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Seiberg–Witten prepotential for E-string theory and random partitions

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Abstract

We find a Nekrasov-type expression for the Seiberg–Witten prepotential for the six-dimensional non-critical E_8 string theory toroidally compactified down to four dimensions. The prepotential represents the BPS partition function of the E_8 strings wound around one of the circles of the toroidal compactification with general winding numbers and momenta. We show that our expression exhibits expected modular properties. In particular, we prove that it obeys the modular anomaly equation known to be satisfied by the prepotential.

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1. Introduction

An intriguing feature of local quantum field theories in six dimensions is the existence of interacting theories that involve self-dual tensor fields and strings. The non-critical E_8 string theory, or the E-string theory, is known as the simplest theory of this kind with (1,0) supersymmetry [1–5]. It includes one tensor multiplet and possesses an E_8 global symmetry. The theory was originally discovered in the study of small E_8 instantons in the $E_8 \times E_8$ heterotic string theory compactified on K3 [1, 2].

While the whole picture of the theory remains still mysterious, toroidal compactification of the theory has been extensively studied. Among others, compactification down to four dimensions is of particular interest [4–11]. In this case, the low energy effective theory admits a description in terms of Seiberg–Witten theory [12, 13]. The Seiberg–Witten prepotential represents the BPS partition function of the E-strings wound around one of the circles of the toroidal compactification with general winding numbers and momenta [3]. Upon compactification one can introduce eight Wilson line parameters which break the E_8 global symmetry [6]. The corresponding Seiberg–Witten curve was constructed in the presence of the most general Wilson line parameters [4, 11]. The prepotential can also be interpreted as the genus zero topological string amplitude for the local $\frac{1}{2}$K3 or as the generating function of the partition functions of $\mathcal{N} = 4$ U(n) topological Yang–Mills theories on $\frac{1}{2}$K3 [5].

In this paper, we present an explicit expression for the Seiberg–Witten prepotential for the E-string theory. We consider the case with no Wilson line parameters. Seiberg–Witten prepotential for this particular case was studied in detail by Minahan, Nemeschansky and Warner [9]. They considered the winding number expansion of the prepotential and computed the expansion coefficients up to certain orders. They elucidated that these coefficients are computed either by solving the parametric relation among period integrals or by recursively solving the modular anomaly equation. Our expression is of Nekrasov type [14] and directly gives these coefficients at all orders. The coefficients obtained from our expression are in perfect agreement with those computed by the above methods (verified for winding numbers $n \leq 15$). We show that our expression exhibits expected modular properties. In particular, we prove that it satisfies the modular anomaly equation of [9].

While the prepotential represents the genus zero topological string amplitude for the local $\frac{1}{2}$K3, the sum over partitions in our expression differs from the all-genus topological string partition function for this Calabi–Yau threefold. We clarify the difference of modular anomalies between them.

The organization of this paper is as follows. In section 2, we recall some basic facts about the Seiberg–Witten prepotential for the E-string theory and briefly review how to compute its winding number expansion by conventional methods. In section 3, we present our Nekrasov-type expression for the prepotential and discuss its structure. In section 4, we show that our expression exhibits expected modular properties. In particular, we prove that it satisfies the modular anomaly equation. We also make a comparison of modular anomalies between our sum over partitions and the all-genus topological string partition function for the local $\frac{1}{2}$K3. Section 5 is devoted to the discussion. Some technical details are relegated to appendices.

2. Seiberg–Witten prepotential for E-string theory

In this section we recall some basic facts about the Seiberg–Witten prepotential for the E-string theory and briefly review how to compute its winding number expansion by two different methods developed by Minahan, Nemeschansky and Warner [9].

The E-string theory in six dimensions includes one tensor multiplet. When the theory is toroidally compactified down to four dimensions, the tensor multiplet turns into a vector multiplet. It contains a complex scalar field, whose vev φ parametrizes the Coulomb branch of the vacuum moduli space of the compactified theory. The low energy effective theory takes the form of a four-dimensional $\mathcal{N}=2$ U(1) gauge theory. The effective action is fully characterized by a prepotential F_0, a holomorphic function of φ, in the same way as the original Seiberg–Witten theories [12, 13]. In the most general situation, the prepotential also depends on the complex structure modulus τ of the torus on which we compactify the theory and eight Wilson line parameters $m_1, \ldots, m_8$ which break the E_8 global symmetry. In this paper, we restrict ourselves to the case with no Wilson line parameters, namely the case of $m_1 = \cdots = m_8 = 0$.

The prepotential represents the BPS partition function of the E-strings wound around one of the circles of the toroidal compactification with general winding numbers and momenta. It can be expressed as

$$F_0(\varphi,\tau) = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} N_{n,k} \sum_{m=1}^{\infty} \frac{p^{mn}q^{mk}}{m^3}, \tag{2.1}$$

where

$$p := e^{2\pi i\varphi}, \qquad q := e^{2\pi i\tau}. \tag{2.2}$$

Integer $N_{n,k}$ represents the multiplicity of BPS states of winding number n and momentum k. The first few of them read [3]

$$\begin{aligned} &N_{1,0}=1, \qquad N_{1,1}=252, \qquad N_{1,2}=5130, \qquad \cdots \\ &N_{2,0}=0, \qquad N_{2,1}=0, \qquad N_{2,2}=-9252, \qquad \cdots . \end{aligned} \tag{2.3}$$

The prepotential can be viewed as the genus zero topological string amplitude for the local $\frac{1}{2}$K3. The above integers are the numbers of rational curves in this Calabi–Yau threefold. Indeed, these integers were computed in [3] by using the mirror map.

In practical computations, it is convenient to express F_0 as a winding number expansion of the following form

$$F_0(\varphi,\tau)=\sum_{n=1}^{\infty} Q^n Z_n(\tau), \tag{2.4}$$

where

$$Q := q^{1/2}p. \tag{2.5}$$

Z_n is interpreted as the partition function of topological $\mathcal{N}=4$ U(n) Yang–Mills theory on $\frac{1}{2}$K3 [5]. The first few of them read

$$Z_1=\frac{E_4}{\eta^{12}}, \qquad Z_2=\frac{E_2E_4^2+2E_4E_6}{24\eta^{24}}, \qquad \cdots . \tag{2.6}$$

Here $E_{2n}(\tau)$ are the Eisenstein series of weight $2n$ and $\eta(\tau)$ is the Dedekind eta function (see Appendix A). For general n, Z_n takes the form

$$Z_n=\frac{P_{6n-2}(E_2,E_4,E_6)}{\eta^{12n}}, \tag{2.7}$$

where $P_{6n-2}(E_2,E_4,E_6)$ denotes a polynomial in E_2,E_4,E_6 of weight $6n-2$. The explicit form of Z_n at low orders can be calculated by either of the methods described below.

The prepotential can be computed from the Seiberg–Witten curve of the theory. In the present case, the Seiberg–Witten curve is given by

$$y^2=4x^3-\frac{1}{12}E_4(\tau)u^4x-\frac{1}{216}E_6(\tau)u^6+4u^5. \tag{2.8}$$

If we think of u and τ as parameters, it is an elliptic curve in the Weierstrass form. An elliptic curve in this form admits the following canonical parametrization

$$y^2=4x^3-\frac{1}{12}\frac{E_4(\tilde{\tau})}{\omega^4}x-\frac{1}{216}\frac{E_6(\tilde{\tau})}{\omega^6}. \tag{2.9}$$

Here $\tilde{\tau}$ is the complex structure modulus and ω (multiplied by 2π) is one of the fundamental periods of the elliptic curve. By comparing these two expressions, one can calculate $\omega(u,\tau), \tilde{\tau}(u,\tau)$ as series expansions in $1/u$. They are related to the scalar vev φ and the prepotential F_0 by

$$\partial_u \varphi = \frac{i}{2\pi}\omega, \tag{2.10}$$
$$\partial_\varphi^2 F_0 = 8\pi^3 i(\tilde{\tau} - \tau). \tag{2.11}$$

These relations parametrically determine the function $F_0(\varphi, \tau)$. The integration constants as well as the normalizations of F_0 and φ are determined so that F_0 gives the BPS partition function described above. For further details of the calculation, see [9, 15].

An alternative way to compute the prepotential is to solve the modular anomaly equation [9]. As is well known, $E_2(\tau)$ is not strictly a modular form, but transforms anomalously. The dependence of the prepotential on E_2 (i.e. the modular anomaly of the prepotential) is governed by the following equation

$$\partial_{E_2} F_0 = \frac{1}{24}\left(\Theta_Q F_0\right)^2, \tag{2.12}$$

where $\Theta_Q := Q\partial_Q = \frac{1}{2\pi i}\partial_\varphi$. This modular anomaly equation was derived in [9] from the Seiberg–Witten description. In terms of Z_n, the equation is written as

$$\partial_{E_2} Z_n = \frac{1}{24}\sum_{k=1}^{n-1} k(n-k) Z_k Z_{n-k}. \tag{2.13}$$

This equation recursively determines Z_n up to a piece which does not contain E_2. Given the general structure (2.7), the remaining ambiguity can be fixed by the gap condition

$$q^{n/2} Z_n = \frac{1}{n^3} + \mathcal{O}(q^n). \tag{2.14}$$

This condition follows from the geometric structure of the local $\frac{1}{2}$K3 [5].

3. Nekrasov-type expression

In this section we present an explicit expression for the Seiberg–Witten prepotential for the E-string theory and discuss its structure.

Let $\boldsymbol{R} = (R_1, \ldots, R_N)$ denote an N-tuple of partitions. Each partition R_k is a nonincreasing sequence of nonnegative integers

$$R_k = \{\mu_{k,1} \geq \mu_{k,2} \geq \cdots \geq \mu_{k,\ell(R_k)} > \mu_{k,\ell(R_k)+1} = \mu_{k,\ell(R_k)+2} = \cdots = 0\}. \tag{3.1}$$

Here the number of nonzero $\mu_{k,i}$ is denoted by $\ell(R_k)$. R_k is represented by a Young tableau. We let $|R_k|$ denote the size of R_k, i.e. the number of boxes in the Young tableau of R_k:

$$|R_k| := \sum_{i=1}^{\infty} \mu_{k,i} = \sum_{i=1}^{\ell(R_k)} \mu_{k,i}. \tag{3.2}$$

Similarly, the size of $\boldsymbol{R}$ is denoted by

$$|\boldsymbol{R}| := \sum_{k=1}^{N} |R_k|. \tag{3.3}$$

We let $R_k^\vee = \{\mu_{k,1}^\vee \geq \mu_{k,2}^\vee \geq \cdots\}$ denote the conjugate partition of R_k. We also introduce the notation

$$h_{k,l}(i,j) := \mu_{k,i} + \mu_{l,j}^\vee - i - j + 1, \tag{3.4}$$

which represents the relative hook-length of a box at (i,j) between the Young tableaux of R_k and R_l.

In our expression we consider a sum over four partitions. For our present purpose, it is convenient to express these partitions as

$$\boldsymbol{R} = (R_1, R_2, R_3, R_4) = (R_{11}, R_{10}, R_{00}, R_{01}). \tag{3.5}$$

The prepotential is then given by

$$F_0 = \left(2\hbar^2 \ln \mathcal{Z}\right)\Big|_{\hbar=0}, \tag{3.6}$$

where

$$\mathcal{Z} = \sum_{\boldsymbol{R}} Q^{|\boldsymbol{R}|} \prod_{a,b,c,d} \prod_{(i,j)\in R_{ab}} \frac{\vartheta_{ab}\left(\frac{1}{2\pi}(j-i)\hbar, \tau\right)^2}{\vartheta_{1-|a-c|,1-|b-d|}\left(\frac{1}{2\pi} h_{ab,cd}(i,j)\hbar, \tau\right)^2}. \tag{3.7}$$

Here the sum is taken over all possible partitions $\boldsymbol{R}$ (including the empty partition). Indices a, b, c, d take values $0, 1$, while a set of indices (i,j) run over the coordinates of all boxes in the Young tableau of R_{ab}. $\vartheta_{ab}(z,\tau)$ are the Jacobi theta functions (see Appendix A). $h_{ab,cd}(i,j)$ is the relative hook-length defined between partitions R_{ab} and R_{cd}.

We find that the above F_0 coincides with the prepotential described in the last section. Explicit forms of Z_n obtained from this expression are in perfect agreement with those computed by either of the methods described in the last section (verified for $n \leq 15$). We will also show in the next section that the above expression exhibits expected modular properties. In the rest of this section let us make a few comments on the structure of our expression.

For any $\boldsymbol{R}$ with $R_{11} \neq \{0\}$, the product in the sum vanishes. This is because the Young tableau of $R_{11} \neq \{0\}$ always contains a box at $(i,j) = (1,1)$, where the theta function in the numerator becomes $\vartheta_{11}(0,\tau) = 0$. Hence, $\mathcal{Z}$ is actually a sum over three partitions (R_{10}, R_{00}, R_{01}). This structure is expected, as we consider the case with no Wilson line parameters. In the presence of the most general Wilson line parameters, the BPS partition function of singly wound E-strings reads $Z_1 = \frac{1}{2}\eta^{-12}\sum_{a,b}\prod_{i=1}^{8}\vartheta_{ab}(m_i,\tau)$ [5]. By setting $m_i = 0$, the product of ϑ_{11} vanishes and the other three products add up to $Z_1 = \eta^{-12}E_4$, as is found in (2.6). We expect that the expression with four partitions will be useful when one considers the cases with nonzero Wilson line parameters.

The above $\mathcal{Z}$ coincides with a special case of the elliptic generalization of the Nekrasov partition function for the SU(4) gauge theory with 8 massless fundamental hypermultiplets [14, 16]. More specifically, $\mathcal{Z}$ can be expressed as

$$\mathcal{Z} = \sum_{\boldsymbol{R}}(-p)^{|\boldsymbol{R}|}\prod_{k=1}^{4}\prod_{(i,j)\in R_k}\frac{\vartheta_1\left(a_k + \frac{1}{2\pi}(j-i)\hbar, \tau\right)^8}{\prod_{l=1}^{4}\vartheta_1\left(a_k - a_l + \frac{1}{2\pi}h_{kl}(i,j)\hbar, \tau\right)^2} \tag{3.8}$$

with a_k being set to half periods of the torus

$$a_1 = 0, \qquad a_2 = \frac{1}{2}, \qquad a_3 = -\frac{1+\tau}{2}, \qquad a_4 = \frac{\tau}{2}. \tag{3.9}$$

While physical implication of this coincidence is yet unclear, it indicates that the analyticity of $\hbar^2 \ln \mathcal{Z}$ at $\hbar = 0$ follows from that of the Nekrasov partition function. Therefore, F_0 given as in (3.6) is indeed well defined.

4. Modular properties and modular anomaly equation

In this section we show that our expression exhibits the modular properties described in section 2. In particular, we derive the modular anomaly equation (2.12) from our expression.

Recall that the Jacobi theta functions can be expressed as

$$\vartheta_1\left(\frac{z}{2\pi}, \tau\right) = e^{-\frac{1}{24}E_2 z^2}\eta^3\sigma(z|2\pi, 2\pi\tau), \tag{4.1}$$

$$\vartheta_{k+1}\left(\frac{z}{2\pi}, \tau\right) = e^{-\frac{1}{24}E_2 z^2}\vartheta_{k+1}\sigma_k(z|2\pi, 2\pi\tau), \quad k = 1, 2, 3. \tag{4.2}$$

Here σ and σ_k are the Weierstrass sigma function and cosigma functions, respectively, associated with the lattice $2\pi\mathbb{Z} + 2\pi\tau\mathbb{Z}$. They are expanded as

$$\sigma(z|2\pi, 2\pi\tau) = z - \frac{E_4}{2880} z^5 - \frac{E_6}{181440} z^7 + \mathcal{O}(z^9), \tag{4.3}$$

$$\sigma_k(z|2\pi, 2\pi\tau) = 1 - \frac{e_k}{2} z^2 + \left(\frac{E_4}{576} - \frac{e_k^2}{8} \right) z^4 + \mathcal{O}(z^6), \tag{4.4}$$

where

$$e_1 = \frac{\vartheta_3^4 + \vartheta_4^4}{12}, \qquad e_2 = \frac{\vartheta_2^4 - \vartheta_4^4}{12}, \qquad e_3 = \frac{-\vartheta_2^4 - \vartheta_3^4}{12}. \tag{4.5}$$

The coefficients of these expansions can be computed up to any order, as explained in Appendix B. For the present purpose, it is enough to know that the expansion coefficients of σ are polynomials in E_4, E_6 while those of σ_k are polynomials in E_4, E_6, e_k.

We are now in a position to look into the modular properties of our expression. As we saw in the last section, (3.7) is actually a sum over three partitions $(R_{10}, R_{00}, R_{01}) = (R_2, R_3, R_4)$. Let us first consider the contribution of the prefactors η^3, ϑ_{k+1} in (4.1), (4.2). For each box in the Young tableau of R_k, one obtains ϑ_k^8 in the numerator as well as $\eta^6\vartheta_2^2\vartheta_3^2\vartheta_4^2 = 4\eta^{12}$ in the denominator after taking the product with respect to indices c, d. From this structure it is clear that the eta functions always appear in $\mathcal{Z}$ through the combination Q/η^{12}. The ϑ_k^8 in the numerator can be expressed as $\vartheta_2^8 = 16(e_2 - e_3)^2$, $\vartheta_3^8 = 16(e_3 - e_1)^2$, $\vartheta_4^8 = 16(e_1 - e_2)^2$. Next, observe that the way theta functions appear in $\mathcal{Z}$ is entirely symmetric under the permutation of $(\vartheta_2, \vartheta_3, \vartheta_4)$. This can be seen easily by renaming the partitions (R_2, R_3, R_4) correspondingly. Together with the expression of the theta functions (4.2), (4.4), this means that the way e_1, e_2, e_3 appear in F_0 is also entirely symmetric. It is also easy to see that they appear in F_0 always as polynomials. Any symmetric polynomial in e_1, e_2, e_3 is generated by the elementary symmetric polynomials, which are identified as

$$e_1 + e_2 + e_3 = 0, \qquad e_1e_2 + e_1e_3 + e_2e_3 = -\frac{1}{48}E_4, \qquad e_1e_2e_3 = \frac{1}{864}E_6. \tag{4.6}$$

Now we see that τ appears in F_0 only through polynomials in E_2, E_4, E_6 or through the combination Q/η^{12}. Notice that modular weights are preserved in (4.1)–(4.4) if one assigns weight -1 to the expansion variable z. As $\mathcal{Z}$ is manifestly of weight 0, it is obvious that F_0 is of weight -2. Hence, we have shown that our expression indeed reproduces the structure of Z_n as in (2.7).

Next let us consider the modular anomaly. The modular anomalies appear through E_2. An important feature of the expressions (4.1)–(4.4) is that E_2 appears only through the exponential prefactors in (4.1), (4.2). After evaluating all the products in (3.7), the total exponential factor becomes

$$\exp\left(\frac{1}{12}|\boldsymbol{R}|^2 E_2 \hbar^2\right) \tag{4.7}$$

for partitions $\boldsymbol{R}$. Here we have used the following combinatorial identity

$$\sum_{k,l=1}^{N} \sum_{(i,j)\in R_k} \left(h_{kl}(i,j)^2 - (j-i)^2\right) = |\boldsymbol{R}|^2. \tag{4.8}$$

We present a proof of this identity in Appendix C. It is now clear that $\mathcal{Z}$ satisfies the following modular anomaly equation

$$\partial_{E_2}\mathcal{Z} = \frac{1}{12}\hbar^2 \Theta_Q^2 \mathcal{Z}. \tag{4.9}$$

By substituting

$$\mathcal{Z} = \exp\left(\frac{1}{2}F_0\hbar^{-2} + \mathcal{O}(\hbar^0)\right), \tag{4.10}$$

we obtain the modular anomaly equation (2.12).

Since our F_0 coincides with the genus zero topological string amplitude for the local $\frac{1}{2}$K3 (evaluated at $m_i = 0$), one may expect that higher order coefficients of the expansion of $\ln\mathcal{Z}$ in $\hbar$ give higher genus topological string amplitudes. However, this is not the case. It is known that the all-genus topological string partition function for the local $\frac{1}{2}$K3

$$\mathcal{Z}^{\frac{1}{2}\mathrm{K3}} = \exp\left(\sum_{g=0}^{\infty}\hbar^{2g-2}F_g^{\frac{1}{2}\mathrm{K3}}\right) \tag{4.11}$$

satisfies the holomorphic anomaly equation [17]

$$\partial_{E_2}\mathcal{Z}^{\frac{1}{2}\mathrm{K3}} = \frac{1}{24}\hbar^2\Theta_Q(\Theta_Q+1)\mathcal{Z}^{\frac{1}{2}\mathrm{K3}}. \tag{4.12}$$

Despite the apparent difference, equations (4.9) and (4.12) give the same modular anomaly equation (2.12) for $F_0 = F_0^{\frac{1}{2}\mathrm{K3}}\big|_{m_i=0}$. This coincidence, however, does not persist at $g \geq 1$. Higher order parts of the expansion (4.10) do not seem to have an immediate connection with $F_g^{\frac{1}{2}\mathrm{K3}}$ at $g \geq 1$ found in [10, 15, 18].

5. Discussion

In this paper we have presented an explicit expression for the Seiberg–Witten prepotential for the six-dimensional E-string theory toroidally compactified down to four dimensions. The expression is of Nekrasov type and directly gives the coefficients of the winding number expansion of the prepotential at all orders. We have shown that the expression exhibits expected modular properties, in particular we have proved that it satisfies the correct modular anomaly equation.

We have pointed out that the sum over partitions in our expression, denoted by $\mathcal{Z}$, can be viewed as a special case of the elliptic generalization of the Nekrasov partition function for the SU(4) gauge theory with 8 fundamental hypermultiplets. Currently, we do not have a good physical explanation of this coincidence. It would be of great interest if this could uncover yet unknown dualities among field theories in six dimensions.

It is also mysterious that the prepotential is obtained as the 'genus zero part' of $\ln \mathcal{Z}$ (up to an overall factor of 2), where the sum $\mathcal{Z}$ differs from the all-genus topological string partition function for the local $\frac{1}{2}$K3. The local $\frac{1}{2}$K3 does not admit a local toric description and no Nekrasov-type partition function is known for such Calabi–Yau threefolds at present. We hope our expression provides us with a new perspective on the combinatorial study of topological string amplitudes for non-toric Calabi–Yau threefolds.

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A. Conventions of special functions

The Jacobi theta functions are defined as

$$\vartheta_{ab}(z,\tau) := \sum_{n\in\mathbb{Z}} \exp\left[\pi i\left(n+\frac{a}{2}\right)^2\tau + 2\pi i\left(n+\frac{a}{2}\right)\left(z+\frac{b}{2}\right)\right], \tag{A.1}$$

where a, b take values $0, 1$. We also use the notation

$$\begin{aligned} &\vartheta_1(z,\tau) := -\vartheta_{11}(z,\tau), &&\vartheta_2(z,\tau) := \vartheta_{10}(z,\tau), \\ &\vartheta_3(z,\tau) := \vartheta_{00}(z,\tau), &&\vartheta_4(z,\tau) := \vartheta_{01}(z,\tau). \end{aligned} \tag{A.2}$$

The Dedekind eta function is defined as

$$\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \tag{A.3}$$

where $q = e^{2\pi i\tau}$. The Eisenstein series are given by

$$E_{2n}(\tau) = 1 + \frac{2}{\zeta(1-2n)}\sum_{k=1}^{\infty}\frac{k^{2n-1}q^k}{1-q^k}. \tag{A.4}$$

The Weierstrass $\wp$-function is defined as

$$\wp(z|2\omega_1, 2\omega_3) := \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2_{\neq(0,0)}}\left[\frac{1}{(z-\Omega_{m,n})^2} - \frac{1}{{\Omega_{m,n}}^2}\right], \tag{A.5}$$

where $\Omega_{m,n} = 2m\omega_1 + 2n\omega_3$.

We often abbreviate $\vartheta_k(0,\tau)$, $\eta(\tau)$, $E_{2n}(\tau)$ as ϑ_k, η, E_{2n}, respectively.

B. Taylor expansions of Jacobi theta functions

In this appendix, we explain how to compute the Taylor expansions of the Jacobi theta functions. Given the expressions (4.1), (4.2), the problem essentially boils down to the expansions of functions σ and σ_k as in (4.3), (4.4). These expansions can be computed by using some basic properties of the Weierstrass $\wp$-function. In the following, we use the abbreviation

$$\sigma(z) := \sigma(z|2\pi, 2\pi\tau), \quad \sigma_k(z) := \sigma_k(z|2\pi, 2\pi\tau), \quad \wp(z) := \wp(z|2\pi, 2\pi\tau). \tag{B.1}$$

Recall that the $\wp$-function with period 2π, $2\pi\tau$ satisfies the following identity

$$\begin{aligned} \wp'(z)^2 &= 4\wp(z)^3 - \frac{E_4}{12}\wp(z) - \frac{E_6}{216} && \text{(B.2)} \\ &= 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3). && \text{(B.3)} \end{aligned}$$

Here e_k are given in (4.5). They are originally defined as

$$e_k := \wp(\omega_k), \tag{B.4}$$

where ω_k are half periods. In the present case we have

$$\omega_1 = \pi, \qquad \omega_2 = -\pi - \pi\tau, \qquad \omega_3 = \pi\tau. \tag{B.5}$$

The $\wp$-function admits an Laurent expansion of the following form

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} c_n z^{2n}. \tag{B.6}$$

The expansion coefficients are determined by the recurrence relation

$$c_1 = \frac{E_4}{240}, \qquad c_2 = \frac{E_6}{6048}, \tag{B.7}$$

$$c_n = \frac{3}{(n-2)(2n+3)} \sum_{k=1}^{n-2} c_k c_{n-k-1} \qquad (n \geq 3). \tag{B.8}$$

This recurrence relation can be easily derived by substituting (B.6) into the identity

$$\wp''(z) = 6\wp(z)^2 - \frac{E_4}{24}, \tag{B.9}$$

which follows from (B.2). Using identities (B.2), (B.9) recursively, one can express higher derivatives $\wp^{(2n)}(z)$ as polynomials in $\wp(z), E_4, E_6$. Similarly, $\wp^{(2n-1)}(z)$ can be expressed as polynomials in $\wp(z), E_4, E_6$ multiplied by $\wp'(z)$. In particular, it follows from (B.3), (B.4) that $\wp^{(2n)}(\omega_k)$ are polynomials in e_k, E_4, E_6, while $\wp^{(2n-1)}(\omega_k) = 0$.

We are now in a position to consider the expansions of $\sigma(z)$ and $\sigma_k(z)$. They are related to $\wp(z)$ by

$$\partial_z^2 \ln \sigma(z) = -\wp(z), \tag{B.10}$$

$$\partial_z^2 \ln \sigma_k(z) = -\wp(z + \omega_k). \tag{B.11}$$

Integrating twice the series expansions of the r.h.s. of these equations, one obtains

$$\sigma(z) = z \exp\left(-\sum_{n=1}^{\infty} \frac{c_n}{(2n+1)(2n+2)} z^{2n+2} \right), \tag{B.12}$$

$$\sigma_k(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{\wp^{(2n-2)}(\omega_k)}{(2n)!} z^{2n} \right), \tag{B.13}$$

where integration constants were chosen accordingly. These exponential forms are actually convenient for the computation of the prepotential. As we explained above, c_n are polynomials in E_4, E_6, while $\wp^{(2n-2)}(\omega_k)$ are polynomials in E_4, E_6, e_k.

C. Proof of combinatorial identity

In this appendix we prove the identity (4.8). The l.h.s. of (4.8) reads

$$\begin{aligned}&\text{l.h.s.}\\ &= \sum_{k,l=1}^{N} \sum_{(i,j)\in R_k} \left[\left(\mu_{k,i} + \mu^{\vee}_{l,j} - i - j + 1 \right)^2 - (j-i)^2 \right] \\ &= \sum_{k,l=1}^{N} \sum_{i=1}^{\ell(R_k)} \sum_{j=1}^{\mu_{k,i}} \left[(\mu_{k,i} - j + 1 - i)^2 + 2\,(\mu_{k,i} - j + 1 - i)\, \mu^{\vee}_{l,j} + \left(\mu^{\vee}_{l,j}\right)^2 - (j-i)^2 \right] .\end{aligned} \tag{C.1}$$

By introducing a new index $\tilde{\jmath} := \mu_{k,i} - j + 1$, the sum over $j = 1, \dots, \mu_{k,i}$ in the first term can be rewritten as the sum of $(\tilde{\jmath} - i)^2$ over $\tilde{\jmath} = 1, \dots, \mu_{k,i}$. We then see that the first term and the last term cancel each other. Thus we have

$$\begin{aligned}\text{l.h.s.} &= \sum_{k,l=1}^{N} \sum_{(i,j)\in R_k} \left[2\,(\mu_{k,i} - j + 1 - i)\, \mu^{\vee}_{l,j} + \left(\mu^{\vee}_{l,j}\right)^2 \right] \\ &= \sum_{k,l=1}^{N} \sum_{j=1}^{\ell(R^{\vee}_k)} \sum_{i=1}^{\mu^{\vee}_{k,j}} \left[(2\mu_{k,i} - 2j + 1)\, \mu^{\vee}_{l,j} + (-2i+1)\mu^{\vee}_{l,j} + \left(\mu^{\vee}_{l,j}\right)^2 \right] .\end{aligned} \tag{C.2}$$

Let us first evaluate the sum of the last two terms. By performing the sum over $i = 1, \dots, \mu^{\vee}_{k,j}$, they become

$$\sum_{k,l=1}^{N} \sum_{j=1}^{\ell(R^{\vee}_k)} \left[-\left(\mu^{\vee}_{k,j}\right)^2 \mu^{\vee}_{l,j} + \mu^{\vee}_{k,j} \left(\mu^{\vee}_{l,j}\right)^2 \right] . \tag{C.3}$$

Since $\mu^{\vee}_{k,j}$ is defined as $\mu^{\vee}_{k,j} = 0$ for $j > \ell(R^{\vee}_k)$, the sum over j in the above expression can be replaced by that over $j = 1, \dots, \infty$. It then becomes clear that the expression in the sum over k, l is actually antisymmetric under the exchange of k and l. Therefore the sum vanishes.

Now, we are left with

$$\text{l.h.s.} = \sum_{k,l=1}^{N} \sum_{j=1}^{\ell(R^{\vee}_k)} \sum_{i=1}^{\mu^{\vee}_{k,j}} (2\mu_{k,i} - 2j + 1)\, \mu^{\vee}_{l,j} . \tag{C.4}$$

It is easy to see that the following relation holds

$$\sum_{i=1}^{\mu^{\vee}_{k,j}} (\mu_{k,i} - j) = \sum_{j'=j+1}^{\ell(R^{\vee}_k)} \mu^{\vee}_{k,j'} . \tag{C.5}$$

Using this relation, we finally obtain

$$\begin{aligned}
\text{l.h.s.} &= \sum_{k,l=1}^{N} \sum_{j=1}^{\ell(R_k^\vee)} \left(2 \sum_{j'=j+1}^{\ell(R_k^\vee)} \mu_{k,j'}^\vee + \mu_{k,j}^\vee \right) \mu_{l,j}^\vee \\
&= \sum_{k,l=1}^{N} \sum_{j=1}^{\infty} \left(2 \sum_{j'=j+1}^{\infty} \mu_{k,j'}^\vee + \mu_{k,j}^\vee \right) \mu_{l,j}^\vee \\
&= \sum_{k,l=1}^{N} \sum_{j=1}^{\infty} \left(\sum_{j'=1}^{\infty} \mu_{k,j'}^\vee - \sum_{j'=1}^{j-1} \mu_{k,j'}^\vee + \sum_{j'=j+1}^{\infty} \mu_{k,j'}^\vee \right) \mu_{l,j}^\vee \\
&= \sum_{k,l=1}^{N} \left(\sum_{j,j'=1}^{\infty} \mu_{k,j'}^\vee \mu_{l,j}^\vee - \sum_{0<j'<j} \mu_{k,j'}^\vee \mu_{l,j}^\vee + \sum_{0<j<j'} \mu_{k,j'}^\vee \mu_{l,j}^\vee \right) \\
&= \sum_{k,l=1}^{N} |R_k||R_l| = \text{r.h.s.}
\end{aligned} \tag{C.6}$$

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