BIADJOINTNESS IN CYCLOTOMIC KHOVANOV-LAUDA-ROUQUIER ALGEBRAS

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Abstract. In this paper, we prove that a pair of functors E_i^Λ and F_i^Λ appearing in the categorification of irreducible highest weight modules of quantum groups via cyclotomic Khovanov-Lauda-Rouquier algebras is a biadjoint pair.

1. Introduction

Lascoux-Leclerc-Thibon ([13]) conjectured that the irreducible representations of Hecke algebras of type A are controlled by the upper global basis ([8, 9]) (or dual canonical basis ([16]) of the basic representation of the affine quantum group $U_q(A_\ell^{(1)})$. Then Ariki ([1]) proved this conjecture by generalizing it to cyclotomic affine Hecke algebras. The crucial ingredient there was the fact that the cyclotomic affine Hecke algebras categorify the irreducible highest weight representations of $U(A_\ell^{(1)})$. Because of the lack of grading on the cyclotomic affine Hecke algebras, these algebras do not categorify the representation of the quantum group.

Then Khovanov-Lauda and Rouquier introduced independently a new family of graded algebras, a generalization of affine Hecke algebras of type A, in order to categorify arbitrary quantum groups ([10, 11, 17]). These algebras are called *Khovanov-Lauda-Rouquier algebras* or *quiver Hecke algebras.*

Let $U_q(\mathfrak{g})$ be the quantum group associated with a symmetrizable Cartan datum and let $\{R(\beta)\}_{\beta\in Q^+}$ be the corresponding Khovanov-Lauda-Rouquier algebras. Then it was shown in [10, 11] that there exists an algebra isomorphism

$$U_{\mathbf{A}}^-(\mathfrak{g}) \simeq \bigoplus_{\beta\in Q^+} K\big(\mathrm{Proj}(R(\beta))\big),$$

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where $U^-_{\mathbf{A}}(\mathfrak{g})$ is the integral form of the half $U^-_q(\mathfrak{g})$ of $U_q(\mathfrak{g})$ with $\mathbf{A} = \mathbf{Z}[q, q^{-1}]$, and $K\bigl(\mathrm{Proj}(R(\beta))\bigr)$ is the Grothendieck group of finitely generated projective graded $R(\beta)$-modules. Moreover, when the generalized Cartan matrix is a symmetric matrix, Varagnolo and Vasserot proved that *lower global basis* introduced by the author or Lusztig's *canonical basis* corresponds to the isomorphism classes of indecomposable projective R-modules under this isomorphism ([18]).

For each dominant integral weight $\Lambda \in P^+$, the algebra $R(\beta)$ has a special quotient $R^\Lambda(\beta)$ which is called the *cyclotomic Khovanov-Lauda-Rouquier algebra*. In [10], Khovanov and Lauda conjectured that $\bigoplus_{\beta \in Q^+} K\bigl(\mathrm{Proj}(R^\Lambda(\beta))\bigr)$ has a $U_{\mathbf{A}}(\mathfrak{g})$-module structure and that there exists a $U_{\mathbf{A}}(\mathfrak{g})$-module isomorphism

$$V_{\mathbf{A}}(\Lambda) \simeq \bigoplus_{\beta \in Q^+} K\bigl(\mathrm{Proj}(R^\Lambda(\beta))\bigr),$$

where $V_{\mathbf{A}}(\Lambda)$ denotes the $U_{\mathbf{A}}(\mathfrak{g})$-module with highest weight Λ. After partial results of Brundan and Stroppel ([4]), Brundan and Kleshchev ([2, 3]) and Lauda and Vazirani ([15]), the conjecture was proved by Seok-Jin Kang and the author for *all* symmetrizable Kac-Moody algebras ([7]).

For each $i \in I$, let us consider the restriction functor and the induction functor:

$$\begin{aligned} E_i^\Lambda &\colon \mathrm{Mod}(R^\Lambda(\beta+\alpha_i)) \longrightarrow \mathrm{Mod}(R^\Lambda(\beta)),\\ F_i^\Lambda &\colon \mathrm{Mod}(R^\Lambda(\beta)) \longrightarrow \mathrm{Mod}(R^\Lambda(\beta+\alpha_i)) \end{aligned}$$

defined by

$$\begin{aligned} E_i^\Lambda(N) &= e(\beta, i)N = e(\beta, i)R^\Lambda(\beta+\alpha_i) \otimes_{R^\Lambda(\beta+\alpha_i)} N,\\ F_i^\Lambda(M) &= R^\Lambda(\beta+\alpha_i)e(\beta, i) \otimes_{R^\Lambda(\beta)} M, \end{aligned}$$

where $M \in \mathrm{Mod}(R^\Lambda(\beta))$, $N \in \mathrm{Mod}(R^\Lambda(\beta+\alpha_i))$. Then these functors categorify the root operators e_i and f_i in the quantum groups.

It is obvious that E_i^Λ is a right adjoint functor of F_i^Λ.

Khovanov-Lauda ([10, 11, 12, 14]) and Rouquier ([17]) conjectured that E_i^Λ and F_i^Λ are biadjoint to each other. Namely E_i^Λ is also a left adjoint of F_i^Λ. Furthermore they gave candidates of the unit and the counit of this adjunction explicitly from the first adjunction. In [12], Khovanov-Lauda proved it in the case of $\mathfrak{sl}_n$. Rouquier proved that the candidate of a counit (resp. unit) is the counit (resp. unit) of an adjunction in a more general framework ([17, Theroem 5.16]). In this paper we prove that these candidates are indeed the unit and the counit of an adjunction for an arbitrary cyclotomic Khovanov-Lauda-Rouquier algebra.

In order to prove this we use a similar method employed in [7]. Namely we use the module $e(\beta, i^2)R(\beta+2\alpha_i)e(\beta+\alpha_i, i)\otimes_{R(\beta+\alpha_i)} R^\Lambda(\beta+\alpha_i)$ in order to study $e(\beta, i^2)R^\Lambda(\beta+2\alpha_i)e(\beta+\alpha_i, i)$. We fully use the fact that this module is a free right module over the ring $\mathbf{k}[x_{n+2}]$ (Lemma 5.3).

Webster proved similar results in [19, Theorem 1.6] by a totally different method beyond the author's comprehension. We also mention that [5] is related to our results.

This paper is organized as follows. In Section 2, we recall the notions of Khovanov-Lauda-Rouquier algebras. In Section 3, we recall the definition of cyclotomic Khovanov-Lauda-Rouquier algebras and the results in [7], and then state our main result (Theorem 3.5). In Section 4, we interpret it in terms of the algebras ((4.1), (4.2) and (4.3)), and we gave their proof in Section 5.

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2. The Khovanov-Lauda-Rouquier algebra

2.1. **Cartan data.** Let I be a finite index set. An integral square matrix $A = (a_{ij})_{i,j\in I}$ is called a *symmetrizable generalized Cartan matrix* if it satisfies (i) $a_{ii} = 2$ $(i \in I)$, (ii) $a_{ij} \le 0$ $(i \ne j)$, (iii) $a_{ij} = 0$ if $a_{ji} = 0$ $(i, j \in I)$, (iv) there is a diagonal matrix $D = \mathrm{diag}(d_i \in \mathbf{Z}_{>0} \mid i \in I)$ such that DA is symmetric.

A *Cartan datum* $(A, P, \Pi, P^\vee, \Pi^\vee)$ consists of

(1) a symmetrizable generalized Cartan matrix A,
(2) a free abelian group P of finite rank, called the *weight lattice*,
(3) $P^\vee := \mathrm{Hom}(P, \mathbf{Z})$, called the *co-weight lattice*,
(4) $\Pi = \{\alpha_i \mid i \in I\} \subset P$, called the set of *simple roots*,
(5) $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$, called the set of *simple coroots*,

satisfying the condition: $\langle h_i, \alpha_j\rangle = a_{ij}$ for all $i, j \in I$.

We denote by

$$P^+ := \{\lambda \in P \mid \langle h_i, \lambda\rangle \in \mathbf{Z}_{\ge 0} \text{ for all } i \in I\}$$

the set of *dominant integral weights*. The free abelian group $Q := \bigoplus_{i\in I} \mathbf{Z}\alpha_i$ is called the *root lattice*. Set $Q^+ = \sum_{i\in I} \mathbf{Z}_{\ge 0}\alpha_i$. For $\alpha = \sum k_i\alpha_i \in Q^+$, we define the *height* $\mathrm{ht}(\alpha)$

of α to be $\mathrm{ht}(\alpha) = \sum k_i$. Let $\mathfrak{h} = \mathbf{Q} \otimes_{\mathbf{Z}} P^\vee$. Since A is symmetrizable, there is a symmetric bilinear form $(\quad|\quad)$ on $\mathfrak{h}^*$ satisfying

$$(\alpha_i|\alpha_j) = d_i a_{ij} \quad (i, j \in I) \quad \text{and } \langle h_i, \lambda\rangle = \frac{2(\alpha_i|\lambda)}{(\alpha_i|\alpha_i)} \text{ for any } \lambda \in \mathfrak{h}^* \text{ and } i \in I.$$

2.2. **Definition of Khovanov-Lauda-Rouquier algebra.** Let $(A, P, \Pi, P^\vee, \Pi^\vee)$ be a Cartan datum. In this section, we recall the construction of Khovanov-Lauda-Rouquier algebra associated with $(A, P, \Pi, P^\vee, \Pi^\vee)$ and its properties. We take as a base ring a graded commutative ring $\mathbf{k} = \bigoplus_{n\in\mathbf{Z}} \mathbf{k}_n$ such that $\mathbf{k}_n = 0$ for any $n < 0$. Let us take a matrix $(Q_{ij})_{i,j\in I}$ in $\mathbf{k}[u, v]$ such that $Q_{ij}(u, v) = Q_{ji}(v, u)$ and $Q_{ij}(u, v)$ has the form

$$Q_{ij}(u,v) = \begin{cases} 0 & \text{if } i = j, \\ \sum\limits_{p,q\geq 0} t_{i,j;p,q} u^p v^q & \text{if } i \neq j, \end{cases} \tag{2.1}$$

where $t_{i,j;p,q} \in \mathbf{k}_{-2(\alpha_i|\alpha_j)-(\alpha_i|\alpha_i)p-(\alpha_j|\alpha_j)q}$ and $t_{i,j} := t_{i,j;-a_{ij},0} \in \mathbf{k}_0^\times$. In particular, we have $t_{i,j;p,q} = 0$ if $(\alpha_i|\alpha_i)p + (\alpha_j|\alpha_j)q > -2(\alpha_i|\alpha_j)$. Note that $t_{i,j;p,q} = t_{j,i;q,p}$.

We denote by $S_n = \langle s_1, \ldots, s_{n-1}\rangle$ the symmetric group on n letters, where $s_i = (i, i+1)$ is the transposition. Then S_n acts on I^n.

Definition 2.1 ([10, 17])**.** *The* Khovanov-Lauda-Rouquier algebra $R(n)$ *of degree* n *associated with a Cartan datum* $(A, P, \Pi, P^\vee, \Pi^\vee)$ *and* $(Q_{ij})_{i,j\in I}$ *is the associative algebra over* $\mathbf{k}$ *generated by* $e(\nu)$ $(\nu \in I^n)$, x_k $(1 \leq k \leq n)$, τ_l $(1 \leq l \leq n-1)$ *satisfying the following defining relations:*

$$\begin{aligned}
&e(\nu)e(\nu') = \delta_{\nu,\nu'} e(\nu), \quad \sum_{\nu\in I^n} e(\nu) = 1, \\
&x_k x_l = x_l x_k, \quad x_k e(\nu) = e(\nu) x_k, \\
&\tau_l e(\nu) = e(s_l(\nu))\tau_l, \quad \tau_k\tau_l = \tau_l\tau_k \quad \textit{if } |k-l| > 1, \\
&\tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}) e(\nu), \\
&(\tau_k x_l - x_{s_k(l)}\tau_k) e(\nu) = \begin{cases} -e(\nu) & \textit{if } l = k, \nu_k = \nu_{k+1}, \\ e(\nu) & \textit{if } l = k+1, \nu_k = \nu_{k+1}, \\ 0 & \textit{otherwise}, \end{cases}
\end{aligned}$$

$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) = \begin{cases} \dfrac{Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}) - Q_{\nu_k,\nu_{k+1}}(x_{k+2}, x_{k+1})}{x_k - x_{k+2}} e(\nu) & \textit{if } \nu_k = \nu_{k+2}, \\ 0 & \textit{otherwise.} \end{cases}$$

Note that $R(n)$ has an anti-involution ψ that fixes the generators x_k, τ_l and $e(\nu)$.

The $\mathbf{Z}$-grading on $R(n)$ is given by

$$\deg e(\nu) = 0, \quad \deg\ x_k e(\nu) = (\alpha_{\nu_k}|\alpha_{\nu_k}), \quad \deg\ \tau_l e(\nu) = -(\alpha_{\nu_l}|\alpha_{\nu_{l+1}}). \tag{2.2}$$

For $a, b, c \in \{1, \dots, n\}$, we define the elements of $R(n)$ by

$$\begin{aligned} e_{a,b} &= \sum_{\nu \in I^n,\, \nu_a = \nu_b} e(\nu), \\ Q_{a,b} &= \sum_{\nu \in I^n} Q_{\nu_a,\nu_b}(x_a, x_b) e(\nu), \\ \overline{Q}_{a,b,c} &= \sum_{\nu \in I^n,\, \nu_a = \nu_c} \frac{Q_{\nu_a,\nu_b}(x_a, x_b) - Q_{\nu_a,\nu_b}(x_c, x_b)}{x_a - x_c} e(\nu) \quad \text{if } a \neq c. \end{aligned} \tag{2.3}$$

Then we have

$$\begin{aligned} &Q_{a,b} = Q_{b,a}, \quad \tau_a^2 = Q_{a,a+1}, \\ &\tau_{a+1}\tau_a\tau_{a+1} = \tau_a\tau_{a+1}\tau_a + \overline{Q}_{a,a+1,a+2}. \end{aligned} \tag{2.4}$$

We define the operators $\partial_{a,b}$ on $\bigoplus_{\nu \in I^n} \mathbf{k}[x_1, \dots, x_n]e(\nu)$. by

$$\partial_{a,b} f = \frac{s_{a,b} f - f}{x_a - x_b} e_{a,b}, \quad \partial_a = \partial_{a,a+1}, \tag{2.5}$$

where $s_{a,b} = (a, b)$ is the transposition.

Thus we obtain

$$\begin{aligned} &\tau_a e_{b,c} = e_{s_a(b),s_a(c)} \tau_a, \\ &\tau_a f - (s_a f)\tau_a = f\tau_a - \tau_a(s_a f) = (\partial_a f) e_{a,a+1}. \end{aligned} \tag{2.6}$$

For $n \in \mathbf{Z}_{\geq 0}$ and $\beta \in Q^+$ such that $\mathrm{ht}(\beta) = n$, we set

$$I^\beta = \{\nu = (\nu_1, \dots, \nu_n) \in I^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta\}.$$

We define

$$\begin{aligned} &e(\beta) = \textstyle\sum_{\nu \in I^\beta} e(\nu), \\ &R(\beta) = R(n)e(\beta) = \bigoplus_{\nu \in I^\beta} R(n)e(\nu). \end{aligned} \tag{2.7}$$

The algebra $R(\beta)$ is called the *Khovanov-Lauda-Rouquier algebra at* β.

For $\ell \geq 0$, we set

$$
\begin{aligned}
&e(\beta, i^\ell) = \textstyle\sum_\nu e(\nu) \in R(\beta + \ell\alpha_i)\\
&\qquad\qquad \text{where } \nu \text{ ranges over the set of } \nu \in I^{\beta+\ell\alpha_i} \text{ such that}\\
&\qquad\qquad \nu_k = i \text{ for } n+1 \leq k \leq n+\ell.
\end{aligned}
\tag{2.8}
$$

We sometimes regard $R(\beta)$ as a $\mathbf{k}$-subalgebra of the $\mathbf{k}$-algebra $e(\beta, i^\ell)R(\beta+\ell\alpha_i)e(\beta, i^\ell)$.

Theorem 2.2 ([7])**.** *Let $\beta \in Q^+$ with $\mathrm{ht}(\beta) = n$ and $i \in I$. Then there exists a natural isomorphism*

$$
\begin{aligned}
&R(\beta)e(\beta - \alpha_i, i) \otimes_{R(\beta-\alpha_i)} \mathbf{k}\tau_n \otimes e(\beta - \alpha_i, i)R(\beta) \bigoplus \mathbf{k}[x_{n+1}] \otimes R(\beta)\\
&\qquad \xrightarrow{\sim} e(\beta, i)R(\beta + \alpha_i)e(\beta, i).
\end{aligned}
\tag{2.9}
$$

Here $R(\beta)e(\beta - \alpha_i, i) \otimes_{R(\beta-\alpha_i)} \mathbf{k}\tau_n \otimes e(\beta - \alpha_i, i)R(\beta) \to e(\beta, i)R(\beta + \alpha_i)e(\beta, i)$ is given by $a \otimes \tau_n \otimes b \mapsto a\tau_n b$.

Here, τ_n in $\mathbf{k}\tau_n$ is a symbolical basis of a free $\mathbf{k}$-module of rank one. We sometimes use such notations in order to make morphisms more explicit.

Note that if $\beta - \alpha_i \notin Q^+$ then $R(\beta)e(\beta - \alpha_i, i) \otimes_{R(\beta-\alpha_i)} \mathbf{k}\tau_n \otimes e(\beta - \alpha_i, i)R(\beta)$ should be understood to be zero.

3. The cyclotomic Khovanov-Lauda-Rouquier algebras

3.1. **Definition of cyclotomic Khovanov-Lauda-Rouquier algebras.** Let $\Lambda \in P^+$ be a dominant integral weight. For each $i \in I$, we shall choose a monic polynomial of degree $\langle h_i, \Lambda\rangle$

$$
a_i^\Lambda(u) = \sum_{k=0}^{\langle h_i,\Lambda\rangle} c_{i;k} u^{\langle h_i,\Lambda\rangle - k}
\tag{3.1}
$$

with $c_{i;k} \in \mathbf{k}_{k(\alpha_i|\alpha_i)}$ and $c_{i;0} = 1$.

For k $(1 \leq k \leq n)$ and $\beta \in Q^+$ with $\mathrm{ht}(\beta) = n$, we set

$$
a^\Lambda(x_k) = \sum_{\nu \in I^\beta} a^\Lambda_{\nu_k}(x_k)e(\nu) \in R(\beta).
\tag{3.2}
$$

Hence $a^\Lambda(x_k)e(\nu)$ is a homogeneous element of $R(\beta)$ with degree $2(\alpha_{\nu_k}|\Lambda)$.

Definition 3.1. *For $\beta \in Q^+$ the* cyclotomic Khovanov-Lauda-Rouquier algebra $R^\Lambda(\beta)$ at β *is defined to be the quotient algebra*

$$R^\Lambda(\beta) = \frac{R(\beta)}{R(\beta)a^\Lambda(x_1)R(\beta)}.$$

In this paper we *forget the grading*, and we denote by $\mathrm{Mod}(R^\Lambda(\beta))$ the abelian category of $R^\Lambda(\beta)$-modules.

For each $i \in I$, we define the functors

$$\begin{aligned} \mathsf{E}_i^\Lambda &\colon \mathrm{Mod}(R^\Lambda(\beta+\alpha_i)) \longrightarrow \mathrm{Mod}(R^\Lambda(\beta)), \\ \mathsf{F}_i^\Lambda &\colon \mathrm{Mod}(R^\Lambda(\beta)) \longrightarrow \mathrm{Mod}(R^\Lambda(\beta+\alpha_i)) \end{aligned}$$

by

$$\begin{aligned} \mathsf{E}_i^\Lambda(N) &= e(\beta,i)N \simeq e(\beta,i)R^\Lambda(\beta+\alpha_i) \otimes_{R^\Lambda(\beta+\alpha_i)} N \\ &\simeq \mathrm{Hom}_{R^\Lambda(\beta+\alpha_i)}(R^\Lambda(\beta+\alpha_i)e(\beta,i), N), \\ \mathsf{F}_i^\Lambda(M) &= R^\Lambda(\beta+\alpha_i)e(\beta,i) \otimes_{R^\Lambda(\beta)} M, \end{aligned} \tag{3.3}$$

where $M \in \mathrm{Mod}(R^\Lambda(\beta))$ and $N \in \mathrm{Mod}(R^\Lambda(\beta+\alpha_i))$.

Then the following result is proved in [7].

Theorem 3.2 ([7])**.** *The module $R^\Lambda(\beta+\alpha_i)e(\beta,i)$ is a projective right $R^\Lambda(\beta)$-module. Similarly, $e(\beta,i)R^\Lambda(\beta+\alpha_i)$ is a projective left $R^\Lambda(\beta)$-module.*

Corollary 3.3. (i) *The functor E_i^Λ sends finitely generated projective modules to finitely generated projective modules.*
(ii) *The functor F_i^Λ is exact.*

3.2. The pair $(\mathsf{F}_i^\Lambda, \mathsf{E}_i^\Lambda)$ has a canonical adjunction : the unit $\eta\colon \mathrm{id} \longrightarrow \mathsf{E}_i^\Lambda\mathsf{F}_i^\Lambda$ and the counit $\varepsilon\colon \mathsf{F}_i^\Lambda\mathsf{E}_i^\Lambda \longrightarrow \mathrm{id}$.

For $\beta \in Q^+$ with $\mathrm{ht}(\beta) = n$, the functors

$$\mathrm{Mod}(R^\Lambda(\beta)) \underset{\mathsf{E}_i^\Lambda}{\overset{\mathsf{F}_i^\Lambda}{\rightleftarrows}} \mathrm{Mod}(R^\Lambda(\beta+\alpha_i))$$

are represented by the kernel bimodules $R^\Lambda(\beta+\alpha_i)e(\beta,i)$ and $e(\beta,i)R^\lambda(\beta+\alpha_i)$ as in (3.3). In the sequel, we denote by $\mathbf{1}_\beta$ the identity functor of the category $\mathrm{Mod}(R^\Lambda(\beta))$, and we denote by $\mathbf{1}_\beta\mathsf{E}_i^\Lambda = \mathsf{E}_i^\Lambda\mathbf{1}_{\beta+\alpha_i}$ the restriction functor $\mathsf{E}_i^\Lambda\colon \mathrm{Mod}(R^\Lambda(\beta+\alpha_i)) \to \mathrm{Mod}(R^\Lambda(\beta))$. Similarly, $\mathsf{F}_i^\Lambda\mathbf{1}_\beta = \mathbf{1}_{\beta+\alpha_i}\mathsf{F}_i^\Lambda$ denotes the induction functor $\mathsf{F}_i^\Lambda\colon \mathrm{Mod}(R^\Lambda(\beta)) \to \mathrm{Mod}(R^\Lambda(\beta+\alpha_i))$.

Let us denote by x the endomorphism of $\mathbf{1}_\beta \mathsf{E}_i^\Lambda$ represented by the left multiplication of x_{n+1} on $e(\beta, i)R^\Lambda(\beta+\alpha_i)$ and by τ the endomorphism of $\mathbf{1}_\beta \mathsf{E}_i^\Lambda \mathsf{E}_i^\Lambda \colon \operatorname{Mod}(R^\Lambda(\beta+2\alpha_i)) \to \operatorname{Mod}(R^\Lambda(\beta))$ represented by the left multiplication of τ_{n+1} on $e(\beta,i)R^\Lambda(\beta+\alpha_i)\otimes_{R^\Lambda(\beta+\alpha_i)} e(\beta+\alpha_i, i)R^\Lambda(\beta+2\alpha_i) \simeq e(\beta, i^2)R^\Lambda(\beta+2\alpha_i)$. Similarly the endomorphism x of $\mathsf{F}_i^\Lambda \mathbf{1}_\beta$ is represented by the right multiplications of x_{n+1} on $R^\Lambda(\beta+\alpha_i)e(\beta,i)$ and the endomorphism τ of $\mathsf{F}_i^\Lambda \mathsf{F}_i^\Lambda \mathbf{1}_\beta \colon \operatorname{Mod}(R^\Lambda(\beta)) \to \operatorname{Mod}(R^\Lambda(\beta+2\alpha_i))$ is represented by the right multiplication of τ_{n+1} on $R^\Lambda(\beta+2\alpha_i)e(\beta+\alpha_i,i)\otimes_{R^\Lambda(\beta+\alpha_i)} R^\Lambda(\beta+\alpha_i)e(\beta,i) \simeq R^\Lambda(\beta+2\alpha_i)e(\beta,i^2)$. Then $x \in \operatorname{End}(\mathsf{F}_i^\Lambda \mathbf{1}_\beta)$ and $x \in \operatorname{End}(\mathbf{1}_\beta \mathsf{E}_i^\Lambda)$ are dual to each other and $\tau \in \operatorname{End}(\mathsf{F}_i^\Lambda \mathsf{F}_i^\Lambda \mathbf{1}_\beta)$ and $\tau \in \operatorname{End}(\mathbf{1}_\beta \mathsf{E}_i^\Lambda \mathsf{E}_i^\Lambda)$ are dual to each other.

By the adjunction, $\tau \in \operatorname{End}(\mathsf{E}_i^\Lambda \mathsf{E}_i^\Lambda)$ induces a morphism

$$\sigma \colon \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_\beta \longrightarrow \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathbf{1}_\beta. \tag{3.4}$$

It is represented by the morphism

$$R^\Lambda(\beta)e(\beta-\alpha_i, i) \otimes_{R^\Lambda(\beta-\alpha_i)} e(\beta-\alpha_i, i)R^\Lambda(\beta) \to e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)$$

given by $x \otimes y \longrightarrow x\tau_n y$.

The following theorem was formulated as one of the axioms for the categorification of representations of quantum groups ([6, 12, 14, 17]), and proved in [7] for an arbitrary Khovanov-Lauda-Rouquier algebra.

Theorem 3.4 ([7])**.** *Set $\lambda := \Lambda - \beta$ and $\lambda_i := \langle h_i, \lambda\rangle$.*

(a) *Assume $\lambda_i := \langle h_i, \lambda\rangle \geq 0$. The the morphism of endofunctors on* $\operatorname{Mod}(R^\Lambda(\beta))$

$$\rho \ : \ \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_\beta \oplus \bigoplus_{k=0}^{\lambda_i - 1} \mathbf{k}x^k \otimes \mathbf{1}_\beta \longrightarrow \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathbf{1}_\beta$$

is an isomorphism. Here $\mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_\beta \to \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathbf{1}_\beta$ is given by σ, and $\mathbf{k}x^k \otimes \mathbf{1}_\beta \to \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_\beta$ is given by $(x^k \mathsf{F}_i^\Lambda) \circ \eta = (\mathsf{E}_i^\Lambda x^k) \circ \eta \colon \mathbf{1}_\beta \to \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathbf{1}_\beta$.

(b) *Assume that $\lambda_i \leq 0$. Then the morphism*

$$\rho \ : \ \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_\beta \longrightarrow \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathbf{1}_\beta \oplus \bigoplus_{k=0}^{-\lambda_i - 1} \mathbf{k}(x^{-1})^k \otimes \mathbf{1}_\beta$$

is an isomorphisms. Here $\mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_\beta \to \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathbf{1}_\beta$ is given by σ, and $\mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_\beta \to \mathbf{k}(x^{-1})^k \otimes \mathbf{1}_\beta$ is given by $\varepsilon \circ (x^k \mathsf{E}_i^\Lambda) = \varepsilon \circ (\mathsf{F}_i^\Lambda x^k) \colon \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_\beta \to \mathbf{1}_\beta$.

In the theorem, x^k in $\mathbf{k}x^k$ and $(x^{-1})^k$ in $\mathbf{k}(x^{-1})^k$ are a symbolical basis of a free $\mathbf{k}$-module.

Now let us define the morphism $\widehat{\eta} \colon \mathbf{1}_\beta \to \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_\beta$ as follows.

(i) If $\lambda_i := \langle h_i, \lambda\rangle \geq 0$, then $\widehat{\eta}$ is given by the commutativity of

$$\begin{tikzcd}
\mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_\beta & \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_\beta \oplus \bigoplus_{k=0}^{\lambda_i-1} \mathbf{k}x^k \otimes \mathbf{1}_\beta \arrow[l, "\text{projection}"'] \arrow[d, "\rho", "\wr"'] \\
\mathbf{1}_\beta \arrow[u, "-\widehat{\eta}"'] \arrow[r, "x^{\lambda_i} F\circ\eta"] & \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda .
\end{tikzcd}$$

Here the top horizontal arrow is the projection. The minus sign in front of $\widehat{\eta}$ should be noted.

(ii) If $\lambda_i < 0$, then $\widehat{\eta}$ is defined as the composition

$$\begin{tikzcd}
\mathbf{1}_\beta \arrow[r, "\widehat{\eta}"] \arrow[d, "\wr"'] & \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_\beta \arrow[d, "\rho", "\wr"'] \\
\mathbf{k}(x^{-1})^{-\lambda_i-1} \otimes \mathbf{1}_\beta \arrow[r, hook] & \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathbf{1}_\beta \oplus \bigoplus_{k=0}^{-\lambda_i-1} \mathbf{k}(x^{-1})^k \otimes \mathbf{1}_\beta .
\end{tikzcd}$$

Here the bottom horizontal arrow is the canonical inclusion and the left vertical arrow is derived from $\mathbf{k} \xrightarrow{\sim} \mathbf{k}(x^{-1})^{-\lambda_i-1}$ ($1 \mapsto (x^{-1})^{-\lambda_i-1}$).

The morphism $\widehat{\varepsilon}$: $\mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathbf{1}_\beta \to \mathbf{1}_\beta$ is defined as follows.

(i) If $\lambda_i > 0$, then $\widehat{\varepsilon}$ is defined as the composition

$$\begin{tikzcd}
\mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_\beta \oplus \bigoplus_{k=0}^{\lambda_i-1} \mathbf{k}x^k \otimes \mathbf{1}_\beta \arrow[r, "\text{projection}"] \arrow[d, "\rho", "\wr"'] & \mathbf{k}x^{\lambda_i-1} \otimes \mathbf{1}_\beta \arrow[d, "\wr"'] \\
\mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathbf{1}_\beta \arrow[r, "\widehat{\varepsilon}"] & \mathbf{1}_\beta .
\end{tikzcd}$$

Here the top horizontal arrow is the canonical projection and the right vertical arrow is induced by $x^{\lambda_i-1} \mapsto 1$.

(ii) If $\lambda_i \leq 0$, then $\widehat{\varepsilon}$ is defined as the composition

$$\begin{tikzcd}
\mathbf{1}_\beta & \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_\beta \arrow[l, "\varepsilon\circ(x^{-\lambda_i}\mathsf{E}_i^\Lambda)"'] \arrow[d, "\rho", "\wr"'] \\
\mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \arrow[u, "\widehat{\varepsilon}"'] \arrow[r, hook] & \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathbf{1}_\beta \oplus \bigoplus_{k=0}^{-\lambda_i-1} \mathbf{k}(x^{-1})^k \otimes \mathbf{1}_\beta .
\end{tikzcd}$$

Here the bottom horizontal arrow is the canonical inclusion.

Now our main result can be stated as follows.

Theorem 3.5. *The pair* $(\mathsf{E}_i^\Lambda, \mathsf{F}_i^\Lambda)$ *is an adjoint pair with* $(\widehat{\eta}, \widehat{\varepsilon})$ *as adjunction. Namely the compositions* $\mathsf{E}_i^\Lambda \xrightarrow{\mathsf{E}_i^\Lambda \widehat{\eta}} \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \xrightarrow{\widehat{\varepsilon}\, \mathsf{E}_i^\Lambda} \mathsf{E}_i^\Lambda$ *and* $\mathsf{F}_i^\Lambda \xrightarrow{\widehat{\eta}\, \mathsf{F}_i^\Lambda} \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \xrightarrow{\mathsf{F}_i^\Lambda \widehat{\varepsilon}} \mathsf{F}_i^\Lambda$ *are equal to the identities.*

We shall prove this theorem in the rest of the paper.

As mentioned in Introduction, we note that Rouquier ([17]) proved that there exists a morphism $\varepsilon' \colon \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \to \mathbf{1}_\beta$ such that $\mathsf{E}_i^\Lambda \xrightarrow{\mathsf{E}_i^\Lambda \widehat{\eta}} \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \xrightarrow{\varepsilon'\, \mathsf{E}_i^\Lambda} \mathsf{E}_i^\Lambda$ and $\mathsf{F}_i^\Lambda \xrightarrow{\widehat{\eta}\, \mathsf{F}_i^\Lambda} \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \xrightarrow{\mathsf{F}_i^\Lambda \varepsilon'} \mathsf{F}_i^\Lambda$ are equal to the identities. Of cource, such an ε' is uniquely determined. However, the identity $\varepsilon' = \widehat{\varepsilon}$ is non trivial.

4. Proof of Theorem 3.5

4.1. We shall first prove that the composition $\mathbf{1}_\beta \mathsf{E}_i^\Lambda \xrightarrow{\mathsf{E}_i^\Lambda \widehat{\eta}} \mathbf{1}_\beta \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \xrightarrow{\widehat{\varepsilon}\, \mathsf{E}_i^\Lambda} \mathbf{1}_\beta \mathsf{E}_i^\Lambda$ is equal to the identity. Here $\beta \in Q^+$ with $\mathrm{ht}(\beta) = n$ and we set $\lambda := \Lambda - \beta$ and $\lambda_i := \langle h_i, \lambda \rangle$.

4.1.1. $\lambda_i \geq 2$ *Case.* We shall first assume that $\lambda_i \geq 2$. Then the composition $\mathbf{1}_\beta \mathsf{E}_i^\Lambda \xrightarrow{\mathsf{E}_i^\Lambda \widehat{\eta}} \mathbf{1}_\beta \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \xrightarrow{\widehat{\varepsilon}\, \mathsf{E}_i^\Lambda} \mathbf{1}_\beta \mathsf{E}_i^\Lambda$ can be described by the kernel bimodules as follows. The morphism $\mathbf{1}_\beta \mathsf{E}_i^\Lambda \xrightarrow{\mathsf{E}_i^\Lambda \widehat{\eta}} \mathbf{1}_\beta \mathsf{E}_i^\Lambda (\mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_{\beta+\alpha_i})$ is given by the $\big(R^\Lambda(\beta), R^\Lambda(\beta+\alpha_i)\big)$-bilinear homomorphism:

$$
\begin{array}{c}
e(\beta,i)R^\Lambda(\beta+\alpha_i) \\
\Big\downarrow {\scriptstyle -x_{n+2}^{\lambda_i-2}} \\
e(\beta,i^2)R^\Lambda(\beta+2\alpha_i)e(\beta+\alpha_i,i) \\
{\scriptstyle \rho}\Big\uparrow \wr \\
e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i) \underset{R^\Lambda(\beta)}{\otimes} \mathbf{k}\tau_{n+1} \otimes e(\beta,i)R^\Lambda(\beta+\alpha_i) \\
\oplus \displaystyle\bigoplus_{k=0}^{k=\lambda_i-3} \mathbf{k}x_{n+2}^k \otimes e(\beta,i)R^\Lambda(\beta+\alpha_i) \\
\Big\downarrow \text{projection} \\
e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i) \underset{R^\Lambda(\beta)}{\otimes} \mathbf{k}\tau_{n+1} \otimes e(\beta,i)R^\Lambda(\beta+\alpha_i).
\end{array}
$$

The morphism $(\mathbf{1}_\beta \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda)\mathsf{E}_i^\Lambda \xrightarrow{\widehat{\varepsilon}\,\mathsf{E}_i^\Lambda} \mathbf{1}_\beta \mathsf{E}_i^\Lambda$ is given by the $\big(R^\Lambda(\beta), R^\Lambda(\beta+\alpha_i)\big)$-bilinear homomorphism:

$$
\begin{array}{c}
e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i) \underset{R^\Lambda(\beta)}{\otimes} \mathbf{k}\tau_{n+1} \otimes e(\beta,i)R^\Lambda(\beta+\alpha_i) \\
\rho \big\uparrow \wr \\
\Big(R^\Lambda(\beta)e(\beta-\alpha_i,i) \underset{R^\Lambda(\beta-\alpha_i)}{\otimes} \mathbf{k}\tau_n \otimes e(\beta-\alpha_i,i)R^\Lambda(\beta) \\
\oplus \bigoplus_{k=0}^{\lambda_i-1} \mathbf{k}x_{n+1}^k \otimes R^\Lambda(\beta)\Big) \otimes_{R^\Lambda(\beta)} \mathbf{k}\tau_{n+1} \otimes e(\beta,i)R^\Lambda(\beta+\alpha_i) \\
\big\downarrow \text{projection} \\
\mathbf{k}x_{n+1}^{\lambda_i-1} \otimes \mathbf{k}\tau_{n+1} \otimes e(\beta,i)R^\Lambda(\beta+\alpha_i) \\
\big\downarrow \wr \\
e(\beta,i)R^\Lambda(\beta+\alpha_i).
\end{array}
$$

Hence in order to see that the composition is the identity, it is enough to show the inclusion

$$
\begin{aligned}
&x_{n+2}^{\lambda_i-2}e(\beta,i^2) + x_{n+1}^{\lambda_i-1}\tau_{n+1}e(\beta,i^2) \\
&\qquad \in R^\Lambda(\beta)\tau_n\tau_{n+1}e(\beta-\alpha_i,i^3)R^\Lambda(\beta+\alpha_i) \\
&\qquad\qquad + \textstyle\sum_{k=0}^{\lambda_i-2} x_{n+1}^k\tau_{n+1}e(\beta,i^2)R^\Lambda(\beta+\alpha_i) \\
&\qquad\qquad\qquad + \textstyle\sum_{k=0}^{\lambda_i-3} x_{n+2}^k e(\beta,i^2)R^\Lambda(\beta+\alpha_i)
\end{aligned}
\tag{4.1}
$$

as an element of $e(\beta,i^2)R^\Lambda(\beta+2\alpha_i)e(\beta+\alpha_i,i)$.

This inclusion is proved in § 5.

4.1.2. Now let us treat the case $\lambda_i = 1$.

The morphism $\mathbf{1}_\beta \mathsf{E}_i^\Lambda \xrightarrow{\mathsf{E}_i^\Lambda \widehat{\eta}} \mathbf{1}_\beta \mathsf{E}_i^\Lambda (\mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_{\beta+\alpha_i})$ is given by

$$\begin{array}{c}
e(\beta,i)R^\Lambda(\beta+\alpha_i)\\
\Big\downarrow{\scriptstyle\text{inclusion}}\\
e(\beta,i^2)R^\Lambda(\beta+2\alpha_i)e(\beta+\alpha_i,i)\oplus e(\beta,i)R^\Lambda(\beta+\alpha_i)\\
\wr\Big\uparrow{\scriptstyle\rho=\Sigma\oplus\widetilde{E}}\\
e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)\otimes_{R^\Lambda(\beta)}e(\beta,i)R^\Lambda(\beta+\alpha_i)\ni u.
\end{array}$$

(See below for Σ and $\widetilde{E}$.)

The morphism $(\mathbf{1}_\beta \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda)\mathsf{E}_i^\Lambda \xrightarrow{\widehat{\varepsilon}\,\mathsf{E}_i^\Lambda} \mathbf{1}_\beta \mathsf{E}_i^\Lambda$ is given by

$$\begin{array}{c}
e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)\otimes_{R^\Lambda(\beta)}e(\beta,i)R^\Lambda(\beta+\alpha_i)\ni u\\
\wr\Big\uparrow{\scriptstyle\rho}\\
\Big(R^\Lambda(\beta)e(\beta-\alpha_i,i)\underset{R^\Lambda(\beta-\alpha_i)}{\otimes}e(\beta-\alpha_i,i)R^\Lambda(\beta)\oplus R^\Lambda(\beta)\Big)\\
\qquad\qquad\qquad\otimes_{R^\Lambda(\beta)}e(\beta,i)R^\Lambda(\beta+\alpha_i)\\
\Big\downarrow{\scriptstyle\text{projection}}\\
e(\beta,i)R^\Lambda(\beta+\alpha_i).
\end{array}$$

Hence in order to see that the composition is the identity, it is enough to show the following existence :

(4.2) There exists $u\in e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)\otimes_{R^\Lambda(\beta)}e(\beta,i)R^\Lambda(\beta+\alpha_i)$ such that
(a) $\Sigma(u)=0$,
(b) $\widetilde{E}(u)=e(\beta,i)$,
(c) $u-e(\beta,i)\otimes e(\beta,i)\in\big(R^\Lambda(\beta)e(\beta-\alpha_i,i^2)\tau_n e(\beta-\alpha_i,i^2)R^\Lambda(\beta)\big)\otimes_{R^\Lambda(\beta)}e(\beta,i)R^\Lambda(\beta+\alpha_i)$.

Here

$$\Sigma\;:\;e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)\underset{R^\Lambda(\beta)}{\otimes}e(\beta,i)R^\Lambda(\beta+\alpha_i)\longrightarrow e(\beta,i^2)R^\Lambda(\beta+2\alpha_i)e(\beta+\alpha_i,i)$$

is given by $\Sigma(a\otimes b)=a\tau_{n+1}b$, and

$$\widetilde{E}\quad:\quad e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)\underset{R^\Lambda(\beta)}{\otimes}e(\beta,i)R^\Lambda(\beta+\alpha_i)\to e(\beta,i)R^\Lambda(\beta+\alpha_i)$$

is given by $\widetilde{E}(a\otimes b)=ab$.

The proof of (4.2) will be given in § 5.

4.1.3. Now we assume that $\lambda_i\le 0$. Then the composition $\mathbf{1}_\beta\mathsf{E}_i^\Lambda\xrightarrow{\mathsf{E}_i^\Lambda\,\widehat{\eta}}\mathbf{1}_\beta\mathsf{E}_i^\Lambda\mathsf{F}_i^\Lambda\mathsf{E}_i^\Lambda\xrightarrow{\widehat{\varepsilon}\,\mathsf{E}_i^\Lambda}\mathbf{1}_\beta\mathsf{E}_i^\Lambda$ can be described by the kernel bimodules as follows.

The morphism $\mathbf{1}_\beta\mathsf{E}_i^\Lambda\xrightarrow{\mathsf{E}_i^\Lambda\,\widehat{\eta}}\mathbf{1}_\beta\mathsf{E}_i^\Lambda(\mathsf{F}_i^\Lambda\mathsf{E}_i^\Lambda\mathbf{1}_{\beta+\alpha_i})$ is given by

$$\begin{array}{c}
e(\beta,i)R^\Lambda(\beta+\alpha_i)\\
\big\downarrow\wr\\
\mathbf{k}(x_{n+1}^{-1})^{1-\lambda_i}\otimes e(\beta,i)R^\Lambda(\beta+\alpha_i)\\
\big\downarrow\text{inclusion}\\
e(\beta,i^2)R^\Lambda(\beta+2\alpha_i)e(\beta+\alpha_i,i)\oplus\displaystyle\bigoplus_{k=0}^{1-\lambda_i}\mathbf{k}(x_{n+1}^{-1})^k\otimes e(\beta,i)R^\Lambda(\beta+\alpha_i)\\
{\scriptstyle\rho=f\oplus\oplus_k H'_k}\big\uparrow\wr\\
e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)\otimes_{R^\Lambda(\beta)}e(\beta,i)R^\Lambda(\beta+\alpha_i).
\end{array}$$

The morphism $(\mathbf{1}_\beta\mathsf{E}_i^\Lambda\mathsf{F}_i^\Lambda)\mathsf{E}_i^\Lambda\xrightarrow{\widehat{\varepsilon}\,\mathsf{E}_i^\Lambda}\mathbf{1}_\beta\mathsf{E}_i^\Lambda$ is given by

$$\begin{array}{c}
e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)\otimes_{R^\Lambda(\beta)}e(\beta,i)R^\Lambda(\beta+\alpha_i)\\
\big\downarrow\text{inclusion}\\
e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)\underset{R^\Lambda(\beta)}{\otimes}e(\beta,i)R^\Lambda(\beta+\alpha_i)\\
\oplus\displaystyle\bigoplus_{k=0}^{-\lambda_i-1}\mathbf{k}(x_n^{-1})^k\otimes e(\beta,i)R^\Lambda(\beta+\alpha_i)\\
{\scriptstyle\rho=g\oplus\underset{k}{\oplus}T_k}\big\uparrow\wr\\
R^\Lambda(\beta)e(\beta-\alpha_i,i)\otimes_{R^\Lambda(\beta-\alpha_i)}e(\beta-\alpha_i,i^2)R^\Lambda(\beta+\alpha_i)\ni v\\
\big\downarrow T_{-\lambda_i}\\
e(\beta,i)R^\Lambda(\beta+\alpha_i).
\end{array}$$

Hence in order to see that the composition is the identity, it is enough to show the following:

$$
(4.3)\quad \begin{cases} \text{There exists } v \in R^\Lambda(\beta)e(\beta-\alpha_i)\otimes_{R^\Lambda(\beta-\alpha_i)} e(\beta,i)R^\Lambda(\beta+\alpha_i) \text{ such that} \\ \text{we have} \\ \qquad \text{(a) } T_k(v)=0 \text{ for } 0\le k\le -\lambda_i-1 \\ \qquad \text{(b) } T_{-\lambda_i}(v)=e(\beta,i), \\ \qquad \text{(c) } G(v)=0, \\ \qquad \text{(d) } H_k(v)=0 \text{ for } 0\le k\le -\lambda_i, \\ \qquad \text{(e) } H_{1-\lambda_i}(v)=e(\beta,i). \end{cases}
$$

Here the homomorphism

$$f\colon e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)\underset{R^\Lambda(\beta)}{\otimes} e(\beta,i)R^\Lambda(\beta+\alpha_i)\to e(\beta,i^2)R^\Lambda(\beta+2\alpha_i)e(\beta+\alpha_i,i)$$

is given by $f(a\otimes b)=a\tau_{n+1}b$,

$$
\begin{aligned}
H'_k\colon e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)&\underset{R^\Lambda(\beta)}{\otimes} e(\beta,i)R^\Lambda(\beta+\alpha_i)\\
&\longrightarrow \mathbf{k}(x_{n+1}^{-1})^k\otimes e(\beta,i)R^\Lambda(\beta+\alpha_i)\simeq e(\beta,i)R^\Lambda(\beta+\alpha_i)
\end{aligned}
$$

is given by $H'_k(a\otimes b)=ax_{n+1}^k b$,

$$g\colon R^\Lambda(\beta)e(\beta-\alpha_i)\underset{R^\Lambda(\beta-\alpha_i)}{\otimes} e(\beta,i)R^\Lambda(\beta+\alpha_i)\to e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)\underset{R^\Lambda(\beta)}{\otimes} e(\beta,i)R^\Lambda(\beta+\alpha_i)$$

is given by $g(a\otimes b)=a\tau_n\otimes b$,

$$T_k\colon R^\Lambda(\beta)e(\beta-\alpha_i)\underset{R^\Lambda(\beta-\alpha_i)}{\otimes} e(\beta,i)R^\Lambda(\beta+\alpha_i)\to e(\beta,i)R^\Lambda(\beta+\alpha_i)$$

is given by $T_k(a\otimes b)=ax_n^k b$,

$$G=f\circ g\colon R^\Lambda(\beta)e(\beta-\alpha_i)\underset{R^\Lambda(\beta-\alpha_i)}{\otimes} e(\beta,i)R^\Lambda(\beta+\alpha_i)\to e(\beta,i^2)R^\Lambda(\beta+2\alpha_i)e(\beta+\alpha_i,i)$$

is given by $G(a\otimes b)=a\tau_n\tau_{n+1}b$, and

$$H_k=H'_k\circ g\colon R^\Lambda(\beta)e(\beta-\alpha_i)\underset{R^\Lambda(\beta-\alpha_i)}{\otimes} e(\beta,i)R^\Lambda(\beta+\alpha_i)\to e(\beta,i)R^\Lambda(\beta+\alpha_i)$$

is given by $H_k(a\otimes b)=a\tau_n x_{n+1}^k b$.

The statement (4.3) is proved in § 5.

4.2. Let us show that the composition $\mathsf{F}_i^\Lambda \xrightarrow{\widehat{\eta}\,\mathsf{F}_i^\Lambda} \mathsf{F}_i^\Lambda\mathsf{E}_i^\Lambda\mathsf{F}_i^\Lambda \xrightarrow{\mathsf{F}_i^\Lambda\,\widehat{\varepsilon}} \mathsf{F}_i^\Lambda$ is equal to the identity by reducing it to the corresponding statement for $\mathsf{E}_i^\Lambda \xrightarrow{\mathsf{E}_i^\Lambda\,\widehat{\eta}} \mathsf{E}_i^\Lambda\mathsf{F}_i^\Lambda\mathsf{E}_i^\Lambda \xrightarrow{\widehat{\varepsilon}\,\mathsf{E}_i^\Lambda} \mathsf{E}_i^\Lambda$.

Let us recall that ψ is the anti-involution of $R^\Lambda(\beta)$ sending the generators $e(\nu)$, x_k, τ_k to themselves. For an $R^\Lambda(\beta)$-module M, we denote by M^ψ the $R^\Lambda(\beta)^{\rm opp}$-module induced by ψ from M, where $R^\Lambda(\beta)^{\rm opp}$ is the opposite ring of $R^\Lambda(\beta)$. We define the bifunctor

$$\Psi_\beta\colon \operatorname{Mod}(R^\Lambda(\beta)) \times \operatorname{Mod}(R^\Lambda(\beta)) \longrightarrow \operatorname{Mod}(\mathbf{k})$$

by

$$\Psi_\beta(M,N) := M^\psi \underset{R^\Lambda(\beta)}{\otimes} N.$$

We have a functorial isomorphism

$$\Psi_\beta(M,N) \simeq \Psi_\beta(N,M) \quad \text{in } M, N \in \operatorname{Mod}(R^\Lambda(\beta)).$$

For two $\mathbf{k}$-linear categories $\mathscr{C}$ and $\mathscr{C}'$, let us denote by $\operatorname{Fct}_{\mathbf{k}}(\mathscr{C},\mathscr{C}')$ be the category of $\mathbf{k}$-linear functors from $\mathscr{C}$ to $\mathscr{C}'$. Then Ψ_β induces a functor

$$\mathsf{H}_\beta\colon \operatorname{Mod}(R^\Lambda(\beta)) \longrightarrow \operatorname{Fct}_{\mathbf{k}}\bigl(\operatorname{Mod}(R^\Lambda(\beta)), \operatorname{Mod}(\mathbf{k})\bigr)$$

by assigning to $M \in \operatorname{Mod}(R^\Lambda(\beta))$ the functor $N \mapsto \Psi_\beta(M,N)$. The following lemma similar to Yoneda lemma is easily proved, and its proof is omitted.

Lemma 4.1. *The functor H_β is fully faithful.*

For $\beta, \beta' \in Q^+$ and a pair of $\mathbf{k}$-linear functors $F\colon \operatorname{Mod}(R^\Lambda(\beta)) \to \operatorname{Mod}(R^\Lambda(\beta'))$ and $G\colon \operatorname{Mod}(R^\Lambda(\beta')) \to \operatorname{Mod}(R^\Lambda(\beta))$, we say that F and G are Ψ-*adjoint* or G is a Ψ-*adjoint* of F if there exists a functorial isomorphism

$$\Psi_{\beta'}(F(M),N) \simeq \Psi_\beta(M,G(N)) \quad \text{in } M \in \operatorname{Mod}(R^\Lambda(\beta)) \text{ and } N \in \operatorname{Mod}(R^\Lambda(\beta')).$$

For a given F, a Ψ-adjoint of F is unique up to a unique isomorphism if it exists. We shall denote by $F^\vee$ the Ψ-adjoint of F (if it exists).

If $\operatorname{Mod}(R^\Lambda(\beta)) \xrightarrow{F} \operatorname{Mod}(R^\Lambda(\beta')) \xrightarrow{F'} \operatorname{Mod}(R^\Lambda(\beta''))$ are functors which admit Ψ-adjoint, then $F^\vee \circ F'^\vee$ is a Ψ-adjoint of $F' \circ F$.

Now let $F_k\colon \operatorname{Mod}(R^\Lambda(\beta)) \to \operatorname{Mod}(R^\Lambda(\beta'))$ $(k=1,2)$ be two functors. Then Lemma 4.1 implies

$$\operatorname{Hom}(F_1,F_2) \simeq \operatorname{Hom}(F_1^\vee,F_2^\vee).$$

For $f \in \mathrm{Hom}(F_1, F_2)$, the corresponding morphism in $\mathrm{Hom}(F_1^\vee, F_2^\vee)$ is called the Ψ-adjoint of f and we denote it by $f^\vee$. By the definition we have a commutative diagram

$$\begin{array}{ccc} \Psi_{\beta'}(F_1(M), N) & \xrightarrow{\sim} & \Psi_\beta(M, F_1^\vee(N)) \\ \Big\downarrow f & & \Big\downarrow f^\vee \\ \Psi_{\beta'}(F_2(M), N) & \xrightarrow{\sim} & \Psi_\beta(M, F_2^\vee(N)). \end{array}$$

Then $(f \circ g)^\vee = f^\vee \circ g^\vee$ for $F_1 \xrightarrow{g} F_2 \xrightarrow{f} F_3$.

The following lemma is elementary and its proof is omitted.

Lemma 4.2. (i) *Let K be a $\big(R^\Lambda(\beta'), R^\Lambda(\beta)\big)$-bimodule and the functor $F\colon \mathrm{Mod}(R^\Lambda(\beta)) \to \mathrm{Mod}(R^\Lambda(\beta'))$ is given by $K \otimes_{R^\Lambda(\beta)} \bullet$. Then F admits a Ψ-adjoint.*

(ii) *Conversely if a $\mathbf{k}$-linear functor $F\colon \mathrm{Mod}(R^\Lambda(\beta)) \to \mathrm{Mod}(R^\Lambda(\beta'))$ admits a Ψ-adjoint, then F is isomorphic to $F(R^\Lambda(\beta)) \otimes_{R^\Lambda(\beta)} \bullet$, and $F^\vee(R^\Lambda(\beta')) \simeq F(R^\Lambda(\beta))^\psi$ as $\big(R^\Lambda(\beta), R^\Lambda(\beta')\big)$-bimodules.*

We can easily see that E_i^Λ and F_i^Λ are Ψ-adjoint. Moreover, $x \in \mathrm{End}(\mathsf{E}_i^\Lambda)$ and $x \in \mathrm{End}(\mathsf{F}_i^\Lambda)$, $\tau \in \mathrm{End}(\mathsf{E}_i^\Lambda \circ \mathsf{E}_i^\Lambda)$ and $\tau \in \mathrm{End}(\mathsf{F}_i^\Lambda \circ \mathsf{F}_i^\Lambda)$ are Ψ-adjoint, respectively. We can also see that $\eta \in \mathrm{Hom}(\mathbf{1}_\beta, \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathbf{1}_\beta)$ is a Ψ-adjoint of itself. Similarly $\varepsilon \in \mathrm{Hom}(\mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_\beta, \mathbf{1}_\beta)$ $\sigma \in \mathrm{Hom}(\mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda, \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda)$ are Ψ-adjoint of themselves. Note that $\mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda$ and $\mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda$ are a Ψ-adjoint of themselves. Hence $\widehat{\eta}$ and $\widehat{\varepsilon}$ are also a Ψ-adjoint of themselves.

Therefore $\mathsf{F}_i^\Lambda \xrightarrow{\widehat{\eta}\,\mathsf{F}_i^\Lambda} \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \xrightarrow{\mathsf{F}_i^\Lambda\,\widehat{\varepsilon}} \mathsf{F}_i^\Lambda$ is a Ψ-adjoint of $\mathsf{E}_i^\Lambda \xrightarrow{\mathsf{E}_i^\Lambda\,\widehat{\eta}} \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \xrightarrow{\widehat{\varepsilon}\,\mathsf{E}_i^\Lambda} \mathsf{E}_i^\Lambda$. Hence if the composition of $\mathsf{E}_i^\Lambda \xrightarrow{\mathsf{E}_i^\Lambda\,\widehat{\eta}} \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \xrightarrow{\widehat{\varepsilon}\,\mathsf{E}_i^\Lambda} \mathsf{E}_i^\Lambda$ is the identity, then the composition of $\mathsf{F}_i^\Lambda \xrightarrow{\widehat{\eta}\,\mathsf{F}_i^\Lambda} \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \xrightarrow{\mathsf{F}_i^\Lambda\,\widehat{\varepsilon}} \mathsf{F}_i^\Lambda$ is also the identity.

Thus we have reduced Theorem 3.5 to the three statements (4.1), (4.2) and (4.3), which will be proved in the next section.

5. Proof of the three statements

5.1. **Intertwiner.** Let us set $\varphi_a \in R(n)$ as follows:

$$\begin{aligned} \varphi_a e(\nu) &= (x_a \tau_a - \tau_a x_a) e(\nu) = (\tau_a x_{a+1} - x_{a+1} \tau_a) e(\nu) \\ &= \big((x_a - x_{a+1})\tau_a + 1\big) e(\nu) = \big(\tau_a (x_{a+1} - x_a) - 1\big) e(\nu) \end{aligned}$$

if $\nu_a = \nu_{a+1}$ and $\varphi_a e(\nu) = \tau_a e(\nu)$ if $\nu_a \neq \nu_{a+1}$. It is called the *intertwiner.*

The following lemma is well-known (for example, it easily follows from by the polynomial representation of Khovanov-Lauda-Rouquier algebras ([10, Proposition 2.3], [17, Proposition 3.12]).

Lemma 5.1. (i) *For $1 \le a \le n$, we have*

$$x_{s_a(b)}\varphi_a = \varphi_a x_b (1 \le b \le n+1).$$

(ii) $\varphi_a^2 = Q_{a,a+1} + e_{a,a+1}$.
(iii) $\{\varphi_k\}_{1 \le k < n}$ *satisfies the braid relation.*
(iv) *For $w \in S_n$ and $1 \le k < n$, if $w(k+1) = w(k)+1$, then $\varphi_w \tau_k = \tau_{w(k)}\varphi_w$.*
(v) *In particular*

$$\begin{aligned}
&\tau_a\varphi_{a+1}\varphi_a = \varphi_{a+1}\varphi_a\tau_{a+1}, \quad \textit{and} \quad \tau_{a+1}\varphi_a\varphi_{a+1} = \varphi_a\varphi_{a+1}\tau_a,\\
&\tau_k\varphi_a\cdots\varphi_{n-1} = \varphi_a\cdots\varphi_{n-1}\tau_{k-1} \quad \textit{for } a < k \le n-1.
\end{aligned}$$

5.2. Let us take $\beta \in Q^+$ with $\mathrm{ht}(\beta) = n$ and $i \in I$. Let p be the number of times that α_i appears in β. The following lemma is proved by repeated use of Theorem 2.2.

Lemma 5.2. *We have*

$$\begin{aligned}
&e(\beta, i^2)R(\beta+2\alpha_i)e(\beta+\alpha_i, i) \otimes_{R(\beta+\alpha_i)} R^\Lambda(\beta+\alpha_i)\\
&\qquad \simeq R(\beta)e(\beta-\alpha_i, i) \otimes \mathbf{k}\tau_n\tau_{n+1} \otimes_{R(\beta-\alpha_i)} e(\beta-\alpha_i, i)R^\Lambda(\beta+\alpha_i)\\
&\qquad\qquad \bigoplus \tau_{n+1}\mathbf{k}[x_{n+2}] \otimes e(\beta, i)R^\Lambda(\beta+\alpha_i)\\
&\qquad\qquad\qquad \bigoplus \mathbf{k}[x_{n+2}] \otimes e(\beta, i)R^\Lambda(\beta+\alpha_i).
\end{aligned}$$

Proof. We have

$$\begin{aligned}
&e(\beta, i^2)R(\beta+2\alpha_i)e(\beta+\alpha_i, i) \otimes_{R(\beta+\alpha_i)} R^\Lambda(\beta+\alpha_i)\\
&= e(\beta, i^2)\Big(R(\beta+\alpha_i)e(\beta, i)\tau_{n+1} \otimes_{R(\beta)} e(\beta, i)R(\beta+\alpha_i) \bigoplus \mathbf{k}[x_{n+2}] \otimes_{\mathbf{k}} R(\beta+\alpha_i)\Big)\\
&\qquad\qquad \otimes_{R(\beta+\alpha_i)} R^\Lambda(\beta+\alpha_i)\\
&= e(\beta, i^2)\Big(R(\beta)e(\beta-\alpha_i, i)\tau_n \otimes_{R(\beta-\alpha_i)} e(\beta-\alpha_i, i)R(\beta) \oplus \mathbf{k}[x_{n+1}] \otimes R(\beta)\Big)\tau_{n+1}\\
&\qquad\qquad \otimes_{R(\beta)} e(\beta, i)R^\Lambda(\beta+\alpha_i)\\
&\qquad \bigoplus \mathbf{k}[x_{n+2}] \otimes_{\mathbf{k}} R^\Lambda(\beta+\alpha_i)\\
&= e(\beta, i^2)R(\beta)e(\beta-\alpha_i, i)\tau_n\tau_{n+1} \otimes_{R(\beta-\alpha_i)} e(\beta-\alpha_i, i^2)R^\Lambda(\beta+\alpha_i)\\
&\qquad \bigoplus \mathbf{k}[x_{n+1}]\tau_{n+1} \otimes e(\beta, i^2)R^\Lambda(\beta+\alpha_i) \bigoplus \mathbf{k}[x_{n+2}] \otimes_{\mathbf{k}} e(\beta, i^2)R^\Lambda(\beta+\alpha_i).
\end{aligned}$$

Then the lemma follows from $\mathbf{k}[x_{n+1}]\tau_{n+1} \oplus \mathbf{k}[x_{n+2}] = \tau_{n+1}\mathbf{k}[x_{n+2}] \oplus \mathbf{k}[x_{n+2}]$. □

We set

$$\begin{aligned} K &:= e(\beta, i^2)R(\beta+2\alpha_i)e(\beta+\alpha_i, i) \otimes_{R(\beta+\alpha_i)} R^\Lambda(\beta+\alpha_i) \\ &\simeq \frac{e(\beta, i^2)R(\beta+2\alpha_i)e(\beta+\alpha_i, i)}{e(\beta, i^2)R(\beta+2\alpha_i)a^\Lambda(x_1)R(\beta+\alpha_i)e(\beta+\alpha_i, i)}. \end{aligned}$$

Then K is an $\bigl(e(\beta, i^2)R(\beta+2\alpha_i)e(\beta, i^2), R^\Lambda(\beta+\alpha_i) \otimes \mathbf{k}[x_{n+2}]\bigr)$-bimodule.

The preceding lemma says

$$\begin{aligned} K = R(\beta)\tau_n\tau_{n+1}e(\beta-\alpha_i, i^3)R^\Lambda(\beta+\alpha_i) + \tau_{n+1}\mathbf{k}[x_{n+2}]e(\beta, i^2)R^\Lambda(\beta+\alpha_i) \\ +\mathbf{k}[x_{n+2}]e(\beta, i^2)R^\Lambda(\beta+\alpha_i). \end{aligned}$$

We define the filtration $\{\Gamma_k\}_{k\in\mathbf{Z}}$ of K by

$$\Gamma_k = \begin{cases} 0 & \text{if } k < -1, \\ R(\beta)\tau_n\tau_{n+1}e(\beta-\alpha_i, i^3)R^\Lambda(\beta+\alpha_i) + e(\beta, i^2)\tau_{n+1}R^\Lambda(\beta+\alpha_i) & \text{if } k = -1, \\ \Gamma_{k-1} + e(\beta, i^2)x_{n+2}^k R^\Lambda(\beta+\alpha_i) + e(\beta, i^2)\tau_{n+1}x_{n+2}^{k+1}R^\Lambda(\beta+\alpha_i) & \text{if } k \geq 0. \end{cases}$$

Note that $\Gamma_k = \Gamma_{k-1} + e(\beta, i^2)x_{n+2}^k R^\Lambda(\beta+\alpha_i) + e(\beta, i^2)x_{n+1}^{k+1}\tau_{n+1}R^\Lambda(\beta+\alpha_i)$ for $k \geq 0$.

Recall that $\mathrm{Gr}_k^\Gamma K := \Gamma_k/\Gamma_{k-1}$. Then we have the following lemma that will be used frequently.

Lemma 5.3. *We have*

(i) *the Γ_k's are $(R(\beta), R^\Lambda(\beta+\alpha_i))$-bimodules,*
(ii) *$\Gamma_k x_{n+2} \subset \Gamma_{k+1}$ for any k,*
(iii) *the right multiplication of x_{n+2} induces an isomorphism $\mathrm{Gr}_k^\Gamma K \xrightarrow{\sim} \mathrm{Gr}_{k+1}^\Gamma K$ for any $k \geq 0$,*
(iv) $\mathrm{Ker}\bigl(x_{n+2} \colon \Gamma_{-1} \to \mathrm{Gr}_0^\Gamma K\bigr) = R(\beta)\tau_n\tau_{n+1}e(\beta-\alpha_i, i^3)R^\Lambda(\beta+\alpha_i)$.

Proof. (i) is obvious.
(ii) follows from

$$\tau_n\tau_{n+1}x_{n+2} = \tau_n(x_{n+1}\tau_{n+1}+1) = (x_n\tau_n+1)\tau_{n+1} + \tau_n. \tag{5.1}$$

(iii) follows from Lemma 5.2.

Let us prove (iv). Set $S := R(\beta)\tau_n\tau_{n+1}e(\beta-\alpha_i, i^3)R^\Lambda(\beta+\alpha_i)$. Then $Sx_{n+2} \subset \Gamma_{-1} + e(\beta, i^2)R^\Lambda(\beta+\alpha_i)$ by (5.1). The homomorphism $\Gamma_{-1}/S \to (\mathrm{Gr}_0^\Gamma K)/\bigl(e(\beta, i^2)R^\Lambda(\beta+\alpha_i)\bigr)$

is an isomorphism since it is isomorphic to $\mathbf{k}\tau_{n+1} \otimes e(\beta, i)R^\Lambda(\beta+\alpha_i) \xrightarrow[x_{n+2}]{\sim} \mathbf{k}\tau_{n+1}x_{n+2} \otimes e(\beta, i)R^\Lambda(\beta+\alpha_i)$. $\square$

As a corollary of the lemma above, we obtain the following

Lemma 5.4. *Let $m \in \mathbf{Z}$ and let $f(x_{n+2}) \in R^\Lambda(\beta+\alpha_i) \otimes \mathbf{k}[x_{n+2}]$ be a monic polynomial of degree $r \geq 0$ in x_{n+2} and $u \in K$. Assume that $uf(x_{n+2}) \in \Gamma_m$. Then we have*

(i) *if $m \geq r-1$, then $u \in \Gamma_{m-r}$,*
(ii) *$ux_{n+2}^k \in \Gamma_{\max(-1, m-r+k)}$ for any $k \geq 0$,*
(iii) *$uf(x_{n+2}) \equiv ux_{n+2}^r \mod \Gamma_{\max(-1,m-1)}$,*
(iv) *if $m < r-1$, then $u \in R(\beta)\tau_n\tau_{n+1}e(\beta-\alpha_i, i^3)R^\Lambda(\beta+\alpha_i)$.*

Proof. (i) It is enough to show that if $u \in \Gamma_k$ and $k > m-r$, then $u \in \Gamma_{k-1}$. For such a u we have $uf(x_{n+2}) \in \Gamma_m \subset \Gamma_{k+r-1}$, and the injectivity of $\operatorname{Gr}_k^\Gamma K \xrightarrow{f(x_{n+2})=x_{n+2}^r} \operatorname{Gr}_{r+k}^\Gamma K$ implies $u \in \Gamma_{k-1}$.

(ii) We have $ux_{n+2}^k f(x_{n+2}) \in \Gamma_{m+k} \subset \Gamma_{r+\max(-1,m-r+k)}$. Hence (i) implies that $ux_{n+2}^k \in \Gamma_{\max(-1,m-r+k)}$.

(iii) follows from (ii).

(iv) By (ii), $u, ux_{n+2} \in \Gamma_{-1}$. Then the assertion follows from Lemma 5.3 (iv). $\square$

Our goal of this subsection is to prove Proposition 5.7 below, and the following lemma is its starting point.

Lemma 5.5. *For $\nu \in I^\beta$ we have, as an element of K*

$$\begin{aligned}&\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\varphi_1\cdots\varphi_{n+1}e(\nu, i^2)\prod_{a\leq n,\ \nu_a=i}(x_a - x_{n+2})\\ &\equiv -\tau_{n+1}a^\Lambda(x_{n+2})\prod_{\nu_a\neq i}Q_{i,\nu_a}(x_{n+2}, x_a)e(\nu, i, i) \mod \Gamma_{-1}.\end{aligned}$$

Proof. We have $\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\varphi_1\cdots\varphi_{n+1} = \tau_{n+1}\cdots\tau_1\varphi_1\cdots\varphi_{n+1}a^\Lambda(x_{n+2})$. We shall show for $a \leq n$

$$\begin{aligned}&\tau_{n+1}\cdots\tau_a\varphi_a\cdots\varphi_{n+1}a^\Lambda(x_{n+2})e(\nu, i^2)\prod_{a\leq k\leq n,\, \nu_k=i}(x_k - x_{n+2})\prod_{k<a,\, \nu_k\neq i}Q_{i,\nu_a}(x_{n+2}, x_k)\\ &\equiv \tau_{n+1}\cdots\tau_{a+1}\varphi_{a+1}\cdots\varphi_{n+1}a^\Lambda(x_{n+2})e(\nu, i^2)\\ &\qquad \prod_{a+1\leq k\leq n,\, \nu_k=i}(x_k - x_{n+2})\prod_{k<a+1,\, \nu_k\neq i}Q_{i,\nu_a}(x_{n+2}, x_k).\end{aligned} \tag{5.2}$$

If $\nu_a \neq i$, it is obvious. Assume that $\nu_a = i$. Then

$$\begin{aligned}
&\tau_{n+1}\cdots\tau_a\varphi_a\cdots\varphi_{n+1}a^{\Lambda}(x_{n+2})e(\nu,i^2)(x_a-x_{n+2})\\
&=\tau_{n+1}\cdots\tau_a(x_{a+1}-x_a)\varphi_a\cdots\varphi_{n+1}a^{\Lambda}(x_{n+2})e(\nu,i^2)\\
&=\tau_{n+1}\cdots\tau_{a+1}(\varphi_a+1)\varphi_a\varphi_{a+1}\cdots\varphi_{n+1}a^{\Lambda}(x_{n+2})e(\nu,i^2)\\
&=\tau_{n+1}\cdots\tau_{a+1}(\varphi_a+1)\varphi_{a+1}\cdots\varphi_{n+1}a^{\Lambda}(x_{n+2})e(\nu,i^2)\\
&=\tau_{n+1}\cdots\tau_{a+1}\varphi_a\varphi_{a+1}\cdots\varphi_{n+1}a^{\Lambda}(x_{n+2})e(\nu,i^2)\\
&+\tau_{n+1}\cdots\tau_{a+1}\varphi_{a+1}\cdots\varphi_{n+1}a^{\Lambda}(x_{n+2})e(\nu,i^2).
\end{aligned}$$

We shall show that for any $f(x_{n+2})$ and $g = g(x_1,\ldots,x_n)$, we have

$$\tau_{n+1}\cdots\tau_{a+1}\varphi_a\varphi_{a+1}\cdots\varphi_{n+1}e(\nu,i^2)f(x_{n+2})g\in\Gamma_{-1}. \tag{5.3}$$

We have

$$\begin{aligned}
&\tau_{n+1}\cdots\tau_{a+1}\varphi_a\varphi_{a+1}\cdots\varphi_{n+1}e(\nu,i^2)f(x_{n+2})\\
&=\tau_{n+1}\cdots\tau_{a+1}f(x_a)\varphi_a\varphi_{a+1}\cdots\varphi_{n+1}e(\nu,i^2)\\
&=f(x_a)\tau_{n+1}\cdots\tau_{a+1}\varphi_a\varphi_{a+1}\cdots\varphi_{n+1}e(\nu,i^2)\\
&=f(x_a)\varphi_a\varphi_{a+1}\cdots\varphi_{n+1}\tau_n\cdots\tau_a e(\nu,i^2).
\end{aligned}$$

We have

$$\begin{aligned}
&\varphi_n\varphi_{n+1}=\varphi_n(x_{n+1}\tau_{n+1}-\tau_{n+1}x_{n+1})\\
&=x_n(x_n\tau_n-\tau_nx_n)\tau_{n+1}-(x_n\tau_n-\tau_nx_n)\tau_{n+1}x_{n+1}\\
&=x_n^2\tau_n\tau_{n+1}-x_n\tau_n\tau_{n+1}x_n-x_n\tau_n\tau_{n+1}x_{n+1}+\tau_n\tau_{n+1}x_nx_{n+1}
\end{aligned}$$

and it belongs to Γ_{-1}. Hence we obtain (5.3). Then the repeated use of (5.2) implies that

$$\begin{aligned}
&\tau_{n+1}\cdots\tau_1\varphi_1\cdots\varphi_{n+1}a^{\Lambda}(x_{n+2})e(\nu,i^2)\prod_{k\le n,\,\nu_k=i}(x_k-x_{n+2})\\
&\equiv\tau_{n+1}\varphi_{n+1}a^{\Lambda}(x_{n+2})e(\nu,i^2)\prod_{\nu_k\neq i}Q_{i,\nu_a}(x_{n+2},x_k).
\end{aligned}$$

Finally $\tau_{n+1}\varphi_{n+1}e(\nu,i^2)=\tau_{n+1}(\tau_{n+1}(x_{a+1}-x_a)-1)e(\nu,i^2)=-\tau_{n+1}e(\nu,i^2)$.

□

Lemma 5.6. *The following equality holds as an element of K.*

$$\begin{aligned}&\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\varphi_1\cdots\varphi_{n+1}e(\nu,i^2)\\&\qquad=\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\tau_1\cdots\tau_{n+1}e(\nu,i^2)\prod_{k\,=\,n+1\ \text{or}\ \nu_k\,=\,i}(x_{n+2}-x_a).\end{aligned}$$

Proof. It is enough to show that

$$\begin{aligned}&\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\tau_1\cdots\tau_{a-1}\varphi_a\cdots\varphi_{n+1}e(\nu,i^2)\\&\quad=\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\tau_1\cdots\tau_a\varphi_{a+1}\cdots\varphi_{n+1}e(\nu,i^2)(x_{n+2}-x_k)^{\delta(a\,=\,n+1\ \text{or}\ \nu_a\,=\,i)}.\end{aligned}\tag{5.4}$$

If $\nu_a\neq i$ it is trivial. If $\nu_a=i$ or $a=n+1$ then we have

$$\begin{aligned}\varphi_a\cdots\varphi_{n+1}e(\nu,i^2)&=(\tau_a(x_{a+1}-x_a)-1)\varphi_{a+1}\cdots\varphi_{n+1}e(\nu,i^2)\\&=\tau_a\varphi_{a+1}\cdots\varphi_{n+1}(x_{n+2}-x_a)e(\nu,i^2)-\varphi_{a+1}\cdots\varphi_{n+1}e(\nu,i^2).\end{aligned}$$

Since

$$\begin{aligned}&\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\tau_1\cdots\tau_{a-1}\varphi_{a+1}\cdots\varphi_{n+1}e(\nu,i^2)\\&\qquad=\tau_{n+1}\cdots\tau_1\varphi_{a+1}\cdots\varphi_{n+1}a^\Lambda(x_1)\tau_1\cdots\tau_{a-1}e(\nu,i^2)\end{aligned}$$

vanishes as an element of K for $a\le n+1$, we obtain (5.4). ∎

Thus we have

$$\begin{aligned}&(-1)^p\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\tau_1\cdots\tau_{n+1}e(\nu,i^2)\prod_{a\,=\,n+1\ \text{or}\ \nu_a\,=\,i}(x_{n+2}-x_a)^2\\&\equiv-\tau_{n+1}a^\Lambda(x_{n+2})\prod_{\nu_a\neq i}Q_{i,\nu_a}(x_{n+2},x_a)e(\nu,i^2)(x_{n+2}-x_{n+1})\mod\Gamma_{-1}.\end{aligned}$$

We have $\tau_{n+1}(x_{n+2}-x_{n+1})\in\Gamma_0$, and hence Lemma 5.4 implies

$$\begin{aligned}&\tau_{n+1}a^\Lambda(x_{n+2})\prod_{\nu_a\neq i}Q_{i,\nu_a}(x_{n+2},x_a)e(\nu,i^2)(x_{n+2}-x_{n+1})\\&\qquad\equiv\tau_{n+1}x_{n+2}^{\langle h_i,\Lambda-\beta\rangle+2p+1}e(\nu,i^2)\prod_{\nu_a\neq i}t_{i\nu_a}\mod\Gamma_{\langle h_i,\Lambda-\beta\rangle+2p-1}.\end{aligned}$$

In particular

$$\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\tau_1\cdots\tau_{n+1}e(\nu,i^2)\prod_{a\,=\,n+1\ \text{or}\ \nu_a\,=\,i}(x_{n+2}-x_a)^2\in\Gamma_{\langle h_i,\Lambda-\beta\rangle+2p}.$$

Hence it is equivalent to $\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\tau_1\cdots\tau_{n+1}e(\nu,i,i)x_{n+2}^{2p+2}$ modulo $\Gamma_{\langle h_i,\Lambda-\beta\rangle+2p-1}$.

Thus we obtain the following proposition.

Proposition 5.7. *For $\beta \in Q^+$, let p be the number of times that α_i appears in β, and set $\lambda := \Lambda - \beta$, $\lambda_i := \langle h_i, \lambda \rangle$. Then there exists $c \in \mathbf{k}_0 * \times$ such that*

$$\tau_{n+1} x_{n+2}^{\lambda_i+2p+1} e(\beta, i^2) \equiv c\tau_{n+1} \cdots \tau_1 a^{\Lambda}(x_1)\tau_1 \cdots \tau_{n+1} e(\beta, i^2) x_{n+2}^{2p+2} \quad \operatorname{mod} \Gamma_{\lambda_i+2p-1}.$$

Note that $\lambda_i + 2p \geq 0$.

5.3. Let us define two homomorphisms

$$P\colon R(\beta)e(\beta-\alpha_i, i) \underset{R(\beta-\alpha_i)}{\otimes} e(\beta-\alpha_i, i^2)R^{\Lambda}(\beta+\alpha_i) \longrightarrow K \quad \text{and}$$
$$E\colon R(\beta)e(\beta-\alpha_i, i) \underset{R(\beta-\alpha_i)}{\otimes} e(\beta-\alpha_i, i^2)R^{\Lambda}(\beta+\alpha_i) \longrightarrow e(\beta, i)R^{\Lambda}(\beta+\alpha_i)$$

by $P(a \otimes b) = a\tau_n\tau_{n+1} \otimes b$ and $E(a \otimes b) = ab$. Then P is injective and Lemma 5.4 implies

$$\operatorname{Im}(P) = \operatorname{Ker}\bigl(x_{n+2}\colon \Gamma_{-1} \longrightarrow \operatorname{Gr}_0^{\Gamma} K\bigr). \tag{5.5}$$

We can see that $R(\beta)e(\beta-\alpha_i, i) \otimes_{R(\beta-\alpha_i)} e(\beta-\alpha_i, i^2)R^{\Lambda}(\beta+\alpha_i)$ has a structure of $(\mathbf{R}(\beta) \otimes \mathbf{k}\langle x_n, x_{n+1}, \tau_n\rangle, \mathbf{k}[x_n] \otimes R^{\Lambda}(\beta+\alpha_i))$-bimodule by

$$(a \otimes b)(x_n \otimes 1) = ax_n \otimes b,$$
$$(1 \otimes \tau_n)(a \otimes b) = a \otimes \tau_n b,$$
$$(1 \otimes x_k)(a \otimes b) = a \otimes x_k b \quad \text{for } k = n, n+1.$$

Here $\mathbf{k}\langle x_n, x_{n+1}, \tau_n\rangle$ is the $\mathbf{k}$-subalgebra of $e(\beta-\alpha_i, i^2)R^{\Lambda}(\beta+\alpha_i)e(\beta-\alpha_i, i^2)$ generated by x_n, x_{n+1}, τ_n, and it is isomorphic to the nil affine Hecke algebra $R(2\alpha_i)$.

Lemma 5.8. *For any $z \in R(\beta)e(\beta-\alpha_i, i) \otimes_{R(\beta-\alpha_i)} e(\beta-\alpha_i, i^2)R^{\Lambda}(\beta+\alpha_i)$, we have*

$$P(z)x_{n+2} = P(z(x_n \otimes 1)) + \tau_{n+1}E(z) + E((1 \otimes \tau_n)z).$$

Proof. For $z = a \otimes b$, we have

$$\begin{aligned} P(a \otimes b)x_{n+2} &= a\tau_n\tau_{n+1}x_{n+2} \otimes b = a\tau_n(x_{n+1}\tau_{n+1} + 1) \otimes b \\ &= a(x_n\tau_n + 1)\tau_{n+1} \otimes b + 1 \otimes a\tau_n b. \end{aligned}$$

∎

Corollary 5.9. *If $z \in R(\beta)e(\beta-\alpha_i, i) \underset{R(\beta-\alpha_i)}{\otimes} e(\beta-\alpha_i, i^2)R^{\Lambda}(\beta+\alpha_i)$ satisfies $P(z)x_{n+2} \in \Gamma_{-1}$, then $E\bigl((1 \otimes \tau_n)z\bigr) = 0$.*

Indeed, $P(z)x_{n+2} \equiv E((1 \otimes \tau_n)z) \mod \Gamma_{-1}$.

Set $K^\Lambda = e(\beta, i^2)R^\Lambda(\beta^+2\alpha_i)e(\beta+\alpha_i, i)$. Hence we have

$$\begin{aligned} K &\simeq \frac{e(\beta, i^2)R(\beta+2\alpha_i)e(\beta+\alpha_i, i)}{e(\beta, i^2)R(\beta^+2\alpha_i)a^\Lambda(x_1)R(\beta+\alpha_i)e(\beta+\alpha_i, i)}, \\ K^\Lambda &\simeq \frac{e(\beta, i^2)R(\beta+2\alpha_i)e(\beta+\alpha_i, i)}{e(\beta, i^2)R(\beta^+2\alpha_i)a^\Lambda(x_1)R(\beta+2\alpha_i)e(\beta+\alpha_i, i)}. \end{aligned}$$

Then there exists a surjective homomorphism

$$p\colon K \to K^\Lambda.$$

Note that

$$p\big(\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\tau_1\cdots\tau_{n+1}e(\beta, i^2)\big) = 0. \tag{5.6}$$

Let us denote by $\{\Gamma_k^\Lambda\}_{k\in\mathbf{Z}}$ the filtration of K^Λ induced by the filtration Γ of K.

5.4. **Proof of (4.1).** Assume that $\lambda_i \geq 2$. The statement (4.1) can be read as

$$x_{n+2}^{\lambda_i-2}e(\beta, i^2) + x_{n+1}^{\lambda_i-1}\tau_{n+1}e(\beta, i^2) \in \Gamma^\Lambda_{\lambda_i-3} \quad \text{as an element of } K^\Lambda.$$

By Proposition 5.7 and Lemma 5.3, we have

$$\tau_{n+1}x_{n+2}^{\lambda_i-1}e(\beta, i^2) \equiv c\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\tau_1\cdots\tau_{n+1}e(\beta, i^2) \mod \Gamma_{\lambda_i-3}$$

as an element of K. Then the desired result holds since

$$\tau_{n+1}x_{n+2}^{\lambda_i-1}e(\beta, i^2) \equiv (x_{n+1}^{\lambda_i-1}\tau_{n+1} + x_{n+2}^{\lambda_i-2})e(\beta, i^2) \mod \Gamma_{\lambda_i-3}.$$

5.5. **Proof of (4.2).** Assume that $\lambda_i = 1$. Set

$$w := \tau_{n+1}e(\beta, i^2) - c\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\tau_1\cdots\tau_{n+1}e(\beta, i^2) \in K.$$

Then Proposition 5.7 together with Lemma 5.4 (iv) implies that w, $wx_{n+2} \in \Gamma_{-1}$ and w belongs to $\mathrm{Im}(P)$. Hence we can write $w = P(z)$ for some $z \in R(\beta)e(\beta - \alpha_i, i) \otimes_{R(\beta-\alpha_i)} e(\beta-\alpha_i, i^2)R^\Lambda(\beta+\alpha_i)$. Then Corollary 5.9 implies that

$$E(1\otimes\tau_n)z) = 0.$$

Let us define the morphism

$$T\colon R(\beta)e(\beta-\alpha_i, i) \underset{R(\beta-\alpha_i)}{\otimes} e(\beta-\alpha_i, i^2)R^\Lambda(\beta+\alpha_i) \to e(\beta, i)R^\Lambda(\beta+\alpha_i) \underset{R^\Lambda(\beta)}{\otimes} e(\beta, i)R^\Lambda(\beta+\alpha_i)$$

by $T(a \otimes b) = (a\tau_n) \otimes b$. Then we have

$$\begin{aligned} \Sigma(T(z)) &= p(P(z)), \\ \widetilde{E}(T(z)) &= E((1 \otimes \tau_n)z) = 0. \end{aligned}$$

Let us show that $u := e(\beta, i) \otimes e(\beta, i) - T(z)$ satisfies the condition (4.2).

(a) $\Sigma(T(z)) = p(P(z)) = \tau_{n+1}e(\beta, i^2)$ as an element of $R^\Lambda(\beta + 2\alpha_i)$.

(b) $\widetilde{E}(u) = e(\beta, i) - E\big(T(z)\big) = e(\beta, i)$.

(c) is obvious.

5.6. **Proof of (4.3).** Assume that $\lambda_i \le 0$. Note that $\ell := -\lambda_i \le 2p$. Then Proposition 5.7 says that, by setting $w := c\tau_{n+1} \cdots \tau_1 a^\Lambda(x_1)\tau_1 \cdots \tau_{n+1}e(\beta, i^2)$, the element $\big(wx_{n+2}^{2+\ell} - \tau_{n+1}x_{n+2}e(\beta, i^2)\big)x_{n+2}^{-\ell+2p}$ of K belongs to $\Gamma_{-\ell+2p-1}$.

Hence we have

$$wx_{n+2}^{\ell+2} - \tau_{n+1}x_{n+2}e(\beta, i^2) \in \Gamma_{-1}.$$

Since $\tau_{n+1}x_{n+2}e(\beta, i^2) \in \Gamma_0$, we have $wx_{n+2}^{\ell+2} \in \Gamma_0$. Hence Lemma 5.4 implies that $wx_{n+2}^k \in \Gamma_{-1}$ for $0 \le k \le \ell + 1$. We set

$$wx_{n+2}^k = P(z_k) + \tau_{n+1}y_k \quad \text{for } 0 \le k \le \ell + 1$$

with $z_k \in R(\beta)e(\beta - \alpha_i, i) \otimes_{R(\beta-\alpha_i)} e(\beta - \alpha_i, i^2)R^\Lambda(\beta + \alpha_i)$ and $y_k \in e(\beta, i)R^\Lambda(\beta + \alpha_i)$. Then we have for $1 \le k \le \ell + 2$

$$\begin{aligned} wx_{n+2}^k &= \big(P(z_{k-1}) + \tau_{n+1}y_{k-1}\big)x_{n+2} \\ &= P(z_{k-1}(x_n \otimes 1)) + \tau_{n+1}E(z_{k-1}) + E((1 \otimes \tau_n)z_{k-1}) + \tau_{n+1}x_{n+2}y_{k-1}. \end{aligned}$$

Hence Lemma 5.2 implies

$$\begin{aligned} &z_k = z_{k-1}(x_n \otimes 1) \quad \text{for } 1 \le k \le \ell + 1, \\ &y_k = E(z_{k-1}) \quad \text{for } 1 \le k \le \ell + 1, \\ &E\big((1 \otimes \tau_n)z_{k-1}\big) = 0 \quad \text{for } 1 \le k \le \ell + 1, \\ &y_{k-1} = 0 \quad \text{for } 1 \le k \le \ell + 1. \end{aligned}$$

Since $wx_{n+2}^{\ell+2} \equiv \tau_{n+1}x_{n+2}e(\beta,i^2) \mod \Gamma_{-1}$, we have $y_{\ell+1} = e(\beta,i)$ and $E\big((1\otimes\tau_n)z_{\ell+1}\big) = 0$. Thus we obtain $z_k = z_0(x_n^k \otimes 1)$ for $0 \le k \le 1+\ell$, and

$$E\big(z_0(x_n^k\otimes 1)\big) = \begin{cases} 0 & \text{for } 0 \le k \le \ell-1 \\ e(\beta,i) & \text{for } k=\ell. \end{cases} \tag{5.7}$$

$$E\big((1\otimes\tau_n)z_0(x_n^k\otimes 1)\big) = 0 \quad 0 \le k \le 1+\ell. \tag{5.8}$$

Let us denote by

$$\begin{aligned} q\colon R(\beta)e(\beta-\alpha_i,i) \underset{R(\beta-\alpha_i)}{\otimes} e(\beta-\alpha_i,i^2)R^\Lambda(\beta+\alpha_i) \\ \longrightarrow R^\Lambda(\beta)e(\beta-\alpha_i,i) \underset{R^\Lambda(\beta-\alpha_i)}{\otimes} e(\beta-\alpha_i,i^2)R^\Lambda(\beta+\alpha_i) \end{aligned}$$

the canonical homomorphism, and set $v = q(z_0)$. Then (a) and (b) in (4.3) follow from $T_k(v) = E\big(z_0(x_n^k\otimes 1)\big)$. The equality $G(v)=0$ follows from $G(v) = p(P(z_0)) = p(w) = 0$.

Finally let us prove (d) and (e). We have

$$E\big((1\otimes x_n^k\tau_n)z_0\big) = E\big((1\otimes\tau_n)z_0(x_n^k\otimes 1)\big) = 0 \quad \text{for } 0 \le k \le \ell+1$$

by (5.8). On the other hand we have $H_k(v) = E\big((1\otimes\tau_n x_{n+1}^k)z_0\big)$. Since $\tau_n x_{n+1}^k = x_n^k\tau_n + \sum_{a+b=k-1} x_{n+1}^a x_n^b$, we obtain

$$\begin{aligned} H_k(v) &= E\big((1\otimes x_n^k\tau_n)z_0\big) + \sum_{a+b=k-1} x_{n+1}^a E\big((1\otimes x_n^b)z_0\big) \\ &= \sum_{a+b=k-1} x_{n+1}^a E\big(z_0(x_n^b\otimes 1)\big). \end{aligned}$$

Hence (5.7) implies that $H_k(v) = 0$ for $0 \le k \le \ell$ and $H_{\ell+1}(v) = e(\beta,i)$.

Thus the proof of (4.1), (4.2) and (4.1) is complete.

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