Elliptic K3 Surfaces as Dynamical Models and their Hamiltonian Monodromy

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Key words K3 dynamical model, completely integrable system, Lagrangian fibration, elliptic K3 surface, monodromy
MSC(2010) 37J35, 14J27, 14J28, 58K10

Abstract

This note deals with Lagrangian fibrations of elliptic K3 surfaces and the associated Hamiltonian monodromy. The fibration is constructed through the Weierstraß normal form of elliptic surfaces. There is given an example of K3 dynamical models with the identity monodromy matrix around 12 elementary singular loci.

1 Introduction

In the sense of Liouville, the completely integrable systems on $2n$-dimensional phase space can be characterized by the existence of n functionally independent first integrals in involution. From the famous Liouville-Arnol'd theorem, there exist action-angle coordinates for such a system, if restricted to regular region. It is, however, not so trivial whether one can take global action-angle coordinates or not. According to the paper [5] by J.J. Duistermaat, there are several topological obstructions to the existence of the global action-angle coordinates, the first of which is the (Hamiltonian) monodromy. Since the appearance of [5], the Hamiltonian monodromy has been calculated for the momentum mapping of many classical completely integrable systems. See, e.g., [4] for several examples.

For the last decades, a lot of interesting researches have been done on the quantum version of Hamiltonian monodromy, quantum monodromy. On the level of semi-classical approximation, one can outline the quantum monodromy as follows: Given a completely integrable system, which is assumed to be with two degrees of

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freedom for simplicity, one can consider the momentum mapping of the symplectic manifold induced by the first integrals. This mapping can be regarded as a special case of Lagrangian fibrations with possible singular fibres. One can dot the points in the image of the momentum mapping, whose corresponding fibres satisfy the Bohr-Sommerfeld condition. These points locally form the structure of integral lattices. Because of the existence of singular fibres of the momentum mapping, however, the whole lattice may carry some defects in it. If one parallelly transports a fundamental parallelogram of one of the local lattices along a contour around defects, then the fundamental parallelogram is presumably expected to be transformed to another parallelogram after a cycle of parallel transport, which can be described as an image of the original parallelogram through a certain monodromy matrix. This is the quantum monodromy. The interpretation of quantum monodromy as a consequence of lattice defects is given by B. Zhilinskií. See [13] for details.

Furthermore, Zhilinskií proposes the inverse problem of relating lattice defects to Hamiltonian monodromy. In the ordinary problem, starting with classical completely integrable systems, one usually has some lattice defects through the semi-classical quantization procedure. Conversely, one can ask whether there would be a classical completely integrable system which gives rise to a given lattice defects or not. In fact, the correspondence between lattice defects and monodromy is not simply one-to-one even in the case of two degrees of freedom. For example, A.V. Bolsinov, H.R. Dullin, and A.P. Veselov considered in [3] the quantization of the geodesic flow on three-dimensional sol-manifolds with respect to the metrics given by the right-invariant Riemannian metrics on the three-dimensional solvable Lie group. They calculated the Hamiltonian and the quantum monodromy for this system, which are respectively in the form $\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} A^{\mathrm{T}} & 0 \\ 0 & 1 \end{bmatrix}$, where $A \in SL(2, \mathbb{Z})$ is a hyperbolic matrix defining the sol-manifold. (Although this system has three degrees of freedom, the lattice defect is essentially two dimensional, since it is a cone over the two dimensional lattice defect given by the matrix A^{T}.) In the case where A is the Arnol'd cat map $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, the quantum monodromy is $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and the fundamental parallelogram is rotated by 2π when it is transported along a counter around the defect, as is shown in Figure 5 of [13]. On the other hand, the Arnol'd cat map can also be given by the lattice defect obtained with the aid of one positive and one negative elementary defects, for which the fundamental parallelogram is transported along a closed curve going around the defect without any 2π-rotation, as is explained in Figure 6 of [13]. Thus, Zhilinskií pointed out that one should also pay attention to the number of 2π-rotations of the fundamental parallelogram during the transportation along a counter enclosing the defects. At the same time, it is natural to ask whether there would be a classical completely integrable system which allows the appearance of a certain lattice defect of the identity monodromy matrix with non-zero number of 2π-rotations of the fundamental parallelogram during the transportation of the fundamental parallelogram around the defects. A concrete example of such a lattice defect can be given by the sunflower pattern which originates in phyllotaxis morphology of botanic sciences, as is described in Section 6 of [13]. For this lattice defect, one can read, in the same manner as the quantum monodromy, the identity monodromy matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with 2π-rotation manifested by the defect. Zhilinskií suggests in his paper that a corresponding classical completely integrable system could be obtained from a K3 surface with some almost toric Lagrangian fibration. However, an explicit construction of such a K3 dynamical model with the identity monodromy matrix representing 2π-rotation is still an open problem.

In this note, an explicit construction is made of a K3 dynamical model with an identity monodromy matrix around 12 nontrivial singular loci by means of the notion of the Weierstraß normal form for elliptic surfaces. After a brief review of completely integrable systems in Section 2, it is shown that K3 surfaces can be regarded as real four-dimensional phase spaces in Section 3. Elliptic K3 surfaces are described as collections of locally-defined completely integrable systems. The explicit construction of K3 dynamical models is given through the Weierstraß normal form of elliptic surfaces in Section 4. Finally, an example of K3 dynamical models having the identity monodromy matrix around 12 nontrivial singular loci is given by using some rational elliptic surface with two singular points of type D_4.

2 Integrable Systems and Lagrangian Fibrations

As is well known, a Hamiltonian system (M, ω, H) over a $2n$-dimensional symplectic manifold (M, ω) is called a completely integrable system, if there are n functionally independent first integrals $f_1, \ldots, f_{n-1}, f_n (= H)$, which mutually commute with respect to the Poisson bracket induced by the symplectic form ω. These first integrals give rise to the momentum mapping $f = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$ onto a subset of $\mathbb{R}^n$. The famous Liouville-Arnol'd theorem states that a fibre $f^{-1}(p)$ for a regular value $p \in \mathbb{R}^n$ of f is a real n-dimensional Lagrangian torus, if it is connected and compact. Moreover, one can take a local trivialization of f around such a fibre, which is usually understood as the existence of the action-angle variables.

From a mathematical point of view, one can generalize the notion of completely integrable systems to that of Lagrangian fibrations as follows:

Definition 2.1. *Let (M, ω) be a $2n$-dimensional symplectic manifold and B an n-dimensional manifold. A mapping $f : M \to B$ is called a Lagrangian fibration, if there is an open dense subset $B_0 \subset B$ over which the restriction $f|_{f^{-1}(B_0)} : f^{-1}(B_0) \to B_0$ is a locally trivial fibre bundle with Lagrangian fibres.*

One can say that a Lagrangian fibration is a collection of locally-defined completely integrable systems except the singular fibres. It is possible that B is a manifold with boundary, although all the base spaces in what follows are without boundary. Since this definition is sometimes too general, we introduce a little bit more moderate notion, following N.C. Leung and M. Symington:

Definition 2.2 ([10]). *A Lagrangian fibration $f : (M, \omega) \to B$ is called an almost toric Lagrangian fibration, if any critical point of f admits a Darboux coordinates $(p, q) = (p_1, \dots, p_n, q_1, \dots, q_n)$ with $\omega = \sum_{i=1}^{n} \mathsf{d}p_i \wedge \mathsf{d}q_i$ such that the projection $f = (f_1, \dots, f_k, f_{k+1}, \dots, f_n)$ is locally expressed as $f_i(p, q) = p_i$ for $i \le k$ and as either $f_j(p, q) = p_j^2 + q_j^2$ or $(f_j, f_l)(p, q) = (p_j q_j + p_l q_l, p_j q_l - p_l q_j)$ for other components. In addition, each fibre is assumed to be compact and connected.*

Put another way, an almost toric Lagrangian fibration is a collection of local structures of completely integrable systems with compact connected fibres admitting only centre-centre and focus-focus equilibria (or their direct products). In Section 4, we will give an explicit construction of K3 dynamical models through the Weierstraß normal form. Some of the models are shown to be almost toric Lagrangian fibrations.

Before closing this section, we make a brief comment on the Hamiltonian monodromy. Let $f : M \to B$ be a Lagrangian fibration of a $2n$-dimensional symplectic manifold. We take an open dense subset $B_0 \subset B$ over which f is locally trivial. For any $b \in B_0$, we have the isomorphism of the homology group: $H_1(f^{-1}(b), \mathbb{Z}) \cong \mathbb{Z}^{\oplus n}$. Fix a reference point $b_0 \in B_0$ and consider a closed path $\gamma : [0, 1] \ni s \mapsto \gamma(s) \in B_0$ starting and ending at b_0. We choose a basis $c_1 \dots, c_n \in H_1(f^{-1}(b_0), \mathbb{Z})$ and pursue the elements $c_1(s), \dots, c_n(s) \in H_1(f^{-1}(\gamma(s)), \mathbb{Z})$ which continuously depend on the parameter s and satisfy $c_i(0) = c_i$. Then, the mapping $(c_1, \dots, c_n) \mapsto (c_1(1), \dots, c_n(1))$ is a homomorphism of $H_1(f^{-1}(b_0), \mathbb{Z})$ represented by a matrix in $GL(n, \mathbb{Z})$. This homomorphism depends only on the homotopy class of γ, so that we have a representation $\rho : \pi_1(B_0, b_0) \to GL(n, \mathbb{Z})$ of the fundamental group of B_0 which is the Hamiltonian monodromy associated to the Lagrangian fibration f.

3 K3 surfaces as real four-dimensional phase spaces

A simply connected compact complex surface M is called a K3 surface if its canonical bundle is trivial, i.e. if it admits a non-vanishing holomorphic two-form Ω. It is easy to obtain a real symplectic structure on M from Ω.

Proposition 3.1. *The real part $\omega := \mathrm{Re}(\Omega) = \frac{\Omega + \overline{\Omega}}{2}$ defines a real symplectic structure on M which is regarded as a real four-dimensional manifold.*

Remark. One can also take the imaginary part to get a real symplectic structure on M, by replacing Ω by $-\sqrt{-1}\Omega$.

With respect to this real symplectic structure, the real Hamiltonian vector field X_h^ω for a real smooth function h on (an open subset of) M is determined through $\iota_{X_h^\omega}\omega = -\mathsf{d}h$.

On the other hand, the holomorphic two-form Ω itself defines a holomorphic symplectic structure on M. Thus, we also have the complex Hamiltonian vector field X_h^Ω with respect to Ω: $\iota_{X_h^\Omega}\Omega = -\mathsf{d}h$. Note that the function h could be complex valued in this case and that X_h^Ω is a real $\mathcal{C}^\infty$-section of TM_h, where TM_h stands for the holomorphic tangent bundle over the domain $M_h \subset M$ of the function h.

We get interested in the relation between the real and the complex Hamiltonian vector fields associated with a holomorphic function h defined over an open subset of M. The following two propositions, which are essentially concerned with local structure, are already realized by H. Flaschka [6] or L.M. Bates and R.H. Cushman [2].

Proposition 3.2 (Cf. [6, 2])**.** *If h is a holomorphic function over an open subset of M, then the complex Hamiltonian vector field X_h^Ω is related to the real Hamiltonian vector fields $X_{\mathrm{Re}(h)}^\omega$ and $X_{\mathrm{Im}(h)}^\omega$ by*

$$X_{\mathrm{Re}(h)}^\omega = \mathrm{Re}(X_h^\Omega), \qquad X_{\mathrm{Im}(h)}^\omega = \mathrm{Im}(X_h^\Omega).$$

Moreover, the two functions $\mathrm{Re}(h)$ and $\mathrm{Im}(h)$ to Poisson commute:

Proposition 3.3 (Cf. [6, 2])**.** *If h is a holomorphic function on an open subset of M, then the real part $\mathrm{Re}(h)$ and the imaginary part $\mathrm{Im}(h)$ commute with respect to the Poisson structure defined by the real symplectic form ω.*

From this proposition, we can conclude that any open subset M_h of a K3 surface M equipped with a holomorphic function h on it determines a locally-defined completely integrable system. The real and the imaginary parts of h might be regarded as action-variables of this system from the viewpoint of Liouville-Arnol'd theorem. However, a holomorphic function defined everywhere over M must be constant by the Liouville theorem in function theory. Thus, if we restrict ourselves to holomorphic functions, the action-variables are defined only locally, and the momentum mapping is not defined globally. We are consequently led to consider Lagrangian fibrations of M instead of momentum mappings. In order to get a globally defined Lagrangian fibration of M, we should take meromorphic functions of M into account. If there is given a collection of holomorphic functions over each element of an open covering of M in a compatible manner, which gives rise to a meromorphic function $f : M \to P_1(\mathbb{C})$, then f defines a Lagrangian fibration of (M, ω) onto the projective line $P_1(\mathbb{C})$. In fact, the mapping $f : M \to P_1(\mathbb{C})$ can be shown to be an elliptic fibration by Zariski's lemma (cf. [1]) and by Theorem 1 in §3 of [12], provided that each regular fibre of f is irreducible. In the next section, we explain an explicit construction of K3 dynamical models through the elliptic K3 surfaces in Weierstraß normal form.

Remark. As to the integrable symplectic structures on K3 surfaces, D.G. Markushevich studies in [11] the integrability problem for K3 surfaces only from the viewpoint of complex geometry. However, we consider the elliptic K3 surfaces as a real symplectic manifold admitting Lagrangian fibrations together with an explicit construction of K3 dynamical models.

4 Elliptic K3 surfaces as Lagrangian fibrations

So far we have explained how one can obtain, as an extension of completely integrable systems, Lagrangian fibrations of K3 surfaces in an abstract manner. In this section, we give an explicit construction of K3 dynamical models. To this end, we start with the notion of the Weierstraß normal form of elliptic surfaces.

Let B be a compact Riemann surface and $L \to B$ a holomorphic line bundle over it. We take two holomorphic sections $g_2 \in H^0(B, L^{\otimes 4})$ and $g_3 \in H^0(B, L^{\otimes 6})$. Let $(x : y : z)$ be the homogeneous fibre coordinates of the $P_2(\mathbb{C})$-bundle $P(L^{\otimes 2} \oplus L^{\otimes 3} \oplus \mathcal{O}_B)$ over B, where $\mathcal{O}_B$ denotes the structure sheaf of B which we identify with the trivial line bundle over B. Now, we consider the hypersurface W of the total space P of the $P_2(\mathbb{C})$-bundle $P(L^{\otimes 2} \oplus L^{\otimes 3} \oplus \mathcal{O}_B)$ defined through

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3.$$

Restricting the natural projection $P \to B$ to W, we obtain an elliptic surface $\pi_W : W \to B$, which is called an elliptic surface in Weierstraß normal form. The discriminant Δ and the functional invariant J are given by $\Delta = g_2^3 - 27g_3^2$ and $J = \frac{g_2^3}{\Delta}$, respectively. Since the elliptic surface W has singularities in general, we have to give the desingularization $\widehat{W}$ of W in order to get a smooth elliptic surface. Note that these singular points are simple singularities. See, e.g., [8] for more details on Weierstraß normal form for elliptic surfaces.

In what follows, we consider the case where $B = P_1(\mathbb{C})$ and $L = \mathcal{O}_{P_1(\mathbb{C})}(2)$. Here, $\mathcal{O}_{P_1(\mathbb{C})}(m)$ stands for the holomorphic line bundle over $P_1(\mathbb{C})$ of the first Chern class m in $H^2(P_1(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}$, i.e. the tensor product of hyperplane section bundles over $P_1(\mathbb{C})$ to the power m. Denote the homogeneous coordinates of $B = P_1(\mathbb{C})$ by $(t_0 : t_1)$. On the open subset of W given by $t_1 \neq 0$, $y \neq 0$, $z \neq 0$, we have a non-vanishing holomorphic two-form

$$\Omega = \frac{\mathsf{d}\overline{x}}{\overline{y}} \wedge \mathsf{d}t, \tag{4.1}$$

where $\overline{x} = \frac{x}{z}$, $\overline{y} = \frac{y}{z}$ are the affine fibre coordinates and $t = \frac{t_0}{t_1}$ is the affine coordinate of B. As to this holomorphic two-form, we have the following proposition.

Proposition 4.1. *The holomorphic two-form Ω given in* (4.1) *can be extended over the open subset W^{reg} of the Weierstraß normal form W consisting of smooth points. Moreover, it gives rise to a non-vanishing holomorphic two-form over the minimal desingularization $\widehat{W}$.*

The proof for the first part is done through a straightforward calculation. The second part can be shown by the simpleness of the singularities of W. By means of the homotopy exact sequence of fibre spaces, we can show that the elliptic surface $\widehat{W}$ is simply connected. Thus, we see that $\widehat{W}$ is an elliptic K3 surface. For the monodromy of the Lagrangian fibration $\pi_{\widehat{W}} : \widehat{W} \to B = P_1(\mathbb{C})$, we can use the classification of the singular fibres for elliptic surfaces together with the conjugacy classes of monodromy matrices by Kodaira [9].

Here, we explain the relation of an elliptic K3 surface to almost toric Lagrangian fibrations, which we mentioned in Section 2. From the classification result of four-dimensional almost toric Lagrangian fibrations by Leung-Symington [10], such a fibration is given by a K3 surface admitting a Lagrangian fibration with 24 nodal (i.e. focus-focus) singularities. Taking into account the classification of the singular fibres for elliptic surfaces by Kodaira [9], we can see that an elliptic K3 surface can be regarded as an almost toric Lagrangian fibration over the Riemann sphere if and only if it merely has singular fibres of type I_b $(b \geq 1)$ in Kodaira's notation.

In closing this section, we give the local expression of the complex Hamilton's equation for the Lagrangian fibration $\pi_{\widehat{W}} : \widehat{W} \to P_1(\mathbb{C})$. On the open subset of $\widehat{W}$ where $t_1 \neq 0$ and $z \neq 0$, the complex Hamilton's equations for the Hamiltonian $t = \frac{t_0}{t_1}$ can be expressed as

$$\begin{cases} \dot{\overline{x}} & = -\overline{y}, \\ \dot{\overline{y}} & = -\frac{12\overline{x}^2 - g_2}{2}. \end{cases} \tag{4.2}$$

Now, we recall the famous formula for the Weierstraß $\wp$-function:

$$\ddot{\wp} = \frac{12\wp^2 - g_2}{2}.$$

See [7] for the theory of elliptic functions. From this formula, we can conclude that the complex Hamilton's equations (4.2) can be solved by setting $\overline{x} = \wp$ and $\overline{y} = -\dot{\wp}$. Since the real Hamiltonian vector fields $X^{\omega}_{\mathrm{Re}(t)}$ and $X^{\omega}_{\mathrm{Im}(t)}$ are exactly the real and the imaginary parts of the complex Hamiltonian vector field X^{Ω}_t by Proposition 3.2, the dynamical system can be solved essentially by the Weierstraß $\wp$-function.

5 Example

In this section, we give an example of the explicit construction of K3 dynamical models through the Weierstraß normal form, which possesses an identity monodromy matrix corresponding to a hoop around 12 nontrivial singular loci. Note that the construction problem for such a K3 dynamical model has been proposed by B. Zhilinskií in relation to the lattice defects associated with the sunflower pattern, as was mentioned in Section 1. See [13] and the references therein for details.

We first consider the following rational elliptic surface in Weierstraß normal form. Let $B = P_1(\mathbb{C}) : (t_0 : t_1)$ be the projective line and choose the sections $g_2 \in H^0(B, \mathcal{O}_B(4))$ and $g_3 \in H^0(B, \mathcal{O}_B(6))$ as the collections $\{t^2, \tau^2\}$ and $\{t^3, \tau^3\}$ of holomorphic functions over the open subsets $t_1 \neq 0$ and $t_0 \neq 0$, respectively. Here, $t = \frac{t_0}{t_1}$ and $\tau = \frac{t_1}{t_0}$ are the affine coordinates of $B = P_1(\mathbb{C})$ defined on the open subsets $t_1 \neq 0$ and $t_0 \neq 0$, respectively. We denote the associated rational elliptic surface in Weierstraß normal form by $\pi_W : W \to P_1(\mathbb{C})$. The discriminant $\Delta = g_2^3 - 27g_3^2$ of this fibration is the section of $\mathcal{O}_B(12)$ given by $\{-26t^6, -26\tau^6\}$, so that the singular fibres of W lie on $t = 0$ (i.e. $\tau = \infty$) and $t = \infty$ (i.e. $\tau = 0$). In fact, we can see that the elliptic surface W has two singular points of type D_4 along these singular fibres. (See [1, III. 7.] for the types of simple singularities of complex surfaces.) After taking the minimal desingularization of the singular points, we have a smooth rational elliptic surface $\pi_{\widehat{W}} : \widehat{W} \to P_1(\mathbb{C})$ whose singular fibres are of type I_0^* in Kodaira's notation and sitting over $t = 0$ and $t = \infty$. The singular fibres are described in Figure 1. Note that the monodromy matrices of these singular fibres are both $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

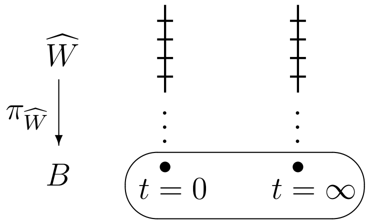

Figure 1.

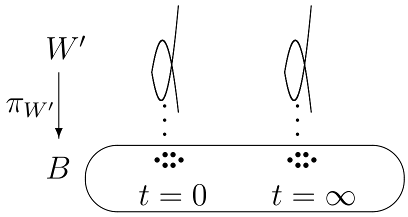

Figure 2.

We proceed to perturb the above rational elliptic surface by changing the sections $g_2 \in H^0(B, \mathcal{O}_B(4))$ and $g_3 \in H^0(B, \mathcal{O}_B(6))$ into $g_2' \in H^0(B, \mathcal{O}_B(4))$ and $g_3' \in H^0(B, \mathcal{O}_B(6))$ which are given by the collections of local holomorphic functions as

$$\begin{aligned} &\{(t - a_1)(t - a_2)(t - a_3)(t - a_4), (1 - a_1\tau)(1 - a_2\tau)(1 - a_3\tau)(1 - a_4\tau)\}, \\ &\{(t - b_1)(t - b_2)(t - b_3)(t - b_4)(t - b_5)(t - b_6), \\ &\qquad (1 - b_1\tau)(1 - b_2\tau)(1 - b_3\tau)(1 - b_4\tau)(1 - b_5\tau)(1 - b_6\tau)\}, \end{aligned}$$

respectively. Here, the complex numbers a_i $(i = 1, \dots, 4)$ and b_j $(j = 1, \dots, 6)$ are chosen generically so as to satisfy

$$0 < |a_1|, |a_2|, |b_1|, |b_2|, |b_3| \ll 1 \ll |a_3|, |a_4|, |b_4|, |b_5|, |b_6| < \infty. \tag{5.1}$$

We denote the perturbed elliptic surface by $\pi_{W'} : W' \to P_1(\mathbb{C})$, which is defined by g_2' and g_3'. Since the discriminant ${g_2'}^3 - 27{g_3'}^2$ has only simple roots for generic parameters, the perturbed fibration $\pi_{W'}$ has 12 singular fibres of type I_1 in Kodaira's notation, six of which are sitting near $t = 0$ and the other six near $t = \infty$, as is described in Figure 2. The singular fibre of type I_1 is a rational curve with one node. From the viewpoint of real torus fibrations, the singular fibre of type I_1 is that obtained from the real two-dimensional torus by pinching one of the cycles as in Figure 3. It is to be noted that the total space W' is a smooth rational surface.

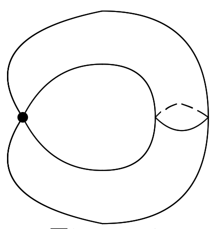

Figure 3.

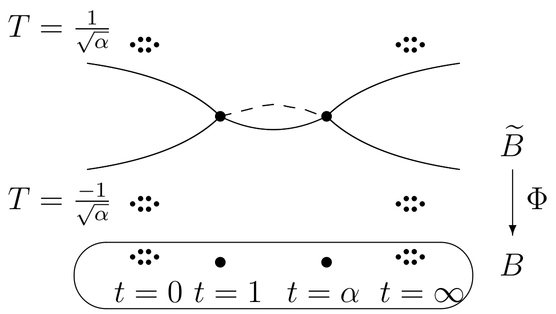

Figure 4.

Now, we take a complex number α with $1 < |\alpha| \ll |a_3|, |a_4|, |b_4|, |b_5|, |b_6|$. Then, we consider the double covering $\Phi : \widetilde{B} \to B$ of the base curve branched at $t = 1$ and $t = \alpha$. The resulting Riemann surface $\widetilde{B}$ is again a projective line. Pulling back the perturbed fibration $\pi_{\widetilde{W'}} : \widetilde{W'} \to B$ through Φ, we obtain an elliptic K3 surface $\pi_{W'} : W' \to \widetilde{B}$ over the projective line. In fact, this elliptic surface is in Weierstraß normal form induced by the two sections $\widetilde{g_2'} \in H^0(\widetilde{B}, \mathcal{O}_{\widetilde{B}}(8))$ and $\widetilde{g_3'} \in H^0(\widetilde{B}, \mathcal{O}_{\widetilde{B}}(12))$ described as

$$\widetilde{g_2'}(T) = \prod_{i=1}^{4}((1 - a_i) - (\alpha - a_i)T^2), \tag{5.2}$$

$$\widetilde{g_3'}(T) = \prod_{j=1}^{6}((1 - b_j) - (\alpha - b_j)T^2), \tag{5.3}$$

where T is the local coordinate of $\widetilde{B}$ satisfying $T^2 = \frac{1-t}{\alpha-t}$. It is obvious that the elliptic K3 surface $\pi_{\widetilde{W'}} : \widetilde{W'} \to \widetilde{B}$ has 24 singular fibres of type I_1. The singular loci of this elliptic fibration are described as in Figure 4. The monodromy matrix corresponding to a closed path going around the six singular loci near $T = \frac{1}{\sqrt{\alpha}}$ and the six near $T = -\frac{1}{\sqrt{\alpha}}$ can be calculated as $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Note that the monodromy is a topological invariant. Thus, we have constructed a K3 dynamical model with the identity monodromy matrix around 12 singular loci. Since elliptic K3 surfaces with 24 singular fibres of type I_1 are almost toric Lagrangian fibrations, as was mentioned in Section 4, we have the following theorem.

Theorem 5.1. *The elliptic K3 surface* $\pi_{\widetilde{W'}} : \widetilde{W'} \to \widetilde{B}$ *associated with* $\widetilde{g_2'}$ *in* (5.2) *and* $\widetilde{g_3'}$ *in* (5.3) *is an almost toric Lagrangian fibration over the projective line, whose equilibria are all focus-focus, for the generically chosen parameters* a_i $(i = 1, \ldots, 4)$ *in* (5.2) *and* b_j $(j = 1, \ldots, 6)$ *in* (5.3) *with the condition* (5.1). *Let* $\widetilde{B}_0$ *denote the open subset of* $\widetilde{B}$ *consisting of all regular loci of* $\pi_{\widetilde{W'}}$. *Then, there is a nontrivial element* $\sigma \in \pi_1(\widetilde{B}_0, *)$ *which gives rise to the identity monodromy matrix, where* σ *can be represented by a closed path enclosing 12 singular loci.*

Acknowledgment. The author is grateful to Professor Boris Zhilinskií for many stimulating discussions. He also thanks Professor Isao Naruki for a lot of useful discussions. He expresses his deep gratitude to Professor Toshihiro Iwai for the valuable suggestions and for the generous and continual encouragement. He thanks the referee for valuable comments on the manuscript.

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