Conformal higher-order viscoelastic fluid mechanics

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ABSTRACT: We present a generally covariant formulation of conformal higher-order viscoelastic fluid mechanics with strain allowed to take arbitrarily large values. We give a general prescription to determine the dynamics of a relativistic viscoelastic fluid in a way consistent with the hypothesis of local thermodynamic equilibrium and the second law of thermodynamics. We then elaborately study the transient time scales at which the strain almost relaxes and becomes proportional to the gradients of velocity. We particularly show that a conformal second-order fluid with all possible parameters in the constitutive equations can be obtained without breaking the hypothesis of local thermodynamic equilibrium, if the conformal fluid is defined as the long time limit of a conformal second-order viscoelastic system. We also discuss how local thermodynamic equilibrium could be understood in the context of the fluid/gravity correspondence.

Contents

1 Introduction

Viscoelasticity is the property shared by almost all continuum materials, showing elasticity at short time scales while behaving as a viscous fluid at long time scales [1, 2]. In papers [3, 4], the present authors constructed relativistic viscoelastic fluid mechanics in a generally covariant form based on Onsager's linear nonequilibrium thermodynamics. We showed there that, for arbitrary parameters in the constitutive equations, viscoelastic fluids thus defined behave as standard first-order viscous fluids (obeying the relativistic Navier-Stokes equations [5, 6]) at long time scales [4]. We also showed that the evolution equations are hyperbolic for a wide range of parameters due to the elasticity at short time scales. In this sense, a relativistic viscoelastic model with such parameters gives a causal completion of standard first-order relativistic fluid mechanics [4].

Recently, various models of relativistic fluids with second-order corrections in the derivative expansion (called second-order fluid mechanics) have been considered in the analysis of heavy-ion collision experiments and also in the study of the holographic duality between the long wavelength dynamics of black hole horizons and the dynamics of viscous fluids (see, e.g., [7, 8] and references therein). For both cases, fluid systems are well approximated to be invariant under conformal transformations. In particular, in [9] and [10],

a model of conformal fluid with full second-order corrections was constructed and shown to be consistent with the second law of thermodynamics for a wide range of parameters [10].

A remarkable point for second-order fluids is that their entropy densities generally contain spatial derivatives of thermodynamic variables. This means that local thermodynamic equilibrium is broken for second-order fluids. In fact, in thermodynamics the coordinates $x = (x^0, \boldsymbol{x})$ represent a coarse-grained spacetime point, whose temporal and spatial resolutions we denote by $\epsilon_{\rm t}$ and $\epsilon_{\rm s}$, respectively [3]. We say that local thermodynamic equilibrium is realized at x with resolution $(\epsilon_{\rm t}, \epsilon_{\rm s})$ if a small spatial region around $\boldsymbol{x}$ (of linear size $\epsilon_{\rm s}$) at time x^0 can be well regarded as being in thermodynamic equilibrium at least for a time duration of $\epsilon_{\rm t}$.[1] This implies that the local entropy in the coarse-grained region around x is already maximized for given values of local thermodynamic variables at x (such as the energy-momentum density and the charge density), and thus that the entropy density at x, $s(x)$, is a function only of these local thermodynamic variables and should not depend on their spatial derivatives (which include contributions from nearby material particles). Thus, when we consider thermodynamics of second-order fluids, we are generally forced to give up the hypothesis of local thermodynamic equilibrium and to use (a variant of) *extended thermodynamics*, where the entropy density can depend on quantities including spatial derivatives (such as dissipative currents) [13–15].

In this paper, introducing the elastic strain tensor as one of local thermodynamic variables, and utilizing the manifestly Weyl-covariant formulation of conformal second-order fluid mechanics developed by Loganayagam [10], we construct conformal higher-order (i.e., not first-order) *viscoelastic* fluid mechanics in the Landau-Lifshitz frame in such a way that local thermodynamic equilibrium and the second law of thermodynamics are manifestly realized. We show that the conformal second-order fluid mechanics of [9, 10] with all possible parameters in the constitutive equations can be fully recovered as the long time limit of our viscoelastic model of second order. Thus, if we define conformal second-order fluid mechanics as the long time limit of a conformal viscoelastic system, conformal second-order fluid mechanics can be constructed without violating the hypothesis of local thermodynamic equilibrium.[2]

This paper is organized as follows. In section 2, we briefly review a part of the manifestly Weyl-covariant formulation of conformal second-order fluid mechanics [10]. In section 3, assuming local thermodynamic equilibrium, we present a general theory describing

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[1] We call such spatial regions *material particles*.

[2] We comment that local thermodynamic equilibrium can also be realized in divergence-type fluid mechanics developed by Geroch and Lindblom [11, 12]. There, a symmetric traceless tensor $\xi_{\mu\nu}$ is introduced as a local dynamical variable in addition to the standard dynamical variables $\xi \equiv \mu/T$ and $\xi_\mu \equiv u_\mu/T$ (T: temperature, u^μ: velocity, μ: chemical potential). In fact, the causally completed extensions of Eckart's fluid mechanics [11] and of Landau-Lifshitz's [12] satisfy local thermodynamic equilibrium in the sense that the entropy density depends only on ξ, ξ_μ and $\xi_{\mu\nu}$. The apparent difference between divergence-type fluid mechanics and viscoelastic fluid mechanics is that the additional dynamical variable in the former is related to the conserved current (often denoted by $A^{\lambda\mu\nu}$), while the strain tensor introduced in the latter has no origin as a conserved quantity. A possible relationship between two theories will be commented in section 5.

relativistic viscoelastic fluids with large strain. In section 4, we construct conformal higher-order viscoelastic fluid mechanics, based on the manifestly Weyl-covariant formalism. In section 5, we investigate the transient time scales at which the strain almost relaxes and becomes proportional to the gradients of velocity. We there verify our claim that a conformal second-order fluid with all possible parameters in the constitutive equations can be obtained as the long time limit of a conformal viscoelastic system that satisfies the hypothesis of local thermodynamic equilibrium. In section 6, we briefly discuss how local thermodynamic equilibrium could be understood in the context of the fluid/gravity correspondence, and point out that its manifest realization may lead to the viscoelasticity/quantum gravity correspondence. Appendix A gives the list of the dimensions and the weights of various local thermodynamic quantities, and appendix B collects useful formulas in the manifestly Weyl-covariant formalism with proofs.

2 Manifestly Weyl-covariant formalism

In this section, in order to fix our notation, we give a brief review on the manifestly Weyl-covariant formulation of second-order fluid mechanics developed in [10] (see also [16, 17]).

We consider a d-dimensional spacetime with background metric $g_{\mu\nu}$ of signature $(-, +, \ldots, +)$. A tensor $Q^{\mu\cdots}_{\nu\cdots}$ is called a conformal tensor of weight w if it transforms as

$$Q^{\mu\cdots}_{\nu\cdots} = \mathrm{e}^{w\phi} \, Q'^{\mu\cdots}_{\nu\cdots} \tag{2.1}$$

under the Weyl transformation[3]

$$g_{\mu\nu} = \mathrm{e}^{-2\phi} \, g'_{\mu\nu} \,. \tag{2.2}$$

We introduce the Weyl connection $\Gamma^\lambda_{(\mathrm{w})\,\mu\nu} \equiv \Gamma^\lambda_{\mu\nu} + W^\lambda_{\mu\nu}$ with

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{1}{2} \, g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right), \quad W^\lambda_{\mu\nu} \equiv g_{\mu\nu} \, \mathcal{A}^\lambda - \delta^\lambda_\mu \, \mathcal{A}_\nu - \delta^\lambda_\nu \, \mathcal{A}_\mu \,. \tag{2.3}$$

$\Gamma^\lambda_{(\mathrm{w})\,\mu\nu}$ is invariant under the Weyl transformation (2.2) if the (non-conformal) vector field $\mathcal{A}_\mu$ transforms as

$$\mathcal{A}_\mu = \mathcal{A}'_\mu - \partial_\mu \phi \,. \tag{2.4}$$

Following [10], we introduce Dirac's *co-covariant derivative* [18] for a conformal tensor $Q^{\mu\cdots}_{\nu\cdots}$ of weight w as

$$\begin{aligned}
\mathcal{D}_\lambda Q^{\mu\cdots}_{\nu\cdots} &\equiv \partial_\lambda Q^{\mu\cdots}_{\nu\cdots} + \Gamma^\mu_{(\mathrm{w})\,\lambda\rho} \, Q^{\rho\cdots}_{\nu\cdots} + \cdots - \Gamma^\rho_{(\mathrm{w})\,\lambda\nu} \, Q^{\mu\cdots}_{\rho\cdots} - \cdots + w \, \mathcal{A}_\lambda \, Q^{\mu\cdots}_{\nu\cdots} \\
&= \nabla_\lambda Q^{\mu\cdots}_{\nu\cdots} + W^\mu_{\lambda\rho} \, Q^{\rho\cdots}_{\nu\cdots} + \cdots - W^\rho_{\lambda\nu} \, Q^{\mu\cdots}_{\rho\cdots} - \cdots + w \, \mathcal{A}_\lambda \, Q^{\mu\cdots}_{\nu\cdots} \,,
\end{aligned} \tag{2.5}$$

which enjoys the following properties [10]:

$$\mathcal{D}_\lambda Q^{\mu\cdots}_{\nu\cdots} = \mathrm{e}^{w\phi} \, \mathcal{D}'_\lambda Q'^{\mu\cdots}_{\nu\cdots} \quad \text{if} \quad Q^{\mu\cdots}_{\nu\cdots} = \mathrm{e}^{w\phi} \, Q'^{\mu\cdots}_{\nu\cdots} \,, \tag{2.6}$$

$$\mathcal{D}_\lambda g_{\mu\nu} = 0 \,, \qquad \mathcal{D}_\lambda g^{\mu\nu} = 0 \,. \tag{2.7}$$

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[3]Note that $g_{\mu\nu}$ itself is a conformal tensor of weight $w = -2$. The dimensions and the weights of various local thermodynamic quantities are listed in appendix A.

Note that if a contravariant vector v^μ has weight $w = d$, the following equality holds:

$$\mathcal{D}_\mu v^\mu = \nabla_\mu v^\mu + (w - d)\, \mathcal{A}_\mu v^\mu = \nabla_\mu v^\mu \quad (\text{when } w = d)\,. \tag{2.8}$$

Similarly, if a $(2,0)$ tensor $Q^{\mu\nu}$ is symmetric traceless and has weight $w = d + 2$, we have

$$\mathcal{D}_\mu Q^{\mu\nu} = \nabla_\mu Q^{\mu\nu} \quad (\text{when } w = d + 2\,,\ Q^{\mu\nu} = Q^{\nu\mu} \text{ and } Q^\mu_\mu = 0)\,. \tag{2.9}$$

Given the velocity field u^μ for a fluid (having unit weight and being normalized as $u^\mu u_\mu = -1$), the vector field $\mathcal{A}_\mu$ can be uniquely determined by requiring the covariant derivative of u^μ to be transverse ($u^\nu \mathcal{D}_\nu u^\mu = 0$) and divergenceless ($\mathcal{D}_\mu u^\mu = 0$) [10]:

$$\mathcal{A}^\mu = a^\mu - \frac{\vartheta}{d - 1}\, u^\mu\,, \tag{2.10}$$

where a^μ and ϑ are the acceleration and the expansion, respectively:

$$a^\mu \equiv \nabla_u u^\mu \equiv u^\nu \nabla_\nu u^\mu\,, \qquad \vartheta \equiv \nabla_\mu u^\mu\,. \tag{2.11}$$

In this paper, we write the (anti-)symmetrization of indices as $Q_{(\mu\nu)} \equiv (1/2)\big(Q_{\mu\nu} + Q_{\nu\mu}\big)$ and $Q_{[\mu\nu]} \equiv (1/2)\big(Q_{\mu\nu} - Q_{\nu\mu}\big)$. We further denote the symmetric, transverse, traceless part of $Q_{\mu\nu}$ by

$$Q_{\langle\mu\nu\rangle} \equiv h_\mu^\alpha h_\nu^\beta Q_{(\alpha\beta)} - \frac{1}{d - 1}\,(h^{\alpha\beta} Q_{\alpha\beta})\, h_{\mu\nu}\,, \tag{2.12}$$

where $h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$ is the metric projected to a surface orthogonal to u^μ. One then can show [10] that $\mathcal{D}_\mu u^\nu$ can be decomposed as

$$\mathcal{D}_\mu u^\nu = \sigma_\mu^{\ \nu} + \omega_\mu^{\ \nu} \tag{2.13}$$

with[4]

$$\sigma_{\mu\nu} \equiv \mathcal{D}_{(\mu} u_{\nu)} = \nabla_{\langle\mu} u_{\nu\rangle}\,, \qquad \omega_{\mu\nu} \equiv \mathcal{D}_{[\mu} u_{\nu]} = h_\mu^\rho h_\nu^\sigma \nabla_{[\rho} u_{\sigma]}\,. \tag{2.14}$$

We introduce the Weyl-covariantized Riemann tensor $\mathcal{R}_{\mu\nu\lambda}^{\ \ \ \sigma}$ [10] as the curvature for the Weyl connection $\Gamma^\lambda_{(\mathrm{w})\,\mu\nu}$:[5]

$$\mathcal{R}_{\mu\nu\lambda}^{\ \ \ \sigma} \equiv -\partial_\mu \Gamma^\sigma_{(\mathrm{w})\,\nu\lambda} + \partial_\nu \Gamma^\sigma_{(\mathrm{w})\,\mu\lambda} - \Gamma^\sigma_{(\mathrm{w})\,\mu\rho} \Gamma^\rho_{(\mathrm{w})\,\nu\lambda} + \Gamma^\sigma_{(\mathrm{w})\,\nu\rho} \Gamma^\rho_{(\mathrm{w})\,\mu\lambda} = \mathcal{R}'_{\mu\nu\lambda}{}^\sigma\,. \tag{2.15}$$

The explicit form is given by

$$\mathcal{R}_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma} + \mathcal{F}_{\mu\nu}\, g_{\lambda\sigma} - 4\, \delta^\alpha_{[\mu} g_{\nu][\lambda} \delta^\beta_{\sigma]} \left(\nabla_\alpha \mathcal{A}_\beta + \mathcal{A}_\alpha \mathcal{A}_\beta - \frac{\mathcal{A}^2}{2}\, g_{\alpha\beta} \right) = e^{-2\phi}\, \mathcal{R}'_{\mu\nu\lambda\sigma}\,. \tag{2.16}$$

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[4]Note that $\sigma_{\mu\nu} = \sigma_{\langle\mu\nu\rangle}$. We will use the abbreviation such as $(\sigma^2)_{\mu\nu} \equiv \sigma_\mu^\alpha \sigma_{\alpha\nu}$ and $\mathrm{tr}(\sigma^2) \equiv \sigma_\alpha^\beta \sigma_\beta^\alpha$. It then holds that $(\sigma^n)_{\mu\nu} = (\sigma^n)_{(\mu\nu)} = h_\mu^\alpha h_\nu^\beta (\sigma^n)_{(\alpha\beta)}$ for $n = 1, 2, \ldots$, and $(\omega^n)_{\mu\nu}$ is $h_\mu^\alpha h_\nu^\beta (\omega^n)_{(\alpha\beta)}$ for even n and $h_\mu^\alpha h_\nu^\beta (\omega^n)_{[\alpha\beta]}$ for odd n.

[5]We follow the convention of [16, 17] where all the curvature tensors are negative of those in [10]. Tensors of non-calligraphic font are the ones constructed from the affine connection $\Gamma^\lambda_{\mu\nu}$ (i.e. the ones obtained by replacing $\mathcal{D}_\mu$ by ∇_μ).

It is easy to see that the following equality holds for a covariant vector v_λ of weight w [10]:

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\, v_\lambda = \mathcal{R}_{\mu\nu\lambda}{}^\sigma\, v_\sigma + w\, \mathcal{F}_{\mu\nu}\, v_\lambda\,, \tag{2.17}$$

$$\mathcal{F}_{\mu\nu} \equiv \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu\,. \tag{2.18}$$

For a contravariant vector v_1^λ of weight w_1, we obtain[6]

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\, v_1^\lambda = -\mathcal{R}_{\mu\nu\sigma}{}^\lambda\, v_1^\sigma + w_1\, \mathcal{F}_{\mu\nu}\, v_1^\lambda\,. \tag{2.19}$$

We also introduce the following conformal tensors:

$$\mathcal{R}_{\mu\nu} \equiv \mathcal{R}_{\mu\alpha\nu}{}^\alpha = R_{\mu\nu} + \mathcal{F}_{\mu\nu} + \nabla_\rho \mathcal{A}^\rho\, g_{\mu\nu} + (d-2)\left(\nabla_\mu \mathcal{A}_\nu + \mathcal{A}_\mu\, \mathcal{A}_\nu - \mathcal{A}^2 g_{\mu\nu}\right) = \mathcal{R}'_{\mu\nu}\,, \tag{2.20}$$

$$\mathcal{R} \equiv \mathcal{R}_\alpha{}^\alpha = R + 2(d-1)\, \nabla_\rho \mathcal{A}^\rho - (d-1)\,(d-2)\, \mathcal{A}^2 = e^{2\phi}\, \mathcal{R}'\,, \tag{2.21}$$

$$\mathcal{G}_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2}\, \mathcal{R}\, g_{\mu\nu}\,,$$

$$= G_{\mu\nu} + \mathcal{F}_{\mu\nu} + (d-2)\left[\nabla_\mu \mathcal{A}_\nu + \mathcal{A}_\mu\, \mathcal{A}_\nu - \left(\nabla_\rho \mathcal{A}^\rho - \frac{d-3}{2}\, \mathcal{A}^2\right) g_{\mu\nu}\right] = \mathcal{G}'_{\mu\nu}\,. \tag{2.22}$$

These tensors have the following symmetry properties for their indices [10]:

$$\mathcal{R}_{(\mu\nu)\,\lambda\sigma} = 0\,, \qquad \mathcal{R}_{\mu\nu\,(\lambda\sigma)} = \mathcal{F}_{\mu\nu}\, g_{\lambda\sigma}\,, \qquad \mathcal{R}_{[\lambda\mu\nu]\sigma} = 0\,, \tag{2.23}$$

$$\mathcal{R}_{\mu\nu\,\lambda\sigma} = \mathcal{R}_{\lambda\sigma\,\mu\nu} + 4\, \delta^\alpha_{[\mu} g_{\nu][\lambda} \delta^\beta_{\sigma]}\, \mathcal{F}_{\alpha\beta} + \mathcal{F}_{\mu\nu}\, g_{\lambda\sigma} - g_{\mu\nu}\, \mathcal{F}_{\lambda\sigma}\,, \tag{2.24}$$

$$v^\alpha\, \mathcal{R}_{\alpha\,[\mu\nu]\,\beta}\, v^\beta = -\frac{1}{2}\, \mathcal{F}_{\mu\nu}\, v^2\,, \qquad \mathcal{R}_{[\mu\nu]} = \frac{d}{2}\, \mathcal{F}_{\mu\nu}\,. \tag{2.25}$$

The Bianchi identity $\big([\mathcal{D}_\lambda, [\mathcal{D}_\mu, \mathcal{D}_\nu]] + (\text{cyclic})\big)\, v_\rho = 0$ gives [10]

$$\mathcal{D}_\lambda \mathcal{R}_{\mu\nu\rho}{}^\sigma + \mathcal{D}_\mu \mathcal{R}_{\nu\lambda\rho}{}^\sigma + \mathcal{D}_\nu \mathcal{R}_{\lambda\mu\rho}{}^\sigma = 0\,, \tag{2.26}$$

$$\nabla_\lambda \mathcal{F}_{\mu\nu} + \nabla_\mu \mathcal{F}_{\nu\lambda} + \nabla_\nu \mathcal{F}_{\lambda\mu} = 0\,. \tag{2.27}$$

Contracting the indices in the first equation, we obtain

$$\mathcal{D}_\alpha \mathcal{R}_{\mu\nu\lambda}{}^\alpha + \mathcal{D}_\mu \mathcal{R}_{\nu\lambda} - \mathcal{D}_\nu \mathcal{R}_{\mu\lambda} = 0\,, \tag{2.28}$$

$$\mathcal{D}_\nu \mathcal{R}^{\mu\nu} = \frac{1}{2}\, \mathcal{D}^\mu \mathcal{R} + \mathcal{D}_\nu \mathcal{F}^{\mu\nu}\,. \tag{2.29}$$

The last equation is equivalent to

$$0 = \mathcal{D}_\mu\big(\mathcal{G}^{\nu\mu} - \mathcal{F}^{\nu\mu}\big) = \mathcal{D}_\mu\Big(\mathcal{G}^{(\mu\nu)} - \frac{d-2}{2}\, \mathcal{F}^{\mu\nu}\Big)\,. \tag{2.30}$$

The Weyl tensor is defined as

$$C_{\mu\nu\lambda\sigma} \equiv R_{\mu\nu\lambda\sigma} + \frac{4}{d-2}\, \delta^\alpha_{[\mu} g_{\nu][\lambda} \delta^\beta_{\sigma]} \left(R_{\alpha\beta} - \frac{R}{2(d-1)}\, g_{\alpha\beta}\right) = e^{-2\phi}\, C'_{\mu\nu\lambda\sigma}\,. \tag{2.31}$$

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[6]Note that $w_1 = w + 2$ if $v_1^\lambda = g^{\lambda\mu}\, v_\mu$.

Although this is given only in terms of the metric, we can rewrite it as a sum of Weyl-covariantized curvature tensors (see eq. (B.6)):

$$C_{\mu\nu\lambda\sigma} = \mathcal{R}_{\mu\nu\lambda\sigma} - \mathcal{F}_{\mu\nu}\, g_{\lambda\sigma} + \frac{4}{d-2}\, \delta^{\alpha}_{[\mu} g_{\nu][\lambda} \delta^{\beta}_{\sigma]} \left(\mathcal{R}_{\alpha\beta} - \frac{\mathcal{R}}{2(d-1)}\, g_{\alpha\beta} + \mathcal{F}_{\alpha\beta} \right). \qquad (2.32)$$

When $\mathcal{A}_\mu$ is constructed from the conformal velocity field u^μ of unit weight as in eq. (2.10), we have the following formulas (see eqs. (B.7) and (B.2)):

$$u^\alpha\, \mathcal{R}_{\alpha\,\langle\mu\nu\rangle\,\beta}\, u^\beta = u^\alpha\, C_{\alpha\mu\nu\beta}\, u^\beta\, + \frac{1}{d-2}\, \mathcal{R}_{\langle\mu\nu\rangle}\,, \qquad (2.33)$$

$$u^\alpha\, \mathcal{R}_{\alpha\beta}\, u^\beta = -\operatorname{tr}(\sigma^2) - \operatorname{tr}(\omega^2)\,. \qquad (2.34)$$

The shear $\sigma_{\mu\nu}$ and the vorticity $\omega_{\mu\nu}$ satisfy the following equalities $\big(\mathcal{D}_u \equiv u^\mu \mathcal{D}_\mu\big)$ (see eqs. (B.12), (B.10), (B.3), (B.7) and (B.4)):

$$\mathcal{D}_\mu \mathcal{D}_\nu \sigma^{\mu\nu} = \mathcal{R}^{\langle\mu\nu\rangle}\, \sigma_{\mu\nu} - \frac{1}{2}\, \mathcal{F}^{\mu\nu} \omega_{\mu\nu} + \frac{1}{2}\, \mathcal{D}_u\, \mathcal{R} + (d-2)\, \mathcal{D}_\mu(\mathcal{F}^{\mu\nu} u_\nu)\,, \qquad (2.35)$$

$$\mathcal{D}_\mu \mathcal{D}_\nu \omega^{\mu\nu} = -\frac{d-3}{2}\, \mathcal{F}^{\mu\nu} \omega_{\mu\nu}\,, \qquad (2.36)$$

$$\mathcal{D}_u \sigma_{\mu\nu} = u^\alpha\, C_{\alpha\mu\nu\beta}\, u^\beta + \frac{1}{d-2}\, \mathcal{R}_{\langle\mu\nu\rangle} - (\sigma^2)_{\langle\mu\nu\rangle} - (\omega^2)_{\langle\mu\nu\rangle}\,, \qquad (2.37)$$

$$\mathcal{D}_u \omega_{\mu\nu} = \frac{1}{2}\, h_\mu{}^\lambda\, h_\nu{}^\sigma\, \mathcal{F}_{\lambda\sigma} - (\sigma\omega + \omega\sigma)_{\mu\nu}\,. \qquad (2.38)$$

One can further show that

$$\mathcal{D}_\mu\big(u_\nu \mathcal{R}^{\nu\mu}\big) = \mathcal{R}^{\mu\nu} \sigma_{\mu\nu} + \frac{1}{2}\, \mathcal{D}_u \mathcal{R} - \frac{d-2}{2}\, \mathcal{F}^{\mu\nu} \omega_{\mu\nu} - \mathcal{D}_\mu(\mathcal{F}^{\mu\nu} u_\nu)$$

$$= \mathcal{D}_\mu \mathcal{D}_\nu \sigma^{\mu\nu} - \frac{d-3}{2}\, \mathcal{F}^{\mu\nu} \omega_{\mu\nu} - (d-1)\, \mathcal{D}_\mu(\mathcal{F}^{\mu\nu} u_\nu)\,, \qquad (2.39)$$

$$\mathcal{D}_\mu\big(u_\nu \mathcal{R}^{\nu\rho} h_\rho{}^\mu\big) = 2\operatorname{tr}(\sigma^3) + 6\operatorname{tr}(\sigma\omega^2) - 2\sigma^{\mu\nu}\, u^\alpha\, C_{\alpha\mu\nu\beta}\, u^\beta - \frac{2}{d-2}\, \mathcal{R}^{\mu\nu} \sigma_{\mu\nu}$$

$$- \frac{d-5}{2}\, \mathcal{F}^{\mu\nu} \omega_{\mu\nu} + \mathcal{D}_\mu \mathcal{D}_\nu \sigma^{\mu\nu} - (d-1)\mathcal{D}_\mu(\mathcal{F}^{\mu\nu} u_\nu)\,. \qquad (2.40)$$

3 Relativistic viscoelastic fluids with large strain

In this paper, we consider a conformal viscoelastic fluid in a d-dimensional spacetime with background metric $g_{\mu\nu}(x)$. We assume in this section that there exists a conserved charge (such as particle number) in addition to energy and momentum, and later will ignore the charge (or set the corresponding chemical potential to be zero) to simplify discussions and expressions. A viscoelastic fluid then has the following local thermodynamic variables [3, 4]:

$$p_\mu(x)\,, \qquad n(x)\,, \qquad g_{\mu\nu}(x)\,, \qquad \varepsilon_{\mu\nu}(x)\,. \qquad (3.1)$$

Here, $p_\mu(x)$ is the energy-momentum vector, $n(x)$ the charge density, and $\varepsilon(x) = \big(\varepsilon_{\mu\nu}(x)\big)$ the strain tensor, which is assumed to be spatial ($\varepsilon_{\mu\nu} u^\nu = 0$) and symmetric ($\varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}$) [3, 4]. We define the proper energy density as

$$e(x) \equiv \sqrt{-g^{\mu\nu}(x)\, p_\mu(x)\, p_\nu(x)}\,, \qquad (3.2)$$

which is essentially the sum of the rest mass energy density and the internal energy density. Note that p_μ are always additive quantities, but e is not in a relativistic theory.

In this paper, we work in the Landau-Lifshitz frame, where the velocity field $u = u^\mu \, \partial_\mu$ is defined as $u^\mu \equiv p^\mu/e$ (note that $u^\mu u_\mu = -1$).[7] The metric projected to a surface orthogonal to u^μ is defined as

$$h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu \, , \tag{3.3}$$

and represents the shape of a material [1, 4]. By writing the velocity field as[8]

$$u = u^\mu \, \partial_\mu = \frac{1}{N} \, \partial_0 + \frac{N^i}{N} \, \partial_i \tag{3.4}$$

with the lapse function N and the shift functions N^i, the metric $g_{\mu\nu}$ can be expressed with the following ADM parametrization:

$$ds^2 = -N^2 \left(dx^0\right)^2 + h_{ij} \left(dx^i - N^i \, dx^0\right) \left(dx^j - N^j \, dx^0\right) . \tag{3.5}$$

Note that $h_{\mu\nu}$ and $h^{\mu\nu}$ are written as

$$h_{\mu\nu} = \begin{pmatrix} h_{00} & h_{0j} \\ h_{i0} & h_{ij} \end{pmatrix} = \begin{pmatrix} h_{kl} \, N^k N^l & - h_{jk} \, N^k \\ - h_{ik} \, N^k & h_{ij} \end{pmatrix} , \quad h^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \left(h^{-1}\right)^{ij} \end{pmatrix} , \tag{3.6}$$

where h^{-1} is the inverse matrix of the $(d-1) \times (d-1)$ matrix $h = (h_{ij})$. Thus, the local volume $\sqrt{h} \equiv \sqrt{\det(h_{ij})}$ satisfies the identity[9]

$$\delta\sqrt{h} = \frac{\sqrt{h}}{2} \left(h^{-1}\right)^{ij} \delta h_{ij} = \frac{\sqrt{h}}{2} \, h^{\mu\nu} \delta h_{\mu\nu} = \frac{\sqrt{h}}{2} \, h^{\mu\nu} \delta g_{\mu\nu} \, . \tag{3.7}$$

We assume that local thermodynamic equilibrium is realized at each spacetime point, so that the local entropy density $s(x)$ at spacetime point x is given as a function of the values of these local thermodynamic variables at the same point x, $s(x) = s\big(p_\mu(x), n(x), g_{\mu\nu}(x), \varepsilon_{\mu\nu}(x)\big)$. In order to treat conserved quantities, it is often convenient to multiply densities by the local volume $\sqrt{h}$, and we write such multiplied densities with placing tilde on the original densities. Then, the fundamental relation for the local entropy $\tilde{s} = \sqrt{h} \, s$ is given by [3, 4]

$$\delta\tilde{s} = -\frac{u^\mu}{T} \, \delta\tilde{p}_\mu - \frac{\mu}{T} \, \delta\tilde{n} + \frac{\sqrt{h}}{2T} \, T_0^{\mu\nu} \, \delta g_{\mu\nu} + \frac{\sqrt{h}}{T} \, S_\nu^\mu \, \delta\varepsilon_\mu^\nu \, . \tag{3.8}$$

Here, T and μ are the temperature and the chemical potential, respectively, and $T_0^{\mu\nu} = e \, u^\mu u^\nu + \tau_0^{\mu\nu}$ is the quasi-conservative energy-momentum tensor with $\tau_0^{\mu\nu}$ the quasi-conservative stress tensor ($\tau_0^{\mu\nu} = \tau_0^{\nu\mu}$, $\tau_0^{\mu\nu} u_\nu = 0$).[10] We write the above with ε_μ^ν (not with $\varepsilon_{\mu\nu}$) for

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[7]In this paper, we lower (or raise) indices always with $g_{\mu\nu}$ (or with its inverse $g^{\mu\nu}$).

[8]We denote spacetime coordinates by $x = (x^\mu) = (x^0, x^i)$ ($\mu = 0, 1, \ldots, d-1; \, i = 1, \ldots, d-1$). Note that we need not assume u^μ to be hypersurface orthogonal when considering only infinitesimal neighbors around a spacetime point.

[9]To obtain the last equality, we have used the identity $h^{\mu\nu} \, \delta\big(u_\mu u_\nu\big) = 0$.

[10]$T_0^{\mu\nu}$ and $\tau_0^{\mu\nu}$ were denoted by $T_{(q)}^{\mu\nu}$ and $\tau_{(q)}^{\mu\nu}$, respectively, in [3, 4].

later convenience. S_ν^μ is assumed to be spatial and symmetric ($S^{\mu\nu} = S^{\nu\mu}$) and represents the entropic (not energetic) force caused by strain.[11] If we rewrite $\delta\tilde{p}_\mu$ and $\delta g_{\mu\nu}$ in eq. (3.8) in terms of $\delta\tilde{e}$, δu_μ and $\delta h_{\mu\nu}$ by using $\tilde{p}_\mu = \tilde{e}\, u_\mu$ and $g_{\mu\nu} = h_{\mu\nu} - u_\mu u_\nu$, then terms proportional to δu_μ are all canceled, and we obtain another expression for the fundamental relation:

$$\delta\tilde{s} = \frac{1}{T}\,\delta\tilde{e} - \frac{\mu}{T}\,\delta\tilde{n} + \frac{\sqrt{h}}{2T}\,\tau_0^{\mu\nu}\,\delta h_{\mu\nu} + \frac{\sqrt{h}}{T}\,S_\nu^\mu\,\delta\varepsilon_\mu^\nu\,. \tag{3.9}$$

We assume the densities with tilde to be extensive:[12]

$$\lambda\,\tilde{s}\big(\tilde{e},\,\tilde{n},\,h_{\mu\nu},\,\varepsilon_\mu^\nu\big) = \tilde{s}\big(\lambda\,\tilde{e},\,\lambda\,\tilde{n},\,\lambda^{2/(d-1)}\,h_{\mu\nu},\,\varepsilon_\mu^\nu\big) \quad (\forall\lambda > 0)\,. \tag{3.10}$$

Then, taking a derivative with respect to λ and setting $\lambda = 1$ afterwards, we obtain the following Euler relation, having the same form as that for a fluid:

$$\tilde{s} = \frac{\tilde{e} + \tilde{P} - \mu\,\tilde{n}}{T} \qquad \text{or} \qquad s = \frac{e + P - \mu\,n}{T}\,. \tag{3.11}$$

Here, $P \equiv (\mathrm{tr}\,\tau_0)/(d-1) \equiv g_{\mu\nu}\,\tau_0^{\mu\nu}/(d-1) = h_{\mu\nu}\,\tau_0^{\mu\nu}/(d-1)$ corresponds to the pressure of an isotropic fluid. From eqs. (3.7), (3.9) and (3.11), we obtain the fundamental relation for s,

$$\delta s = \frac{1}{T}\,\delta e - \frac{\mu}{T}\,\delta n + \frac{1}{2T}\,\tau_0^{\langle\mu\nu\rangle}\,\delta h_{\mu\nu} + \frac{1}{T}\,S_\nu^\mu\,\delta\varepsilon_\mu^\nu\,, \tag{3.12}$$

together with the Gibbs-Duhem relation,

$$0 = e\,\delta\Big(\frac{1}{T}\Big) + n\,\delta\Big(-\frac{\mu}{T}\Big) + \delta\Big(\frac{P}{T}\Big) - \frac{1}{2T}\,\tau_0^{\langle\mu\nu\rangle}\,\delta h_{\mu\nu} - \frac{1}{T}\,S_\nu^\mu\,\delta\varepsilon_\mu^\nu\,. \tag{3.13}$$

We define the *order* of a local quantity as the order of derivatives necessary to be taken in order to construct the quantity from local thermodynamic variables p_μ, n, $g_{\mu\nu}$, and $\varepsilon_{\mu\nu}$:

	order
p_μ, n, $g_{\mu\nu}$, $\varepsilon_{\mu\nu}$, e, u^μ, T, μ, $\tau_0^{\mu\nu}$, s	0
$\sigma_{\mu\nu}$, $\omega_{\mu\nu}$, ...	1
$\mathcal{R}_{\mu\nu\lambda\sigma}$, $\mathcal{R}_{\mu\nu}$, $\mathcal{R}$, $C_{\mu\nu\lambda\sigma}$, ...	2
$\vdots$	$\vdots$

$$\tag{3.14}$$

Accordingly, the energy-momentum tensor of a viscoelastic fluid has the following expansion in the Landau-Lifshitz frame:

$$\begin{aligned} T^{\mu\nu} &= e\,u^\mu u^\nu + \tau^{\mu\nu} \\ &= e\,u^\mu u^\nu + \tau_0^{\mu\nu} + \tau_1^{\mu\nu} + \tau_2^{\mu\nu} + \cdots \qquad \big(\tau_i^{\mu\nu} u_\nu = 0;\ i = 0, 1, 2, \cdots\big)\,, \end{aligned} \tag{3.15}$$

where the subscript $0, 1, 2, \ldots$ of a term stands for its order.

---

[11]We call S_ν^μ the entropic force since it works such as to maximize the local entropy $\tilde{s}$. This is in contrast to the (energetic) elastic force that is exerted on a material particle through the spatial divergence of the stress tensor (see Eq. (4.11)).

[12]If we use instead $\varepsilon_{\mu\nu}$ as an independent variable, then it should be scaled in the same way as that for $h_{\mu\nu}$; $\varepsilon_{\mu\nu} \to \lambda^{2/(d-1)}\,\varepsilon_{\mu\nu}$.

4 Conformal viscoelastic fluids with large strain

4.1 Definition

We say that a viscoelastic fluid is conformal if (1) its energy-momentum tensor is traceless ($T^\mu_\mu = 0$) and has weight $w = d + 2$ and (2) its equations of motion are Weyl covariant. We assume that the strain $\varepsilon = (\varepsilon_{\mu\nu})$ has the same weight as that of the metric $g_{\mu\nu}$ (i.e., $w = -2$). We further assume that the trace of $T^{\mu\nu}$ vanishes at each order, obtaining the equalities

$$e = \operatorname{tr}\tau_0 = (d-1)\,P\,, \qquad \operatorname{tr}\tau_i = 0 \quad (i = 1, 2, \ldots)\,. \tag{4.1}$$

Thus, the conformal energy-momentum tensor is generically written in the following form:

$$\begin{aligned}
T^{\mu\nu} &= e\,u^\mu u^\nu + \frac{e}{d-1}\,h^{\mu\nu} + \tau^{\langle\mu\nu\rangle} \\
&= e\,u^\mu u^\nu + \frac{e}{d-1}\,h^{\mu\nu} + \tau_0^{\langle\mu\nu\rangle} + \tau_1^{\langle\mu\nu\rangle} + \tau_2^{\langle\mu\nu\rangle} + \cdots\,.
\end{aligned} \tag{4.2}$$

To avoid discussions (and expressions) from being unnecessarily messy, we will set $\mu = 0$ in what follows.[13]

For a system of infinite size with free boundary condition, a system which has neither energy-momentum transfers nor strains at spatial infinity, a global equilibrium is characterized by the conditions that (i) $\partial\tilde{s}/\partial\tilde{p}_\mu = -u^\mu/T$ be spatially constant and (ii) $\varepsilon_{\mu\nu} = 0$ [3]. The Weyl-covariant expression for the condition (i) is given by

$$0 = h_\mu{}^\alpha\, \mathcal{D}_\alpha\Big(\frac{u^\nu}{T}\Big) = \frac{1}{T}\,\mathcal{D}_\mu u^\nu + \Big[h_\mu{}^\alpha\,\mathcal{D}_\alpha\Big(\frac{1}{T}\Big)\Big]u^\nu \qquad \text{(in global equilibrium)}, \tag{4.3}$$

or

$$\begin{cases} \text{(i-a)}\ \ \mathcal{D}_\mu u_\nu = \sigma_{\mu\nu} + \omega_{\mu\nu} = 0 \\ \text{(i-b)}\ \ h_\mu{}^\alpha\,\mathcal{D}_\alpha T\ \big(= h_\mu{}^\alpha\,(\partial_\alpha T + a_\alpha\,T)\big) = 0 \end{cases} \qquad \text{(in global equilibrium)}. \tag{4.4}$$

We know from (i-b) that the nonvanishing of spatial covariant derivatives of T indicates departure of the system from global equilibrium,[14] so that it should be more natural to express thermodynamic quantities with the pair (T, h, ε) than with (e, h, ε) when making the derivative expansion around a global equilibrium state. The temperature T is then the only dimensionful quantity for a conformal viscoelastic fluid, and we can express various thermodynamic quantities as follows ($\sigma = (\sigma_{\alpha\beta})$, $\omega = (\omega_{\alpha\beta})$):

$$s = T^{d-1}\,\bar{s}(h, \varepsilon)\,, \qquad e = T^d\,\bar{e}(h, \varepsilon)\,, \qquad S^\mu_\nu = T^d\,\bar{S}^\mu_\nu(h, \varepsilon)\,, \tag{4.5}$$

$$\tau_0^{\mu\nu} = T^d\,\bar{\tau}_0^{\mu\nu}(h, \varepsilon)\,, \qquad \tau_1^{\mu\nu} = T^d\,\bar{\tau}_1^{\mu\nu}(h, \varepsilon,\, \sigma/T,\, \omega/T)\,, \quad \ldots \tag{4.6}$$

---

[13]For a conformal fluid (in the absence of strain), one can show from the Gibbs-Duhem relation (3.13) that e and n are written in the form $e = \bar{e}\,T^k\mu^{d-k}$ and $n = \big((d-k)/(d-1)\big)\,\bar{e}\,T^k\mu^{d-k-1}$ with some constants $\bar{e}$ and k.

[14]The acceleration $a_\mu = u^\nu\nabla_\nu u_\mu = u^\nu\,(\partial_\nu u_\mu - \partial_\mu u_\nu)$ can be written as $a_\mu = N^{-1}\,h_\mu{}^\nu\,\partial_\nu N$ for the ADM parametrization (3.4) and (3.5). Thus, the condition (i-b) can be rewritten as $h_\mu{}^\alpha\,\partial_\alpha(N\,T) = 0$, which is equivalent to $\partial_i(N\,T) = 0$. Note that $T_0 \equiv N\,T$ is the temperature conjugate to the energy density measured with x^0 (not to the proper energy density measured with the local proper time) which includes the gravitational potential (Tolman's law), and that it is T_0 which becomes spatially constant in global equilibrium when the gravitational field exists [3, 14, 19].

From the fundamental relation (3.12) and the Euler relation (3.11) (with $\mu = 0$ and $P = e/(d-1)$), we obtain the following equations:

$$\bar{s} = \frac{d}{d-1}\,\bar{e}\,, \qquad \delta\bar{s} = \frac{d}{d-1}\,\delta\bar{e} = \frac{d}{2}\,\bar{\tau}_0^{\langle\mu\nu\rangle}\,\delta h_{\mu\nu} + d\,\bar{S}_\nu^\mu\,\delta\varepsilon_\mu^\nu\,. \tag{4.7}$$

4.2 Equations of motion

The equations of motion comprise (1) the conservation law of $T^{\mu\nu}$ and (2) the rheology equations [3, 4, 20, 21]. The former, $\nabla_\mu T^{\mu\nu} = 0$, determines the evolution of the energy-momentum $p^\mu = e\,u^\mu$ (or equivalently, that of T and u^μ) and describes the motion of material particles of a given material, while the latter determine the evolution of the strain $\varepsilon = (\varepsilon_{\mu\nu})$ and describe the plastic deformation of the material.

(1) conservation law

Since $T^{\mu\nu}$ is symmetric traceless and has weight $w = d + 2$, the conservation law can be written in a Weyl-covariant form by using the equality (2.9) [10]:[15]

$$\mathcal{D}_\mu T^{\mu\nu} = \nabla_\mu T^{\mu\nu} = 0\,. \tag{4.8}$$

The longitudinal component can be further rewritten as follows:

$$\begin{aligned}
0 = -\,u_\nu\,\mathcal{D}_\mu T^{\mu\nu} &= -\mathcal{D}_\mu(T^{\mu\nu}u_\nu) + T^{\mu\nu}\,\mathcal{D}_\mu u_\nu \\
&= \mathcal{D}_\mu(eu^\mu) + \left[e\,u^\mu u^\nu + \frac{e}{d-1}\,h^{\mu\nu} + \tau^{\langle\mu\nu\rangle}\right]\sigma_{\mu\nu} \\
&= \mathcal{D}_u e + \tau^{\langle\mu\nu\rangle}\,\sigma_{\mu\nu}\,.
\end{aligned} \tag{4.9}$$

This determines the time evolution of e. Setting $\delta = \mathcal{D}_u$ in the Gibbs-Duhem relation (3.13) (with $\mu = 0$ and $P = e/(d-1)$), and using the identity $\mathcal{D}_u g_{\mu\nu} = \mathcal{D}_u h_{\mu\nu} = 0$, we can also write down the equation determining the time evolution of T:

$$\begin{aligned}
\frac{\mathcal{D}_u T}{T} &= \frac{1}{d}\frac{\mathcal{D}_u e}{e} - \frac{d-1}{d}\frac{S^{\mu\nu}}{e}\,\mathcal{D}_u\varepsilon_{\mu\nu} \\
&= -\frac{1}{d\,\bar{e}}\left[\bar{\tau}^{\langle\mu\nu\rangle}\,\sigma_{\mu\nu} + (d-1)\,\bar{S}^{\mu\nu}\,\mathcal{D}_u\varepsilon_{\mu\nu}\right].
\end{aligned} \tag{4.10}$$

The transverse component of the conservation law can be interpreted in two ways. When it is written with the standard covariant derivative ∇_μ, the equation $h^\mu_\nu \mathcal{D}_\alpha T^{\alpha\nu} = 0$ gives the equation

$$a^\mu = u^\nu\nabla_\nu u^\mu = -\frac{1}{e}\,h^\mu_\nu\,\nabla_\alpha\tau^{\alpha\nu}\,, \tag{4.11}$$

which determines the time evolution of u^μ and describes how a material particle moves under the influence of the stress $\tau^{\mu\nu}$. The second interpretation is obtained when the equation is written in a manifestly Weyl-covariant form:

$$0 = h^\mu_\nu\,\mathcal{D}_\alpha T^{\alpha\nu} = h^\mu_\nu\,\mathcal{D}_\alpha\tau^{\alpha\nu} = h^\mu_\nu\,\mathcal{D}_\alpha\left(T^d\,\bar{\tau}^{\alpha\nu}\right)\,, \tag{4.12}$$

---

[15]See [3] for the derivation of the conservation law $\nabla_\mu T^{\mu\nu} = 0$ on the basis of Onsager's linear regression theory.

or

$$0 = h^\mu{}_\nu \, \mathcal{D}_\alpha \bar\tau^{\alpha\nu} + d \, \bar\tau^{\mu\nu} \left(h_\nu{}^\alpha \frac{\mathcal{D}_\alpha T}{T} \right), \tag{4.13}$$

from which the spatial derivatives of the temperatures, $h_\mu{}^\alpha \mathcal{D}_\alpha T/T$, can be expressed in terms of other local variables.

Equations (4.10) and (4.13) and the identity $\mathcal{D}_u h_{\mu\nu} = 0$ thus allow us to concentrate our consideration only on the time evolution of the strain.

(2) rheology equations

The rheology equations, on the other hand, have the following form in the derivative expansion:

$$\mathcal{D}_u \varepsilon_{\mu\nu} = [\mathcal{D}_u \varepsilon_{\mu\nu}]_0 + [\mathcal{D}_u \varepsilon_{\mu\nu}]_1 + [\mathcal{D}_u \varepsilon_{\mu\nu}]_2 + \cdots. \tag{4.14}$$

Their explicit form will be given in section 5 assuming that the strain is small (see eqs. (5.10) and (5.11)).

4.3 Entropy production and the second law of thermodynamics

The entropy current s^μ consists of the convective part $s\,u^\mu$ and the dissipative part $s^\mu_{(\mathrm{d})}$,

$$s^\mu = s\,u^\mu + s^\mu_{(\mathrm{d})}, \tag{4.15}$$

from which the total entropy density is defined as

$$s_{\mathrm{tot}} \equiv -u_\mu \, s^\mu = s + \Delta s \quad \left(\Delta s \equiv -u_\mu \, s^\mu_{(\mathrm{d})} \right). \tag{4.16}$$

Under the hypothesis of local thermodynamic equilibrium (assumed throughout the present paper), Δs is a local function of thermodynamic variables (not depending on their spatial derivatives) and can be absorbed into the original entropy density s. We, however, leave the possibility of the existence of Δs in order to make easier the comparison of the results to be obtained with those obtained in other references based on extended thermodynamics.

The entropy production rate is then given by

$$\nabla_\mu s^\mu = \mathcal{D}_\mu s^\mu = \mathcal{D}_u s + \mathcal{D}_\mu s^\mu_{(\mathrm{d})}. \tag{4.17}$$

The first term of (4.17) can be expressed explicitly by using eqs. (3.12) and (4.9) as

$$\mathcal{D}_u s = \frac{1}{T} \mathcal{D}_u e + \frac{1}{T} S^\mu_\nu \mathcal{D}_u \varepsilon^\nu_\mu = -\frac{1}{T} \tau^{\langle\mu\nu\rangle} \sigma_{\mu\nu} + \frac{1}{T} S^{\mu\nu} \mathcal{D}_u \varepsilon_{\mu\nu}$$
$$= T^{d-1} \left(-\bar\tau^{\langle\mu\nu\rangle} \sigma_{\mu\nu} + \bar S^{\mu\nu} \mathcal{D}_u \varepsilon_{\mu\nu} \right), \tag{4.18}$$

where we again have used the identity $\mathcal{D}_u g_{\mu\nu} = \mathcal{D}_u h_{\mu\nu} = 0$. The second term of (4.17) can be calculated once $s^\mu_{(\mathrm{d})}$ is given explicitly, whose generic form in the derivative expansion is:

$$s^\mu_{(\mathrm{d})} = T^{d-1} \left(\bar m \, u^\mu + \bar n^\mu \right) \quad (\bar n^\mu u_\mu = 0)$$
$$= T^{d-1} \left[\left(\bar m_0 + \bar m_1 + \bar m_2 + \cdots \right) u^\mu + \bar n^\mu_0 + \bar n^\mu_1 + \bar n^\mu_2 + \cdots \right]. \tag{4.19}$$

Here, $\bar n^\mu_0 = 0$ since a dimensionless transverse vector cannot be constructed only from $h_{\mu\nu}$, $\varepsilon_{\mu\nu}$ and u^μ. We will set $\bar m_0 = 0$ in the following discussions since $\bar m_0$ can be absorbed into $\bar s = s/T^{d-1}$. We leave the possibility of the existence of $\bar m_i$ $(i = 1, 2, \ldots)$ for the reason stated below eq. (4.16).

5 Conformal viscoelastic fluids at transient time scales

5.1 Small strain expansion

We denote the shear (i.e. traceless) component of the strain tensor $\varepsilon = (\varepsilon_{\mu\nu})$ by $\varepsilon_S = (\varepsilon^S_{\mu\nu}) \equiv (\varepsilon_{\langle\mu\nu\rangle})$, and decompose the strain as

$$\varepsilon_{\mu\nu} = \varepsilon^S_{\mu\nu} + \frac{1}{d-1}\left(\operatorname{tr}\varepsilon\right)h_{\mu\nu}. \tag{5.1}$$

The rheology equations then take the following form in the derivative expansion:[16]

$$\mathcal{D}_u \varepsilon^S_{\mu\nu} = \text{const.}\, T\, \varepsilon^S_{\mu\nu} + \text{const.}\, \sigma_{\mu\nu} + \cdots, \tag{5.2}$$

$$\mathcal{D}_u \operatorname{tr}\varepsilon = \text{const.}\, T\, \operatorname{tr}\varepsilon + \text{const.}\, T\, \operatorname{tr}(\varepsilon^2_S) + \text{const.}\, \operatorname{tr}(\varepsilon^S \sigma) + \text{const.}\, T^{-1}\, \operatorname{tr}(\sigma^2) + \cdots. \tag{5.3}$$

As time elapses, the strain gets relaxed and the rate of change becomes small. If the time scale of observation is sufficiently long, then the left hand sides of eqs. (5.2) and (5.3) become negligible compared to the right hand sides, and we obtain[17]

$$\varepsilon^S_{\mu\nu} \rightarrow \text{const.}\, T^{-1}\, \sigma_{\mu\nu} + \cdots, \tag{5.4}$$

$$\operatorname{tr}\varepsilon \rightarrow \text{const.}\, \operatorname{tr}(\varepsilon^2_S) + \text{const.}\, T^{-1}\, \operatorname{tr}(\varepsilon_S\, \sigma) + \text{const.}\, T^{-2}\, \operatorname{tr}(\sigma^2) + \cdots$$

$$\simeq \text{const.}\, T^{-2}\, \operatorname{tr}(\sigma^2) + \cdots. \tag{5.5}$$

Thus, in order to investigate a conformal viscoelastic fluid at the transient time scales where $\mathcal{D}_u \varepsilon_{\mu\nu}$ is small but still not negligible, it is convenient to expand the equations of motion based on the following *semi-long time (SLT) order*:

	SLT-order
$e, T, \ldots$	0
$\sigma_{\mu\nu}, \omega_{\mu\nu}, \varepsilon^S_{\mu\nu}, \ldots$	1
$\operatorname{tr}\varepsilon, \mathcal{R}_{\langle\mu\nu\rangle}, \ldots$	2
$\vdots$	$\vdots$

(5.6)

For the rest of this section, we investigate a conformal viscoelastic fluid at the transient time scales to second SLT-order in the constitutive equations.

We first expand the energy-momentum tensor

$$T^{\mu\nu} = T^d \left[\bar{e}\, u^\mu u^\nu + \frac{\bar{e}}{d-1}\, h^{\mu\nu} + \bar{\tau}^{\langle\mu\nu\rangle} \right] \tag{5.7}$$

---

[16] An explicit parametrization is given in eqs. (5.10) and (5.11).

[17] Note that, unlike the shear part (5.4), the trace part, $\operatorname{tr}\varepsilon$, cannot contain terms proportional to the gradient of velocity, $\vartheta = \nabla_\mu u^\mu$, in the derivative expansion, since ϑ is not a conformal scalar. This is consistent with the absence of bulk viscosity in conformal fluids.

to second SLT-order as[18]

$$\bar{\tau}^{\langle\mu\nu\rangle} = a_1\, T^{-1}\, \sigma^{\mu\nu} + a_2\, T^{-2}\, (\sigma^2)^{\langle\mu\nu\rangle} + a_3\, T^{-2}\, (\sigma\,\omega)^{\langle\mu\nu\rangle} + a_4\, T^{-2}\, (\omega^2)^{\langle\mu\nu\rangle}$$
$$+ a_5\, T^{-2}\, u_\alpha\, C^{\alpha\langle\mu\nu\rangle\beta}\, u_\beta + a_6\, T^{-2}\, \mathcal{R}^{\langle\mu\nu\rangle}$$
$$+ b_1\, \varepsilon_S^{\mu\nu} + b_2\, (\varepsilon_S^2)^{\langle\mu\nu\rangle} + b_3\, T^{-1}\, (\varepsilon_S\,\sigma)^{\langle\mu\nu\rangle} + b_4\, T^{-1}\, (\varepsilon_S\,\omega)^{\langle\mu\nu\rangle} + \cdots. \tag{5.8}$$

We also expand the entropic force caused by the strain, $S^{\mu\nu} = T^d\, \bar{S}^{\mu\nu}$, as

$$\bar{S}^{\mu\nu} = \bar{S}^{\langle\mu\nu\rangle} + \frac{1}{d-1}\,(\mathrm{tr}\,\bar{S})\, h^{\mu\nu} = \ell_1\, \varepsilon_S^{\mu\nu} + \ell_2\,(\varepsilon_S^2)^{\langle\mu\nu\rangle} + \cdots + \left(\ell_3\,\mathrm{tr}\,\varepsilon + \cdots\right) h^{\mu\nu}, \tag{5.9}$$

and the rheology equations as[19]

$$\mathcal{D}_u \varepsilon_S^{\mu\nu} = c_1\, \sigma^{\mu\nu} + c_2\, T^{-1}\, (\sigma^2)^{\langle\mu\nu\rangle} + c_3\, T^{-1}\, (\sigma\,\omega)^{\langle\mu\nu\rangle} + c_4\, T^{-1}\, (\omega^2)^{\langle\mu\nu\rangle}$$
$$+ c_5\, T^{-1}\, u_\alpha\, C^{\alpha\langle\mu\nu\rangle\beta}\, u_\beta + c_6\, T^{-1}\, \mathcal{R}^{\langle\mu\nu\rangle}$$
$$+ d_1\, T\, \varepsilon_S^{\mu\nu} + d_2\, T\, (\varepsilon_S^2)^{\langle\mu\nu\rangle} + d_3\, (\varepsilon_S\,\sigma)^{\langle\mu\nu\rangle} + d_4\, (\varepsilon_S\,\omega)^{\langle\mu\nu\rangle} + \cdots, \tag{5.10}$$
$$\mathcal{D}_u\,\mathrm{tr}\,\varepsilon = f_0\, T\,\mathrm{tr}\,\varepsilon + f_1\, T\,\mathrm{tr}(\varepsilon_S^2) + f_2\,\mathrm{tr}(\varepsilon_S\,\sigma) + f_3\, T^{-1}\,\mathrm{tr}(\sigma^2) + f_4\, T^{-1}\,\mathrm{tr}(\omega^2)$$
$$+ f_5\, T\,\mathrm{tr}(\varepsilon_S^3) + f_6\,\mathrm{tr}(\varepsilon_S^2\sigma) + f_7\, T^{-1}\,\mathrm{tr}(\varepsilon_S\sigma^2)$$
$$+ f_8\, T^{-1}\,\mathrm{tr}(\varepsilon_S\,\sigma\,\omega) + f_9\, T^{-1}\,\mathrm{tr}(\varepsilon_S\,\omega^2) + f_{10}\, T^{-2}\,\mathrm{tr}(\sigma^3) + f_{11}\, T^{-2}\,\mathrm{tr}(\sigma\,\omega^2)$$
$$+ f_{12}\, T^{-1}\, \varepsilon_S^{\mu\nu}\, u^\alpha\, C_{\alpha\langle\mu\nu\rangle\beta}\, u^\beta + f_{13}\, T^{-2}\, \sigma^{\mu\nu}\, u^\alpha\, C_{\alpha\langle\mu\nu\rangle\beta}\, u^\beta$$
$$+ f_{14}\, T^{-1}\, \varepsilon_S^{\mu\nu}\, \mathcal{R}_{\langle\mu\nu\rangle} + f_{15}\, T^{-2}\, \sigma^{\mu\nu}\, \mathcal{R}_{\langle\mu\nu\rangle}$$
$$+ f_{16}\, T^{-1}\, \mathcal{D}_\mu\mathcal{D}_\nu\varepsilon_S^{\mu\nu} + f_{17}\, T^{-2}\, \mathcal{D}_\mu\mathcal{D}_\nu\sigma^{\mu\nu} + \cdots. \tag{5.11}$$

Then, by using eq. (4.18), $\mathcal{D}_u s$ can be expressed explicitly as follows:[20]

$$\mathcal{D}_u s = T^d\,\big[\,\ell_1\, d_1\,\mathrm{tr}(\varepsilon_S^2) + (-b_1 + \ell_1\, c_1)\, T^{-1}\,\mathrm{tr}(\varepsilon_S\,\sigma) - a_1\, T^{-2}\,\mathrm{tr}(\sigma^2)$$
$$+ (\ell_2\, d_1 + \ell_1\, d_2)\,\mathrm{tr}(\varepsilon_S^3) + (-b_2 + \ell_2\, c_1 + \ell_1\, d_3)\, T^{-1}\,\mathrm{tr}(\varepsilon_S^2\,\sigma)$$
$$+ (-b_3 + \ell_1\, c_2)\, T^{-2}\,\mathrm{tr}(\varepsilon_S\,\sigma^2) + (b_4 + \ell_1\, c_3)\, T^{-2}\,\mathrm{tr}(\varepsilon_S\,\sigma\,\omega) + \ell_1\, c_4\, T^{-2}\,\mathrm{tr}(\varepsilon_S\,\omega^2)$$
$$- a_2\, T^{-3}\,\mathrm{tr}(\sigma^3) - a_4\, T^{-3}\,\mathrm{tr}(\sigma\,\omega^2) + \ell_1\, c_5\, T^{-2}\, \varepsilon_S^{\mu\nu}\, u^\alpha\, C_{\alpha\langle\mu\nu\rangle\beta}\, u^\beta$$
$$- a_5\, T^{-3}\, \sigma^{\mu\nu}\, u^\alpha\, C_{\alpha\langle\mu\nu\rangle\beta}\, u^\beta + \ell_1\, c_6\, T^{-2}\, \varepsilon_S^{\mu\nu}\, \mathcal{R}_{\langle\mu\nu\rangle} - a_6\, T^{-3}\, \sigma^{\mu\nu}\, \mathcal{R}_{\langle\mu\nu\rangle}\,\big]. \tag{5.12}$$

As for the dissipative part $s_{(\mathrm{d})}^\mu$ in the entropy current $s^\mu = s\, u^\mu + s_{(\mathrm{d})}^\mu$, we need to

---

[18] As can be seen from eqs. (2.33) and (2.37), only two are independent among the symmetric, transverse, traceless tensors of weight two; $\mathcal{D}_u\sigma_{\mu\nu}$, $u^\alpha\mathcal{R}_{\alpha\langle\mu\nu\rangle\beta}u^\beta$, $\mathcal{R}_{\langle\mu\nu\rangle}$, $u^\alpha C_{\alpha\mu\nu\beta}u^\beta$ [10]. We will take $\mathcal{R}_{\langle\mu\nu\rangle}$ and $u^\alpha C_{\alpha\mu\nu\beta}u^\beta$ as such two in the following discussions.

[19] The equations generalize those given in [4, 20, 21] (where only c_1 and d_1 are nonvanishing).

[20] Since $(\varepsilon_S)^\mathrm{T} = \varepsilon_S$, $\sigma^\mathrm{T} = \sigma$, $\omega^\mathrm{T} = -\omega$, we have $\mathrm{tr}(\varepsilon_S\sigma\omega) = (1/2)\,\mathrm{tr}\big((\varepsilon_S\sigma + \sigma\varepsilon_S)\omega\big) = (1/2)\,\mathrm{tr}\big(\varepsilon_S(\sigma\omega - \omega\sigma)\big)$.

expand it to second SLT-order:[21]

$$
\begin{aligned}
s^{\mu}_{(\mathrm{d})} = T^{d-1} \Big\{ & \big[\, A_1\, T^{-1}\, \mathrm{tr}(\varepsilon_S\, \sigma) + (A_2/2)\, T^{-2}\, \mathrm{tr}(\sigma^2) + (A_3/2)\, T^{-2}\, \mathrm{tr}(\omega^2) \\
& + (A_4/2)\, T^{-1}\, \mathrm{tr}(\varepsilon_S^2\, \sigma) + A_5\, T^{-2}\, \mathrm{tr}(\varepsilon_S\, \sigma^2) + A_6\, T^{-2}\, \mathrm{tr}(\varepsilon_S\, \sigma\, \omega) \\
& + A_7\, T^{-2}\, \mathrm{tr}(\varepsilon_S\, \omega^2) + A_8\, T^{-2}\, \mathcal{R} + \cdots \big]\, u^{\mu} \\
& + A_9\, T^{-1}\, \mathcal{D}_{\nu}\varepsilon_S^{\mu\nu} + A_{10}\, T^{-2}\, \mathcal{D}_{\nu}\sigma^{\mu\nu} + A_{11}\, T^{-2}\, \mathcal{D}_{\nu}\omega^{\mu\nu} + A_{12}\, T^{-2}\, u_{\nu}\mathcal{R}^{\nu\rho}h_{\rho}{}^{\mu} + \cdots \Big\} \,.
\end{aligned}
$$
(5.13)

As mentioned before, the scalar in front of u^{μ} represents the correction Δs to the original entropy density s; $\Delta s = s_{\mathrm{tot}} - s$. Local thermodynamic equilibrium is inevitably broken when any of these coefficients A_i $(i = 1, \ldots, 8)$ do not vanish. Although we eventually set $A_i = 0$ $(i = 1, \ldots, 8)$ later, we leave them for a while for comparison with other references.

The entropy production rate can then be written in the following form (note that $\mathcal{D}_{\mu}s^{\mu} = \nabla_{\mu}s^{\mu}$ due to eq. (2.8)):

$$
\begin{aligned}
\mathcal{D}_{\mu}s^{\mu} = \mathcal{D}_{\mu}\big(s\, u^{\mu} + s^{\mu}_{(\mathrm{d})}\big) &= \mathcal{D}_u s + \mathcal{D}_{\mu}s^{\mu}_{(\mathrm{d})} \\
= T^d \Big\{ & \ell_1\, d_1\, \mathrm{tr}(\varepsilon_S^2) + (-b_1 + \ell_1\, c_1 + A_1\, d_1)\, T^{-1}\, \mathrm{tr}(\varepsilon_S\, \sigma) + (-a_1 + A_1\, c_1)\, T^{-2}\, \mathrm{tr}(\sigma^2) \\
& + (\ell_2\, d_1 + \ell_1\, d_2)\, \mathrm{tr}(\varepsilon_S^3) + (-b_2 + \ell_2\, c_1 + A_1\, d_2 + A_4\, d_1 + \ell_1\, d_3)\, T^{-1}\, \mathrm{tr}(\varepsilon_S^2\, \sigma) \\
& + (-b_3 + A_4\, c_1 + \ell_1\, c_2 + A_5\, d_1 + A_1\, d_3 - A_1)\, T^{-2}\, \mathrm{tr}(\varepsilon_S\, \sigma^2) \\
& + (b_4 + \ell_1\, c_3 + A_6\, d_1 - A_1\, d_4)\, T^{-2}\, \mathrm{tr}(\varepsilon_S\, \sigma\, \omega) + (\ell_1\, c_4 + A_7\, d_1 - A_1)\, T^{-2}\, \mathrm{tr}(\varepsilon_S\, \omega^2) \\
& + (-a_2 + A_1\, c_2 + A_5\, c_1 - A_2 + 2A_{12})\, T^{-3}\, \mathrm{tr}(\sigma^3) \\
& + (-a_4 + A_7\, c_1 + A_1\, c_4 - A_2 - 2A_3 + 6A_{12})\, T^{-3}\, \mathrm{tr}(\sigma\, \omega^2) \\
& + (\ell_1\, c_5 + A_1)\, T^{-2}\, \varepsilon_S^{\mu\nu}\, u^{\alpha}\, C_{\alpha\langle\mu\nu\rangle\beta}\, u^{\beta} + (-a_5 + A_1\, c_5 + A_2 - 2A_{12})\, T^{-3}\, \sigma^{\mu\nu}\, u^{\alpha}\, C_{\alpha\langle\mu\nu\rangle\beta}\, u^{\beta} \\
& + \left(\ell_1\, c_6 + \frac{A_1}{d-2}\right) T^{-2}\, \varepsilon_S^{\mu\nu}\, \mathcal{R}_{\langle\mu\nu\rangle} \\
& + \left(-a_6 + A_1\, c_6 + \frac{A_2}{d-2} - 2A_8 - \frac{2}{d-2}\, A_{12}\right) T^{-3}\, \sigma^{\mu\nu}\, \mathcal{R}_{\langle\mu\nu\rangle} + A_9\, T^{-2}\, \mathcal{D}_{\mu}\mathcal{D}_{\nu}\varepsilon_S^{\mu\nu} \\
& + (2A_8 + A_{10} + A_{12})\, T^{-3}\, \mathcal{D}_{\mu}\mathcal{D}_{\nu}\sigma^{\mu\nu} + \cdots \Big\} \,.
\end{aligned}
$$
(5.14)

Here we have used eq. (2.40) and neglected terms proportional to $\mathcal{D}^{\mu}(\mathcal{F}_{\mu\nu}\, u^{\nu})$ or $\mathcal{D}_{\mu}\mathcal{D}_{\nu}\omega^{\mu\nu} = -\big((d-3)/2\big)\mathcal{F}_{\mu\nu}\, \omega^{\mu\nu}$ since they are at least forth-order derivatives (see eqs. (B.13) and (B.17)). We also have used the fact that T can be treated as being covariantly constant to this order ($\mathcal{D}_{\mu}T \simeq 0$) since both $\mathcal{D}_u T$ and $h_{\mu}{}^{\alpha}\mathcal{D}_{\alpha}T$ are at least of second SLT-order (see eqs. (4.10) and (4.13)).

Now, if we set

$$
A_9 = 0 \,, \qquad 2A_8 + A_{10} + A_{12} = 0 \,,
$$
(5.15)

the entropy production rate takes the following bilinear form in our approximation:

$$
\mathcal{D}_{\mu}s^{\mu} = T^d\, \vec{V}^{\mathrm{T}}_{\mu\nu}\, \mathcal{M}\, \vec{V}^{\mu\nu}
$$
(5.16)

---

[21]We have neglected terms proportional to $T^{d-1}\, u^{\mu}\, \mathrm{tr}\,\varepsilon$ or $T^{d-1}\, u^{\mu}\, \mathrm{tr}(\varepsilon_S^2)$ since they can be absorbed into s (see a comment below eq. (4.19)). We also have neglected terms proportional to $\mathcal{F}^{\mu\nu}\, u_{\nu}$ since it is at least of third-order derivative (see eq. (B.13)).

with

$$\mathcal{M} = \begin{pmatrix} \ell_1 \, d_1 & \frac{\ell_1 \, c_1 - b_1 + A_1 \, d_1}{2} & * \\ \frac{\ell_1 \, c_1 - b_1 + A_1 \, d_1}{2} & -a_1 + A_1 \, c_1 & * \\ * & * & *_{8 \times 8} \end{pmatrix}, \qquad \vec{V}^{\mu\nu} = \begin{pmatrix} \varepsilon_S^{\mu\nu} \\ \sigma^{\mu\nu}/T \\ (\varepsilon_S^2)^{\langle \mu\nu \rangle} \\ (\varepsilon_S \, \sigma)^{\langle \mu\nu \rangle}/T \\ (\varepsilon_S \, \omega)^{\langle \mu\nu \rangle}/T \\ (\sigma^2)^{\langle \mu\nu \rangle}/T^2 \\ (\sigma \, \omega)^{\langle \mu\nu \rangle}/T^2 \\ (\omega^2)^{\langle \mu\nu \rangle}/T^2 \\ u_\alpha \, C^{\alpha \langle \mu\nu \rangle \beta} \, u_\beta /T^2 \\ \mathcal{R}^{\langle \mu\nu \rangle}/T^2 \end{pmatrix}. \qquad (5.17)$$

The second law of thermodynamics is then certified if the coefficient matrix $\mathcal{M}$ is positive semi-definite.

The conformal fluid mechanics of [10] is obtained by setting the parameters as follows:

$$a_1 = \eta_1, \quad a_2 = -\eta_2 + \eta_4, \quad a_3 = -2\,\eta_3, \quad a_4 = -\eta_2 + \eta_5,$$

$$a_5 = \eta_2 + \eta_6, \quad a_6 = \frac{\eta_2}{d-2}, \quad b_i = c_i = d_i = f_i = 0 \quad (i = 1, 2, \cdots),$$

$$A_2 = \eta_2 + \frac{d-4}{d-2}\,\eta_6, \quad A_3 = -\frac{1}{2}\left(\eta_5 + \frac{d+2}{d-2}\,\eta_6\right), \quad A_8 = \frac{\eta_6}{2(d-2)}, \quad A_{11} = \frac{\eta_5 + 3\eta_6}{2(d-3)},$$

$$A_{12} = -\frac{\eta_6}{d-2}, \qquad A_1 = A_4 = A_5 = A_6 = A_7 = A_9 = A_{10} = 0. \qquad (5.18)$$

A calculation based on the fluid/gravity correspondence shows that $A_2 \neq 0$, $A_3 \neq 0$, $A_8 \neq 0$ [9, 10, 16], and thus local thermodynamic equilibrium is broken for such conformal fluids. In the next subsection, we show that the breakdown of local thermodynamic equilibrium can be avoided if a conformal fluid is always defined as the long time limit of a conformal viscoelastic system.

5.2 Long time limit and second-order fluid mechanics

We now consider a conformal higher-order viscoelastic system when the time scale of observation is much longer than the relaxation times of the strain:

$$\mathcal{D}_u \varepsilon_S^{\mu\nu} \ll d_1 \, T \, \varepsilon_S^{\mu\nu}, \qquad \mathcal{D}_u \, \mathrm{tr} \, \varepsilon \ll f_0 \, T \, \mathrm{tr} \, \varepsilon. \qquad (5.19)$$

As in first-order viscoelastic systems (see, e.g., [4]), our second-order viscoelastic system comes to behave as a second-order viscous fluid (which is conformal now). In fact, the

rheology equations (5.10) and (5.11) gives the following equations in the long time limit:[22]

$$
\begin{aligned}
\varepsilon_S^{\mu\nu} \to &-\frac{1}{d_1}\Big[\, c_1\, T^{-1}\, \sigma^{\mu\nu} + c_2\, T^{-2}\, (\sigma^2)^{\langle\mu\nu\rangle} + c_3\, T^{-2}\, (\sigma\,\omega)^{\langle\mu\nu\rangle} \\
&\quad + c_4\, T^{-2}\, (\omega^2)^{\langle\mu\nu\rangle} + c_5\, T^{-2}\, u_\alpha\, C^{\alpha\langle\mu\nu\rangle\beta}\, u_\beta + c_6\, T^{-2}\, \mathcal{R}^{\langle\mu\nu\rangle} \\
&\quad + d_2\, (\varepsilon_S^2)^{\langle\mu\nu\rangle} + d_3\, T^{-1}\, (\varepsilon_S\,\sigma)^{\langle\mu\nu\rangle} + d_4\, T^{-1}\, (\varepsilon_S\,\omega)^{\langle\mu\nu\rangle} + \cdots\Big] \\
\simeq &-\frac{c_1}{d_1}\, T^{-1}\, \sigma^{\mu\nu} - \Big[\frac{c_2}{d_1} + \frac{d_2}{d_1}\Big(\frac{c_1}{d_1}\Big)^2 - \frac{d_3 c_1}{d_1^2}\Big]\, T^{-2}\, (\sigma^2)^{\langle\mu\nu\rangle} - \Big(\frac{c_3}{d_1} - \frac{d_4 c_1}{d_1^2}\Big)\, T^{-2}\, (\sigma\,\omega)^{\langle\mu\nu\rangle} \\
&- \frac{c_4}{d_1}\, T^{-2}\, (\omega^2)^{\langle\mu\nu\rangle} - \frac{c_5}{d_1}\, T^{-2}\, u_\alpha\, C^{\alpha\langle\mu\nu\rangle\beta}\, u_\beta - \frac{c_6}{d_1}\, T^{-2}\, \mathcal{R}^{\langle\mu\nu\rangle} + \cdots, \quad\quad (5.20)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tr}\,\varepsilon \to &-\frac{1}{f_0}\Big[\, f_1\, \mathrm{tr}(\varepsilon_S^2) + f_2\, T^{-1}\, \mathrm{tr}(\varepsilon_S\sigma) + f_3\, T^{-2}\, \mathrm{tr}(\sigma^2) + f_4\, T^{-2}\, \mathrm{tr}(\omega^2) + \cdots\Big] \\
\simeq &-\Big[\frac{f_1}{f_0}\Big(\frac{c_1}{d_1}\Big)^2 - \frac{f_2}{f_0}\frac{c_1}{d_1} + \frac{f_3}{f_0}\Big]\, T^{-2}\, \mathrm{tr}(\sigma^2) - \frac{f_4}{f_0}\, T^{-2}\, \mathrm{tr}(\omega^2) + \cdots. \quad\quad (5.21)
\end{aligned}
$$

Substituting this into the constitutive equations (5.7) and (5.8), we obtain the energy-momentum tensor in the long time limit:

$$
\begin{aligned}
T_{(\mathrm{long})}^{\mu\nu} = &\, a_0\, T^d\, (g^{\mu\nu} + d\, u^\mu u^\nu) + \frac{a_1\, d_1 - b_1\, c_1}{d_1}\, T^{d-1}\, \sigma^{\mu\nu} \\
&+ \frac{a_2\, d_1^3 + b_1\, (c_1\, d_3\, d_1 - c_1^2\, d_2 - c_2\, d_1^2) + b_2\, c_1^2\, d_1 - b_3\, c_1\, d_1^2}{d_1^3}\, T^{d-2}\, (\sigma^2)^{\langle\mu\nu\rangle} \\
&+ \frac{a_3\, d_1^2 + b_1\, (c_1\, d_4 - c_3\, d_1) - b_4\, c_1\, d_1}{d_1^2}\, T^{d-1}\, (\sigma\,\omega)^{\langle\mu\nu\rangle} + \frac{a_4\, d_1 - b_1\, c_4}{d_1}\, T^{d-2}\, (\omega^2)^{\langle\mu\nu\rangle} \\
&+ \frac{a_5\, d_1 - b_1\, c_5}{d_1}\, T^{d-2}\, u_\alpha\, C^{\alpha\langle\mu\nu\rangle\beta}\, u_\beta + \frac{a_6\, d_1 - b_1\, c_6}{d_1}\, T^{d-2}\, \mathcal{R}^{\langle\mu\nu\rangle} + \cdots. \quad\quad (5.22)
\end{aligned}
$$

Here, we have set $\bar{e} = (d-1)\, a_0$ (a_0: constant) since $\bar{e}$ becomes constant in the long time limit (Stefan-Boltzmann law). Equation (5.22) has the same form as the energy-momentum tensor of a generic conformal second-order fluid [10, 16, 17].

Interestingly, even if we start from a viscoelastic system with manifest local thermodynamic equilibrium (i.e. a system with $A_i = 0$ ($i = 1, \ldots, 8$) so that $\Delta s = 0$), all terms can appear in $T_{(\mathrm{long})}^{\mu\nu}$ with any possible values, in a way consistent with the second law of thermodynamics. In fact, there are no constraints on parameters in the constitutive equations in order for the strain $\varepsilon_{\mu\nu}$ to be converted to spatial derivatives (such as $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$) in the long time limit (see eqs. (5.20) and (5.21)). Furthermore, we can also understand the appearance of spatial derivatives in the entropy density of a conformal fluid as a result of the same conversion mechanism, which is now applied to the entropy density with manifest local thermodynamic equilibrium:

$$
s = s(p_\mu, g_{\mu\nu}, \varepsilon_{\mu\nu}) = s(p_\mu, g_{\mu\nu}, 0) + \mathrm{const.}\, T^{d-1}\, \mathrm{tr}(\varepsilon_S^2) + \mathrm{const.}\, T^{d-1}\, \mathrm{tr}\,\varepsilon + \cdots, \quad (5.23)
$$

that transmutes in the long time limit into the one with spatial derivatives:

$$
s_{(\mathrm{long})} = s(p_\mu, g_{\mu\nu}, 0) + \mathrm{const.}\, T^{d-3}\, \mathrm{tr}(\sigma^2) + \mathrm{const.}\, T^{d-3}\, \mathrm{tr}(\omega^2) + \mathrm{const.}\, T^{d-3}\, \mathcal{R} + \cdots.
$$
$$
(5.24)
$$

---

[22]Note that the long time limit of shear strain, Eq. (5.20), has the same form as the additional dynamical variable $\xi_{\mu\nu}$ in divergence-type conformal fluid mechanics (see Eq. (98) of [12]).

Thus, even though the hypothesis of local thermodynamic equilibrium holds at short time scales for our viscoelastic system, it is seemingly broken when the system is observed at long time scales and is treated as a viscous fluid.

6   Conclusion and discussions

In this paper, we defined conformal higher-order viscoelastic fluid mechanics. We wrote down the equations of motion in such a way that the evolution is consistent with the second law of thermodynamics. We further showed that any conformal second-order fluid with arbitrary parameters in the constitutive equations can be obtained by taking the long time limit of a viscoelastic conformal fluid, without violating the hypothesis of local thermodynamic equilibrium.

On the other hand, if one trusts the fluid/gravity correspondence, the entropy current s^μ of a conformal fluid can be computed in the gravity side. The result [10, 16] shows that the total entropy density s_{tot} contains spatial derivative terms with nonvanishing coefficients, and thus we know that local thermodynamic equilibrium is violated even at short distance scales for such conformal fluids that have gravity duals.[23]

As was argued in [3], even when local thermodynamic equilibrium is realized for a system with resolution (ϵ_{t}, ϵ_{s}), spatial derivative terms are naturally induced in the total entropy density (as the *entropy functional* in the language of [3] or as in eq. (5.24)) if we observe the system at larger scales in both the temporal and the spatial directions. Since viscoelastic fluids allow a description with manifest local thermodynamic equilibrium, we expect that the fluid/gravity correspondence is an already coarse-grained correspondence between viscoelastic fluid mechanics and a more microscopic description of gravity. If this is the case, it then should give an important clue to finding fundamental degrees of freedom in quantum gravity to try to formulate such "viscoelasticity/quantum gravity correspondence." A study along this line is now in progress and will be reported elsewhere [22].

As another direction of future research, it would be interesting to apply viscoelastic fluid mechanics to the phenomenology of heavy-ion collision experiments. In fact, relativistic viscoelastic model gives a causal completion of relativistic fluid mechanics (the latter being defined as the long time limit of the former) [4], and thus it is tempting to assume that there is a phase of viscoelasticity prior to the stage of viscous fluidity. Then, it is important to investigate how elasticity at short time scales affects the dynamics of states right after collisions. In particular, one should investigate whether elasticity drives the system to an ideal fluid more rapidly than in the standard second-order fluid mechanics, as has been observed in divergence-type fluid mechanics [12].

---

[23]One should be careful about this statement. In fact, the local entropy current is defined in the fluid/gravity correspondence by pulling-back the area form on the horizon to the boundary on which fluid mechanics is defined. However, there are ambiguities in the definition [16] (e.g., the ambiguity in making the boundary-to-horizon map, and the ambiguity of whether the area form is constructed on the event horizon or on the apparent horizon). Thus, it might be possible to find a suitable definition of the local entropy such that local thermodynamic equilibrium is not violated.

Acknowledgments

A Weights of local thermodynamic variables

The Weyl weight of a (p, q) tensor $Q^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q}$ of dimension Δ is given by $w = \Delta + p - q$. We list below the dimensions and the weights of various local thermodynamic quantities.

	dimension Δ	weight w	order	SLT-order
$\mathcal{D}_\mu$	1	0	1	N/A
u^μ	0	1	0	0
$\varepsilon^S_{\mu\nu} = \varepsilon_{\langle\mu\nu\rangle}$	0	-2	0	1
$\operatorname{tr} \varepsilon$	0	0	0	2
$g_{\mu\nu}$	0	-2	0	0
$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$	0	-2	0	0
$\mathcal{R}_{\mu\nu\lambda\sigma}$	2	-2	2	
$C_{\mu\nu\lambda\sigma}$	2	-2	2	
$\mathcal{R}_{\mu\nu} = \mathcal{R}_{\mu\alpha\nu}{}^\alpha$	2	0	2	
$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$	2	0	2	
$\mathcal{R} = \mathcal{R}_\alpha{}^\alpha$	2	2	2	
p_μ	d	$d - 1$	0	0
$e = \sqrt{-g^{\mu\nu} p_\mu p_\nu}$	d	d	0	0
$s = s(e, \varepsilon_{\mu\nu})$	$d - 1$	$d - 1$	0	0
$T = (\partial s / \partial e)^{-1}$	1	1	0	0
$s^\mu = s\, u^\mu + s^\mu_{(\mathrm{d})}$	$d - 1$	d	N/A	N/A
$T^{\mu\nu} = e\, u^\mu u^\nu + \tau^{\mu\nu}$	d	$d + 2$	N/A	N/A

$$(\text{A.1})$$

B Useful formulas

In this appendix, we prove a few useful formulas which are used in the main text.

For the Weyl-covariantized Riemann tensor (2.15)–(2.18), the following equality holds:

$$
\begin{aligned}
u^\alpha u^\beta h_\mu^{\;\rho} h_\nu^{\;\sigma} \mathcal{R}_{\rho\alpha\sigma\beta} &= u^\alpha u^\beta h_\nu^{\;\sigma} \mathcal{R}_{\mu\alpha\sigma\beta} = u^\alpha \mathcal{R}_{\mu\alpha\nu}{}^\sigma u_\sigma + u^\alpha u^\beta u^\sigma u_\nu \mathcal{R}_{\mu\alpha\,(\sigma\beta)} \\
&= u^\alpha \big(\mathcal{R}_{\mu\alpha\nu}{}^\sigma - \mathcal{F}_{\mu\alpha}\,\delta_\nu^\sigma \big) u_\sigma = u^\alpha \left[\mathcal{D}_\mu, \mathcal{D}_\alpha \right] u_\nu = -\mathcal{D}_\mu u^\alpha\, \mathcal{D}_\alpha u_\nu - \mathcal{D}_u \mathcal{D}_\mu u_\nu \\
&= -(\sigma^2)_{\mu\nu} - (\sigma\,\omega)_{\mu\nu} - (\omega\,\sigma)_{\mu\nu} - (\omega^2)_{\mu\nu} - \mathcal{D}_u(\sigma_{\mu\nu} + \omega_{\mu\nu}), \qquad\qquad (\text{B.1})
\end{aligned}
$$

where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ and $\mathcal{D}_u = u^\mu \mathcal{D}_\mu$. By decomposing eq. (B.1) into the trace part, the symmetric traceless part, and the antisymmetric part, we obtain the following identities:[24]

$$u^\alpha \mathcal{R}_{\alpha\beta} u^\beta = -\operatorname{tr}(\sigma^2) - \operatorname{tr}(\omega^2), \tag{B.2}$$

$$u^\alpha \mathcal{R}_{\alpha\langle\mu\nu\rangle\beta} u^\beta = (\sigma^2)_{\langle\mu\nu\rangle} + (\omega^2)_{\langle\mu\nu\rangle} + \mathcal{D}_u \sigma_{\mu\nu}, \tag{B.3}$$

$$\frac{1}{2} h_\mu{}^\rho h_\nu{}^\sigma \mathcal{F}_{\rho\sigma} = (\sigma\,\omega + \omega\,\sigma)_{\mu\nu} + \mathcal{D}_u \omega_{\mu\nu}, \tag{B.4}$$

where we have used the relation $\mathcal{R}_{\mu\alpha\,(\sigma\beta)} = \mathcal{F}_{\mu\alpha}\, g_{\sigma\beta}$ and $u^\alpha \mathcal{R}_{\alpha[\mu\nu]\beta} u^\beta = (1/2)\,\mathcal{F}_{\mu\nu}$ (see eqs. (2.23) and (2.25)).

The Weyl tensor (2.31),

$$C_{\mu\nu\lambda\sigma} \equiv R_{\mu\nu\lambda\sigma} + \frac{4}{d-2}\, \delta^\alpha_{[\mu} g_{\nu][\lambda} \delta^\beta_{\sigma]} \left(R_{\alpha\beta} - \frac{R}{2(d-1)}\, g_{\alpha\beta} \right), \tag{B.5}$$

can be rewritten as a sum of Weyl-covariantized curvature tensors with the use of eqs. (2.16), (2.20) and (2.22):

$$C_{\mu\nu\lambda\sigma} = \mathcal{R}_{\mu\nu\lambda\sigma} - \mathcal{F}_{\mu\nu}\, g_{\lambda\sigma} + \frac{4}{d-2}\, \delta^\alpha_{[\mu} g_{\nu][\lambda} \delta^\beta_{\sigma]} \left(\mathcal{R}_{\alpha\beta} - \frac{\mathcal{R}}{2(d-1)}\, g_{\alpha\beta} + \mathcal{F}_{\alpha\beta} \right). \tag{B.6}$$

This exhibits that $C_{\mu\nu\lambda\sigma}$ is a conformal tensor of weight -2. Since the tensor $L^{\lambda\sigma}_{\alpha\mu\nu\beta} \equiv 4\,\delta^\lambda_{[\alpha} g_{\mu][\nu} \delta^\sigma_{\beta]}$ satisfies $u^\alpha L^{\lambda\sigma}_{\alpha\langle\mu\nu\rangle\beta} u^\beta = -\delta^\lambda_{\langle\mu} \delta^\sigma_{\nu\rangle}$, we have[25]

$$u^\alpha C_{\alpha\mu\nu\beta} u^\beta \left(= u^\alpha C_{\alpha\,\langle\mu\nu\rangle\,\beta} u^\beta\right) = u^\alpha \mathcal{R}_{\alpha\,\langle\mu\nu\rangle\,\beta} u^\beta - \frac{1}{d-2}\, \mathcal{R}_{\langle\mu\nu\rangle}. \tag{B.7}$$

Multiplying eqs. (B.3) and (B.4) by $\sigma^{\mu\nu}$ and $\omega^{\mu\nu}$, respectively, and using eq. (B.7), we obtain the following formulas:

$$\sigma^{\mu\nu} u^\alpha \mathcal{R}_{\alpha\langle\mu\nu\rangle\beta} u^\beta = \sigma^{\mu\nu} u^\alpha C_{\alpha\mu\nu\beta} u^\beta + \frac{1}{d-2}\, \mathcal{R}^{\langle\mu\nu\rangle} \sigma_{\mu\nu}$$

$$= \operatorname{tr}(\sigma^3) + \operatorname{tr}(\sigma\,\omega^2) + \sigma^{\mu\nu} \mathcal{D}_u \sigma_{\mu\nu}, \tag{B.8}$$

$$\frac{1}{2}\, \mathcal{F}^{\mu\nu} \omega_{\mu\nu} = -2\operatorname{tr}(\sigma\,\omega^2) + \omega^{\mu\nu} \mathcal{D}_u \omega_{\mu\nu}. \tag{B.9}$$

---

[24]We can show that eqs. (B.2)–(B.4) are equivalent to the well-known evolution equations of ϑ, $\sigma_{\mu\nu}$, and $\omega_{\mu\nu}$ (such as the Raychaudhuri equation) by using the following equations:

$$u^\alpha \mathcal{R}_{\alpha\beta} u^\beta = u^\alpha R_{\alpha\beta} u^\beta + \nabla_u \vartheta - \nabla_\mu a^\mu + \frac{\vartheta^2}{d-1},$$

$$\mathcal{D}_u \sigma_{\mu\nu} = h_\mu{}^\alpha h_\nu{}^\beta \mathcal{D}_u \sigma_{\alpha\beta} = h_\mu{}^\alpha h_\nu{}^\beta \nabla_u \sigma_{\alpha\beta} + \frac{\vartheta}{d-1}\, \sigma_{\mu\nu},$$

$$\mathcal{D}_u \omega_{\mu\nu} = h_\mu{}^\alpha h_\nu{}^\beta \mathcal{D}_u \omega_{\alpha\beta} = h_\mu{}^\alpha h_\nu{}^\beta \nabla_u \omega_{\alpha\beta} + \frac{\vartheta}{d-1}\, \omega_{\mu\nu},$$

$$\frac{1}{2} h_\mu{}^\rho h_\nu{}^\sigma \mathcal{F}_{\rho\sigma} = h_\mu{}^\alpha h_\nu{}^\beta \nabla_{[\alpha} a_{\beta]} - \frac{\vartheta}{d-1}\, \omega_{\mu\nu}.$$

[25]Note that $u^\alpha C_{\alpha\mu\nu\beta} u^\beta = u^\alpha C_{\alpha\,\langle\mu\nu\rangle\,\beta} u^\beta$ since $C_{\alpha\mu\nu\beta} = C_{\beta\nu\mu\alpha}$ and $u^\alpha u^\mu C_{\alpha\mu\nu\beta} = 0 = C_{\alpha\mu\nu\beta} u^\nu u^\beta$.

We can also show

$$\mathcal{D}_\mu \mathcal{D}_\nu \omega^{\mu\nu} = \frac{1}{2}\left[\mathcal{D}_\mu, \mathcal{D}_\nu\right]\omega^{\mu\nu} = -\mathcal{R}_{[\mu\nu]}\,\omega^{\mu\nu} + \frac{3}{2}\,\mathcal{F}_{\mu\nu}\,\omega^{\mu\nu}$$

$$= -\frac{d-3}{2}\,\mathcal{F}^{\mu\nu}\omega_{\mu\nu}\,, \tag{B.10}$$

$$\mathcal{D}_\mu\big[u_\nu\big(\mathcal{G}^{\nu\mu} - \mathcal{F}^{\nu\mu}\big)\big] = (\mathcal{D}_\mu u_\nu)\left(\mathcal{G}^{(\mu\nu)} - \frac{d-2}{2}\,\mathcal{F}^{\mu\nu}\right) = \mathcal{G}^{\langle\mu\nu\rangle}\sigma_{\mu\nu} - \frac{d-2}{2}\,\mathcal{F}^{\mu\nu}\omega_{\mu\nu}$$

$$= \mathcal{R}^{\langle\mu\nu\rangle}\sigma_{\mu\nu} - \frac{d-2}{2}\,\mathcal{F}^{\mu\nu}\omega_{\mu\nu}\,, \tag{B.11}$$

$$\mathcal{D}_\mu \mathcal{D}_\nu \sigma^{\mu\nu} = \mathcal{D}_\mu \mathcal{D}_\nu \mathcal{D}^\mu u^\nu - \mathcal{D}_\mu \mathcal{D}_\nu \omega^{\mu\nu} = \mathcal{D}^\mu[\mathcal{D}_\nu, \mathcal{D}_\mu]\,u^\nu + \frac{d-3}{2}\,\mathcal{F}^{\mu\nu}\omega_{\mu\nu}$$

$$= \mathcal{D}^\mu\big[(\mathcal{R}_{\mu\nu} - \mathcal{F}_{\mu\nu})\,u^\nu\big] + \frac{d-3}{2}\,\mathcal{F}^{\mu\nu}\omega_{\mu\nu}$$

$$= \mathcal{D}^\mu\big[u^\nu\big(\mathcal{G}_{\nu\mu} - \mathcal{F}_{\nu\mu}\big)\big] + \frac{1}{2}\,\mathcal{D}_u\mathcal{R} + (d-2)\,\mathcal{D}^\mu\big[\mathcal{F}_{\mu\nu}\,u^\nu\big] + \frac{d-3}{2}\,\mathcal{F}^{\mu\nu}\omega_{\mu\nu}$$

$$= \mathcal{R}^{\langle\mu\nu\rangle}\sigma_{\mu\nu} + \frac{1}{2}\,\mathcal{D}_u\mathcal{R} - \frac{1}{2}\,\mathcal{F}^{\mu\nu}\omega_{\mu\nu} + (d-2)\,\mathcal{D}^\mu\big[\mathcal{F}_{\mu\nu}\,u^\nu\big]\,. \tag{B.12}$$

We can show that the spatial vector $\mathcal{F}_{\mu\nu}\,u^\nu$ vanishes up to third-order derivatives:[26]

$$\mathcal{F}_{\mu\nu}\,u^\nu = -\big(\sigma_{\mu\nu} + \omega_{\mu\nu}\big)\,a^\nu - h_{\mu\nu}\,\nabla_u a^\nu + \frac{1}{d-1}\,h_\mu{}^\nu\,\partial_\nu\vartheta = O(\epsilon^3)\,. \tag{B.13}$$

In fact, from eqs. (4.9) and (4.11), we find

$$\frac{1}{e}\,\mathcal{D}_u e = \nabla_u \ln e + \frac{d}{d-1}\,\vartheta = O(\epsilon^2)\,, \qquad a^\mu = -\frac{1}{d}\,h^{\mu\alpha}\,\partial_\alpha \ln e + O(\epsilon^2)\,, \tag{B.14}$$

and thus have

$$h_{\mu\nu}\,\nabla_u a^\nu = -\frac{1}{d}\,h_{\mu\nu}\,\nabla_u\Big(h^{\nu\alpha}\,\partial_\alpha \ln e\Big) + O(\epsilon^3)$$

$$= -\frac{1}{d}\,h_{\mu\nu}\,(\nabla_u h^{\nu\alpha})\,\partial_\alpha \ln e - \frac{1}{d}\,h_\mu{}^\alpha\,\nabla_u\partial_\alpha \ln e + O(\epsilon^3)$$

$$= -\frac{1}{d}\,h_{\mu\nu}\,\big(\nabla_u(u^\nu u^\alpha)\big)\,\partial_\alpha \ln e - \frac{1}{d}\,h_\mu{}^\alpha\,\nabla_\alpha\nabla_u \ln e + \frac{1}{d}\,h_\mu{}^\alpha\,(\nabla_\alpha u^\beta)\,\partial_\beta \ln e + O(\epsilon^3)$$

$$= \frac{1}{d-1}\,a_\mu\,\vartheta + \frac{1}{d-1}\,h_\mu{}^\alpha\,\partial_\alpha\vartheta + \Big(\sigma_\mu{}^\alpha + \omega_\mu{}^\alpha + \frac{1}{d-1}\,\vartheta\,h_\mu{}^\alpha\Big)\partial_\beta \ln e + O(\epsilon^3)$$

$$= \frac{1}{d-1}\,h_\mu{}^\alpha\,\partial_\alpha\vartheta - (\sigma_\mu{}^\alpha + \omega_\mu{}^\alpha)\,a_\alpha + O(\epsilon^3)\,. \tag{B.15}$$

Similarly, using eq. (B.14), we can show that the spatial components of $\mathcal{F}_{\mu\nu}$ vanish up to third-order derivatives:

$$\frac{1}{2}\,h_\mu{}^\alpha\,h_\nu{}^\beta\,\mathcal{F}_{\alpha\beta} = h_\mu{}^\alpha h_\nu{}^\beta \nabla_{[\alpha}a_{\beta]} - \frac{\vartheta}{d-1}\,\omega_{\mu\nu}$$

$$= -\frac{1}{d}\,h_\mu{}^\alpha h_\nu{}^\beta\,\nabla_{[\alpha}\big(h_{\beta]}^\lambda\,\partial_\lambda \ln e\big) - \frac{\vartheta}{d-1}\,\omega_{\mu\nu} + O(\epsilon^3)$$

$$= -\frac{1}{d}\,h_\mu{}^\alpha h_\nu{}^\beta\,(\nabla_{[\alpha}u_{\beta]})\,\nabla_u \ln e - \frac{\vartheta}{d-1}\,\omega_{\mu\nu} + O(\epsilon^3)$$

$$= O(\epsilon^3)\,, \tag{B.16}$$

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[26] We denote terms of nth and higher order derivatives by $O(\epsilon^n)$.

from which we have

$$\mathcal{F}^{\mu\nu}\,\omega_{\mu\nu} = O(\epsilon^4)\,. \tag{B.17}$$

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