i,j=1,2).\end{aligned} \tag{20}$$

This in particular implies that (19) holds, and it, together with Theorem 5, immediately leads to the fact that a causal LTI separator in the lifted framework is eligible under one lifting-timing l if and only if it is under every timing.

If a separator $\widehat{\Theta}(z)$ satisfies (19), we say that it is shift-invariant (with respect to lifting-timing). Similarly, we say that the separator class $\widehat{\boldsymbol{\Theta}}_0(z)$ is shift-invariant if $\widehat{\boldsymbol{\Theta}}_0(z) = \{\mathcal{S}(z)^*\widehat{\Theta}(z)\mathcal{S}(z) \mid \widehat{\Theta}(z) \in \widehat{\boldsymbol{\Theta}}_0(z)\}$. In particular, if every $\widehat{\Theta}(z) \in \widehat{\boldsymbol{\Theta}}_0(z)$ is shift-invariant, we say that the separator class $\widehat{\boldsymbol{\Theta}}_0(z)$ is strongly shift-invariant. Having introduced these terms, we readily see that any class consisting of causal LTI separators is strongly shift-invariant, and thus causal LTI scaling in the lifted framework leads to the same analysis results regardless of the lifting-timing l.

(b) **Causal LPTV scaling** We next consider causal LPTV scaling. Then, it turns out that the eligibility of a causal LPTV separator given by Definition 1 depends generally on lifting-timing. This is because the systems $\widehat{V}_1$, $\widehat{V}_2$ and $\widehat{\Lambda}$ in Definition 1 are the lifted representations of N-periodic systems for which (20) fails, in general. However, this only means that the eligibility of a given causal LPTV separator depends on lifting-timing, and does not necessarily mean that the eligibility of a class of causal LPTV separators does. For example, let us take the class $\widehat{\boldsymbol{\Theta}}^{\text{causal}}_{\text{static}}$ of static LPTV separators, and consider the static causal LPTV scaling based on $\widehat{\boldsymbol{\Theta}}^{\text{causal}}_{\text{static}}$. By Definition 1, this class coincides with the set of matrices in the form of

$$\widehat{\Theta}_{\text{static}} = \left(\text{diag}[X_{ij}^1, X_{ij}^2, \cdots, X_{ij}^N]\right)_{i,j=1,2} \tag{21}$$

where X_{ij}^k $(i,j=1,2, k=1,\cdots,N)$ are constant matrices of the same size. Hence

$$\mathcal{S}(z)^*\widehat{\Theta}_{\text{static}}\mathcal{S}(z) = \left(\text{diag}[X_{ij}^2, \cdots, X_{ij}^N, X_{ij}^1]\right)_{i,j=1,2} \in \widehat{\boldsymbol{\Theta}}^{\text{causal}}_{\text{static}}, \tag{22}$$

and thus $\widehat{\boldsymbol{\Theta}}^{\text{causal}}_{\text{static}}$ is a (non-strongly) shift-invariant class. This means that whether $\widehat{\boldsymbol{\Theta}}^{\text{causal}}_{\text{static}}$ is eligible does not depend on lifting-timing. Even when we consider *dynamic* causal LPTV separators, a naturally constructed separator class would also become (non-strongly) shift-invariant unless the LPTV systems V_1 and V_2 and the constant matrix $\widehat{\Lambda}$ in the causal LPTV separator are restricted to some "distorted sets" that fail to be invariant under the one-step shift in the lifting-free time axis k in (1); considering such distorted sets would never sound sensible, and thus would be unnatural. Hence, the eligibility of a natural class of causal LPTV separators is independent of lifting-timing, and hence causal LPTV scaling also leads virtually to the same analysis results regardless of the lifting-timing l.

(c) **Noncausal LPTV scaling** We have so far discussed the relationship of lifting-timing to two types of causal scaling approaches. We have then observed that all natural classes of causal separators are shift-invariant, and thus their eligibility is virtually independent of the lifting-timing l. However, noncausal LPTV scaling exhibits a different aspect, which could be attributed to the fact that taking a general LTI systems $\widehat{V}$ in noncausal LPTV separators (recall Definition 2) corresponds to ignoring causality to some limited extent, where causality is meant here with respect to the original time axis k in (1) rather than κ in (3). To confirm the different aspect, let us take, for example, the class $\widehat{\boldsymbol{\Theta}}^{\text{noncausal}}_{\text{static}}$ of static noncausal LPTV separators, and

Table 1. Robust stability analysis considering lifting-timing l.

Timing l	Noncausal-(D,G)	Causal-(D,G)
0	0.5292	0.4567
1	0.6032	0.4567
2	0.6293	0.4567

consider the static noncausal LPTV scaling based on $\widehat{\boldsymbol{\Theta}}_{\mathrm{static}}^{\mathrm{noncausal}}$. By Definition 2, this class coincides with the set of constant matrices. Hence, it follows that the z-dependent factors in $\mathcal{S}(z)^*\widehat{\Theta}_{\mathrm{static}}\mathcal{S}(z)$ do not vanish, in general, and thus it does not belong to $\widehat{\boldsymbol{\Theta}}_{\mathrm{static}}^{\mathrm{noncausal}}$. That is, the class $\widehat{\boldsymbol{\Theta}}_{\mathrm{static}}^{\mathrm{noncausal}}$ of static noncausal LPTV separators is not shift-invariant with respect to lifting-timing. This fact is much different from that for static *causal* LPTV separators shown in (22).

4.3 *Numerical confirmation of the properties of causal/noncausal LPTV scaling*

An outcome of the property of static noncausal LPTV scaling about the lack of shift-invariance, stated in the previous subsection, should also be easy to confirm numerically by observing the dependency of analysis results on the lifting-timing l. In fact, however, an example in that direction can never be constructed if Σ is confined to be LTI. This is because the static noncausal LPTV separator $\widehat{\Theta}$ is eligible if and only if $\mathcal{S}(z)^*\widehat{\Theta}\mathcal{S}(z)$ is, since $\widehat{G}^{(1)} = \widehat{G}^{(0)}$ and $\widehat{\Delta}^{(1)} = \widehat{\Delta}^{(0)}$ in Theorem 5 when Σ is LTI; due to this coincidence, the lack of shift-invariance in the class of static noncausal LPTV separator does not lead to difference in the analysis results with respect to the shift in the lifting-timing. We thus consider an example with an LPTV system Σ.
Example: Consider the 3-periodic system G given by

$$A_0=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0.1&0.1&-0.8&-1\end{bmatrix},\ A_1=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0.2&-0.4&0.01&0.2\end{bmatrix},\ A_2=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-0.4&-0.3&0.3&0.5\end{bmatrix},$$

$$\begin{aligned}&B_0=B_1=B_2=\begin{bmatrix}0&0&0&1\end{bmatrix}^T,\ C_0=\begin{bmatrix}0.3&0.2&0.5&0.1\end{bmatrix},\ C_1=\begin{bmatrix}0.3&0.3&0.3&0.4\end{bmatrix},\\&C_2=\begin{bmatrix}0.2&0.6&0.4&0.2\end{bmatrix},\ D_0=D_1=D_2=0,\end{aligned}\tag{23}$$

which we can confirm to be internally stable. In addition, we assume that the corresponding scalar uncertainty $\Delta = \delta$ is static and LTI. The problem we study here is to find (a lower bound of) the maximum $\overline{\delta}$ such that the closed-loop system Σ is robustly stable against the uncertainty set $\boldsymbol{\Delta} = \{\delta : |\delta| < \overline{\delta}\}$.

The analysis results of the maximum $\overline{\delta}$ obtained by static causal/noncausal LPTV scaling are shown in Table 1, where we confined ourselves to the class of (D,G)-scaling type separators (Fan et al. 1991) in both approaches. Such analysis can be carried out through the KYP lemma (Rantzer 1996) and LMI optimization (see Hagiwara and Ohara (2010)).

Table 1 shows that the analysis results of $\overline{\delta}$ obtained by noncausal LPTV scaling depend much on the lifting-timing l. In contrast, we can confirm, also numerically, that the analysis results of $\overline{\delta}$ obtained by causal LPTV scaling are completely independent of l.

5 Shift-invariant reconstruction of separator classes in noncausal LPTV scaling and its implication

This section introduces the idea of shift-invariant reconstruction of separator classes. With this idea, the properties of noncausal LPTV scaling applied to LTI systems are clarified further, particularly from the viewpoint of its possible ability in replacing the conventional frequency-dependent (i.e., dynamic causal LTI) scaling.

5.1 *Shift-invariant reconstruction of separator classes*

The previous section discussed by introducing the (backward) timing-shift matrix the properties and effectiveness of noncausal LPTV scaling applied to LPTV systems. In particular, a central issue there was on the difference in the properties between noncausal LPTV scaling and causal LPTV or LTI scaling, and it was studied from the viewpoint of lifting-timing and its shift. An important key in that study was whether the separator class taken in noncausal LPTV scaling is shift-invariant with respect to lifting-timing. Motivated by this observation, suppose we are given a (not necessarily shift-invariant) class of noncausal LPTV separators denoted by $\widehat{\boldsymbol{\Theta}}_0^{\text{noncausal}}(z)$, and let us construct the separator class

$$\widehat{\boldsymbol{\Theta}}(z) := \left\{ \frac{1}{N} \sum_{l=0}^{N-1} (\mathcal{S}(z)^l)^* \widehat{\Theta}(z) \mathcal{S}(z)^l \;\middle|\; \widehat{\Theta}(z) \in \widehat{\boldsymbol{\Theta}}_0^{\text{noncausal}}(z) \right\}. \tag{24}$$

Then, every separator in $\widehat{\boldsymbol{\Theta}}(z)$ is a noncausal LPTV separator by (11), and is shift-invariant by (13). Hence, $\widehat{\boldsymbol{\Theta}}(z)$ is strongly shift-invariant. Furthermore, it is easy to see that $\widehat{\boldsymbol{\Theta}}(z) = \widehat{\boldsymbol{\Theta}}_0^{\text{noncausal}}(z)$ if and only if $\widehat{\boldsymbol{\Theta}}_0^{\text{noncausal}}(z)$ is strongly shift-invariant. Hence, $\widehat{\boldsymbol{\Theta}}(z) \neq \widehat{\boldsymbol{\Theta}}_0^{\text{noncausal}}(z)$ whenever $\widehat{\boldsymbol{\Theta}}_0^{\text{noncausal}}(z)$ is not shift-invariant. We thus call the separator class $\widehat{\boldsymbol{\Theta}}(z)$ the (strongly) shift-invariant reconstruction of the separator class $\widehat{\boldsymbol{\Theta}}_0^{\text{noncausal}}(z)$. Similarly, we call

$$\frac{1}{N} \sum_{l=0}^{N-1} (\mathcal{S}(z)^l)^* \widehat{\Theta}(z) \mathcal{S}(z)^l \tag{25}$$

the shift-invariant reconstruction of the separator $\widehat{\Theta}(z)$.

This section is primarily concerned with the case when Σ is LTI, and provides some discussions related to shift-invariant reconstruction so that further properties of noncausal LPTV scaling applied to LTI systems can be clarified. In particular, we discuss the relationship between noncausal LPTV scaling based on $\widehat{\boldsymbol{\Theta}}(z)$ and causal LTI scaling based on the separator class $\boldsymbol{\Theta}(\zeta)$ we have introduced earlier in (10). Note that both $\widehat{\boldsymbol{\Theta}}(z)$ and $\boldsymbol{\Theta}(\zeta)$ are constructed from the same class $\widehat{\boldsymbol{\Theta}}_0^{\text{noncausal}}(z)$. What we establish in this section is that, even though a direct relationship between the classes $\boldsymbol{\Theta}(\zeta)$ and $\widehat{\boldsymbol{\Theta}}_0^{\text{noncausal}}(z)$ is still open, a direct relationship between the former class $\boldsymbol{\Theta}(\zeta)$ and the shift-invariant reconstruction $\widehat{\boldsymbol{\Theta}}(z)$ of the latter class can be clarified completely. Implications of the success in this direction will also be discussed.

Remark 3 *There is no inclusion relation between* $\widehat{\boldsymbol{\Theta}}_0^{\text{noncausal}}(z)$ *and its shift-invariant reconstruction* $\widehat{\boldsymbol{\Theta}}(z)$*, in general (see Fig. 3). This can be seen by considering the case* $\widehat{\boldsymbol{\Theta}}_0^{\text{noncausal}}(z) = \widehat{\boldsymbol{\Theta}}_{\text{static}}^{\text{noncausal}}$*. In this case,* $\widehat{\boldsymbol{\Theta}}_{\text{static}}^{\text{noncausal}}$ *includes static noncausal LPTV separators that are not shift-invariant (hence do not belong to* $\widehat{\boldsymbol{\Theta}}(z)$*) and result in dynamic separators (hence do not belong to* $\widehat{\boldsymbol{\Theta}}_{\text{static}}^{\text{noncausal}}$*) when shift-invariant reconstruction is applied to them. This does imply the lack of mutual inclusion, but the intersection* $\widehat{\boldsymbol{\Theta}}(z) \cap \widehat{\boldsymbol{\Theta}}_{\text{static}}^{\text{noncausal}}$ *is nonempty because it equals the class* $\widehat{\boldsymbol{\Theta}}_{\text{static}}^{\text{LTI}}$ *of static causal LTI separators (in the lifted framework).*

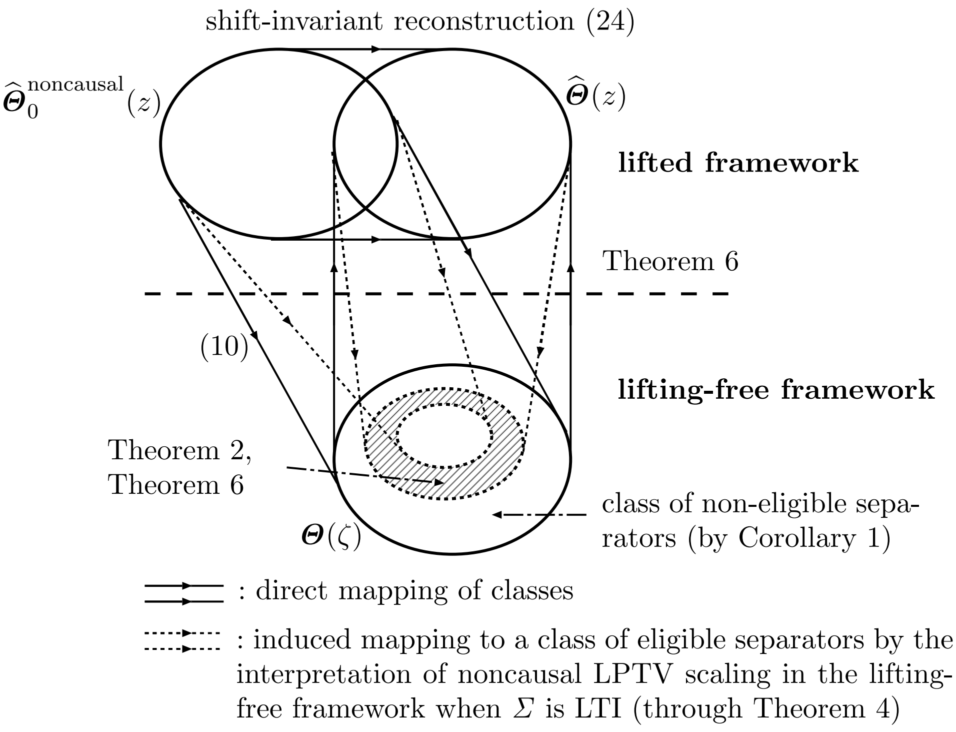

Figure 3. Relationship between some classes of separators and its implication in noncausal LPTV scaling.

5.2 *Properties of shift-invariant reconstruction in noncausal LPTV scaling*

The following theorem, whose proof is given in Appendix C, plays a crucial role in this section.

Theorem 6 *Given a noncausal LPTV separator $\widehat{\Theta}(z)$, consider the causal LTI separator $\Theta(\zeta)$ induced in the lifting-free framework by (8). Then, the equivalent separator $\widehat{\mathrm{E}}[\Theta(\zeta)]$ in the lifted framework coincides with the shift-invariant reconstruction (25) of $\widehat{\Theta}(z)$.*

This theorem together with (10) and (24) implies that the shift-invariant reconstruction $\widehat{\boldsymbol{\Theta}}(z)$ of the separator class $\widehat{\boldsymbol{\Theta}}_0^{\text{noncausal}}(z)$ is nothing but the class $\{\widehat{\mathrm{E}}[\Theta(\zeta)] \mid \Theta(\zeta) \in \boldsymbol{\Theta}(\zeta)\}$ of equivalent embedded causal LTI separators in the lifted framework (see Fig. 3).

Combining our preceding arguments, we are led immediately to the following result about the robust stability analysis of the LTI system Σ.

Corollary 1 *Suppose that G is LTI and $\boldsymbol{\Delta}$ is a set satisfying Assumption 1_{LTI}. Given a noncausal LPTV separator $\widehat{\Theta}(z)$, the associated $\Theta(\zeta)$ in (8) is eligible in the lifting-free framework if and only if the shift-invariant reconstruction (25) of $\widehat{\Theta}(z)$ is eligible in the lifted framework. In particular, given $\widehat{\boldsymbol{\Theta}}_0^{\text{noncausal}}(z)$, $\boldsymbol{\Theta}(\zeta)$ is eligible in the lifting-free framework if and only if $\widehat{\boldsymbol{\Theta}}(z)$ is in the lifted framework.*

Proof Necessity follows from Theorem 2 and Theorem 6, while sufficiency follows from Theorem 3 and Theorem 6. □

5.3 *Implication of the properties of shift-invariant reconstruction in noncausal LPTV scaling*

In the above, we have shown Theorem 6 and Corollary 1 as the main results in this section. We next discuss some facts revealed by these new results so that the significance of these results can be demonstrated. We refer to Fig. 3 to this end, in which the upper part is related to the use of

Theorem 1 (i.e., the lifted framework), while the lower part is related to the use of Theorem 1_{LTI} (i.e., the lifting-free framework).

5.3.1 The classes $\boldsymbol{\Theta}_0(\zeta)$ and $\boldsymbol{\Theta}_1(\zeta)$, their inclusion relation, and their relevance to shift-invariant reconstruction

Let us consider the two subclasses contained in $\boldsymbol{\Theta}(\zeta)$ in this figure. The inner subclass is denoted by $\boldsymbol{\Theta}_0(\zeta)$ while the outer by $\boldsymbol{\Theta}_1(\zeta)$. By definition, $\boldsymbol{\Theta}_0(\zeta)$ is such a set of the eligible separators in the lifting-free framework that are obtained by applying (8) to all *eligible separators* $\widehat{\Theta}(z) \in \widehat{\boldsymbol{\Theta}}_0^{\mathrm{noncausal}}(z)$ in the lifted framework. $\boldsymbol{\Theta}_1(\zeta)$ is defined similarly by replacing $\widehat{\boldsymbol{\Theta}}_0^{\mathrm{noncausal}}(z)$ with its shift-invariant reconstruction $\widehat{\boldsymbol{\Theta}}(z)$. The introduction of the class $\boldsymbol{\Theta}_0(\zeta)$ is motivated by Theorem 4, but it is not ensured that $\boldsymbol{\Theta}_0(\zeta)$ coincides with the subset consisting of all eligible separators in $\boldsymbol{\Theta}(\zeta)$. This is because of the lack of the converse assertion in this theorem. This implies that the lifted framework with $\widehat{\boldsymbol{\Theta}}_0^{\mathrm{noncausal}}(z)$ is not always equivalent to the lifting-free framework with $\boldsymbol{\Theta}(\zeta)$, but could in fact be more conservative. As we have discussed in Section 3, we have been interested in analyzing such a gap. The purpose of this subsection is to show that the preceding arguments in this section about shift-invariant reconstruction successfully lead to clarifying the gap. In fact, it will turn out that the separator class $\boldsymbol{\Theta}_1(\zeta)$ (and thus the shift-invariant reconstruction of $\widehat{\boldsymbol{\Theta}}_0^{\mathrm{noncausal}}(z)$) plays a crucial role in characterizing the gap.

Before proceeding, we first remark that the inclusion between $\boldsymbol{\Theta}_0(\zeta)$ and $\boldsymbol{\Theta}_1(\zeta)$ implicitly asserted in Fig. 3 is not trivial since there is generally no inclusion between $\widehat{\boldsymbol{\Theta}}_0^{\mathrm{noncausal}}(z)$ and $\widehat{\boldsymbol{\Theta}}(z)$ (Remark 3). However, we indeed have $\boldsymbol{\Theta}_0(\zeta) \subset \boldsymbol{\Theta}_1(\zeta)$, and Theorem 6 plays a crucial role in its proof as follows. Let us take an arbitrary separator $\Theta(\zeta) \in \boldsymbol{\Theta}_0(\zeta)$. By the definition of $\boldsymbol{\Theta}_0(\zeta)$, there exists an eligible separator $\widehat{\Theta}(z) \in \widehat{\boldsymbol{\Theta}}_0^{\mathrm{noncausal}}(z)$ such that the above $\Theta(\zeta)$ is represented by (8). On the other hand, the equivalent separator $\widehat{\mathrm{E}}[\Theta(\zeta)]$ in the lifted framework corresponding to $\Theta(\zeta)$ eligible in the lifting-free framework is eligible by Theorem 2, and it belongs to $\widehat{\boldsymbol{\Theta}}(z)$ by Theorem 6. By the definition of $\boldsymbol{\Theta}_1(\zeta)$, this implies that $\boldsymbol{\Theta}_0(\zeta) \subset \boldsymbol{\Theta}_1(\zeta)$. This inclusion implies that we can carry out robust stability analysis in a less conservative fashion by dealing with the shift-invariant reconstruction of separator classes in the lifted framework (at the possible expense of increased complexity in the search of eligible separators).

5.3.2 Implication and significance of shift-invariant reconstruction

Regarding the significance suggested above about the introduction of the shift-invariant reconstruction $\widehat{\boldsymbol{\Theta}}(z)$ and the corresponding class $\boldsymbol{\Theta}(\zeta)$ in the lifting-free framework, we can reveal a much more important and stronger result. In fact, it follows immediately from Corollary 1 that $\boldsymbol{\Theta}_1(\zeta)$ coincides with the class of all eligible separators in $\boldsymbol{\Theta}(\zeta)$. The implication of this fact on the properties of noncausal LPTV scaling is as follows.

As we have discussed earlier, this paper is motivated by the possible ability of (static) noncausal LPTV scaling in replacing the conventional frequency-dependent (i.e., dynamic causal LTI) scaling, as suggested by Theorem 4. Due to the lack of the converse assertion of that theorem, however, the degree of such an ability has not been fully clarified. In particular, it has not been clear if (under some conditions) noncausal LPTV scaling with the separator class $\widehat{\boldsymbol{\Theta}}_0^{\mathrm{noncausal}}(z)$ is equivalent to the conventional causal LTI scaling with the separator class $\boldsymbol{\Theta}(\zeta)$ derived from $\widehat{\boldsymbol{\Theta}}_0^{\mathrm{noncausal}}(z)$ as in (10). What is established by the above observation is that, even though a direct answer to the above question is still open, replacing $\widehat{\boldsymbol{\Theta}}_0^{\mathrm{noncausal}}(z)$ with its shift-invariant reconstruction $\widehat{\boldsymbol{\Theta}}(z)$ in noncausal LPTV scaling does lead, when it is interpreted in the lifting-free framework, equivalently to causal LTI scaling with $\boldsymbol{\Theta}(\zeta)$.

Before closing this section, we note that we could also derive the following result that is somewhat relevant to the inclusion $\boldsymbol{\Theta}_0(\zeta) \subset \boldsymbol{\Theta}_1(\zeta)$, as an immediate consequence of Theorem 5.

Corollary 2 *Suppose that G is LTI, $\boldsymbol{\Delta}$ satisfies Assumption 1_{LTI}, and a class $\widehat{\boldsymbol{\Theta}}_0^{\mathrm{noncausal}}(z)$ of noncausal LPTV separators is given. If the class $\widehat{\boldsymbol{\Theta}}_0^{\mathrm{noncausal}}(z)$ is eligible, the class $\widehat{\boldsymbol{\Theta}}(z)$ is also eligible.*

We have seen in the above the significance of the main results of this paper (Theorem 6 and Corollary 1) in clarifying the ability of noncausal LPTV scaling applied to the robust stability analysis of LTI systems. This clearly demonstrates that the lifting-timing, timing-shift matrix and shift-invariant reconstruction of separator classes are quite important also in the robust stability analysis of the LTI system Σ, in spite of the fact that the lifted representations of LTI systems are independent of lifting-timing. As such, the arguments of this paper are expected to provide a basis for further studies on the properties and effectiveness of noncausal LPTV scaling applied to LTI systems as well as LPTV systems.

5.4 *Numerical confirmation of the inclusion relationship*

This subsection numerically confirms the inclusion relationship $\boldsymbol{\Theta}_0(\zeta) \subset \boldsymbol{\Theta}_1(\zeta)$ and the equivalence relationship between noncausal LPTV scaling based on $\widehat{\boldsymbol{\Theta}}(z)$ and causal LTI scaling based on $\boldsymbol{\Theta}(\zeta)$ in terms of conservativeness in robust stability analysis, which are shown in Fig. 3.

We consider the internally stable LTI system G given by

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccccc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0.1 \\ -0.2 & -0.62 & 0.01 & 0.6 & -0.7 & 0.1 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 0.1 & 0.2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]. \tag{26}$$

In addition, we assume the corresponding uncertainties Δ are static LTI and structured as given by $\Delta = \mathrm{diag}[\delta_1 \ \delta_2]$. The purpose here is to compute (a lower bound of) the maximum $\overline{\delta}$ such that the closed-loop system Σ is robustly stable against the uncertainty set $\boldsymbol{\Delta} = \{\Delta : \|\Delta\| < \overline{\delta}\}$. In such analysis, we employ the idea of the (D,G)-scaling approach (Fan et al. 1991) and take the following four types of separator classes: the first one is the class $\widehat{\boldsymbol{\Theta}}_{\mathrm{static},(D,G)}^{\mathrm{noncausal}}$ of static noncausal LPTV separators of the (D,G)-scaling type under a prescribed N (which we take equal to 6), which we view as $\widehat{\boldsymbol{\Theta}}_0^{\mathrm{noncausal}}(z)$ in the preceding arguments. The other three are the class $\boldsymbol{\Theta}(\zeta)$ constructed from the above class through (10), the class $\widehat{\boldsymbol{\Theta}}(z)$ constructed from the same class through (24), and the class $\boldsymbol{\Theta}_{\mathrm{static},(D,G)}$ of static causal LTI separators of the (D,G)-scaling type.

Note that the second and fourth are separator classes for the lifting-free framework; the reason why we take the fourth is as follows: since the second is induced by *static separators* in the lifted framework (and the third is asserted to have an ability equivalent to the second), we also take, for reference, the fourth one so that we can also demonstrate the advantage of the lifted framework over the lifting-free framework *under the common setting using only static separators.*

Remark 4 *Separators in $\boldsymbol{\Theta}(\zeta)$ are dynamic, in general, and thus this class is less tractable than the above two classes of static separators. However, since free parameters in such separators are contained only in its "numerator part" by (8), no essential difference arises from the case of static separators. That is, eligibility of $\boldsymbol{\Theta}(\zeta)$ can be checked without conservativeness through the KYP lemma (Rantzer 1996) applied to the inequality $(4)_{\mathrm{LTI}}$, because $(5)_{\mathrm{LTI}}$ is satisfied automatically for any $\widehat{\Theta} \in \widehat{\boldsymbol{\Theta}}_{\mathrm{static},(D,G)}^{\mathrm{noncausal}}$, thanks to the properties of (D,G)-scaling. To avoid distraction and concentrate on demonstrating the significance of the preceding arguments in this section, however,*

Table 2. Robust stability analysis with each class of separators ($N = 6$).

Separator class	$\overline{\delta}$
$\boldsymbol{\Theta}_{\mathrm{static},(D,G)}$	1.2115
$\widehat{\boldsymbol{\Theta}}^{\mathrm{noncausal}}_{\mathrm{static},(D,G)}$	1.4091
$\boldsymbol{\Theta}(\zeta)$	1.4942
$\widehat{\boldsymbol{\Theta}}(z)$	1.4942

the details are omitted about the numerical search process of separators (see Hosoe and Hagiwara (2011) for details). Essentially the same comment applies to the treatment of $\widehat{\boldsymbol{\Theta}}(z)$.

The numerical results of the analysis of $\overline{\delta}$ are shown in Table 2, from which we can first confirm that static noncausal LPTV scaling is less conservative than static causal LTI scaling in the lifting-free framework. However, we also see that the former is still more conservative than the dynamic causal LTI scaling in the lifting-free framework based on $\boldsymbol{\Theta}(\zeta)$. In other words, this is an example in which noncausal LPTV scaling with the separator class $\widehat{\boldsymbol{\Theta}}_0^{\mathrm{noncausal}}(z)$ fails to equivalently check eligibility of the associated separator class $\boldsymbol{\Theta}(\zeta)$ in the lifting-free framework. However, it can be also confirmed from Table 2 that the result obtained through noncausal LPTV scaling based on $\widehat{\boldsymbol{\Theta}}(z)$, which is the shift-invariant reconstruction of $\widehat{\boldsymbol{\Theta}}^{\mathrm{noncausal}}_{\mathrm{static},(D,G)}$, is no more conservative than that of causal LTI scaling based on $\boldsymbol{\Theta}(\zeta)$. This confirms not only the inclusion relationship $\boldsymbol{\Theta}_0(\zeta) \subset \boldsymbol{\Theta}_1(\zeta)$ but also the equivalence in the ability of the two scaling approaches with the separator classes $\boldsymbol{\Theta}(\zeta)$ and $\widehat{\boldsymbol{\Theta}}(z)$ defined in the lifting-free framework and lifted framework, respectively.

6 Conclusion

This paper developed a new direction for studying the properties of noncausal LPTV scaling, which has been introduced as a new method for robust stability analysis by applying the separator-type robust stability theorem and the discrete-time lifting technique. A key idea in this direction was to consider the timing of lifting, and shift-invariance of separator classes was introduced as a key notion relevant to the lifting-timing. It was then discussed that, compared with causal LPTV scaling and the conventional frequency-dependent scaling (i.e., causal LTI scaling), noncausal LPTV scaling has, in general, different properties about the shift-invariance of the separator classes it deals with. It was also discussed how taking a different lifting-timing in noncausal LPTV scaling could affect the robust stability analysis of LPTV systems. The robust stability analysis of LTI systems with noncausal LPTV scaling, on the other hand, is not affected by the lifting-timing. This, however, never implies that the idea of lifting-timing is meaningful only in the treatment of LPTV systems. Instead, it was established that the idea of shift-invariant reconstruction of separator classes plays an important role in clarifying further properties of noncausal LPTV scaling applied to LTI systems. In particular, we have given a partial answer to the open question about static noncausal LPTV scaling, i.e., how substantial and promising its ability is in equivalently inducing frequency-dependent scaling in the conventional lifting-free framework.

References

Bittanti, S. and Colaneri, P. (2000), "Invariant representations of discrete-time periodic systems," *Automatica*, **36**-12, pp. 1777–1793.

Bittanti, S. and Colaneri, P. (2009), *Periodic systems: filtering and control*, Springer.

Fan, M.K.H., Tits, A.L., and Doyle, J.C. (1991), "Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics," *IEEE Trans. Automat. Contr.*, **36**-1, pp. 25–38.
Hagiwara, T. and Ohara, Y. (2007), "Noncausal linear periodically time-varying scaling for discrete-time systems," in *IFAC Workshop on Periodic Control Systems*.
Hagiwara, T. and Ohara, Y. (2010), "Noncausal linear periodically time-varying scaling for robust stability analysis of discrete-time systems: frequency-dependent scaling induced by static separators," *Automatica*, **46**-1, pp. 167–173.
Hosoe, Y. and Hagiwara, T. (2010a), "Robust stabilizing controller synthesis based on discrete-time noncausal linear periodically time-varying scaling (in Japanese)," *Transactions of the Society of Instrument and Control Engineers*, **46**-4, pp. 219–228.
Hosoe, Y. and Hagiwara, T. (2010b), "Synthesis of robust performance controllers based on discrete-time noncausal linear periodically time-varying scaling," in *IFAC Workshop on Periodic Control Systems*, Fr2B-79.
Hosoe, Y. and Hagiwara, T. (2011), "Relationship between Noncausal Linear Periodically Time-Varying Scaling and Causal Linear Time-Invariant Scaling for Discrete-Time Systems," in *IFAC World Congress*, to appear.
Iwasaki, T. and Hara, S. (1998), "Well-posedness of feedback systems: Insights into exact robustness analysis and approximate computations," *IEEE Trans. Automat. Contr.*, **43**-5, pp. 619–630.
Rantzer, A. (1996), "On the Kalman-Yakubovich-Popov lemma," *Systems & Control Letters*, **28**-1, pp. 7–10.
Vaidyanathan, P.P. (1993), *Multirate Systems and Filter Banks*, Prentice Hall, New Jersey.
Zhou, K. and Doyle, J.C. (1998), *Essentials of Robust Control*, Prentice Hall.

Appendix A: Proof of Theorem 2

Let $\phi := \exp(2\pi \mathrm{i}/N)$, where i denotes the imaginary unit. Let us define the matrix

$$U_p(\zeta) := \frac{1}{\sqrt{N}} \left[T_p(\zeta) \; T_p(\phi\zeta) \; \cdots \; T_p(\phi^{N-1}\zeta) \right], \tag{A.1}$$

where $T_p(\zeta)$ is given in (9). It follows that $U_p(\zeta)$ is a unitary matrix for $\zeta \in \partial\mathbf{D}$. Since

$$\widehat{G}(\zeta^N) T_p(\zeta) = T_p(\zeta) G(\zeta) \tag{A.2}$$

holds (Bittanti and Colaneri 2000, Vaidyanathan 1993), we immediately see that

$$\widehat{G}(\zeta^N) U_p(\zeta) = U_p(\zeta) \underline{G(\zeta)}, \tag{A.3}$$

where, given a ζ-dependent matrix $M(\zeta)$, we use the shorthand notation

$$\underline{M(\zeta)} = \mathrm{diag}[M(\zeta), M(\phi\zeta), \cdots, M(\phi^{N-1}\zeta)]. \tag{A.4}$$

We now proceed to the proof of Theorem 2. Let us take an eligible causal LTI separator given by (6), where V_1 and V_2 are LTI systems with p inputs. Since it satisfies $(4)_{\mathrm{LTI}}$ and $(5)_{\mathrm{LTI}}$ and since $\phi \in \partial\mathbf{D}$, it also satisfies these two inequalities with ζ replaced by $\phi^i\zeta$ $(i = 1, \cdots, N-1)$.

In other words, we have

$$\underline{\begin{bmatrix} I \\ G(\zeta) \end{bmatrix}^* \Theta(\zeta) \begin{bmatrix} I \\ G(\zeta) \end{bmatrix}} \le 0 \quad (\forall \zeta \in \partial \mathbf{D}) \tag{A.5}$$

$$\underline{\begin{bmatrix} \Delta(\zeta) \\ I \end{bmatrix}^* \Theta(\zeta) \begin{bmatrix} \Delta(\zeta) \\ I \end{bmatrix}} > 0 \quad \begin{pmatrix} \forall \Delta \in \boldsymbol{\Delta}, \\ \forall \zeta \in \partial \mathbf{D} \end{pmatrix}. \tag{A.6}$$

Through appropriate permutations of rows and columns, (A.5) and (A.6) are equivalently transformed into

$$\begin{bmatrix} \underline{I} \\ \underline{G(\zeta)} \end{bmatrix}^* \left(\underline{\Theta_{ij}(\zeta)}\right)_{i,j=1,2} \begin{bmatrix} \underline{I} \\ \underline{G(\zeta)} \end{bmatrix} \le 0 \quad (\forall \zeta \in \partial \mathbf{D}) \tag{A.7}$$

$$\begin{bmatrix} \underline{\Delta(\zeta)} \\ \underline{I} \end{bmatrix}^* \left(\underline{\Theta_{ij}(\zeta)}\right)_{i,j=1,2} \begin{bmatrix} \underline{\Delta(\zeta)} \\ \underline{I} \end{bmatrix} > 0 \quad \begin{pmatrix} \forall \Delta \in \boldsymbol{\Delta}, \\ \forall \zeta \in \partial \mathbf{D} \end{pmatrix}. \tag{A.8}$$

Let us define

$$\mathcal{U}(\zeta) = \mathrm{diag}[U_p(\zeta), U_p(\zeta)]. \tag{A.9}$$

Then, by applying the congruence transformation with $U_p(\zeta)^*$ on (A.7) and (A.8), and by noting the relation (A.3), we have

$$\begin{bmatrix} I \\ \widehat{G}(\zeta^N) \end{bmatrix}^* \mathcal{U}(\zeta)\underline{\Theta}(\zeta)\mathcal{U}(\zeta)^* \begin{bmatrix} I \\ \widehat{G}(\zeta^N) \end{bmatrix} \le 0 \quad (\forall \zeta \in \partial \mathbf{D}) \tag{A.10}$$

$$\begin{bmatrix} \widehat{\Delta}(\zeta^N) \\ I \end{bmatrix}^* \mathcal{U}(\zeta)\underline{\Theta}(\zeta)\mathcal{U}(\zeta)^* \begin{bmatrix} \widehat{\Delta}(\zeta^N) \\ I \end{bmatrix} > 0 \quad \begin{pmatrix} \forall \Delta \in \boldsymbol{\Delta}, \\ \forall \zeta \in \partial \mathbf{D} \end{pmatrix}. \tag{A.11}$$

Regarding the separator in (A.10) and (A.11), we have the following again from (A.3).

$$\begin{aligned} U_p(\zeta)\underline{\Theta_{ij}(\zeta)}U_p(\zeta)^* &= U_p(\zeta)\underline{V_i(\zeta)}^*\underline{\Lambda}\underline{V_j(\zeta)}U_p(\zeta)^* \\ &= \widehat{V}_i(\zeta^N)^* U_p(\zeta)\underline{\Lambda}U_p(\zeta)^*\widehat{V}_j(\zeta^N) \\ &= \widehat{V}_i(\zeta^N)^*\widehat{\Lambda}U_p(\zeta)U_p(\zeta)^*\widehat{V}_j(\zeta^N) \\ &= \widehat{V}_i(\zeta^N)^*\widehat{\Lambda}\widehat{V}_j(\zeta^N) \quad (\forall \zeta \in \partial \mathbf{D}). \end{aligned} \tag{A.12}$$

This implies that the separator (7) is eligible in the lifted framework. This completes the proof.

Appendix B: Proof of Theorem 3

Suppose that $\widehat{\Theta}(z) = \widehat{\mathrm{E}}[\Theta(\zeta)]$ given by (7) satisfies (4) and (5). Then, by post-multiplying (resp. pre-multiplying) these inequalities with $T_p(\zeta)$ (resp. its complex conjugate transpose) and by noting (A.2), we have

$$\begin{bmatrix} I \\ G(\zeta) \end{bmatrix}^* \Theta_{T_p}(\zeta) \begin{bmatrix} I \\ G(\zeta) \end{bmatrix} \le 0 \quad (\forall \zeta \in \partial \mathbf{D}) \tag{B.1}$$

$$\begin{bmatrix} \Delta(\zeta) \\ I \end{bmatrix}^* \Theta_{T_p}(\zeta) \begin{bmatrix} \Delta(\zeta) \\ I \end{bmatrix} > 0 \quad \begin{pmatrix} \forall \Delta \in \boldsymbol{\Delta}, \\ \forall \zeta \in \partial \mathbf{D} \end{pmatrix}, \tag{B.2}$$

where

$$\Theta_{T_p}(\zeta) = \left(\frac{1}{N} T_p(\zeta)^* \widehat{V}_i(\zeta^N)^* \widehat{\Lambda} \widehat{V}_j(\zeta^N) T_p(\zeta)\right)_{i,j=1,2}. \tag{B.3}$$

Regarding the above separator $\Theta_{T_p}(\zeta)$, we have

$$\frac{1}{N} V_i(\zeta)^* T_p(\zeta)^* \widehat{\Lambda} T_p(\zeta) V_j(\zeta) = V_i(\zeta)^* \Lambda V_j(\zeta) \quad (\forall \zeta \in \partial \mathbf{D}) \tag{B.4}$$

by (A.2), since $\widehat{V}_i$ $(i=1,2)$ are the lifted representations of the LTI systems V_i $(i=1,2)$, and $\widehat{\Lambda} = \mathrm{diag}[\Lambda, \cdots, \Lambda]$. That is, $\Theta_{T_p}(\zeta)$ is nothing but the causal LTI separator in the lifting-free framework underlying (7). This completes the proof.

Appendix C: Proof of Theorem 6

The noncausal LPTV separator $\widehat{\Theta}(z)$ can be described by

$$\widehat{\Theta}(z) = \left(\widehat{\Theta}_{ij}(z)\right)_{i,j=1,2}, \quad \widehat{\Theta}_{ij}(z) = \widehat{V}_i(z)^* \Gamma \widehat{V}_j(z) \tag{C.1}$$

where $\widehat{V}(z) =: [\widehat{V}_1(z)\ \widehat{V}_2(z)]$. Hence, $\Theta(\zeta)$ given by (8) is described by

$$\Theta(\zeta) = (\Theta_{ij}(\zeta))_{i,j=1,2} = \left(T_p(\zeta)^* \widehat{V}_i(\zeta^N)^* \Gamma \widehat{V}_j(\zeta^N) T_p(\zeta)\right)_{i,j=1,2}. \tag{C.2}$$

According to the discussion in Appendix A, the derivation of the equivalent causal LTI separator $\widehat{\Theta}(z) = \widehat{\mathrm{E}}[\Theta(\zeta)]$ in the lifted framework corresponding to $\Theta(\zeta)$ amounts to representing $(1/N) U_p(\zeta) \underline{\Theta_{ij}(\zeta)} U_p(\zeta)^*$ in terms of ζ^N. Regarding this issue, we have

$$\begin{aligned}
U_p(\zeta) \underline{\Theta_{ij}(\zeta)} U_p(\zeta)^* &= U_p(\zeta) \underline{T_p(\zeta)^*}\, \underline{\widehat{V}_i(\zeta^N)^* \Gamma \widehat{V}_j(\zeta^N)}\, \underline{T_p(\zeta)} U_p(\zeta)^* \\
&= \widehat{T}_p(\zeta^N)^* U_p(\zeta) \underline{\widehat{V}_i(\zeta^N)^* \Gamma \widehat{V}_j(\zeta^N)} U_p(\zeta)^* \widehat{T}_p(\zeta^N) \\
&= \widehat{T}_p(\zeta^N)^* \{I_N \otimes (\widehat{V}_i(\zeta^N)^* \Gamma \widehat{V}_j(\zeta^N))\} \widehat{T}_p(\zeta^N) && \text{(C.3)} \\
&= \widehat{T}_p(\zeta^N)^* \{I_N \otimes \widehat{\Theta}_{ij}(\zeta^N)\} \widehat{T}_p(\zeta^N) \quad (\forall \zeta \in \partial \mathbf{D}), && \text{(C.4)}
\end{aligned}$$

where $\otimes$ denotes the Kronecker product; in the reduction to (C.3), note that $\widehat{V}_i(\zeta^N)^* \Gamma \widehat{V}_j(\zeta^N)$ is invariant under the replacement of ζ by $\phi\zeta$ and that its size is $p \times p$, which is the same as that of the identity matrices in $T_p(\cdot)$ contained in $U_p(\zeta)$. Hence, the causal LTI separator $\widehat{\mathrm{E}}[\Theta(\zeta)]$ in the lifted framework equivalent to $\Theta(\zeta)$ in the lifting-free framework is given by

$$\widehat{\Theta}(z) = \left(\frac{1}{N} \widehat{T}_p(z)^* \{I_N \otimes \widehat{\Theta}_{ij}(z)\} \widehat{T}_p(z)\right)_{i,j=1,2}. \tag{C.5}$$

To describe this separator in a simpler form, we first aim at giving an explicit form of $\widehat{T}_p(z)$. By the definition of $T_p(\zeta)$, it can be realized with

$$\left[\begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array}\right] := \left[\begin{array}{cc|c} & I_{(N-2)p} & 0_{(N-2)p\times p} \\ 0_p & & I_p \\ \hline I_p & & 0_p \\ & I_{(N-2)p} & 0_{(N-2)p\times p} \\ & 0_{p\times(N-2)p} & I_p \end{array}\right]. \tag{C.6}$$

Hence, by the definition of lifting of systems, $\widehat{T}_p(z)$ can be realized with $(\widehat{A}_T, \widehat{B}_T, \widehat{C}_T, \widehat{D}_T)$ given by

$$\left[\begin{array}{c|cccc} A_T^N & A_T^{N-1}B_T & A_T^{N-2}B_T & \dots & B_T \\ \hline C_T & D_T & & & \\ C_T A_T & C_T B_T & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ C_T A_T^{N-1} & C_T A_T^{N-2} B_T & \dots & C_T B_T & D_T \end{array}\right]. \tag{C.7}$$

By direct calculations, we can obtain

$$A_T^m = \begin{bmatrix} & I_{(N-1-m)p} \\ 0_{mp} & \end{bmatrix}, \quad A_T^m B_T = \begin{bmatrix} 0_{(N-2-m)p\times p} \\ I_p \\ 0_{mp\times p} \end{bmatrix}, \tag{C.8}$$

$$C_T A_T^m = \begin{bmatrix} & I_{(N-1-m)p} \\ 0_{(m+1)p} & \end{bmatrix}, \quad C_T A_T^m B_T = \begin{bmatrix} 0_{(N-2-m)p\times p} \\ I_p \\ 0_{(m+1)p\times p} \end{bmatrix}. \tag{C.9}$$

In particular, $A_T^N = 0$. Therefore, we obtain

$$\widehat{T}_p(z) = z^{-1}\widehat{C}_T\widehat{B}_T + \widehat{D}_T = \begin{bmatrix} S_p(z)^{N-1} \\ \vdots \\ S_p(z) \\ I_{Np} \end{bmatrix}. \tag{C.10}$$

Substituting (C.10) into (C.5) immediately leads to (25). This completes the proof.