<table>
<thead>
<tr>
<th>Title</th>
<th>Supplement to classification of threefold divisorial contractions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kawakita, Masayuki</td>
</tr>
<tr>
<td>Citation</td>
<td>Nagoya Mathematical Journal (2012), 206: 67-73</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2012-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/160035">http://hdl.handle.net/2433/160035</a></td>
</tr>
<tr>
<td>Right</td>
<td>© Editorial Board, Nagoya Mathematical Journal; This is not the published version. Please cite only the published version. この論文は出版社版でありません。引用の際には出版社版をご確認ご利用ください。</td>
</tr>
<tr>
<td>Type</td>
<td>Journal Article</td>
</tr>
<tr>
<td>Textversion</td>
<td>author</td>
</tr>
</tbody>
</table>

Kyoto University
SUPPLEMENT TO CLASSIFICATION OF
THREE-FOLD DIVISORIAL CONTRACTIONS

MASAYUKI KAWAKITA

ABSTRACT. Every three-fold divisorial contraction to a non-Gorenstein point is
a weighted blow-up.

This supplement finishes the explicit description of a three-fold divisorial con-
traction whose exceptional divisor contracts to a non-Gorenstein point. Contrac-
tions to a quotient singularity were treated by Kawamata in [8]. The author’s study
[7], based on the singular Riemann–Roch formula, provided the classification ex-
cept for the case of small discrepancy. On the other hand, Hayakawa classified
those with discrepancy at most one in [1], [2], [3], by the fact that there exist only
a finite number of divisors with such discrepancy over a fixed singularity. The only
case left was when it is a contraction to a $cD/2$ point with discrepancy two. We
demonstrate its classification in Theorem 2 by the method in [7]. It turns out that
every contraction is a weighted blow-up.

Theorem 1. Let $f : Y \to X$ be a three-fold divisorial contraction whose excep-
tional divisor $E$ contracts to a non-Gorenstein point $P$. Then $f$ is a weighted blow-up of
the singularity $P \in X$ embedded into a cyclic quotient of a smooth five-fold.

Our method of the classification is to study the structure of the bi-graded ring
$\bigoplus_{i,j} f^* O_Y(iK_Y + jE)/f^* O_Y(iK_Y + jE - E)$. We find local coordinates at $P$ to meet
this structure and verify that $f$ should be a certain weighted blow-up. The choice of
local coordinates is restricted by the action of the cyclic group, which makes easier
the classification in the non-Gorenstein case. We do not know if this method is suf-
ficient to settle all the remaining Gorenstein cases in [4], [5], [6] with discrepancy
at most four.

By a three-fold divisorial contraction to a point, we mean a projective morphism
$f : (Y \supset E) \to (X \ni P)$ between terminal three-folds such that $-K_Y$ is $f$-ample
and the exceptional locus $E$ is a prime divisor contracting to a point $P$. We shall treat $f$
on the germ at $P$ in the complex analytic category. According to [7, Theorems 1.2,
1.3], the only case left is

type $e_1$ with $P = cD/2$, the discrepancy $a/n = 4/2$
in [7, Table 3]. We shall prove the following theorem.

Theorem 2. Suppose that $f$ is a divisorial contraction of type $e_1$ to a $cD/2$ point
with discrepancy 2. Then $f$ is the weighted blow-up with $\text{wt}(x_1, x_2, x_3, x_4, x_5) =
(\frac{r+1}{2}, \frac{r-1}{2}, 2, 1, r)$ with $r \geq 7$, $r \equiv \pm 1 \mod 8$ for a suitable identifica-
tion

\[
P \in X \simeq o \in \left(\frac{x_1^2 + x_4 x_5 + p(x_2, x_3, x_4) = 0}{x_3^2 + q(x_1, x_3, x_4) + x_5 = 0}\right) \subset C_{x_1,x_2,x_3,x_4,x_5}^{3,4} \times \frac{1}{2}(1,1,0,0),
\]

2010 Mathematics Subject Classification. 14E30, 14J30.
such that \( p \) is of weighted order \( > r \) and \( q \) is weighted homogeneous of weight \( r - 1 \) for the weights distributed above.

The proof is along the argument in [7, Section 7]. Henceforth \( f : (Y \supseteq E) \to (X \ni P) \) is a divisorial contraction of type \( e \) to a \( cD/2 \) point with discrepancy 2. By [7, Table 3], \( Y \) has only one singular point \( Q \) say at which \( E \) is not Cartier. \( Q \) is a quotient singularity of type \( \frac{1}{2}(1,-1,r+4) \) with \( r \geq 7, r \equiv \pm 1 \mod 8 \).

We set vector spaces \( V_i = V_i^0 \oplus V_i^1 \) with

\[
V_i^0 := f_* \mathcal{O}_Y(-iE)/f_* \mathcal{O}_Y(-(i+1)E),
\]

\[
V_i^1 := f_* \mathcal{O}_Y(K_Y - (i+2)E)/f_* \mathcal{O}_Y(K_Y - (i+3)E).
\]

They are zero for negative \( i \), and we have the (bi-)graded ring \( \bigoplus V_i \) by a local isomorphism \( \mathcal{O}_X(2K_X) \simeq \mathcal{O}_X \). To study its structure in lower-degree part, we first compute the dimensions of \( V_i^j \) in terms of the finite sets

\[
N_i := \{ (l_1,l_2,l_3,l_4,l_5) \in \mathbb{Z}^5_{\geq 0} | \frac{r+1}{2}l_1 + \frac{r-1}{2}l_2 + 2l_3 + l_4 + rl_5 = i, l_1,l_2 \leq 1 \}.
\]

\( N_i \) is decomposed into \( N_i^0 \cup N_i^1 \) with \( N_i^j := \{ (l_1,l_2,l_3,l_4,l_5) \in N_i | l_1 + l_2 + l_3 \equiv j \mod 2 \} \).

**Lemma 3.** \( \dim V_i^j = \#N_i^j \).

**Proof.** We follow the notation in [7]. \((r_Q,b_Q,v_Q) = (2r,r+4,2)\) and \( E^3 = 1/r \) by [7, Tables 2, 3]. By \( \dim V_i^j = d(j, -i - 2j) \) for \( j \geq -2 \) in [7, (2.8)], the equality [7, (2.6)] for \((j, -i - 2j)\) implies that for \( i \geq 0 \),

\[
\dim V_i^j - \dim V_{i-1}^{j-1} = \frac{2i+1}{r} + B_{2r}(2i + rj + 2) - B_{2r}(2i + rj).\]

Here \( B_{2r}(k) = (k \cdot 2r - k)/2r \) and \( - \) denotes the residue modulo \( 2r \). On the other hand, by \( N_i^j = (N_i^{j-1} + (0,0,1,0,0)) \cup \{ (l_1,l_2,0,l_4,l_5) \in N_i^j \} \),

\[
\#N_i^j - \#N_{i-1}^{j-1} = \begin{cases} 
\# \{(0,0,0,l_4,l_5) \in N_i^0\} + \# \{(1,0,l_4,l_5) \in N_i^0\} & \text{for } j = 0, \\
\# \{(0,1,0,l_4,l_5) \in N_i^1\} + \# \{(1,0,l_4,l_5) \in N_i^1\} & \text{for } j = 1. 
\end{cases}
\]

The lemma follows by verifying the coincidence of their right-hand sides. q.e.d.

We shall find bases of \( V_i \) starting with an arbitrary identification

\[
P \in X \simeq o \in (\phi = 0) \subset \mathbb{C}^4_{x_1,x_2,x_3,x_4}/\frac{1}{2}(1,1,1,0).
\]

For a semi-invariant function \( h \), \( \ord E h \) denotes the order of \( h \) along \( E \).

**Lemma 4.**

(i) \( \ord E x_4 = 1 \) and \( \ord E x_i \geq 2 \) for \( i = 1,2,3 \). There exists some \( k \) with \( \ord E x_k = 2 \). We set \( x_k = x_3 \) by permutation.

(ii) For \( i < \frac{r+1}{2} \), the monomials \( x_3^{l_3}x_4^{l_4} \) for \((0,0,l_3,l_4,0) \in N_i \) form a basis of \( V_i \). In particular for \( k = 1,2 \), \( \ord E \tilde{x}_k \geq \frac{r+1}{2} \) for \( \tilde{x}_k := x_k + \sum c_{kl}x_3^{l_3}x_4^{l_4} \) with some \( c_{kl} \in \mathbb{C} \), with summation over \((0,0,l_3,l_4,0) \in \bigcup_{i < \frac{r+1}{2}} N_i^1 \).

(iii) There exists some \( k \) with \( \ord E \tilde{x}_k = \frac{r+1}{2} \) such that the monomials \( \tilde{x}_k \) and \( x_3^{l_3}x_4^{l_4} \) for \((0,0,l_3,l_4,0) \in N_r \) form a basis of \( V_{r+1} \). We set \( \tilde{x}_k = \tilde{x}_2 \) by permutation, then \( \ord E \tilde{x}_1 \geq \frac{r+1}{2} \) for \( \tilde{x}_1 := \tilde{x}_1 + \sum c_{1l}x_3^{l_3}x_4^{l_4} \) with some \( c_{1l} \in \mathbb{C} \), with summation over \((0,l_2,l_3,l_4,0) \in N_{r+1} \).
(iv) \( \text{ord}_E \hat{x}_1 = \frac{r+1}{2} \). For \( i < r - 1 \), the monomials \( x_1^{r_1} x_2^{r_2} x_3^{r_3} x_4^{r_4} \) for \( (l_1, l_2, l_3, l_4, 0) \) form a basis of \( V_r \).

(v) \( \text{Set } \bar{N}_i := \{(l_1, l_2, l_3, l_4, 0) \in \mathbb{Z}_{\geq 0}^5 \mid \frac{r+1}{2} l_1 + \frac{r-1}{2} l_2 + 2 l_3 + l_4 + r l_5 = i \} \) and \( \bar{N}_i^0 := \{(l_1, l_2, l_3, l_4, 0) \in \bar{N}_i \mid l_1 + l_2 + l_3 \text{ even} \} \). The monomials \( x_1^{r_1} x_2^{r_2} x_3^{r_3} x_4^{r_4} \) for \( (l_1, l_2, l_3, l_4, 0) \in \bar{N}_r^0 \) have one non-trivial relation, say \( \psi \), in \( V_r^0 \).

The natural exact sequence below is exact.

\[
0 \to \mathbb{C} \psi \to \bigoplus_{(l_1, l_2, l_3, l_4, 0) \in \bar{N}_r} \mathbb{C} x_1^{r_1} x_2^{r_2} x_3^{r_3} x_4^{r_4} \to V_{r-1} \to 0.
\]

(vi) \( \text{ord}_E \psi = r \). The natural exact sequence below is exact.

\[
0 \to \mathbb{C} x_4 \psi \to \bigoplus_{(l_1, l_2, l_3, l_4, 0) \in \bar{N}_r} \mathbb{C} x_1^{r_1} x_2^{r_2} x_3^{r_3} x_4^{r_4} \psi^5 \to V_r \to 0.
\]

**Proof.** We follow the proof of [7, Lemma 7.2], with using the computation of Lemma 3. (i) follows from \( \dim V_1^0 = 1 \), \( \dim V_1^1 = 0 \) and \( \dim V_2^0 = 1 \). Then \( V_2^0 \) is spanned by \( x_3^2 \) and \( x_4^4 \), which should form a basis of \( V_2^0 \) by \( \dim V_2^0 = 2 \). Now (ii) to (v) follow from the same argument as in [7, Lemma 7.2]. We obtain the sequence in (vi) also, which is exact possibly except for the middle. Its exactness is verified by comparing dimensions. q.e.d.

**Corollary 5.** We distribute weights \( \text{wt}(\hat{x}_1, \hat{x}_2, x_3, x_4) = (\frac{r+1}{2}, \frac{r-1}{2}, 2, 1) \) to the coordinates \( \hat{x}_1, \hat{x}_2, x_3, x_4 \) obtained in Lemma 4. Then \( \phi \) in (1) is of form

\[
\phi = c x_4 \psi + \phi_{> r} (\hat{x}_1, \hat{x}_2, x_3, x_4)
\]

with \( c \in \mathbb{C} \) and a function \( \phi_{> r} \) of weighted order \( > r \), where \( \psi \) is as in Lemma 4(v).

**Proof.** Decompose \( \phi = \phi_{\leq r} + \phi_{> r} \) into the part \( \phi_{\leq r} \) of weighted order \( \leq r \) and \( \phi_{> r} \) of weighted order \( > r \). Then \( \text{ord}_E \phi_{\leq r} = \text{ord}_E \phi_{> r} \) > \( r \), so \( \phi_{\leq r} \) is mapped to zero by the natural homomorphism

\[
\bigoplus_{(l_1, l_2, l_3, l_4, 0) \in \bar{N}_r} \mathbb{C} x_1^{r_1} x_2^{r_2} x_3^{r_3} x_4^{r_4} \to \mathcal{O}_X / f_\gamma (-r+1)E,
\]

whose kernel is \( \mathbb{C} x_4 \psi \) by Lemma 4(iv)-(vi). q.e.d.

We shall derive an expression of the germ \( P \in X \) in Theorem 2. By [9, Remark 23.1], the \( cD/2 \) point \( P \in X \) has an identification in (1) with \( \phi \) either of

(A) \[
\phi = x_2^2 + x_2 x_3 x_4 + x_2^2 \alpha + x_3^2 \beta + x_4^2
\]

(B) \[
\phi = x_2^2 + x_2^2 x_4 + x_2 x_3 x_4 \alpha + x_2^2 \beta + x_3^2 \gamma + g(x_3, x_4),
\]

with \( \alpha, \beta \geq 2, \gamma \geq 3, \lambda \in \mathbb{C} \) and \( g \in (x_3^4, x_3^2 x_4^2, x_4^3) \). As its general elephant has type \( D_k \) with \( k \geq 2r \) by [7, Lemma 5.2(ii)], we have

(2) \[
\gamma \geq r \quad \text{in (A)}, \quad \text{ord}_E(0, x_4) \geq r \quad \text{in (B)}.
\]

**Lemma 6. The case (A) does not happen.**

**Proof.** By Lemma 4(i), \( \text{ord}_E x_4 = 1 \), \( \text{ord}_E x_i = 2 \) for \( i = 1, 2, 3 \) and some \( \text{ord}_E x_i = 2 \). \( \text{ord}_E x_1 \geq 3 \) by the relation \( -x_1^2 = x_2 x_3 x_4 + x_2^2 x_4 + x_3^2 \beta + x_4^2 \gamma \) and (2). Thus we may set \( \text{ord}_E x_3 = 2 \) by permutation, and construct \( \hat{x}_1, \hat{x}_2 \) as in Lemma 4(ii).

Let \( W_{\perp} \) be the subspace of \( V_{r-1} \) spanned by the monomials in \( x_3, x_4 \). If \( \hat{x}_1 \notin W_{\perp} \), the triple \( (\hat{x}_1, x_3, x_4) \) plays the role of \( (\hat{x}_2, x_3, x_4) \) in Lemma 4(iii). We construct \( \hat{x}_2 \) as in Lemma 4(iii) to obtain a quartuple \( (\hat{x}_2, \hat{x}_1, x_3, x_4) \), and distribute
By the same reason as in the proof of Lemma 6, we obtain $\bar{E}$ as in Corollary 5. Set $\bar{x}_1 = x_1 + p_1(x_3, x_4)$, $\bar{x}_2 = x_2 + p_2(\bar{x}_1, x_4)$ and rewrite $\phi$ as

$$\phi = (\bar{x}_1 - p_1)^2 + (\bar{x}_2 - p_2)x_3x_4 + (\bar{x}_2 - p_2)^2\alpha^2 + x_3^2 + x_4^2.$$\[\phi \text{ has the term } x_3^2 \text{ of weight } r - 1, \text{ which contradicts Corollary 5.}\]

Hence $\bar{x}_1 \in W_{\frac{r+1}{2}}$, and we obtain a quartuple $(\bar{x}_1, \bar{x}_2, x_3, x_4)$ by $\bar{x}_1 = x_1 + p_1(x_3, x_4)$, $\bar{x}_2 = x_2 + p_2(x_3, x_4)$ as in Lemma 4. Distribute $\text{wt}(\bar{x}_1, \bar{x}_2, x_3, x_4) = (\frac{r+1}{2}, \frac{r-1}{2}, 2, 1)$ and rewrite $\phi$ as

$$\phi = (\bar{x}_1 - p_1)^2 + (\bar{x}_2 - p_2)x_3x_4 + (\bar{x}_2 - p_2)^2\alpha^2 + x_3^2 + x_4^2.$$\[\phi \text{ has the term } \bar{x}_2 x_3 x_4 \text{ of weight } \frac{r+5}{2}, \text{ whereas } \frac{r+5}{2} \geq r \text{ by Corollary 5, a contradiction to } r \geq 7.\] q.e.d.

**Lemma 7.** The germ $P \in X$ has an expression in Theorem 2, with $q$ not of form $(x_3s(x_3^2, x_4))$, such that each $\text{ord}_E x_i$ coincides with $\text{wt} x_i$, distributed in Theorem 2.

**Proof.** We have the case (B) by Lemma 6. $\text{ord}_E x_4 = 1$ and $\text{ord}_E x_3 \geq 3$ as in (A), then $\text{ord}_E x_2 \geq 3$ and $\text{ord}_E x_3 = 2$. We construct $\bar{x}_1, \bar{x}_2$ as in Lemma 4(ii). By the same reason as in the proof of Lemma 6, we obtain $\bar{x}_1 \in W_{\frac{r+1}{2}}$ and a quartuple $(\bar{x}_1, \bar{x}_2, x_3, x_4)$ by $\bar{x}_1 = x_1 + p_1(x_3, x_4)$, $\bar{x}_2 = x_2 + p_2(x_3, x_4)$. Distribute $\text{wt}(\bar{x}_1, \bar{x}_2, x_3, x_4) = (\frac{r+1}{2}, \frac{r-1}{2}, 2, 1)$ and rewrite $\phi$ as

$$\phi = (\bar{x}_1 - p_1)^2 + (\bar{x}_2 - p_2)x_3x_4 + (\bar{x}_2 - p_2)^2\alpha^2 + x_3^2 + x_4^2.$$\[\phi \text{ has the term } \bar{x}_2 x_3 x_4 \text{ of weight } r \text{ and should be of form}\]

$$\phi = (\bar{x}_1^2 + h(\bar{x}_1, \bar{x}_2, x_3, x_4))x_4 + \phi_{>r}(\bar{x}_1, \bar{x}_2, x_3, x_4)$$

as in Corollary 5 with $\psi = \bar{x}_2^2 + h(\bar{x}_1, \bar{x}_2, x_3, x_4)$. In particular $p_2 = 0$ as otherwise $p_2 \bar{x}_2 x_4$ would be of weighted order $< r$, and one can write

$$\phi = \bar{x}_1^2 + x_4 \psi + p(\bar{x}_2, x_3, x_4), \quad \psi = \bar{x}_2^2 + q(\bar{x}_1, x_3, x_4),$$

where $p$ is of weighted order $> r$ and $q$ is weighted homogeneous of weight $r - 1$. A desired expression is derived by setting $x_5 := -\psi$ and replacing $x_4$ with $-x_4$. $q$ is not of form $(x_3s(x_3^2, x_4))$ by Lemma 4(iii) and $\text{ord}_E(\bar{x}_2^2 + q) = r$. q.e.d.

Take an expression of the germ $P \in X$ in Theorem 2 by Lemma 7. We shall apply the extension of [7, Lemma 6.1] to the case when $X$ is embedded into a cyclic quotient of $\mathbb{C}^5$. Let $g: Z \to X$ be the weighted blow-up with $\text{wt} x_i = \text{ord}_E x_i$. By direct calculation, we verify the assumptions of [7, Lemma 6.1] and that $Z$ is smooth outside the strict transform of $x_1x_2x_3x_4x_5 = 0$. We need the condition $q \neq (x_3s)^2$ to check that the restriction $\bar{g} \cap Z$ of the exceptional locus in the ambient space defines an irreducible reduced 2-cycle on $Z$. Therefore $f$ should coincide with $g$ by [7, Lemma 6.1], and Theorem 2 is completed.

**Remark 8.** Using $H \cap E \simeq \mathbb{P}^1$ in the proof of [7, Theorem 5.4], one can show that

(i) if $r \equiv 1 \mod 8, x_3^{(r+3)/4}$ appears in $p$ and $x_3^{(r-1)/2}$ appears in $q$.

(ii) if $r \equiv 7 \mod 8, x_3^{(r+1)/2}$ appears in $p$ and $x_1x_3^{(r-1)/4}$ appears in $q$.

Theorem 1 follows from [1], [2], [3], [7], [8] and Theorem 2.

**Acknowledgements.** I was motivated to write this supplement by a question of Professor J. A. Chen. He, with Professor T. Hayakawa, informed me that only one case was left. Partial support was provided by Grant-in-Aid for Young Scientists (A) 20684002.
REFERENCES

3. T. Hayakawa, Divisorial contractions to 3-dimensional terminal singularities with discrepancy one, J. Math. Soc. Japan 57 (2005), 651-668
5. M. Kawakita, Divisorial contractions in dimension three which contract divisors to compound $A_1$ points, Compos. Math. 133 (2002), 95-116
8. Y. Kawamata, Divisorial contractions to 3-dimensional terminal quotient singularities, Higher-dimensional complex varieties, Walter de Gruyter (1996), 241-246

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

E-mail address: masayuki@kurims.kyoto-u.ac.jp