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SUPPLEMENT TO CLASSIFICATION OF
THREE-FOLD DIVISORIAL CONTRACTIONS

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ABSTRACT. Every three-fold divisorial contraction to a non-Gorenstein point is a weighted blow-up.

This supplement finishes the explicit description of a three-fold divisorial contraction whose exceptional divisor contracts to a non-Gorenstein point. Contractions to a quotient singularity were treated by Kawamata in [8]. The author’s study [7], based on the singular Riemann–Roch formula, provided the classification except for the case of small discrepancy. On the other hand, Hayakawa classified those with discrepancy at most one in [1], [2], [3], by the fact that there exist only a finite number of divisors with such discrepancy over a fixed singularity. The only case left was when it is a contraction to a \( cD/2 \) point with discrepancy two. We demonstrate its classification in Theorem 2 by the method in [7]. It turns out that every contraction is a weighted blow-up.

Theorem 1. Let \( f : Y \to X \) be a three-fold divisorial contraction whose exceptional divisor \( E \) contracts to a non-Gorenstein point \( P \). Then \( f \) is a weighted blow-up of the singularity \( P \in X \) embedded into a cyclic quotient of a smooth five-fold.

Our method of the classification is to study the structure of the bi-graded ring \( \bigoplus_{i,j} \mathcal{O}_Y(iK_Y + jE)/f_* \mathcal{O}_Y(iK_Y + jE - E) \). We find local coordinates at \( P \) to meet this structure and verify that \( f \) should be a certain weighted blow-up. The choice of local coordinates is restricted by the action of the cyclic group, which makes easier the classification in the non-Gorenstein case. We do not know if this method is sufficient to settle all the remaining Gorenstein cases in [4], [5], [6] with discrepancy at most four.

By a three-fold divisorial contraction to a point, we mean a projective morphism \( f : (Y \supset E) \to (X \ni P) \) between terminal three-folds such that \(-K_Y \) is \( f \)-ample and the exceptional locus \( E \) is a prime divisor contracting to a point \( P \). We shall treat \( f \) on the germ at \( P \) in the complex analytic category. According to [7, Theorems 1.2, 1.3], the only case left is type \( e1 \) with \( P = cD/2 \), the discrepancy \( a/n = 4/2 \) in [7, Table 3]. We shall prove the following theorem.

Theorem 2. Suppose that \( f \) is a divisorial contraction of type \( e1 \) to a \( cD/2 \) point with discrepancy \( 2 \). Then \( f \) is the weighted blow-up with \( \text{wt}(x_1, x_2, x_3, x_4, x_5) = (r+1, r-1, 2, 1, r) \) with \( r \geq 7, r \equiv \pm 1 \text{ mod } 8 \) for a suitable identification

\[
P \in X \simeq o \in \left( \begin{array}{c} x_1^2 + x_4x_5 + p(x_2, x_3, x_4) = 0 \\ x_2^2 + q(x_1, x_3, x_4) + x_5 = 0 \end{array} \right) \subset \mathbb{C}^5_{x_1, x_2, x_3, x_4, x_5}/ \frac{1}{2}(1, 1, 1, 0, 0),
\]

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such that $p$ is of weighted order $> r$ and $q$ is weighted homogeneous of weight $r - 1$ for the weights distributed above.

The proof is along the argument in [7, Section 7]. Henceforth $f: (Y \supset E) \to (X \ni P)$ is a divisorial contraction of type $e_1$ to a $C_D/2$ point with discrepancy $2$. By [7, Table 3], $Y$ has only one singular point $Q$ say at which $E$ is not Cartier. $Q$ is a quotient singularity of type $\frac{1}{2}(1, -1, r + 4)$ with $r \geq 7, r \equiv \pm 1 \mod 8$.

We set vector spaces $V_i = V_i^0 \oplus V_i^1$ with
\[
V_i^0 := f_* \mathcal{O}_Y(-iE)/f_* \mathcal{O}_Y(-(i+1)E),
\]
\[
V_i^1 := f_* \mathcal{O}_Y(K_Y - (i+2)E)/f_* \mathcal{O}_Y(K_Y - (i+3)E).
\]

They are zero for negative $i$, and we have the (bi-)graded ring $\bigoplus V_i$ by a local isomorphism $\mathcal{O}_X(2K_X) \simeq \mathcal{O}_X$. To study its structure in lower-degree part, we first compute the dimensions of $V_i^j$ in terms of the finite sets
\[
N_i := \{(l_1, l_2, l_3, l_4, l_5) \in \mathbb{Z}_{\geq 0}^5 \mid \frac{r+1}{2}l_1 + \frac{r-1}{2}l_2 + 2l_3 + 4l_4 + rl_5 = i, \ l_1, l_2 \leq 1\}.
\]

$N_i$ is decomposed into $N_i^0 \sqcup N_i^1$ with $N_i^0 := \{(l_1, l_2, l_3, l_4, l_5) \in N_i \mid l_1 + l_2 + l_3 \equiv j \mod 2\}$.

**Lemma 3.** $\dim V_i^j = \#N_i^j$.

**Proof.** We follow the notation in [7]. $(r_Q, b_Q, v_Q) = (2r, r + 4, 2)$ and $E^3 = 1/r$ by [7, Tables 2, 3]. By $\dim V_i^j = d(j, -i - 2j)$ for $i \geq -2$ in [7, (2.8)], the equality [7, (2.6)] for $(j, -i - 2j)$ implies that for $i \geq 0$,
\[
\dim V_i^j - \dim V_i^{j-1} = \frac{2i + 1}{r} + B_2(2i + rj + 2) - B_2(2i + rj).
\]

Here $B_2(k) = (k \cdot 2r - k)/2r$ and $\bar{}$ denotes the residue modulo $2r$. On the other hand, by $N_i^j = (N_i^{j-1} + (0, 0, 1, 0, 0)) \sqcup \{(l_1, l_2, 0, l_4, l_5) \in N_i^j\}$,
\[
\#N_i^j - \#N_i^{j-1} = \begin{cases} \#\{(0, 0, 0, l_4, l_5) \in N_i^0\} + \#\{(1, 1, 0, l_4, l_5) \in N_i^0\} & \text{for } j = 0, \\ \#\{(0, 1, 0, l_4, l_5) \in N_i^1\} + \#\{(1, 0, 0, l_4, l_5) \in N_i^1\} & \text{for } j = 1. \end{cases}
\]

The lemma follows by verifying the coincidence of their right-hand sides. q.e.d.

We shall find bases of $V_i$ starting with an arbitrary identification
\[
(1) \quad P \in X \simeq o \in (\phi = 0) \subset \mathbb{C}_x^{4|\sum_{x_3|x_4}|}, \frac{1}{2}(1, 1, 1, 0).
\]

For a semi-invariant function $h$, $\text{ord}_E h$ denotes the order of $h$ along $E$.

**Lemma 4.**
(i) $\text{ord}_E x_3 = 1$ and $\text{ord}_E x_i \geq 2$ for $i = 1, 2, 3$. There exists some $k$ with $\text{ord}_E x_k = 2$. We set $x_k = x_3$ by permutation.

(ii) For $i < \frac{r-1}{2}$, the monomials $x_3^{l_3} x_4^{l_4}$ for $(0, 0, l_3, l_4, 0) \in N_i$ form a basis of $V_i$.

In particular for $k = 1, 2$, $\text{ord}_E x_k \geq \frac{r+1}{2}$ for $\bar{x}_k := x_k + \sum c_{kl_3} x_3^{l_3} x_4^{l_4}$ with some $c_{kl_3} \in \mathbb{C}$, with summation over $(0, 0, l_3, l_4, 0) \in \bigcup_{r < \frac{-1}{2}} N_i^1$.

(iii) There exists some $k$ with $\text{ord}_E \bar{x}_k = \frac{r+1}{2}$ such that the monomials $\bar{x}_k$ and $x_3^{l_3} x_4^{l_4}$ for $(0, 0, l_3, l_4, 0) \in N_i^1$ form a basis of $V_i^{1/2}$. We set $\bar{x}_k = x_2$ by permutation, then $\text{ord}_E \bar{x}_1 \geq \frac{r+1}{2}$ for $\bar{x}_1 := \bar{x}_1 + \sum c_{l_3l_4} x_3^{l_3} x_4^{l_4}$ with some $c_{l_3l_4} \in \mathbb{C}$, with summation over $(0, l_2, l_3, l_4, 0) \in N_i^{1/2}$.
(iv) \( \text{ord}_E \hat{x}_1 = \frac{r+1}{2} \). For \( i < r - 1 \), the monomials \( \hat{x}_1^i \hat{x}_2^j \hat{x}_3^k \hat{x}_4^l \) for \( (l_1, l_2, l_3, l_4, 0) \in N_i \) form a basis of \( V_i \).

(v) Set \( \bar{N}_1 := \{ (l_1, l_2, l_3, l_4, l_5) \in \mathbb{Z}_{\geq 0}^5 \mid \frac{r+1}{2} l_1 + \frac{r-1}{2} l_2 + 2l_3 + l_4 + rl_5 = i \} \) and \( \bar{N}_0 := \{ (l_1, l_2, l_3, l_4, l_5) \in \bar{N}_1 \mid l_1 + l_2 + l_3 \text{ even} \} \). The monomials \( \hat{x}_1^i \hat{x}_2^j \hat{x}_3^k \hat{x}_4^l \) for \( (l_1, l_2, l_3, l_4, 0) \in \bar{N}_0^{-1} \) have one non-trivial relation, say \( \psi \), in \( V_0^{-1} \).

The natural exact sequence below is exact.

\[
0 \to \mathbb{C} \psi \to \bigoplus_{(l_1, l_2, l_3, l_4, 0) \in \bar{N}_0} \mathbb{C} \hat{x}_1^i \hat{x}_2^j \hat{x}_3^k \hat{x}_4^l \to V_{-1} \to 0.
\]

(vi) \( \text{ord}_E \psi = r \). The natural exact sequence below is exact.

\[
0 \to \mathbb{C} x_4 \psi \to \bigoplus_{(l_1, l_2, l_3, l_4, 0) \in \bar{N}_0} \mathbb{C} \hat{x}_1^i \hat{x}_2^j \hat{x}_3^k \hat{x}_4^l \psi^g \to V_r \to 0.
\]

\[\text{Proof.}\] We follow the proof of [7, Lemma 7.2], with using the computation of Lemma 3. (i) follows from \( \dim V_0^1 = 1 \), \( \dim V_1^1 = 0 \) and \( \dim V_2^1 = 1 \). Then \( V_0^0 \) is spanned by \( x_3^2 \) and \( x_4^4 \), which should form a basis of \( V_0^0 \) by \( \dim V_0^0 = 2 \). Now (ii) to (v) follow from the same argument as in [7, Lemma 7.2]. We obtain the sequence in (vi) also, which is exact possibly except for the middle. Its exactness is verified by comparing dimensions. \[\text{q.e.d.}\]

**Corollary 5.** We distribute weights \( \text{wt}(\hat{x}_1, \hat{x}_2, x_3, x_4) = (\frac{r+1}{2}, \frac{r-1}{2}, 2, 1) \) to the coordinates \( \hat{x}_1, \hat{x}_2, x_3, x_4 \) obtained in Lemma 4. Then \( \phi \) in (1) is of form

\[\phi = cx_4 \psi + \phi_{>r}(\hat{x}_1, \hat{x}_2, x_3, x_4)\]

with \( c \in \mathbb{C} \) and a function \( \phi_{>r} \) of weighted order \( > r \), where \( \psi \) is as in Lemma 4(v).

\[\text{Proof.}\] Decompose \( \phi = \phi_{\leq r} + \phi_{>r} \) into the part \( \phi_{\leq r} \) of weighted order \( \leq r \) and \( \phi_{>r} \) of weighted order \( > r \). Then \( \text{ord}_E \phi_{\leq r} = \text{ord}_E \phi_{>r} > r \), so \( \phi_{\leq r} \) is mapped to zero by the natural homomorphism

\[
\bigoplus_{(l_1, l_2, l_3, l_4, 0) \in \bigcup_{l_5} \bar{N}_0} \mathbb{C} \hat{x}_1^i \hat{x}_2^j \hat{x}_3^k \hat{x}_4^l \to \mathcal{O}_X / f_i \mathcal{O}_Y (-(r+1)E),
\]

whose kernel is \( \mathbb{C} \psi \) by Lemma 4(iv)-(vi). \[\text{q.e.d.}\]

We shall derive an expression of the germ \( P \in X \) in Theorem 2. By [9, Remark 23.1], the \( cD/2 \) point \( P \in X \) has an identification in (1) with \( \phi \) either of

(A) \[\phi = x_2^2 + x_2 x_4 + x_2^{2\alpha} + x_3^{2\beta} + x_4^2,\]

(B) \[\phi = x_2^2 + x_2 x_4 + \lambda x_2 x_3^{2\alpha-1} + g(x_3, x_4),\]

with \( \alpha, \beta \geq 2, \gamma \geq 3, \lambda \in \mathbb{C} \) and \( g \in (x_3^4, x_3^2 x_4^2, x_4^3) \). As its general elephant has type \( D_k \) with \( k \geq 2r \) by [7, Lemma 5.2(i)], we have

(2) \[\gamma \geq r \ \text{in (A), \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ Ord_0(0, x_4) \geq r \ \text{in (B).}\]

**Lemma 6.** The case (A) does not happen.

\[\text{Proof.}\] By Lemma 4(i), \( \text{ord}_E x_4 = 1 \), \( \text{ord}_E x_i = 2 \) for \( i = 1, 2, 3 \) and some \( \text{ord}_E x_i = 2 \). \( \text{ord}_E x_3 \geq 3 \) by the relation \( -x_2^2 = x_2 x_3 x_4 + x_2^{2\alpha} + x_3^{2\beta} + x_4^2 + x_3^\gamma \) and (2). Thus we may set \( \text{ord}_E x_3 = 2 \) by permutation, and construct \( \hat{x}_1, \hat{x}_2 \) as in Lemma 4(ii).

Let \( W_{-1} \) be the subspace of \( V_{-1} \) spanned by the monomials in \( x_3, x_4 \). If \( \hat{x}_1 \not\in W_{-1} \), the triple \( (\hat{x}_1, x_3, x_4) \) plays the role of \( (\hat{x}_2, x_3, x_4) \) in Lemma 4(iii). We construct \( \hat{x}_2 \) as in Lemma 4(iii) to obtain a quartuple \( (\hat{x}_2, \hat{x}_1, x_3, x_4) \), and distribute
\[ \text{wt}(\hat{x}_2, \hat{x}_1, x_3, x_4) = (\frac{r+1}{2}, \frac{r-1}{2}, 2, 1) \] as in Corollary 5. Set \( \hat{x}_1 = x_1 + p_1(x_3, x_4) \), \( \hat{x}_2 = x_2 + p_2(\hat{x}_1, x_3, x_4) \) and rewrite \( \phi \) as
\[ \phi = (\hat{x}_1 - p_1)^2 + (\hat{x}_2 - p_2)x_3x_4 + (\hat{x}_2 - p_2)^2x_3^2 + x_4^\gamma. \]
\( \phi \) has the term \( x_4^2 \) of weight \( r - 1 \), which contradicts Corollary 5.

Hence \( \hat{x}_1 \in W_{\frac{r+1}{2}} \), and we obtain a quartuple \((\hat{x}_1, \hat{x}_2, x_3, x_4)\) by \( \hat{x}_1 = x_1 + p_1(x_3, x_4) \), \( \hat{x}_2 = x_2 + p_2(x_3, x_4) \) as in Lemma 4. Distribute \( \text{wt}(\hat{x}_1, \hat{x}_2, x_3, x_4) = (\frac{r+1}{2}, \frac{r-1}{2}, 2, 1) \) and rewrite \( \phi \) as
\[ \phi = (\hat{x}_1 - p_1)^2 + (\hat{x}_2 - p_2)x_3x_4 + (\hat{x}_2 - p_2)^2x_3^2 + x_4^\gamma. \]
\( \phi \) has the term \( \hat{x}_2x_3x_4 \) of weight \( \frac{r+5}{2} \), whence \( \frac{r+5}{2} \geq r \) by Corollary 5, a contradiction to \( r \geq 7 \). q.e.d.

**Lemma 7.** The germ \( P \in X \) has an expression in Theorem 2, with \( q \) not of form \((x_3s(x_3^2, x_4))^2\), such that each \( \text{ord}_E x_i \) coincides with \( \text{wt}_i \) distributed in Theorem 2.

**Proof.** We have the case (B) by Lemma 6. \( \text{ord}_E x_4 = 1 \) and \( \text{ord}_E x_1 \geq 3 \) as in (A), then \( \text{ord}_E x_2 \geq 3 \) and \( \text{ord}_E x_3 = 2 \). We construct \( \hat{x}_1, \hat{x}_2 \) as in Lemma 4(ii). By the same reason as in the proof of Lemma 6, we obtain \( \hat{x}_1 \in W_{\frac{r+1}{2}} \) and a quartuple \((\hat{x}_1, \hat{x}_2, x_3, x_4)\) by \( \hat{x}_1 = x_1 + p_1(x_3, x_4) \), \( \hat{x}_2 = x_2 + p_2(x_3, x_4) \). Distribute \( \text{wt}(\hat{x}_1, \hat{x}_2, x_3, x_4) = (\frac{r+1}{2}, \frac{r-1}{2}, 2, 1) \) and rewrite \( \phi \) as
\[ \phi = (\hat{x}_1 - p_1)^2 + (\hat{x}_2 - p_2)x_3x_4 + (\hat{x}_2 - p_2)^2x_3^2 + x_4^\gamma. \]
\( \phi \) has the term \( \hat{x}_2x_3x_4 \) of weight \( r \) and should be of form
\[ \phi = (\hat{x}_2 + h(\hat{x}_1, \hat{x}_2, x_3, x_4))x_4 + \phi_{>r}(\hat{x}_1, \hat{x}_2, x_3, x_4) \]
as in Corollary 5 with \( \psi = \hat{x}_2 + h(\hat{x}_1, \hat{x}_2, x_3, x_4) \). In particular \( p_2 = 0 \) as otherwise \( p_2\hat{x}_2x_4 \) would be of weighted order \( < r \), and one can write
\[ \phi = \chi_1 + x_4\psi + p(\hat{x}_2, x_3, x_4), \quad \psi = \hat{x}_2 + q(\hat{x}_1, x_3, x_4), \]
where \( p \) is of weighted order \( > r \) and \( q \) is weighted homogeneous of weight \( r - 1 \). A desired expression is derived by setting \( x_5 := -\psi \) and replacing \( x_4 \) with \( -x_4 \). \( q \) is not of form \((x_3s(x_3^2, x_4))^2\) by Lemma 4(iii) and \( \text{ord}_E(\hat{x}_2^2 + q) = r \). q.e.d.

Take an expression of the germ \( P \in X \) in Theorem 2 by Lemma 7. We shall apply the extension of [7, Lemma 6.1] to the case when \( X \) is embedded into a cyclic quotient of \( \mathbb{C}^5 \). Let \( g: Z \to X \) be the weighted blow-up with \( wt_x = \text{ord}_E x_i \). By direct calculation, we verify the assumptions of [7, Lemma 6.1] and that \( Z \) is smooth outside the strict transform of \( x_1x_2x_3x_4x_5 = 0 \). We need the condition \( q \neq (x_3s)^2 \) to check that the restriction \( F \cap Z \) of the exceptional locus in the ambient space defines an irreducible reduced 2-cycle on \( Z \). Therefore \( f \) should coincide with \( g \) by [7, Lemma 6.1], and Theorem 2 is completed.

**Remark 8.** Using \( H \cap E \simeq \mathbb{P}^1 \) in the proof of [7, Theorem 5.4], one can show that
(i) if \( r \equiv 1 \mod 8 \), \( x_3^{(r+3)/4} \) appears in \( p \) and \( x_3^{(r-1)/2} \) appears in \( q \).
(ii) if \( r \equiv 7 \mod 8 \), \( x_3^{(r+1)/2} \) appears in \( p \) and \( x_3x_4^{(r-3)/4} \) appears in \( q \).

Theorem 1 follows from [1], [2], [3], [7], [8] and Theorem 2.

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- weighted homogeneous germ
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- weighted blow-up
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[A. Chen](Author's details)
THREE-FOLD DIVISORIAL CONTRACTIONS

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