On Polish Groups of Finite Type

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Abstract

Sorin Popa initiated the study of Polish groups which are embeddable into the unitary group of a separable II_1 factor. Such groups are called of finite type or said to belong to the class $\mathscr{U}_{\mathrm{fin}}$. We give necessary and sufficient conditions for Polish groups to be of finite type, and construct exmaples of such groups from I_∞ and II_∞ algebras. We also discuss permanence properties of finite type groups under various algebraic operations. Finally we close the paper with some questions concerning Polish groups of finite type.

Keywords bi-invariant metric, class $\mathscr{U}_{\mathrm{fin}}$, finite type group, Polish group, positive definite functions, SIN-group, II_1 factors

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Contents

1 Introduction

In this paper we consider the following problem. Denote by $\mathcal{U}(M)$ the unitary group of a von Neumann algebra M.

Problem 1.1. Determine the necessary and sufficient condition for a Polish group G to be isomorphic as a topological group onto a strongly closed subgroup of some $\mathcal{U}(M)$, where M is a separable finite von Neumann algebra.

S. Popa defined a Polish group to be of finite type if it has this embedding property. Denote by $\mathscr{U}_{\text{fin}}$ the class of all finite type Polish groups. He initiated the study of this class in an attempt to enrich the study of rapidly developing cocycle superrigidity theory (cf. [7, 16, 20]). In particular, he proposed in [20] the problem of studying and characterizing the class $\mathscr{U}_{\text{fin}}$.

Secondly, this problem is motivated from our previous work [1] on infinite-dimensional Lie algebras associated with such groups: Let M be a finite von Neumann algebra on a Hilbert space $\mathcal{H}$. Let G be a strongly closed subgroup of $\mathcal{U}(M)$ and $\overline{M}$ be a set of all densely defined closed operators on $\mathcal{H}$ which are affiliated to M. It is proved that the set

$$\mathrm{Lie}(G) := \{A^* = -A \in \overline{M}; e^{tA} \in G \text{ for all } t \in \mathbb{R}\}$$

is a complete topological Lie algebra with respect to the strong resolvent topology (see also the related work of D. Beltita [3]). Since these Lie algebras turn out to be non-locally convex in general when M is non-atomic, they are quite exotic as a Lie algebra and their properties are still unknown. Therefore it would be interesting to find non-trivial examples of such groups.

We give an answer in Theorem 2.7 to the above Problem by the aid of positive definite functions on groups and their GNS representations, and characterize locally compact groups or amenable Polish groups of finite type via compatible bi-invariant metrics in Proposition 2.20 and Theorem 2.22 (the former is known, but we give a new proof). Combining with Popa's result [20], Theorem 2.7 gives a necessary and sufficient condition for a Polish group to be isomorphic onto a closed subgroup of the unitary group of a separable II_1 factor. We then give examples of Polish groups G of finite type using noncommutative integration of E. Nelson [18]. Finally we discuss some hereditary properties of finite type groups and pose some questions concerning Polish groups of finite type.

Notation. In this paper we often say a von Neuman algebra M is *separable* if it has a separable predual, especially when the Hilbert space on which M acts is implicit. This is known to be equivalent to the condition that M has a faithful representation on a separable Hilbert space. We denote by $\mathrm{Proj}(M)$ the lattice of all projections in M. A von Neumann algebra is said to be *finite* if it admits no non-unitary isometry. When we consider a group G, its identity is denoted as e_G. However, we also use 1 as the identity when we consider a concrete subgroup of the unitary group of a von Neumann algebra. We always regard the unitary group of a von Neumann algebra as a topological group with the strong operator topology.

2 Polish Groups of Finite Type and its Characterization

In this section, we characterize Polish groups of finite type via positive definite functions. We then characterize when locally compact groups or amenable Polish groups are of finite type via compatible bi-invariant metrics. To this end, we review notions of SIN-groups, bi-invariant metrics and unitary representability.

2.1 Polish Groups of Finite Type

Recall that a Polish space is a separable completely metrizable topological space, and a Polish group is a topological group whose topology is Polish.

We now introduce finite type groups after Popa [20].

Definition 2.1. A Hausdorff topological group is called of *finite type* if it is isomorphic as a topological group onto a closed subgroup of the unitary group of a finite von Neumann algebra.

Remark 2.2. Popa [20] requires the topological group of finite type to be Polish, whereas our definition of finiteness does not require any countability. We will show in Theorem 2.7 that a Polish group G of finite type in our sense coincides with Popa's definition of finite type group. That is, G is isomorphic onto a closed subgroup of the unitary group of a finite von Neumann algebra acting on a separable Hilbert space.

All of second countable locally compact Hausdorff groups, the unitary group of a von Neumann algebra acting on a separable Hilbert space are Polish groups. Furthermore, separable Banach spaces are Polish groups as an additive group. We denote the class of all Polish groups of finite type by $\mathscr{U}_{\text{fin}}$.

Note that since a von Neumann algebra is finite if and only if its unitary group is complete with respect to the left uniform structure, Polish groups of finite type are necessarily complete. Thus we have the following simple consequence.

Proposition 2.3. *The unitary group of a von Neumann algebra M acting on a separable Hilbert space is of finite type if and only if M is finite.*

Another examples of Polish groups of finite type are given later.

2.2 Positive Definite Functions

A complex valued function f on a Hausdorff topological group G is called *positive definite* if for all $g_1, \cdots, g_n \in G$ and for all $c_1, \cdots, c_n \in \mathbb{C}$,

$$\sum_{i,j=1}^{n} \bar{c}_i c_j f(g_i^{-1} g_j) \geq 0$$

holds. Moreover if a complex valued function f is invariant under inner automorphisms, that is

$$f(hgh^{-1}) = f(g), \quad \forall g, h \in G,$$

then f is called *a class function*.

It is well-known that there is an one-to-one correspondence between the set of all continuous positive definite functions on a topological group and the set of unitary equivalence classes of all cyclic unitary representations of it. more precisely, for each continuous positive definite function f on a topological group G, there exists a triple $(\pi_f, \mathcal{H}_f, \xi_f)$ consisting of a cyclic unitary representation π_f in a Hilbert space $\mathcal{H}_f$ and a cyclic vector ξ_f in $\mathcal{H}_f$ such that

$$f(g) = \langle \xi_f, \pi_f(g)\xi_f \rangle, \quad g \in G,$$

and this triple is unique up to unitary equivalence. This triple is called the *GNS triple* associated to f. Note that if G is separable, then so is $\mathcal{H}_f$.

The GNS triple is of the following form for each continuous positive definite class function.

Lemma 2.4. *Let f be a continuous positive definite class function on a topological group G and $(\pi, \mathcal{H}, \xi)$ be its GNS triple. Then the von Neumann algebra M generated by $\pi(G)$ is finite and the linear functional*

$$\tau(x) := \langle \xi, x\xi \rangle, \qquad x \in M,$$

is a faithful normal tracial state on M. In particular M is countably decomposable.

Proof. It is clear that τ is a normal state on M. Since f is a class function, it is easy to see that τ is tracial on the strongly dense *-subalgebra of M spanned by $\pi(G)$. Therefore by normality, τ is tracial on M. Therefore we have only to check the faithfulness of τ. Assume $\tau(x^*x) = 0$. Since τ is a trace, we have

$$\|x\pi(g)\xi\|^2 = \tau(\pi(g)^* x^* x \pi(g)) = 0,$$

for all $g \in G$. By the cyclicity of ξ, x must be 0. □

Example 2.5 (I. J. Schoenberg [21])**.** Let $\mathcal{H}$ be a complex Hilbert space. Note that $\mathcal{H}$ is an additive group. Then a function f defined by $f(\xi) := e^{-\|\xi\|^2}$ $(\xi \in \mathcal{H})$ is a positive definite (class) function on $\mathcal{H}$.

Example 2.6 (I. J. Schoenberg [21])**.** For all $1 \leq p \leq 2$ a function f_p defined by $f_p(a) := e^{-\|a\|_p^p}$ $(a \in l^p)$ is a positive definite (class) function on a separable Banach space l^p.

For more details about positive definite class functions, see [12].

2.3 The First Characterization

We now characterize Polish groups of finite type.

Theorem 2.7. *For a Polish group G the following are equivalent.*

(i) G is of finite type.

(ii) G is isomorphic as a topological group onto a closed subgroup of the unitary group of a finite von Neumann algebra acting on a separable Hilbert space.

(iii) A family $\mathcal{F}$ of continuous positive definite class functions on G generates a neighborhood basis of the identity e_G of G. That is, for each neighborhood V of the identity, there are functions $f_1, \cdots, f_n \in \mathcal{F}$ and open sets $\mathcal{O}_1, \cdots, \mathcal{O}_n$ in $\mathbb{C}$ such that

$$e_G \in \bigcap_{i=1}^{n} f_i^{-1}(\mathcal{O}_i) \subset V.$$

(iv) There exists a positive, continuous positive definite class function which generates a neighborhood basis of the identity of G.

(v) A family $\mathcal{F}$ of continuous positive definite class functions on G separates the identity of G and closed subsets A with $A \not\ni e_G$. That is, for each closed subset A with $A \not\ni e_G$, there exists a continuous positive definite class function $f \in \mathcal{F}$ such that

$$\sup_{x \in A} |f(x)| < |f(e_G)|.$$

(vi) There exists a positive continuous positive definite class function which separates the identity of G and closed subsets A with $A \not\ni e_G$.

Proof. (iv)$\Leftrightarrow$(vi)$\Rightarrow$(v)$\Rightarrow$(iii) and (ii)$\Rightarrow$(i) are trivial.

(iii)$\Rightarrow$(ii). Since G is first countable, there exists a countable subfamily $\{f_n\}_n$ of $\mathcal{F}$ which generates a neighborhood basis of the identity of G. Let $(\pi_n, \xi_n, \mathcal{H}_n)$ be the GNS triple associated to f_n and M_n be a von Neumann algebra generated by $\pi_n(G)$. Since each M_n is finite, the direct sum $M := \bigoplus_n M_n$ is also finite and acts on a separable Hilbert space $\mathcal{H} := \bigoplus_n \mathcal{H}_n$ (see the remark above Lemma 2.4). Put $\pi := \bigoplus_n \pi_n$, then π is an embedding of G into $\mathcal{U}(M)$. The image of π is closed in $\mathcal{U}(M)$, as both G and $\mathcal{U}(M)$ are Polish.

(i)$\Rightarrow$(iii). Let π be an embedding of G into the unitary group of a finite von Neumann algebra M. Since each finite von Neumann algebra is the direct sum of countably decomposable finite von Neumann algebras, we can take of a family of countably decomposable finite von Neumann algebras $\{M_i\}_{i\in I}$ with $M = \bigoplus_{i\in I} M_i$. In this case π is also of the form $\pi = \bigoplus_{i\in I} \pi_i$, where each $\pi_i : G \to \mathcal{U}(M_i)$ is a continuous group homomorphism. Let τ_i be a faithful normal tracial state on M_i and $(\rho_i, \xi_i, \mathcal{H}_i)$ be its GNS triple as a C^*-algebra. Here each ρ_i is an isomorphism from M_i into $\mathbb{B}(\mathcal{H}_i)$ and

$$\tau_i(x) = \langle \xi_i, \rho_i(x)\xi_i \rangle, \qquad x \in M_i,$$

holds. Now set $f_i := \tau_i \circ \pi_i$. Then each f_i is a continuous positive definite class functions on G and $\{f_i\}_{i\in I}$ generates a neighborhood basis of the identity e_G of G.

(iii)$\Rightarrow$(iv). Let $\{f_n\}_n$ be a countable family of continuous positive definite class functions generating a neighborhood basis of the identity of G with $f_n(e_G) = 1$. Set

$$\begin{aligned} f_n'(g) &:= e^{\mathrm{Re}(f_n(g))-1} \\ &= e^{-1} \sum_{k=0}^{\infty} \frac{1}{k!} \left[\mathrm{Re}(f_n(g))\right]^k, \qquad g \in G, \end{aligned}$$

then $\{f_n'\}_n$ is not only a family of continuous positive definite class functions generating a neighborhood basis of the identity of G with $f_n'(e_G) = 1$ but also a family of positive functions. Define a positive, continuous positive definite class function by $f(g) := \sum_n f_n'(g)/2^n$ ($g \in G$). It is easy to see that f generates a neighborhood basis of the identity of G. $\square$

Remark 2.8. The proof of the above theorem is inspired by Theorem 2.1 of S. Gao [9].

Remark 2.9. Popa (Lemma 2.6 of [20]) showed that a Polish group G is of finite type if and only if it is isomorphic onto a closed subgroup of the unitary group of a separable II_1 factor. Therefore Theorem 2.7 gives a necessary and sufficient condition for a Polish group to be isomorphic onto a closed subgroup of the unitary group of a separable II_1 factor.

2.4 SIN-groups and Bi-invariant Metrics

To discuss further properties of finite type groups, we consider the following notions, say SIN-groups, bi-invariant metrics and unitarily representability.

A neighborhood V at the identity of a topological group G is called *invariant* if it is invariant under all inner automorphisms, that is, $gVg^{-1} = V$ holds for all $g \in G$. *A SIN-group* is a topological group which has a neighborhood basis of the identity consisting of invariant identity neighborhoods. Note that a locally compact Hausdorff SIN-group is unimodular.

A bi-invariant metric on a group G is a metric d which satisfies

$$d(kg, kh) = d(gk, hk) = d(g, h), \quad \forall g, h, k \in G.$$

It is known that a first countable Hausdorff topological group is SIN if and only if it admits a compatible bi-invariant metric.

As Popa [20] pointed out, one of the most important fact of Polish groups of finite type is an existence of a compatible bi-invariant metric.

Lemma 2.10. *Each Polish group of finite type has a compatible bi-invariant metric. In particular, it is SIN.*

Proof. It is enough to show that for every finite von Neumann algebra M acting on a separable Hilbert space $\mathcal{H}$ the unitary group $\mathcal{U}(M)$ has a compatible bi-invariant metric. For this let τ be a faithful normal tracial state on M. Then a metric d defined by

$$d(u, v) := \tau((u - v)^*(u - v))^{\frac{1}{2}}, \quad u, v \in \mathcal{U}(M),$$

is a compatible bi-invariant metric on $\mathcal{U}(M)$. □

2.5 Unitary Representability

A Hausdorff topological group is called *unitarily representable* if it is isomorphic as a topological group onto a subgroup of the unitary group of a Hilbert space. All locally compact Hausdorff groups are unitarily representable via the left regular representation. It is clear that a Polish group of finite type is necessarily unitarily representable. The following characterization of unitary representability has been considered by specialists and can be seen in e.g., Gao [9].

Lemma 2.11. *For a Polish group G the following are equivalent.*

(i) G is unitarily representable.

(ii) There exists a positive, continuous positive definite function which separates the identity of G and closed subsets A with $A \not\ni e_G$.

2.6 Simple Examples

All of the following examples are well-known. The first three examples are locally compact groups.

Example 2.12. Any compact metrizable group is a Polish group of fintie type. This follows from the Peter-Weyl theorem.

Example 2.13. Any abelian second countable locally compact Hausdorff group is a Polish group of finite type. Indeed its left regular representation is an embedding into the unitary group of a Hilbert space and the von Neumann algebra generated by its image is commutative (in particular, finite).

Example 2.14. Any countable discrete group is a Polish group of finite type. For its left regular representation is an embedding into the unitary group of a finite von Neumann algebra.

The following two examples suggest there are few other examples of locally compact groups of finite type.

Example 2.15. Let $G := \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{K})\ ;\ x \in \mathbb{K}^\times, y \in \mathbb{K} \right\}$ be the $ax+b$ group, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. By easy computations, we have

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} x & -bx+ay+b \\ 0 & 1 \end{pmatrix},$$

so that the conjugacy class $C\left(\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\right)$ of $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ is

$$C\left(\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\right) = \begin{cases} \left\{ \begin{pmatrix} x & \sharp \\ 0 & 1 \end{pmatrix}\ ;\ \sharp \in \mathbb{K} \right\} & (x \neq 1), \\ \left\{ \begin{pmatrix} 1 & \sharp \\ 0 & 1 \end{pmatrix}\ ;\ \sharp \in \mathbb{K}^\times \right\} & (x = 1,\ y \neq 0), \\ \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} & (x = 1,\ y = 0). \end{cases}$$

Thus for each $n \in \mathbb{N}$ there exists a matrix $h_n \in G$ such that $h_n g_n h_n^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, where $g_n := \begin{pmatrix} 1 & 1/n \\ 0 & 1 \end{pmatrix}$. Clearly, $g_n \to 1$ and $h_n g_n h_n^{-1} \not\to 1$. This implies that the $ax+b$ group does not admit a compatible bi-invariant metric. Hence it is not of finite type.

Example 2.16. The special linear group $SL(n, \mathbb{K})$ $(n \geq 2)$ is not of finite type since the map $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ is an embedding of the $ax + b$ group into $SL(2, \mathbb{K})$. Thus the general linear group $GL(n, \mathbb{K})$ $(n \geq 2)$ is also not of finite type.

Next we consider abelian groups. Note that an abelian topological group is of finite type if and only if it is unitarily representable.

Example 2.17. Any separable Hilbert space is a Polish group of finite type. This follows from Example 2.5 and Theorem 2.7.

Example 2.18. A separable Banach space l^p $(1 \leq p \leq \infty)$ is a Polish group of finite type if and only if $1 \leq p \leq 2$. The "if" part follows from Example 2.6 and Theorem 2.7, but the "only if" part is non-trivial. For details, see [15].

Here is another counter example.

Example 2.19. Separable Banach space $C[0, 1]$ of all continuous functions on the interval $[0, 1]$ is a Polish group but not of finite type. For, since every separable Banach space is isometrically isomorphic to a closed subspace of $C[0, 1]$, if $C[0, 1]$ is of finite type, then any separable Banach space is a Polish group of finite type. But this is a contradiction to the previous example.

2.7 Application to Locally Compact Groups

It is known that a second countable locally compact group is of finite type if and only if it is a SIN-group (see e.g., Theorem 13.10.5 of J. Dixmier [**?**]). We give a new proof of this fact using Theorem 2.7. We thank the referee for letting us know the above literature.

Proposition 2.20. *A second countable locally compact Hausdorff group is of finite type if and only if it is SIN.*

Proof. Let G be a second countable locally compact Hausdorff SIN-group, μ be the Haar measure on it and λ be its left-regular representation. For each compact invariant neighborhood U of the identity, we define a continuous positive definite function φ_U on G by

$$\varphi_U(g) := \langle \chi_U, \lambda(g)\chi_U \rangle = \mu\left(U \cap gU\right), \quad g \in G.$$

Note that, for each $g, h, x \in G$, we have

$$h^{-1}x \in U \Leftrightarrow x \in hU = Uh \Leftrightarrow xh^{-1} \in U,$$

and

$$(gh)^{-1}x \in U \Leftrightarrow x \in ghU = gUh \Leftrightarrow xh^{-1} \in gU.$$

Also note that a locally compact SIN-group is unimodular. Thus we see that

$$\begin{aligned}\varphi_U(h^{-1}gh) &= \langle \lambda(h)\chi_U, \lambda(gh)\chi_U\rangle \\ &= \int_G \chi_U(h^{-1}x)\chi_U((gh)^{-1}x)d\mu(x) \\ &= \int_G \chi_U(xh^{-1})\chi_{gU}(xh^{-1})d\mu(x) \\ &= \int_G \chi_U(x)\chi_{gU}(x)d\mu(x) \\ &= \int_G \chi_U(x)\chi_U(g^{-1}x)d\mu(x) \\ &= \varphi_U(g).\end{aligned}$$

This implies φ_U is a class function. It is not hard to check that a family $\{\varphi_U\}_U$ generates a neighborhood basis of the identity of G. This completes the proof by Theorem 2.7. □

Remark 2.21. (1) R. V. Kadison and I. Singer [14] proved that every connected locally compact Hausdorff SIN group is isomorphic as a topological group onto a topological group of the form $\mathbb{R}^n \times K$, where K is a compact Hausdorff group. (2) K. Hofmann, S. Morris and M. Stroppel [13] proved that every totally disconnected locally compact Hausdorff group is SIN if and only if it is a strict projective limit of discrete groups.

2.8 A Characterization for Amenable Groups

Next, we characterize (not necessarily locally compact) amenable Polish groups of finite type. Recall that a Hausdorff topological group G is amenable if $\mathrm{RUCB}(G)$ admits a left-translation invariant positive functional $m \in \mathrm{RUCB}(G)^*$ with $m(1) = 1$, where $\mathrm{RUCB}(G)$ is a complex Banach space of all right-uniformly continuous bounded functions on G. Such a m is called an *invariant mean.*

Theorem 2.22. *A unitarily representable amenable Polish group is of finite type if and only if it is SIN.*

Proof. Let G be a unitarily representable amenable Polish SIN-group and let f be a positive, continuous positive definite function on G which separates the identity of G and closed subsets A with $A \not\ni e_G$ (see Lemma 2.11). We may and do assume $f(e_G) = 1$. For each $x \in G$, we define a positive function $\Psi_{x,f} : G \to [0, 1]$ by

$$\Psi_{x,f}(g) := f(g^{-1}xg), \quad g \in G.$$

We show that $\Psi_{x,f} \in \mathrm{RUCB}(G)$. Fix an arbitrary $\varepsilon > 0$. Since the positive definite function f is right-uniformly continuous, there exists a neighborhood V of e_G such that

$$|f(g) - f(h)| < \varepsilon$$

holds whenver $g, h \in G$ satisfy $hg^{-1} \in V$. There exists a neighborhood W of e_G such that $W = W^{-1}$ and $W \cdot W \subset V$ holds. Since G is SIN, there exists an invariant neighborhood U of e_G with $U \subset W$. Let $g, h \in G$ satisfy $hg^{-1} \in U$. By the invariance of U, it holds that $h \in Ug = gU$ and therefore that $g^{-1}h \in U$. Then we see that

$$\begin{aligned}(h^{-1}xh)(g^{-1}xg)^{-1} &= h^{-1}xhg^{-1}x^{-1}g \in h^{-1}xUx^{-1}g \\ &= h^{-1}Ug = Uh^{-1}g \\ &= U(g^{-1}h)^{-1} \subset W \cdot W^{-1} \\ &\subset V,\end{aligned}$$

which implies

$$|\Psi_{x,f}(h) - \Psi_{x,f}(g)| = |f(h^{-1}xh) - f(g^{-1}xg)| < \varepsilon.$$

Hence $\Psi_{x,f}$ is right-uniformly continuous and we have $\Psi_{x,f} \in \mathrm{RUCB}(G)_+$. Let $m \in \mathrm{RUCB}(G)^*$ be an invariant mean. Put

$$\psi_f(x) := m(\Psi_{x,f}), \qquad x \in G,$$

then $\psi_f(x)$ is clearly a positive, positive definite class function on G with $\psi_f(e_G) = 1$. We show that ψ_f is continuous. Since m is continuous, it suffices to show that $G \ni x \mapsto \Psi_{x,f} \in \mathrm{RUCB}(G)_+$ is continuous. Let $x, y \in G$. By Krein's inequality, we have

$$\begin{aligned}||\Psi_{x,f} - \Psi_{y,f}||^2 &= \sup_{g \in G} |f(g^{-1}xg) - f(g^{-1}yg)|^2 \\ &\leq 2 \sup_{g \in G} |1 - \mathrm{Re} f(g^{-1}yx^{-1}g)| \\ &= 2 \sup_{g \in G} |1 - f(g^{-1}yx^{-1}g)|.\end{aligned}$$

Fix $\varepsilon > 0$. Since f is right-uniformly continuous, there exists an invariant neighborhood V of e_G such that $|f(x) - f(y)| < \varepsilon$ holds for $x, y \in G$ with $yx^{-1} \in G$. Then for $x, y \in G$ with $yx^{-1} \in V$, we have $g^{-1}yx^{-1}g \in g^{-1}Vg = V$. Therefore it holds that

$$|1 - f(g^{-1}yx^{-1}g)| = |f(e_G) - f(g^{-1}yx^{-1}g)| < \varepsilon.$$

Hence we have

$$||\Psi_{x,f} - \Psi_{y,f}||^2 \leq 2\varepsilon.$$

Therefore $G \ni x \mapsto \Psi_{x,f} \in \mathrm{RUCB}(G)_+$ is continuous, hence so is ψ_f. We next show that ψ_f separates the identity of G and closed subsets A with $A \not\ni e_G$. Fix such a closed set A. Since $A^c = G \setminus A$ is an open neighborhood of e_G, there exists an open invariant neighborhood V of e_G contained in A^c. Then we have $A \subset V^c$ and $e_G \notin V^c$. Since f separates e_G and V^c, we have

$$\delta := \sup_{g \in V^c} |f(g)| < 1.$$

It then follows, by the invariance of V^c, that for $x \in V^c$,

$$||\Psi_{x,f}|| = \sup_{g \in G} |f(g^{-1}xg)| \leq \sup_{g \in V^c} |f(g)| \leq \delta,$$

which implies

$$\sup_{x \in A} |\psi_f(x)| \leq \sup_{x \in V^c} |\psi_f(x)| = \sup_{x \in V^c} |m(\Psi_{x,f})| \leq \sup_{x \in V^c} ||\Psi_{x,f}|| \leq \delta < 1.$$

Therefore ψ_f separates A and e_G. This completes the proof by Theorem 2.7. $\square$

Remark 2.23. The above proof is inspired by the proof of Theorem 2.13 of J. Galindo [8].

3 More Examples of Finite Type Groups

In this section we will give nother examples of Polish groups of finite type. To construct such examples we need to start not from finite von Neumann algebras, but from semifinite von Neumann algebras, say of type I_∞ or of type II_∞. In the end of this section we also review other known examples of Polish groups of finite type.

3.1 L^2-unitary Groups $\mathcal{U}(M)_2$

Let M be a semifinite von Neumann algebra on a Hilbert space $\mathcal{H}$ equipped with a normal faithful semifinite trace τ. A densely defined, closed operator T on $\mathcal{H}$ is said to be *affiliated* to M if for all $u \in \mathcal{U}(M')$, $uTu^* = T$ holds. Denote by $\overline{M}$ the set of all densely defined, closed operators on $\mathcal{H}$ which are affiliated to M. Recall that $L^2(M,\tau)$ is a Hilbert space completion of the space $\mathfrak{n}_\tau := \{x \in M; \tau(x^*x) < \infty\}$ by the inner product

$$\langle x, y \rangle := \tau(x^*y), \qquad x, y \in \mathfrak{n}_\tau.$$

We define $||x||_2 := \tau(x^*x)^{\frac{1}{2}}$ for $x \in L^2(M,\tau)$.

Definition 3.1. We call $\mathcal{U}(M)_2 := \{u \in \mathcal{U}(M); 1 - u \in L^2(M,\tau)\}$ the *L^2-unitary group* of (M,τ).

Note that when M is not a factor, $\mathcal{U}(M)_2$ depends on the choice of τ too. In the sequel we show the following theorem.

Theorem 3.2. *Let M be a separable semifinite von Neumann algebra with a normal faithful semifinite trace τ. Then $\mathcal{U}(M)_2$ is a Polish group of finite type, where the topology is determined by the following metric d,*

$$d(u, v) := ||u - v||_2, \qquad u, v \in \mathcal{U}(M)_2.$$

To prove the theorem, we need some preparations. In the sequel we consider M to be represented on $\mathcal{H} = L^2(M,\tau)$ by left multiplication. Recall that a closed operator $T \in \overline{M}$ on $L^2(M,\tau)$ is called *τ-measurable* if for any $\varepsilon > 0$, there exists a projection $p \in M$ with $\operatorname{ran}(p) \subset \operatorname{dom}(T)$ and $\tau(1-p) < \varepsilon$. Note that $L^2(M,\tau)$ can be identified with the set of closed, densely defined and τ-measurable operators T such that

$$||T||_2^2 := \tau(|T|^2) = \int_0^\infty \lambda^2 d\tau(e(\lambda)) < \infty,$$

where $e(\cdot)$ is a spectral resolution of $|T| = (T^*T)^{\frac{1}{2}}$ and $T = u|T|$ is the polar decomposition of T (for more details about non-commutative integration, see vol II of [22]).

Lemma 3.3. *Let M be a semifinite von Neumann algebra with a normal faithful semifinite trace τ. Then $\mathcal{U}(M)_2$ is a topological group.*

Proof. This can be shown directly, using the equalities:

$$||x^*||_2 = ||x||_2, \quad ||uxv||_2 = ||x||_2,$$

for all $x \in L^2(M,\tau)$ and $u, v \in \mathcal{U}(M)$. □

Lemma 3.4. *Let M be a semifinite von Neumann algebra with a normal faithful semifinite trace τ. Let U be a densely defined closed τ-measurable operator on $L^2(M,\tau)$ affiliated to M. Then $\operatorname{dom}(U) \cap M$ is dense in $L^2(M,\tau)$.*

Proof. Let $\varepsilon > 0$. Let $\xi \in L^2(M,\tau)$. Since $M \cap L^2(M,\tau)$ is dense, there exists $\xi_0 \in M \cap L^2(M,\tau)$ such that $||\xi - \xi_0||_2 < \varepsilon$. On the other hand, the measurability of U implies the existence of an increasing sequence $\{p_n\}_{n=1}^\infty$ of projections in M such that $p_n L^2(M,\tau) \subset \operatorname{dom}(U)$ for all n and $p_n \nearrow 1$ strongly. Therefore there exists $n_0 \in \mathbb{N}$ such that

$$||\xi_0 - p_{n_0}\xi_0||_2 < \varepsilon.$$

By the choice of ξ_0, $p_{n_0}\xi_0 \in \operatorname{dom}(U) \cap M$ and

$$\begin{aligned} ||\xi - p_{n_0}\xi_0||_2 &\leq ||\xi - \xi_0||_2 + ||\xi_0 - p_{n_0}\xi_0||_2 \\ &\leq \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, it follows that $\operatorname{dom}(U) \cap M$ is dense in $L^2(M,\tau)$. □

Lemma 3.5. *Let M be a semifinite von Neumann algebra with a normal faithful semifinite trace τ. d is a complete metric on $\mathcal{U}(M)_2$.*

Proof. Suppose $\{u_n\}_{n=1}^\infty$ is a d-Cauchy sequence in $\mathcal{U}(M)_2$. Since $L^2(M,\tau)$ is complete, there exists $V \in L^2(M,\tau)$ such that $||(1-u_n) - V||_2 \to 0$. Define $U := 1 - V$. Then $||U - u_n||_2 \to 0$. We show that U is bounded and moreover $U \in \mathcal{U}(M)_2$. Since U is closed and $\operatorname{dom}(U) \cap M$ is dense by Lemma 3.4, to prove the boundedness of U it suffices to show that U is isometric on $\mathrm{dom}(U) \cap M$. Let $\xi \in \mathrm{dom}(U) \cap M$. Since ξ is bounded, we have

$$\begin{aligned} ||(U - u_n)\xi||_2^2 &= \tau(\xi^*(U - u_n)^*(U - u_n)\xi) \\ &= \tau((U - u_n)\xi\xi^*(U - u_n)^*) \\ &\leq ||\xi||^2 \tau((U - u_n)(U - u_n)^*) \\ &= ||\xi||^2 ||U - u_n||_2^2 \to 0, \end{aligned}$$

which implies

$$||U\xi||_2 = \lim_{n\to\infty} ||u_n\xi||_2 = ||\xi||_2,$$

for all $\xi \in \mathrm{dom}(U) \cap M$. Therefore $U|_{\mathrm{dom}(U)\cap M}$ is isometric and U is bounded. Since $||U^* - u_n^*||_2 = ||U - u_n||_2$, it holds that U^* is an isometry too, which means U is a unitary. Finally, it is clear that $U = 1 - V \in \mathcal{U}(M)_2$. □

Proof of Theorem 3.2. Since M is separable, the separability of $\mathcal{U}(M)_2$ follows from the separability of $L^2(M, \tau)$. Therefore by Lemma 3.5, $\mathcal{U}(M)_2$ is a Polish group. By Schoenberg's theorem (see Example 2.5),

$$\varphi(u) := e^{-||1-u||_2^2}, \quad u \in \mathcal{U}(M)_2,$$

is a continuous, positive definite class function on $\mathcal{U}(M)_2$. It is easy to see that φ generates a neighborhood basis of the identity of $\mathcal{U}(M)_2$. Therefore the claim follows from Theorem 2.7. □

Remark 3.6. $\mathcal{U}(M)_2'' = M$.

Proof. Clearly $\mathcal{U}(M)_2'' \subset M$. Let p be a finite projection in M. Then $2p \in L^2(M, \tau)$ and $1 - 2p \in \mathcal{U}(M)_2$. Therefore $p \in \mathcal{U}(M)_2''$. Since M is semifinite, M is generated by finite projections. Therefore $\mathcal{U}(M)_2'' = M$. □

When $M = \mathbb{B}(\mathcal{H})$, $\mathcal{U}(M)_2$ is the well-known example of a Hilbert-Lie group and is denoted as $\mathcal{U}(\mathcal{H})_2$.

3.2 Non-isomorphic Properties of $\mathcal{U}(M)_2$

J. Feldman [**?**] gave a complete description of a group isomorphism between the unitary groups of type II_1 von Neumann algebras. In particular, in the proof of Theorem 4 of [**?**], he uses the following simple observation: let p be a projection in a von Neumann algebra M, then $u_p := 1 - 2p$ is a self-adjoint unitary in M. Using this correspondence, he deduced that the group isomorphism $\pi : \mathcal{U}(M_1) \to \mathcal{U}(M_2)$ between type II_1 von Neumann algebras M_1, M_2 induces an order isomorphism between their projection lattices, thereby proving that the isomorphism π is lifted to a ring *-isomorphism $\overline{\pi} : M_1 \to M_2$ (which may not preserve the scalar multiplication) in such a way that

$$\overline{\pi}(u) = \theta(u)\pi(u), \quad \text{for all } u \in \mathcal{U}(M_1)$$

holds, where θ is a multiplicative map from $\mathcal{U}(M_1)$ to $Z(\mathcal{U}(M_2))$. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Using his idea, we show that when M is a II_∞ factor and N is a finite von Neumann algebra, then $\mathcal{U}(M)_2$, $\mathcal{U}(\mathcal{H})_2$ and $\mathcal{U}(N)$ are mutually non-isomorphic. In this subsection, no separability assumptions are required.

Proposition 3.7. *Let M be a II_∞ factor. Then $\mathcal{U}(M)_2$ is not isomorphic onto $\mathcal{U}(\mathcal{H})_2$.*

Proof. Let τ be a normal faithful semifinite trace on M, Tr be the usual operator trace on $\mathcal{H}$. We denote their corresponding trace 2-norms by $||\cdot||_{2,\tau}$ and $||\cdot||_{2,\mathrm{Tr}}$, respectively. We prove the claim by contradiction. Suppose there exists a topological group isomorphism $\varphi : \mathcal{U}(M)_2 \to \mathcal{U}(\mathcal{H})_2$. Let p be a nonzero finite-rank projection in $\mathbb{B}(\mathcal{H})$. Then $1 - 2p \in \mathcal{U}(\mathcal{H})_2$ and let

$$q := \frac{1}{2}(1 - \varphi^{-1}(1 - 2p)).$$

It is easy to see that $q \in L^2(M, \tau)$ is a nonzero finite projection in M. Let $k \in \mathbb{N}$. Since M is a II_∞ factor, there exists a projection $0 < q_k \le q$ in M such that $\lim_{k\to\infty} \tau(q_k) = 0$. Define $p_k := \frac{1-\varphi(1-2q_k)}{2}$. Since

$$||q_k||_{2,\tau}^2 = \tau(q_k) \to 0 \ (k \to \infty),$$

$1 - 2q_k \to 1$ holds in $\mathcal{U}(M)_2$, which in turn means

$$1 - 2p_k = \varphi(1 - 2q_k) \to \varphi(1) = 1 \ \text{ in } \mathcal{U}(\mathcal{H})_2.$$

However, since the topology of $\mathcal{U}(\mathcal{H})_2$ is given by the operator trace 2-norm, it holds that

$$2 \le ||2p_k||_{2,\mathrm{Tr}} = ||1 - (1 - 2p_k)||_{2,\mathrm{Tr}} \to 0 \ (k \to \infty).$$

This is clearly a contradiction. Therefore $\mathcal{U}(M)_2 \ncong \mathcal{U}(\mathcal{H})_2$. □

Proposition 3.8. *Let M be a type I_∞ or type II_∞ factor, N be a finite von Neumann algebra. Then $\mathcal{U}(M)_2$ is not isomorphic onto $\mathcal{U}(N)$.*

Proof. Let τ be a normal faithful semifinite trace on M. Let $u \in Z(\mathcal{U}(M)_2)$ be an element of $\mathcal{U}(M)_2$ which commutes with every element in $\mathcal{U}(M)_2$. Then for any finite projection $p \in M$, $u(1-2p) = (1-2p)u$ holds. Therefore u commutes with all finite projections in M. Since M is generated by its finite projections, $u \in Z(M) = \mathbb{C}1$ holds. Since $u - 1 \in L^2(M, \tau)$, this forces $u = 1$. Therefore the center of $\mathcal{U}(M)_2$ is $\{1\}$, while the center of $\mathcal{U}(N)$ contains $\mathbb{C}1$. □

Remark 3.9. We thank the referee for telling us the above simple proof and the literature [**?**].

3.3 Other Known Examples

The class $\mathscr{U}_{\rm fin}$ has not been studied well. However, there are some known examples other than the ones presented in §2.6.

Example 3.10. Normalizer groups $\mathcal{N}_M(A)$ and $\mathcal{N}(E)$
Let A be an abelian von Neumann subalgebra of a separable II_1 factor M. The *normalizer group* $\mathcal{N}_M(A)$ of A, defined by

$$\mathcal{N}_M(A) := \{u \in \mathcal{U}(M); uAu^* = A\},$$

is clearly a strongly closed subgroup of $\mathcal{U}(M)$ and hence belongs to $\mathscr{U}_{\rm fin}$. This group has been drawn much attention to specialists, especially when A is maximal abelian and $\mathcal{N}_M(A)$ generates M as a von Neumann algebra. In such a case, A is called a *Cartan subalgebra*. Similarly, the *normalizer group* $\mathcal{N}(E)$ for a normal faithful conditional expectation $E : M \to N$ onto a von Neumann subalgebra N,

$$\mathcal{N}(E) := \{u \in \mathcal{U}(M); uE(x)u^* = E(uxu^*),\ \text{for all}\ x \in M\}$$

is also of finite type.

Example 3.11. The full group $[\mathcal{R}]$
Let $\mathcal{R}$ be a II_1 countable equivalence relation on a standard probability space (X, μ). A. Furman showed that the full group $[\mathcal{R}]$ equipped with so-called *uniform topology* is a Polish group of finite type (see §2 of Furman [7]).

4 Hereditary Properties of Finite Type Groups

In this section, we discuss several permanence properties of the class $\mathscr{U}_{\rm fin}$ under several algebraic operations. In summary, we will observe the following permanence properties of finite type groups.

Operation	$\mathscr{U}_{\rm fin}$?
Closed subgroup $H < G$	YES
Countable direct product $\prod_{n\geq 1} G_n$	YES
Semidirect product $G \rtimes H$	NO
Quotient G/N	NO
Extension $1 \to N \to G \to K \to 1$	NO
Projective limit $\varprojlim G_n$	YES

As can be seen from the above table, finiteness property is delicate and can easily be broken under natural operations.

Remark 4.1. (On the ultraproduct of metric groups) Let $\{(G_n, d_n)\}_{n=1}^{\infty}$ be a sequence of finite type Polish groups with a compatible bi-invariant metric. It is not difficult to show that the ultraproduct (G_ω, d_ω) of $\{(G_n, d_n)\}_{n=1}^{\infty}$ along a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ is a completely metrizable topological group of finite type, but not Polish in general. We will discuss topological groups which are embeddable into the unitary group of a (not necessarily separable) finite von Neumann algebra elsewhere.

4.1 Closed Subgroup and Countable Direct Product

It is clear the class $\mathscr{U}_{\text{fin}}$ is closed under taking a closed (or even G_δ) subgroup. Since a countable direct sum of separable finite von Neumann algebras is again separable and finite, the class $\mathscr{U}_{\text{fin}}$ is closed under countable direct product.

4.2 Extension and Semidirect Product

The class $\mathscr{U}_{\text{fin}}$ is not closed under extension nor semidirect product.

Proposition 4.2. *There exits a Polish group G not of finite type, which has a closed normal subgroup N such that N and the quotient group G/N are of finite type.*

Proof. Let G be the $ax+b$ group (see Example 2.15). Since G does not have a compatible bi-invariant metric, it is not of finite type. On the other hand, G can be written as a semidirect product $G = \mathbb{K} \rtimes \mathbb{K}^\times$, where $\mathbb{K}^\times$ acts on $\mathbb{K}$ as a multiplication. There fore the exact sequence

$$0 \longrightarrow \mathbb{K} \longrightarrow G \longrightarrow \mathbb{K}^\times \longrightarrow 1$$

gives a counter example for extension case. □

Note that the above example also shows that the class $\mathscr{U}_{\text{fin}}$ is not closed under semidirect product.

4.3 Quotient

The class $\mathscr{U}_{\text{fin}}$ is not closed under quotient.

Proposition 4.3. *There exists an abelian Polish groups of finite type G such that the quotient G/N of G by its closed subgroup is not of finite type.*

Proof. Consider the separable Banach space $A := l^3$ as an additive Polish group. As we saw in Example 2.18, $l^p (1 \le p \le \infty)$ is unitarily representable if and only if $1 \le p \le 2$. On the other hand, every separable Banach space is isomorphic onto a quotient Banach space of ℓ^1 (see e.g., Theorem 5.1 of [6]). In particular, although not of finite type, $A = \ell^3$ is a quotient of $G := \ell^1$ by its closed subgroup N. □

Remark 4.4. Note that even for abelian Polish groups, the situation can be worst possible. It is known (chapter 4 of [2]) that there exists an abelian Polish group A which has no non-trivial unitary representation. Such a group is called *strongly exotic*. On the other hand, S. Gao and V. Pestov [10] proved that any abelian Polish group is a quotient of ℓ^1 by a closed subgroup N. Therefore, strongly exotic groups are also quotients of finite type Polish groups.

4.4 Projective Limit

The class $\mathscr{U}_{\text{fin}}$ is closed under projective limit.

Proposition 4.5. *Let $\{G_n, j_{m,n} : G_m \to G_n (n \leq m)\}_{n,m=1}^{\infty}$ be a projective system of Polish groups of finite type. Then $G = \varprojlim G_n$ is a Polish group of finite type.*

Proof. Since the connecting map $\{j_{m,n}\}$ is continuous, it is clear that G can be seen as a closed subgroup of $\prod_{n\in\mathbb{N}} G_n$. Since finiteness property passes to direct product, $\prod_{n\in\mathbb{N}} G_n$ is also a Polish group of finite type. Therefore its closed subgroup G is also a Polish of finite type. $\square$

5 Some Questions

Finally let us discuss some questions to which we do not have answers at this stage. Let $\mathscr{U}_{\text{inv}}$ denote the class of Polish groups with a compatible bi-invariant metric. As we saw in Example 2.18, $\mathscr{U}_{\text{inv}}$ is strictly larger than $\mathscr{U}_{\text{fin}}$ (l^3 is in $\mathscr{U}_{\text{inv}}$ but not in $\mathscr{U}_{\text{fin}}$). Therefore the unitarily representability is indispensable (this was also pointed out by Popa). Furthermore, there exists a more interesting example. Recently L. van den Dries and S. Gao [5] constructed a Polish group G with a compatible bi-invariant metric, which does not have Lie sum (see [5] for the definition). On the other hand, we proved in [1] that if G belongs to the class $\mathscr{U}_{\text{fin}}$, then G has Lie sum. Thus G is not of finite type. Therefore it would be desirable to consider the following questions (the latter was posed in Popa[20], §6.5):

Question 5.1. Is van den Dries-Gao's Polish group unitarily representable?

Question 5.2 (Popa)**.** Is a unitarily representable Polish SIN-group of finite type?

Hopefully Theorem 2.7 will play the role for solving the above questions. Also, since l^p belongs to $\mathscr{U}_{\text{fin}}$ if and only if $1 \leq p \leq 2$, it is worth considering whether

Question 5.3. Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space. Does $\mathcal{U}(\mathcal{H})_p := \{u \in \mathcal{U}(\mathcal{H}); 1 - u \in S^p(\mathcal{H})\}$ belong to $\mathscr{U}_{\text{fin}}$ for some $1 \leq p < 2$? Here $S^p(\mathcal{H})$ denotes the space of Schatten p-class operators.

Finally, let us remind that there is another candidate for a counterexample to Question 5.2. Recall that a finite von Neumann algebra N equipped with a normal faithful tracial state τ is said to have *property (T)* if for each $\varepsilon > 0$, there exists a finite set $\mathcal{F} \subset N$ and $\delta > 0$ with the property that whenever $\varphi : N \to N$ is a unital completely positive τ-preserving map satisfying $||\varphi(x) - x||_2 < \delta$ for all $x \in \mathcal{F}$, then $||\varphi(a) - a||_2 \leq \varepsilon ||a||$ holds for all $a \in N$.

Let M be a separable II_1 factor with property (T), $\mathrm{Aut}(M)$ be a Polish group of all *-automorphisms of M equipped with the pointwise $|| \cdot ||_2$-convergence topology. Due to the property (T), this topology coincides with the topology of uniform $||\cdot||_2$-convergence on the closed unit ball M_1. Since the latter topology is given by the bi-invariant metric d defined by

$$d(\alpha,\beta) := \sup_{x\in M_1} ||\alpha(x)-\beta(x)||_2, \quad \alpha,\beta\in \mathrm{Aut}(M),$$

$\mathrm{Aut}(M)$ is a Polish SIN-group. By considering the standard representation, $\mathrm{Aut}(M)$ is unitarily representable as well. Therefore it would be interesting to check if $\mathrm{Aut}(M)$ is actually of finite type or not.

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References

[1] H. Ando, Y. Matsuzawa, Lie Group-Lie Algebra correspondences of unitary groups in finite von Neumann algebras, to appear in *Hokkaido Math. J.*

[2] Wojciech Banaszczyk, Additive subgroups of topological vector spaces, Springer LNM **1466** (1991)

[3] D. Beltita, Lie theoretic significance of the measure topologies associated with a finite trace, *Forum Math.* **22** (2010), 241–253.

[4] S. Dierolf, W. Roelcke, Uniform structures on topological groups and their quotients, *McGrow-Hill* (1981)

[5] L. van den Dries, S. Gao, A Polish group without Lie sums, *Abh. Math. Sem. Hambourg* **79** (2009) 135–147.

[6] M. Fabian, P. Habala, P Hajek, V. Montesinos, V. Zizler, Banach Space Theory: The Basis for Linear and Nonlinear Analysis, *CMS Books in Mathematics*, Springer-Verlag (2010)

[7] A. Furman, On Popa's Cocycle Superrigidity Theorem, *Inter. Math. Res. Notices IMRN*, (2007), 1–46.

[8] J. Galindo, On group and semigroup compactifications of topological groups, Online Note (http://www.mat.ucm.es/imi/documents/topological Groups/JGalindo.pdf).

[9] S. Gao, Unitary group actions and Hilbertian Polish metric spaces. in *Logic and its applications*, *Contemp. Math.* **380**, AMS Providence, RI (2005), 53–72.

[10] S. Gao, V. Pestov, On a universality property of some abelian Polish groups, *Fund. Math.* **179** no. 1 (2003), 1–15

[11] U. Haagerup, C. Winsløw, The Effros-Maréchal Topology in the space of von Neumann algebras II, *J. Funct. Anal.* **171** no. **2** (2000), 401–431.

[12] T. Hirai, E, Hirai, Positive definite class functions on a topological group and characters of factor representations, *J. Math. Kyoto. Univ.* (2005), 355–379

[13] K. Hofmann, S. Morris and M. Stroppel, Varieties of topological groups, Lie groups and SIN-groups, *Colloq. Math.* **70** (1996), 151–163.

[14] R. V. Kadison, I. M. Singer, Some remarks on representations of connected groups, *Proc. Nat. Acad. Sci.* **38** (1952), 419–423.

[15] M. Megrelishvili, Reflexively but not unitarily representable topological groups, *Topol. Proc.* **25** (2000), 615–625.

[16] J. Peterson, T. Sinclair, On cocycle superrigidity for Gaussian actions, arXiv0910.3958

[17] F. Murray, J. von Neumann, On Rings of Operators, *Ann. Math.* **37** (1936), 116–229.

[18] E. Nelson, Notes on non-commutative integration, *J. Funct. Anal.* **15** (1974), 103–116.

[19] G. Pedersen, C*-algebras and their automorphism groups, Academic Press (1979)

[20] S. Popa, Cocycle and orbit equivalence superrigidity for malleable actions of w-rigid groups, *Invent. Math.* **170** (2007), 243–295.

[21] I. J. Schoenberg, Metric spaces and positive definite functions, *Trans. Amer. Math. Soc.* **44** (1938) no 3, 522–536

[22] M. Takesaki, Theory of Operator Algebras I (1979), II and III (2002), Springer-Verlag, Berlin.