Sparsity and Connectivity of Medial Graphs: Concerning Two Edge-disjoint Hamiltonian Paths in Planar Rigidity Circuits

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Abstract

A simple undirected graph $G = (V, E)$ is a rigidity circuit if $|E| = 2|V| - 2$ and $|E_G[X]| \leq 2|X| - 3$ for every $X \subset V$ with $2 \leq |X| \leq |V| - 1$, where $E_G[X]$ denotes the set of edges connecting vertices in X. It is known that a rigidity circuit can be decomposed into two edge-disjoint spanning trees. Graver, Servatius and Servatius (1993) asked if any rigidity circuit with maximum degree 4 can be decomposed into two edge-disjoint Hamiltonian paths. This paper presents infinitely many counterexamples for the question. Counterexamples are constructed based on a new characterization of a 3-connected plane graph in terms of the sparsity of its medial graph and a sufficient condition for the connectivity of medial graphs.

Keywords: Rigidity circuits; Edge-disjoint Hamiltonian paths; Medial graphs; Sparsity; Connectivity

1 Introduction

A simple undirected graph $G = (V_G, E_G)$ is a *rigidity circuit* if it satisfies $|E_G| = 2|V_G| - 2$ and the following *sparsity condition*:

$$|E_G[X]| \leq 2|X| - 3 \qquad \text{for every } X \subset V_G \text{ with } 2 \leq |X| \leq |V_G| - 1, \tag{1}$$

where $E_G[X]$ denotes the set of edges connecting vertices in X. Rigidity circuits arose in the study of combinatorial rigidity; see [6].

Any vertex of a rigidity circuit has degree at least 3. This implies that, if G is a rigidity circuit with degree at most 4, G has exactly four vertices of degree 3 and the other vertices have degree 4. Any rigidity circuit is known to be decomposable into two edge-disjoint spanning trees by Nash-Williams' forest-partition-theorem [15]. Motivated by these facts, Graver, Servatius and Servatius [6, Exercise 4.69 (Open Question)] posed the question whether any rigidity circuit with maximum degree 4 can be decomposed into two edge-disjoint Hamiltonian paths. In this paper we present counterexamples to the question, even in a restricted case.

Theorem 1. *There are infinitely many 3-connected planar rigidity circuits with maximum degree 4 which cannot be decomposed into two edge-disjoint Hamiltonian paths.*

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For the proof we present a simple construction of counterexamples based on classical results on the Hamiltonian decomposability of regular graphs as well as a new characterization of 3-connected plane graphs in terms of the sparsity of *medial graphs* (Theorem 2) and a sufficient condition for the connectivity of medial graphs (Theorem 6). In Section 2 we present these new properties of medial graphs. The proof of Theorem 1 will be given in Section 3. We conclude the paper with remarks on the rigidity of medial graphs in Section 4. For more detail on the Hamiltonian decomposability, see e.g., [5].

Just before this submission, we learned that counterexamples were also discovered by [12] independently of us. We remark that [12] mainly concerns with longest paths of rigidity circuits while the main contribution of this paper is to clarify the relation between medial graphs and planar rigidity circuits.

2 Sparsity and Connectivity of Medial Graphs

2.1 Preliminaries

Throughout the paper the vertex set and the edge set of an undirected graph G are denoted by V_G and E_G, respectively. For $F \subseteq E_G$, $V_G(F)$ denotes the set of endvertices of F. G is called *simple* if G has neither loops nor parallel edges.

A vertex subset $S \subset V_G$ (resp., an edge subset $S \subset E_G$) is called a *separator* (resp., a *cut*) if the removal of S disconnects G. G is called *k-connected* (resp., *k-edge-connected*) if the size of any separator (resp., any cut) is at least k. A separator (resp., a cut) is called *nontrivial* if its removal disconnects G into at least two nontrivial connected components, where a connected component is called trivial if it consists of a single vertex. G is called *essentially k-connected* (resp., *essentially k-edge-connected*) if the size of any nontrivial separator (resp., any nontrivial cut) is at least k.

If G satisfies the sparsity condition (1), G is said to be *sparse*. Similarly, a simple graph G is called *weakly sparse* if it satisfies the following weak sparsity condition:

$$|E_G[X]| \leq 2|X| - 3 \qquad \text{for every } X \subset V_G \text{ with } 2 \leq |X| \leq |V_G| - 2. \tag{2}$$

Let G be a plane graph. A *corner* $\{e, f\}$ of a face is a pair of consecutive edges in the face boundary, where $e = f$ may hold if they are incident to a vertex of degree 1. The *medial graph* $G^\star$ of G is defined as a graph whose vertex set is E_G and whose edge set is the set of all corners in G. Namely, two vertices are joined by an edge if they form a corner of a face in G. If G has an edge e incident to a vertex of degree 1, then the vertex corresponding to e is incident to a loop in $G^\star$. Also, if two edges e and f are incident at a vertex of degree 2, then $G^\star$ contains parallel edges between the corresponding two vertices. See Figure 1 or Figure 3 for an example. Notice that $G^\star$ always becomes 4-regular. We also remark that every simple 4-regular plane graph is the medial graph of a plane graph (see e.g., [2]).

To avoid ambiguity, a vertex (resp., a vertex subset) of $G^\star$ corresponding to $e \in E_G$ (resp., $F \subseteq E_G$) is denoted by $e^\star$ (resp., $F^\star$) throughout the paper. Observe that, for each edge $e^\star f^\star \in E_{G^\star}$, there is the unique vertex $v \in V_G$ that is incident with e and f in G. We define $\phi : E_{G^\star} \to V_G$ as a surjective map from $e^\star f^\star \in E_{G^\star}$ to this unique vertex $v \in V_G$, and let $\phi^{-1}(v) := \{e^\star f^\star \in E_{G^\star} : \phi(e^\star f^\star) = v\}$ for each $v \in V_G$.

2.2 Sparsity Theorem

Theorem 2. *Let G be a simple plane graph without isolates. Then, the medial graph $G^\star$ is weakly sparse if and only if G is 3-connected.*

Proof. ("If"-part:) Note that $G^\star$ is simple because G is 3-connected. Suppose for a contradiction that there exists $X \subset V_{G^\star}$ for which the weak sparsity condition is violated. Then, by taking an inclusionwise-minimal violating set, we can find a rigidity circuit $C^\star$ in $G^\star$ (since edge sets of rigidity circuits are minimal dependent sets of the rigidity matroid). Since $V_{C^\star}$ violates (2), we have

$$|V_{C^\star}| \leq |V_{G^\star}| - 2. \tag{3}$$

We divide V_G into three subsets V_1, V_2, V_3 as follows: $v \in V_1$ iff $e^\star \in V_{C^\star}$ for all edges e incident with v in G; $v \in V_3$ iff $e^\star \notin V_{C^\star}$ for all edges e incident with v in G; $V_2 := V \setminus (V_1 \cup V_3)$. Notice that G has no edge between V_1 and V_3 from the definition.

Claim 3. $|V_2| \leq 2$.

Proof. Since $G^\star$ is 4-regular, the maximum degree of $C^\star$ is at most four. This implies that $C^\star$ has exactly four vertices of degree 3 and the others being degree 4 since C^* is a rigidity circuit. Let $e_1^\star, e_2^\star, e_3^\star, e_4^\star$ be these four vertices of degree 3 in $C^\star$. By the 4-regularity of $G^\star$ again, $G^\star$ has the edge $e_i^\star f_i^\star$ such that $e_i^\star f_i^\star \notin E_{C^\star}$ for each $i = 1, \ldots, 4$. Notice that, among all of $E_{G^\star}$, only $e_i^\star f_i^\star$ can connect between $V_{C^\star}$ and $V_{G^\star} \setminus V_{C^\star}$.

Let $v_i = \phi(e_i^\star f_i^\star)$ for $i = 1, \ldots, 4$. Recall that each vertex of V_2 is incident to some edges e and f with $e^\star \in V_{C^\star}$ and $f^\star \notin V_{C^\star}$. We thus have $V_2 \subseteq \{v_i : i = 1, \ldots, 4\}$. Moreover, since $\phi^{-1}(v_i)$ forms a cycle, there must be at least one edge $e^\star f^\star$ in $\phi^{-1}(v_i) \setminus \{e_i^\star f_i^\star\}$ such that $e^\star \in V_{C^\star}$ and $f^\star \notin V_{C^\star}$. This implies that there is an index j with $j \neq i$ such that $v_i = v_j$ for each $i = 1, \ldots, 4$. Consequently we obtain $|V_2| \leq 2$ from $V_2 \subseteq \{v_i : i = 1, \ldots, 4\}$. □

Claim 4. $V_1 \neq \emptyset$ *and* $V_3 \neq \emptyset$.

Proof. To see $V_1 \neq \emptyset$, consider an edge $e^\star f^\star \in E_{C^\star}$. By Claim 3, at least one of endpoints of e or f does not belong to V_2. Since $e^\star \in V_{C^\star}$ and $f^\star \in V_{C^\star}$, this endpoint cannot belong to V_3, implying $V_1 \neq \emptyset$.

Suppose $V_3 = \emptyset$. Then, since $|V_2| \leq 2$ and G is simple, every edge of G except for (at most) one edge is incident to some vertex in V_1. This implies $|V_{C^\star}| \geq |V_{G^\star}| - 1$, contradicting (3). □

Recall that G has no edge between V_1 and V_3. Thus, Claim 4 implies that V_2 is a separator of G. This contradicts the 3-connectivity of G by Claim 3. (If $V_2 = \emptyset$, G is not connected.)

("Only-if"-part:) Suppose G is not 3-connected. We show that $G^\star$ has a loop or $G^\star$ has a nontrivial cut S of size at most 4. If $G^\star$ has a loop, then G is not weakly sparse by definition. If $G^\star$ has such a cut S, then for the vertex set X of a connected component of $G^\star - S$ we have $|E_{G^\star}[X]| \geq 2|X| - 2$ by the 4-regularity of $G^\star$. Hence $G^\star$ is not weakly sparse.

By definition $G^\star$ contains a loop if and only if G has a vertex of degree 1. Hence, throughout the rest of the proof, we assume that G does not have a vertex of degree 1 and prove that $G^\star$ has a nontrivial cut of size at most 4. For a corner $\{e, f\}$ of a face of G and a subgraph H of G, we say that $\{e, f\}$ is *partially included* in H if $e \in E_H$ and $f \notin E_H$. To prove the existence of a nontrivial cut of size at most 4 in $G^\star$, it is sufficient to find a subgraph H of G such that (i) $2 \leq |E_H| \leq |E_G| - 2$ and (ii) at most four corners of G are partially included in H. We now prove G contains a subgraph H satisfying (i) and (ii).

If G is not connected, then a connected component H satisfies (i) and (ii) since any component of G is nontrivial. If G is connected but not 2-connected, then G has a separator $\{v\}$. Let $\mathbb{Z}_k = \{1, \ldots, k\}$ be the cyclic group of order k, and let $N_G(v) = \{v_1, v_2, \ldots, v_k\}$ be the neighbors of v indexed by elements in $\mathbb{Z}_k$ in the consecutive ordering around v. The connected components of $G - v$ induce a partition of $N_G(v)$, and the planarity implies this partition is non-crossing (i.e., if v_i and v_k belong to a component and v_j and v_l belong to the other component, then i, j, k, l are not arranged in the order $ijkl$). In other words, $G - v$ has a connected component H' with $V_{H'} \cap N_G(v) = \{v_i, \ldots, v_j\}$, where the indices are arranged consecutively.

Let H be the subgraph of G induced by $V_{H'} \cup \{v\}$. We may assume $2 \leq |E_H| \leq |E_G| - 2$, since otherwise G contains a vertex of degree 1. Observe that, among all corners of G, only $\{vv_{i-1}, vv_i\}$ and $\{vv_j, vv_{j+1}\}$ are partially included in H. Thus H is a subgraph satisfying (i) and (ii).

If G is 2-connected, then G has a separator S of size 2. Let $S = \{u, v\}$ and let their neighbors be $N_G(u) = \{u_1, \ldots, u_k\}$ and $N_G(v) = \{v_1, \ldots, v_l\}$ indexed by elements in $\mathbb{Z}_k$ and elements in $\mathbb{Z}_l$ in the consecutive order around u and v, respectively. Take any connected component H' of $G - S$, and let H'' be a subgraph of G induced by $V_{H'} \cup S$. Let $H = H'' - uv$ if $uv \in E_G$; otherwise let $H = H''$. Observe that, since G is 2-connected and plane, the vertices of $V_{H'} \cap N_G(u)$ are consecutively indexed. Thus, exactly two corners of G incident to u are partially included in H. Symmetrically, exactly two corners incident to v are partially included in H. Thus, exactly four corners of G are partially included in H, and we found a subgraph H satisfying (i) and (ii). This completes the proof. □

Corollary 5. *Let G be a simple 3-connected plane graph. Then, the graph $G^\star - e^\star$ obtained from the medial graph $G^\star$ by removing any vertex $e^\star \in V_{G^\star}$ is a rigidity circuit.*

Proof. By Theorem 2, $G^\star - e^\star$ is sparse. Also, since $G^\star$ is 4-regular, we have $|E_{G^\star - e^\star}| = |E_{G^\star}| - 4 = 2|V_{G^\star}| - 4 = 2|V_{G^\star - e^\star}| - 2$, implying that $G^\star - e^\star$ is a rigidity circuit. □

2.3 Connectivity Theorem

We also have a sufficient condition for the connectivity of $G^\star$. The result for $k = 3$ will be used in the proof of Theorem 1.

Theorem 6. *Let G be a simple plane graph and k be an integer with $1 \leq k \leq 3$. If G is k-connected, essentially $(k+1)$-edge-connected and k-cycle-free, then $G^\star$ is $(k+1)$-connected.*

Proof. Suppose $G^\star$ is not $(k+1)$-connected. Then, there is a separator $S^\star \subset V_{G^\star}$ of $G^\star$ such that $|S^\star| \leq k$. We may assume that $S^\star$ is a minimum separator of $G^\star$. The removal of $S^\star$ disconnects $G^\star$ into two nonempty parts, whose vertex sets are denoted by $E_1^\star, E_2^\star \subset V_{G^\star}$. Note that $\{S, E_1, E_2\}$ is a partition of E_G into nonempty subsets.

Claim 7. *Every vertex $v \in V_G(E_1) \cap V_G(E_2)$ is incident to at least two edges in S.*

Proof. Note that, from $v \in V_G(E_1) \cap V_G(E_2)$, v is incident to an edge in E_i for each $i = 1, 2$. Since $\phi^{-1}(v)$ forms a cycle, at least two elements of $\phi^{-1}(v)$ need to be deleted to separate $E_1^\star$ and $E_2^\star$ in $G^\star$. This implies the claim. □

Since G has no k-cycle and $|S| \leq k \leq 3$, S is cycle-free. If any two edges of S do not share a vertex, then $V_G(E_1)$ and $V_G(E_2)$ are disjoint from Claim 7. Since $|V_G(E_i)| \geq 2$, S is a nontrivial cut of G with $|S| \leq k$, contradicting the essential $(k+1)$-edge-connectivity of G.

Thus, let us assume that S is not vertex-disjoint. In this case, we may assume $k \geq 2$ by $|S| \leq k$, and hence G is 2-connected. We define a subset X of $V_G(S)$ as follows. Consider the graph $(V_G(S), S)$ edge-induced by S. For each connected component of $(V_G(S), S)$, if it consists of a single edge then insert arbitrary one endvertex of the edge into X; otherwise insert all vertices incident to at least two elements of S into X (in other words, insert all vertices except for leaf nodes). Since some elements of S share a vertex, there is a connected component in $(V_G(S), S)$ which consists of at least two edges and in which at least two vertices do not belong to X. This implies $|X| \leq |S| - 1 \leq k - 1$. Moreover $V_G(E_1) \cap V_G(E_2) \subseteq X$ by Claim 7, and also S is dominated by X (i.e., every edge in S is incident to a vertex in X). Therefore, if $V_G(E_i) \setminus X \neq \emptyset$ for $i = 1, 2$, X is a separator of G with $|X| \leq k - 1$.

Suppose $V_G(E_i) \subseteq X$ for some $i \in \{1, 2\}$. Since $|X| \leq k - 1$ and $V_G(E_i) \geq 2$, we must have $k = 3$. When $k = 3$, we have $|V_G(E_i)| \leq |X| = 2$ and E_i consists of a single edge e. Since $|S| \leq k = 3$, an endpoint v of e is incident to at most one edge of S. Claim 7 thus implies $v \notin V_G(E_{\bar{i}})$ for $\bar{i} \in \{1, 2\} - i$, and hence v is a vertex of degree at most 2 in G, contradicting the 3-connectivity of G. Consequently, we obtain $V_G(E_i) \setminus X \neq \emptyset$ for $i \in \{1, 2\}$, and hence X is a separator of G with $|X| \leq 2$. This contradicts the 3-connectivity of G. □

The following theorem is a corresponding statement for the case of $k = 4$.

Theorem 8. *Let G be a simple plane graph. If G is 4-connected, essentially 5-edge-connected and 4-cycle-free, then $G^\star$ is essentially 5-connected.*

Proof. The proof strategy is basically the same as that of Theorem 6. Suppose $G^\star$ is not essentially 5-connected. Then, there is a nontrivial separator $S^\star \subset V_{G^\star}$ of $G^\star$ such that $|S^\star| \leq 4$, and we may assume that $S^\star$ is a minimum separator. The removal of $S^\star$ disconnects $G^\star$ into two nonempty parts, whose vertex sets are denoted by $E_1^\star, E_2^\star \subset V_{G^\star}$. As in the proof of Theorem 6, $\{S, E_1, E_2\}$ forms a partition of E_G into nonempty subsets, and Claim 7 holds by the exactly same argument. For $k = 4$ we also need the following property of S.

Claim 9. *If S contains a cycle, then S is exactly a 3-cycle.*

Proof. Suppose S contains a cycle. Since G is 4-cycle-free, S cannot be a 4-cycle. Since G is simple, either S is exactly a 3-cycle, or $|S| = 4$ and S contains a 3-cycle. Suppose the latter case. Let e, f, g denote edges of S forming the 3-cycle, and let h denote the remaining edge of S. If this 3-cycle is not a face of G, then the 4-connectivity of G implies that there are at least two edges inside and outside of the 3-cycle in G, respectively, and hence $\{e^\star, f^\star, g^\star\}$ is a nontrivial separator of $G^\star$, contradicting the minimality of S.

Thus, we may assume that the 3-cycle efg forms a face of G. Then, $G^\star$ has the 3-cycle face connecting the vertices $e^\star, f^\star$ and $g^\star$. Let $e_1^\star$ and $e_2^\star$ be the other two vertices in the neighbors of $e^\star$ in $G^\star$. Also $f_i^\star$ and $g_i^\star$ are defined analogously.

From the minimality of $S^\star$, adding the two edges $e^\star e_1^\star$ and $e^\star e_2^\star$ makes $G^\star - S^\star$ connected, and hence $G^\star - S^\star$ consists of two connected components, one containing $e_1^\star$ and the other containing $e_2^\star$. The similar things hold for f and g, and thus none of e_i, f_i and g_i is equal to h. Let $C_1^\star$ and $C_2^\star$ be the connected components of $G^\star - S^\star$. From the symmetry, we may assume without loss of generality that $e_i^\star$, $f_i^\star$ and $g_i^\star$ belong to $C_i^\star$ for each $i = 1, 2$. This however contradicts the planarity of $G^\star$ as $G^\star$ has a 3-cycle face connecting $e^\star, f^\star$ and $g^\star$. □

By Claim 9, we split the proof of Theorem 8 into two cases.

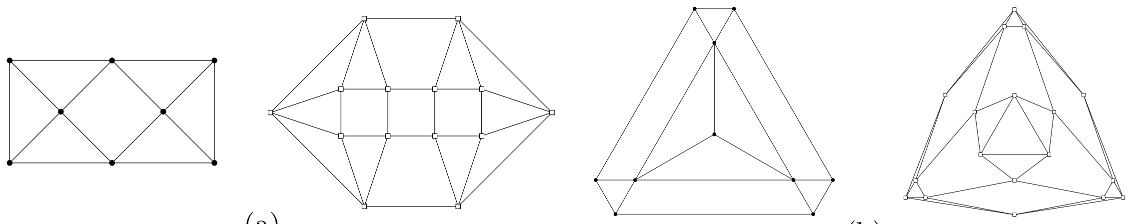
Case 1. If S is a 3-cycle, let $X = V_G(S)$. Notice that Claim 7 implies $(V_G(E_1) \setminus X) \cap (V_G(E_2) \setminus X) = \emptyset$. If $V_G(E_i) \setminus X \neq \emptyset$ for $i = 1, 2$, then X is a separator of G, and it contradicts the 4-connectivity of G. If $V_G(E_i) \subseteq X$, then we have $E_i \subseteq S$ because G is simple and S is a 3-cycle on X with $|X| = 3$. This however contradicts that $\{S, E_1, E_2\}$ is a partition of E.

Case 2. Let us consider the case when S is cycle-free. If any two edges of S do not share a vertex, then $V_G(E_1)$ and $V_G(E_2)$ are disjoint from Claim 7. Since $|V_G(E_i)| \geq 2$, S is a nontrivial cut of G with $|S| \leq 4$, contradicting the essential 5-edge-connectivity of G.

Let us assume that S is not vertex-disjoint. We define a subset X of $V_G(S)$ as in the proof of Theorem 6: For each connected component of $(V_G(S), S)$, if it consists of a single edge then insert arbitrary one endvertex of the edge into X; otherwise insert all vertices incident to at least two elements of S into X. As shown in the proof of Theorem 8, X has the following three properties: (i) $|X| \leq |S| - 1 \leq k - 1 = 3$, (ii) $V_G(E_1) \cap V_G(E_2) \subseteq X$, and (iii) S is dominated by X. These imply that, if $V_G(E_i) \setminus X \neq \emptyset$ for $i = 1, 2$, X is a separator of G with $|X| \leq 3$.

Suppose contrary $V_G(E_i) \subseteq X$ for some $i \in \{1, 2\}$. We have $|V_G(E_i)| \leq |X| \leq 3$. Also, since $|E_i| \geq 2$ as S^* is a nontrivial separator, we have $|V_G(E_i)| = |X| = 3$. Since $|S| \leq 4$ and $|E_i| \geq 2$, there is a vertex v in $V_G(E_i)$ that is incident to at most one edge of S. Claim 7 thus implies $v \notin V_G(E_{\bar{i}})$ for $\bar{i} \in \{1, 2\} - i$, and hence v is a vertex of degree at most 3 in G, contradicting the 4-connectivity of G. Consequently, $V_G(E_i) \setminus X \neq \emptyset$ for $i \in \{1, 2\}$, but this also contradicts the 4-connectivity of G. □

We have three remarks on Theorems 6 and 8: The converse direction of Theorem 6 is not true in general (see Figure 1(a)); The k-cycle-freeness is necessary (see Figure 1(b)); Theorem 8 cannot be extended to the cases $k \geq 5$ (see Figure 2).



(a) (b)

Figure 1: A graph (left) and the medial graph (right). (a) A non-3-connected plane graph whose medial graph is 4-connected. (b) A 3-connected and essentially 4-edge-connected plane graph whose medial graph is not 4-connected.

Figure 2: A subgraph of a 5-connected, essentially 6-edge-connected, 5-cycle-free graph G whose medial graph is not essentially 6-connected.

3 Proof of Theorem 1

For a plane graph G, let F_G be the set of faces of G. Grinberg's criterion [7] asserts that if a plane graph is Hamiltonian then there is a bipartition $\{F_1, F_2\}$ of F_G such that $\sum_{f \in F_1}(d(f) - 2) =$

$\sum_{f'\in F_2}(d(f') - 2)$, where $d(f)$ denotes the degree of a face f. In [2], Bondy and Häggkvist observed an extension of Grinberg's criterion to the decomposability into two edge-disjoint Hamiltonian cycles. As a corollary, they also mentioned the following.

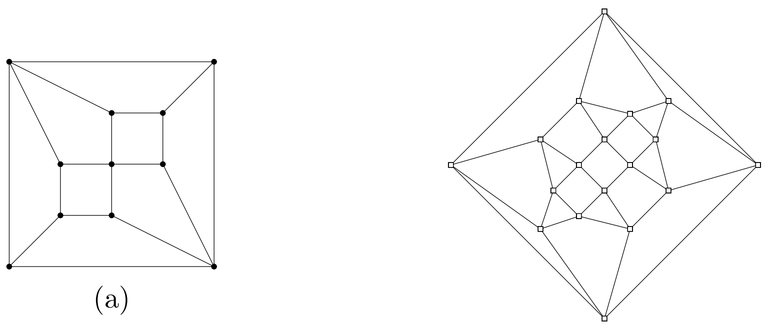
Theorem 10 (Bondy and Häggkvist [2])**.** *Suppose a plane graph G does not satisfy Grinberg's criterion. Then, the medial graph $G^\star$ cannot be decomposed into two edge-disjoint Hamiltonian cycles.*

We remark that there is a non-Hamiltonian plane graph G for which $G^\star$ can be decomposed into two edge-disjoint Hamiltonian cycles [2].

Corollary 5 and Theorem 10 yield the following.

Corollary 11. *Suppose G is a simple 3-connected plane graph and does not satisfy Grinberg's criterion. Then, for any vertex $e^\star \in V_{G^\star}$, $G^\star - e^\star$ is a rigidity circuit that cannot be decomposed into two edge-disjoint Hamiltonian paths.*

The Herschel graph is a minimum non-Hamiltonian 3-connected planar graph, which also violates Grinberg's criterion [2]. Figure 3 shows the Herschel graph and its medial graph. Since the Herschel graph is 3-connected, essentially 4-edge-connected and 3-cycle-free, its medial graph is indeed weakly sparse and 4-connected, and therefore any graph obtained by removing a vertex is a 3-connected rigidity circuit that cannot be decomposed into two edge-disjoint Hamiltonian paths, by Theorem 6 and Corollary 11.



(a)

(b)

Figure 3: (a)The Herschel graph and (b)the medial graph.

To construct infinitely many indecomposable rigidity circuits, we need the following observation taken from the Herschel graph.

Proposition 12. *Let G be a plane graph such that $|F_G|$ is odd and each face is a 4-cycle. Then, G does not satisfy Grinberg's criterion.*

Proof. Suppose there is a bipartition $\{F_1, F_2\}$ of F_G that satisfies the Grinberg's criterion. Then, from $\sum_{f\in F_i}(d(f) - 2) = 2|F_i|$, we have $|F_1| = |F_2|$. This contradicts the parity of $|F_G|$. □

This leads to the following construction of graphs. Suppose we are given 3-connected and essentially 4-edge-connected plane graphs G_0, G_1 and G_2 each of which has the property that the number of faces is odd and each face is a 4-cycle. Take two internal faces f_1 and f_2 from G_0, and replace f_1 and f_2 by G_1 and G_2 as shown in Figure 4. Let G be the resulting plane graph. Clearly, every face of G is a 4-cycle, and $|F_G| = (|F_{G_0}|-2)+(|F_{G_1}|-1)+(|F_{G_2}|-1) = \sum_{i=0,1,2}|F_{G_i}|-4$, which is odd. By Proposition 12, G does not satisfy the Grinberg's criterion. Also, G is clearly

3-connected, essentially 4-edge-connected, and 3-cycle-free, implying that $G^\star$ is 4-connected by Theorem 6. In total, for any vertex $e^\star \in V_{G^\star}$, $G^\star - e^\star$ is a 3-connected rigidity circuit that is indecomposable into two edge-disjoint Hamiltonian paths.

This completes the proof of Theorem 1. □

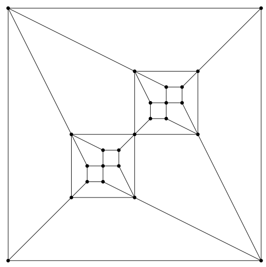


Figure 4: Recursive construction of counterexamples. Here, all of G_0, G_1 and G_2 are the Hershel graphs. The two internal squares of Figure 3(a) are replaced.

Motivated by Nash-Williams' conjecture, Martin [14] and Grünbaum and Malkevitch [9] showed how to construct 4-connected planar 4-regular graphs from cyclically 4-connected planar non-Hamiltonian cubic graphs (see e.g., [2, 5]). Although the construction presented above is much simpler, another sequence of counterexamples can be constructed based on these results. Let us briefly explain it since the approach can be applied to non-planar case.

The *line graph* $L(G)$ of G is the graph on the vertex set E_G where two vertices are connected if and only if the corresponding edges share a vertex in the original graph G. If G is a 3-regular plane graph, then $L(G)$ is equal to the medial graph $G^\star$. Similar to Theorem 2, we have a characterization of 3-connected 3-regular graphs in terms sparsity.

Theorem 13. *Let G be a simple 3-regular graph. Then, $L(G)$ is weakly sparse if and only if G is 3-connected.*

Proof. Observe that, in the proof of sufficiency of Theorem 2, we did not use the planarity of $G^\star$ (except for defining the medial graphs), and we can apply the exactly same proof to show the sufficiency.

To see the necessity, suppose G is not 3-connected. As in the proof of Theorem 2, it is sufficient to show that $L(G)$ has a nontrivial cut of size at most 4. If G is 2-connected, G has a separator $S = \{u, v\}$ of size 2. Let X be the vertex set of a connected component in $G - S$, and let H be the subgraph of G induced by $X \cup \{u, v\}$. Observe that $L(G)$ has at most four edges between E_H and $E \setminus E_H$, and hence $L(G)$ has a cut of size at most 4. By the 3-regularity of G, this cut is nontrivial in $L(G)$. By the same argument we can find a nontrivial cut of size at most two in $L(G)$ if G is not 2-connected. Therefore, $L(G)$ is not weakly sparse. □

Kotzig [13] proved that a 3-connected 3-regular graph G is Hamiltonian if and only if $L(G)$ can be decomposed into two edge-disjoint Hamiltonian cycles. (Similar but weaker statements were also claimed in [9, 14].) We thus obtain a characterization of the decomposability for a certain family of rigidity circuits by Theorem 13.

Corollary 14. *A simple 3-regular graph G is 3-connected and Hamiltonian if and only if $L(G) - v$ is a rigidity circuit which can be decomposed into two edge-disjoint Hamiltonian paths for every vertex v in $L(G)$.*

Corollary 14 has a computational implication. Namely, the problem of deciding whether a planar rigidity circuit can be decomposed into two Hamiltonian paths or not is NP-complete since the problem of computing a Hamiltonian cycle in a planar 3-connected 3-regular graph is NP-complete [4].

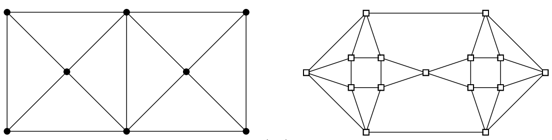
4 Concluding Remarks

A graph G is called *minimally rigid* if it satisfies (1) with $|E_G| = 2|V_G| - 3$, and G is called *rigid* if it contains a minimally rigid subgraph with $|V_G|$ vertices. G is further called *redundantly rigid* if $G - e$ is rigid for every $e \in E_G$. From a combinatorial characterization of 2-dimensional generic global rigidity by Connelly [3] and Jackson and Jordán [10], G is said to be *globally rigid* if it is 3-connected and redundantly rigid. Since any rigidity circuit is redundantly rigid by definition, our counterexamples given in Theorem 1 are in fact globally rigid.

From Corollary 5, it can be easily checked that $G^\star$ is redundantly rigid for any 3-connected plane graph G. We also know that $G^\star$ is 3-connected if G is 3-connected by Theorem 6 with $k = 2$, and hence if G is 3-connected plane graph then $G^\star$ is globally rigid. The converse implication is however not true in general. Figure 5 shows examples of non-3-connected graphs where the medial graphs of (a) and (b) are globally rigid and not rigid, respectively.

Jordán [11] recently proved that the line graph $L(G)$ of a 3-regular graph G is globally rigid if and only if G is 3-edge-connected. Figure 5(b) however indicates that $G^\star$ may not be rigid even when G has high edge-connectivity. (Replace each unit of the wheel of five vertices by a highly edge-connected graph.)

Medial graphs have been also appeared in a proof [8] of Steiniz's theorem for realizations as 1-skeletons of convex 3-polytopes. Originating from Cauchy's rigidity theorem, 3-connected planar graphs have a strong relation to the rigidity of graphs when viewing them as 1-skeletons of convex 3-polytopes (see e.g., [6, Chapter 1.2]). Theorem 2 thus combinatorially connects two separated notions in rigidity theory, rigidity of 3-polytopes and generic global rigidity in the plane, although a direct geometric connection is not clear.


(a)

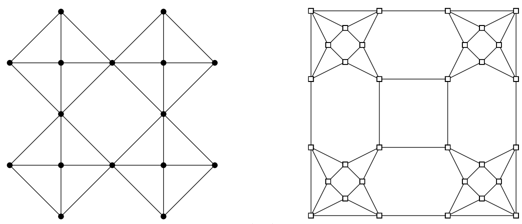

(b)

Figure 5: Non-3-connected plane graphs (left) and the medial graphs (right): (a) globally rigid, (b) non-rigid.

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