STUDIES ON SINGULAR PERTURBATIONS OF OPTIMAL CONTROL SYSTEMS WITH APPLICATIONS TO NUCLEAR REACTOR CONTROL

KOICHI ASATANI

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Preface

It is urged that effective techniques be developed to approximate large systems to systems with reduced model order. When more precise results are desired, one encouters the difficulty of increasing dimensionality, throughout the three steps of model making, analysis of the system and synthesis of the system. Moreover, the estimation of the error in the method adopted, is of great importance and in general more difficult than obtaining the approximate solution itself.

Systems can be classified into two classes, i.e., differential systems (dynamical systems) and algebraic systems. For both systems, several methods have been developed to reduce the dimension, in order to avoid the difficulty of increasing dimensionality. The singular perturbation theory, ε-coupling system, block diagonalization, sparse matrix theory, etc. are more or less approximation methods using subsystems which are generated from the original system and are more easily dealt with than the original. The singular perturbation theory is efficient in treating differential systems of a certain class.

The perturbation method in general is very useful in treating rather complex systems which are not very different from known systems. The conventional perturbation method treats the "regular system", where the perturbing term does not change the dimension of the system. The dynamical system involving a singularly perturbing parameter whose presence causes the dimension of the system to increase, cannot be dealt with by the regular perturbations. Such systems of singular perturbation type appear in many fields, e.g., network theory, fluid mechanics, magnethydrodynamics, etc.

In this thesis, devoted to the studies of the singular perturbation theory in connection with two-point boundary value problems arising in the optimal control systems, emphasis is made upon the error estimations of the derived approximate solution in the form of asymptotic expansions. The singular perturbation theory proves to be basic and widely applicable to real physical systems of large size which usually happen to involve stiff differential equations. The manner of the description of this thesis is as follows; firstly, the method of constructing formal asymptotic solutions is presented, and then the basic theorems are offered establishing the validity of the method proposed.

Abstract

This thesis consists of six chapters. The historical survey of the singular perturbation theory is given in the first chapter. The outline and basic idea of the theory are also presented. Further, the well-known formula of Vasil'eva giving the initial conditions of the higher order recursive set of equations and the matching law of the method of matched asymptotic expansions are proved to be identical. The relation is first shown explicitly by the author.

In Chapter 2, a finite dimensional model is treated, which is described by a set of ordinary differential equations. Such models may offer a description of a broad class of realistic physical systems. Three problems occuring typically in optimization problems are considered in three sections. The results in these sections may cover many variations. In Section 2.2, we derive basic results on tracking problems. The results can be extended to the regulator problems, yielding theorems under hypotheses, different from those of Yackel and Kokotović [Y.73]. In Section 2.3, a fixed-end-point, fixed-end-time and minimum energy problem will be treated. The generalized Riccati transformation is introduced to treat the ill-conditioned two-point boundary value problems generated by the fixed-terminal problems. Section 2.4 is concerned with the system involving two singularly perturbing parameters of different orders of smallness. The results of this section can be easily applied to the systems involving more parameters. The last section is devoted to the study of the asymptotic behaviour of the performance index.

Chapter 3 deals with the nuclear reactor control as a lumped parameter system. Such a lumped parameter model is well known in the field of reactor engineering as a "point reactor". The results in the previous chapter are successfully applied to the analysis and synthesis of suboptimal control of the point reactor. This chapter consists of two sections. The early section is concerned with the

error estimation of prompt jump approximation with the aid of the singular perturbation theory and some numerical examples are shown. The latter section treats the regulator problem of the reactor, also with numerical examples.

Chapter 4 is concerned with near-optimum control of systems involving spatially distributed parameters. Such systems are usually described by partial differential equations, integral equations or integro-differential equations. The system considered is represented by partial differential equations of parabolic type.

It is to be noted that the practical treatment of partial differential equations usually cannot be made without involving an approximation, spatial discritization, modal expansion, etc., since the equations can be regarded as of infinite dimensional. Hence one should realize the importance of the reduction of the number of independent variables if possible.

This chapter consists of two parts, the former one is devoted to the study of the singular perturbations of a certain class of a set of evolution equations of parabolic type. The latter deals with near-optimum control problems on the basis of the results derived in the former one.

Chapter 5 is concerned with the nuclear reactor control as a distributed system. The importance of this treatment has been increasing, since the size of reactors is becoming larger and the spatial effects cannot be neglected. For instance, control of the neutron flux distribution, burnup control with consideration of hot-spot-factor, Xenon oscillation, etc. should be treated by considering distributed parameters.

The method proposed in this chapter is the singular perturbation theory connected with one of the modal expansion, the Helmholtz modal expansion. The Helmholtz mode gives the complete basis in the space L^2. The theoretical foundation, upon which the validity of the method presented relies, has been given in Chapter 4.

In the last chapter, the results obtained are summarized.

Additional remarks and future works to be done are also involved.

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Chapter 1 Singular Perturbation Theory

In this chapter, an outline and survey consisting of four sections will be made. In Section 1.1, the aspects or situations of the singular perturbation theory are introduced by showing one of the simplest example. In Sections 1.2 and 1.3, a brief survey is made of the singular perturbation theory available at present to optimal control problems with lumped and distributed parameters.

The survey is not comprehensive if viewed from a purely mathematical point. The more complete review can be found in Vasil'eva's review paper [Vs.63.1], Wasow's laborious work [Ws.65], or in O'Malley [O.68.2]. The fourth section will be devoted to the description of preliminary results of the singular perturbations of ordinary differential equations. The proofs are omitted here because our main object lies only in that basic ideas and techniques of the singular perturbations are required to be demonstrated.

1.1 Introduction

Physicists or engineers often meet systems described by ordinary differential equations with parasitic parameters appearing as coefficients. Perturbation method is known well to be a good tool to investigate such systems. The method has its origin in the natural intuition, and in the experimental knowledge that there is a little difference between the trajectories of generating system and that of perturbed system. Analysis of the perturbed system is usually more intractable to be handled than the generating system. According to the "regular" perturbation method, the unknown quantities are expanded into power series with respect to a small parameter multiplying the perturbing term, and the coefficients of the expansions are determined stepwise by solving recursive equations, usually of the linear form. The method is powerful and practical in the fields of applied mathematics (Cole [Cl.68]), fluid mechanics (Van Dyke [Vd.64]), nonlinear oscillations (Hayashi [Hy.64], Nishikawa [N.64]), etc.

But among systems to be considered, there are peculiar ones including small parameters whose presence changes the order of the system. Such systems cannot be treated by the "regular" perturbation method. The "singular" perturbation theory is needed to investigate such singularly perturbed systems.

The question may arise how the "singular" perturbation theory differs from the "regular" perturbation theory. The following simplest example may show the singularity and complexity.

A scalar differential equation

$$\varepsilon \frac{d^2}{dt^2} x + \frac{d}{dt} x = 0, \tag{1.1.1}$$

for instance, has the general solution

$$x = C_1 \exp(-t/\varepsilon) + C_2, \tag{1.1.2}$$

which is not continuous with respect to ε at $\varepsilon = 0$, unless $t = 0$. The quantities C_1 and C_2 are determined by subsidiary conditions of the problem. We consider here two-point boundary conditions, for example,

$$x(0) = \alpha, \qquad x(1) = \beta. \tag{1.1.3}$$

The exact solution is

$$x(t,\varepsilon) = \{1 - \exp(-\frac{1}{\varepsilon})\}^{-1}\{(\alpha-\beta)\exp(-\frac{t}{\varepsilon}) + \beta - \alpha\exp(-\frac{1}{\varepsilon})\} . \tag{1.1.4}$$

Since for the positive ε, we have

$$\lim_{\varepsilon\to 0} x(t,\varepsilon) = \beta, \quad \text{for } t \in]0,1]. \tag{1.1.5}$$

The convergence is uniform in any closed t-interval $0 < \delta \leqq t \leqq 1$ but not in the whole interval $0 \leqq t \leqq 1$. The solution $x(t,0) = \beta$ is that of the reduced problem, obtained by letting $\varepsilon = 0$,

$$\frac{d}{dt} x = 0, \quad x(1) = \beta. \tag{1.1.6}$$

In a narrow interval of width $O(\varepsilon)$ the solution (1.1.4) changes rapidly from $x(0,\varepsilon) = \alpha$ to a certain value neighboring $x(t,0) = \beta$ with difference of the order of ε. The outline of the situation is shown in Fig.1.1.1. Pearson [P.68] provides many examples of the boundary problems with illustrations.

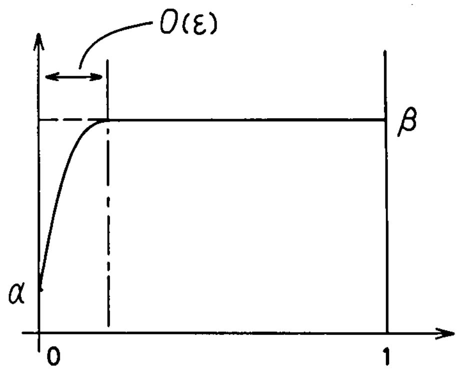


Fig. 1.1.1 A schematic figure of a boundary layer.

Such an interval is called a "boundary layer", which is originally a terminology of fluid dynamics, because of a mathematical analogy with the boundary layers of fluid dynamics (see for example Van Dyke [Vd. 64]). For the negative ε we find that

$$\lim_{\varepsilon \to 0} x(t,\varepsilon) = \alpha, \tag{1.1.7}$$

and the boundary layer occurs at $t = 1$.

In the preceding description, it is shown that the reduced solution does not generally satisfy the conditions of both ends. Also we can show in the same way that if initial value problems are considered, the reduced solution does not satisfy all the prescribed initial conditions. In other words, some of the prescribed conditions are lost in the reduced problem. Here we are confronted with the problem of what conditions are lost in the reduced problem, and how we can make corrections to the reduced solution. The singular perturbation theory provides us with right answers to these questions.

1.2 Singular perturbation theory for ordinary differential equations

Let us consider singular perturbations of a set of differential equations of the first order. In general, such systems can be written in the form

$$\begin{aligned} \frac{d}{dt} x &= f(x, y, t, \varepsilon), \qquad & x(0,\varepsilon) &= \xi(\varepsilon), \\ \varepsilon \frac{d}{dt} y &= g(x, y, t, \varepsilon), \qquad & y(0,\varepsilon) &= \eta(\varepsilon), \end{aligned} \tag{1.2.1}$$

where ε is a small positive parameter and $x, f \in E^n$, $y, g \in E^m$.

The reduced system (degenerate system) is obtained by setting $\varepsilon = 0$ in (1.2.1) and by cancelling the initial condition for y.

$$\begin{aligned} \frac{d}{dt} x &= f(x, y, t, \varepsilon), \qquad & x(0,0) &= \xi(0), \\ 0 &= g(x, y, t, \varepsilon). \end{aligned} \tag{1.2.2}$$

The problem (1.2.1) is called the "full problem" and is said to degenerate regularly at $(x_0(t), y_0(t))$, the solution of the "reduced problem" (1.2.2) on [0,T], if the solution of (1.2.1) converges as $\varepsilon \to +0$ to (x_0, y_0) uniformly on closed subsets of [0,T].

The singular perturbation theory has two aspects, the one is to investigate the regular degeneration of (1.2.1) and the other is to construct an asymptotic expansion of the solution of the full problem (1.2.1).

In both the aspects, the Jacobian matrix

$$\frac{\partial g}{\partial y} (x_0(t), y_0(t), t, 0) \tag{1.2.3}$$

plays a principal role, the nonsingularity of which establishes the convergence and the solvability of the recursive equations in the outer region, which can be obtained through the outer problem:

$$\frac{d}{dt}x = f(x, y, t, \varepsilon), \qquad x(0,\varepsilon) = \xi(\varepsilon),$$
$$\varepsilon\frac{d}{dt}y = g(x, y, t, \varepsilon). \tag{1.2.4}$$

A solution of (1.2.4), $x(t,\varepsilon)$, $y(t,\varepsilon)$, satisfies

$$x(t,0) = x_0(t), \qquad y(t,0) = y_0(t). \tag{1.2.5}$$

The inner problem

$$\frac{d}{d\tau}X = \varepsilon f(X, Y, \varepsilon\tau, \varepsilon), \qquad X(0,\varepsilon) = \xi(\varepsilon),$$
$$\frac{d}{d\tau}Y = g(X, Y, \varepsilon\tau, \varepsilon), \qquad Y(0,\varepsilon) = \eta(\varepsilon), \tag{1.2.6}$$

derived through the stretching transformation $\tau = t/\varepsilon$, describes the solution of (1.2.1) for t near zero, or in the boundary layer.

The associated boundary layer problem

$$\frac{d}{d\tau}Y = g(x_0(t), Y, 0, 0), \qquad Y(0) = \eta(0), \tag{1.2.7}$$

determines the asymptotic behavior of the solution of the inner problem (1.2.6).

Many papers on the regular degeneration of (1.2.1) require various stability properties of the system (1.2.7). Tikhonov [Tk.52] stated that if a certain solution of (1.2.7) is asymptotically stable as $\tau \to \infty$ then (1.2.1) degenerates regularly for each t restricted in a compact interval. His assumption was inadequate partly and was corrected by Hoppensteadt [Hp.67]. Levin and Levinson [Lv.54] gave the condition that all eigenvalues of (1.2.3) have negative real parts for $t \in [0,T]$. This condition is identical with the asymptotic stability of a solution of (1.2.7). Hoppensteadt [Hp.66], [Hp.67] gave a proof of Tikhonov's theorem and extended the result to the case $T = \infty$.

Flatto and Levinson [Fl.55] and Levin [Lv.56], [Lv.59], studied the regular degeneration of (1.2.1) under the conditional stability of the system (1.2.7). Levin showed that the full system (1.2.1) degenerates regularly if the initial data are given on the suitable manifold, called initial stable manifold. The conditional stability of (1.2.7) means that all the eigenvalues of (1.2.3) have nonzero real parts, or the system (1.2.3) possesses an exponential dichotomy (Coppel [Co.67]). Other studies of the conditionally stable case were made by Coppel and Chang [Cn.68], [Cn.69.1], [Cn.69.2]. Their results [Cn.69.1], [Cn.69.2] refine those of [Fl.55], [Lv.56], [Lv.59], and [Cn.68] by removing certain smoothness conditions. Hoppensteadt [Hp. 71.1] also studied the conditional stability case, paying attention to the differentiability of the solution of (1.2.1) with respect to ε, while in [Fl.55] - [Cn.69.2], the main object is with the continuity as $\varepsilon \to 0$ of the difference between the solution of (1.2.1) and that of (1.2.2).

As for the second aspect, i.e. constructing the asymptotic expansion solution, several researches have been made in various ways. From practical points of view, the second aspect is more important, which has been emphasized from many applications. Vasil'eva [Vs.63.1], Kaplun [Ka.61], Hoppensteadt [Hp.71.1], and O'Malley [O.71.1] have taken several approaches. Vasil'eva's method is similar to the popular method of inner and outer expansions. Kaplun [Ka.61] obtained expansion of the solution using the method of matched asymptotic expansions [Fk.69, Vd.64, Ku.67], which is directly applicable to partial differential equations (cf. Hendry and Bell [Hn.69, Hn.70, Hn.71]). In Hoppensteadt [Hp.71.1] and O'Malley [O.71.1] the solution of (1.2.1) was constructed by using boundary layer method with a little difference between them. All the expansions appearing in literatures [Vs. 63.1] - [O.71.1] can be shown to reduce to one another. Throughout these papers only the case where all eigenvalues of (1.2.3) have negative real parts was studied. Tupčiev [Tp.62.1] extended Vasil'eva's result to the conditionally stable case.

Concerning two-point boundary value problems, several approaches have been made, e.g., by Harris [Hr.60], [Hr.62], Tupčiev [Tp.62.1], Vishik and Lyusternik [Vi.57], [Vi.58], Macki [Mc.67], O'Malley and Keller [O.68.1], O'Malley [O.69], [O.70.1], [O.70.2], Hadlock [Hd.70.2], and Hoppensteadt [Hp.70], [Hp.71.1]. Compared with initial value problems, comprehensive methods have not yet been derived. Harris [Hr.60], [Hr.62] obtained some condition that ascertains regular degeneration of homogeneous two-point boundary value problem of the first order system. In O'Malley and Keller [O.68.1], certain cancellation law of boundary conditions was considered as to higher order scalar equations. The regular degeneration of a two-point boundary value problem arising in optimal control theory was also considered by Hadlock [Hd.70.2].

An asymptotic series solution was obtained in Tupčiev [Tp.62.1], Macki [Mc.67] for the nonlinear system of the first order equations. Vishik and Lyustenik [Vi.58], and O'Malley [O.70.1], [O.70.2] also considered asymptotic series solutions of a certain class of quasi-linear system equations. Linear scalar equations were studied by Vishik and Lyusternik [Vi.57], O'Malley [O.69], etc. The comprehensive bibliography and survey on the subjects can be found in O'Malley [O.68.2].

1.3 Singular perturbation theory for abstract evolution equations

The results of singular perturbations of ordinary differential equations have been extended to abstract evolution equations, including parabolic, hyperbolic, and integro-differential equations. (On the abstract evolution equation, parabolic equation etc, Dunford and Schwartz [Df.58] may be referred to.)

In Trenogin [Tr.63], the Cauchy problem in a real Banach space

$$\varepsilon \frac{d}{dt} y(t,\varepsilon) = F(y, t, \varepsilon), \qquad 0<t\leq T,$$

$$y(0,\varepsilon) = 0, \tag{1.3.1}$$

is considered, where both the regular degeneration and asymptotic solution of (1.3.1) are studied, by expanding the nonlinear operator $F(y, t, \varepsilon)$ into a Taylor series in an appropriate sense with respect to y and ε. Krein [Kr.67, Ch.4] investigated the asymptotic method on the Cauchy problem

$$\varepsilon \frac{d}{dt} y(t,\varepsilon) = A(t,\varepsilon)y(t,\varepsilon) + f(t), \tag{1.3.2}$$

The more general Cauchy problem was also considered by Hoppensteadt [Hp.69.1], [Hp.69.2] of the form,

$$\varepsilon \frac{d}{dt} y(t,\varepsilon) + A(t,\varepsilon)y(t,\varepsilon) = f(t, y, \varepsilon), \tag{1.3.3}$$

where $-A(t,\varepsilon)$ is the generator of an analytic semigroup of bounded operators in E (parabolic case) [Hp.69.1]. In Hoppensteadt [Hp.69.2], both parabolic and hyperbolic cases were stated without proof.

In these studies [Tr.63] - [Hp.69.2], each problem is regarded as a Cauchy problem of an infinite dimensional ordinary differential equation in a Banach space, which is tractable on the basis of singular perturbations developed for ordinary differential equations reviewed in the preceding section.

Another approach to singular perturbations of partial differential equations, taken by Hoppensteadt [Hp.71.2] for the initial boundary value problem, appears as

$$\varepsilon \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x, t, u, \nabla u, \varepsilon)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x, t, u, \nabla u, \varepsilon) \tag{1.3.4}$$

$$u(x, t, \varepsilon) = 0 \qquad x \in \text{bdry}\ \Omega, \quad 0 \leqq t \leqq T,$$

$$u(x, 0, \varepsilon) = \mathring{u}(x,\varepsilon) \qquad x \in \overline{\Omega}$$

where the matrix (a_{ij}) is symmetric and positive definite uniformly in its arguments, and $\Omega \in E^n$. He suggested that the results obtained in [Hp.71.2] can be extended to the more general problems, e.g., systems of parabolic type, higher order parabolic equations with more general boundary conditions, and parabolic equations coupled with integro-differential equations.

The method of matched asymptotic expansion can be applied to partial differential equations (see, e.g., Hendry [Hn.69], [Hn.70], [Hn.71]) but a rigorous mathematical proof has not yet been derived for partial differential equations. However a heuristic theoretical consideration and many numerical experiments on the subject support the method [Dm.69].

The more general problems, however,

$$\begin{aligned} \frac{d}{dt} x(t,\varepsilon) &= F(x, y, t, \varepsilon), \\ \varepsilon \frac{d}{dt} y(t,\varepsilon) &= G(x, y, t, \varepsilon), \end{aligned} \tag{1.3.5}$$

have rarely been treated. A certain class of such systems will be treated in the following Chapter 4.

1.4 Asymptotic solutions and basic theorems

Consider the initial value problem

$$\begin{aligned} \frac{d}{dt} x &= f(x, y, t, \varepsilon), \qquad & x(0,\varepsilon) &= \xi(\varepsilon), \\ \varepsilon \frac{d}{dt} y &= g(x, y, t, \varepsilon), \qquad & y(0,\varepsilon) &= \eta(\varepsilon), \end{aligned} \tag{1.4.1}$$

where $x, f \in E^n$, and $y, g \in E^m$.

In order to construct asymptotic expansions of solutions, we expand x and y into Taylor series in ε:

$$x = x_0 + \varepsilon x_1 + \frac{\varepsilon^2}{2} x_2 + \ldots\ldots,$$
$$y = y_0 + \varepsilon y_1 + \frac{\varepsilon^2}{2} y_2 + \ldots\ldots, \tag{1.4.2}$$

Substituting these series into Eq.(1.4.1), and equating the coefficients of like powers of ε, we derive the recursive equation in the outer region, i.e., on $t \in [0 + \delta, T]$, $\delta = O(\varepsilon)$.

$$\frac{d}{dt} x_0 = f(x_0, y_0, t, 0),$$
$$0 = g(x_0, y_0, t, 0), \tag{1.4.3$_0$}$$

$$\frac{d}{dt} x_r = f_x x_r + f_y y_r + p_{r-1},$$
$$\frac{d}{dt} y_{r-1} = g_x x_r + g_y y_r + q_{r-1}, \tag{1.4.3$_r$}$$

where the remainders p_{r-1} and q_{r-1} depend only on t, $x_0(t)$, $y_0(t)$, ..., $x_{r-1}(t)$, $y_{r-1}(t)$.

The asymptotic solution (1.4.2) is valid in the interval except for the boundary layer $0 \leq t \leq O(\varepsilon)$. The boundary layer correction is required here in order to make the solution valid uniformly in the whole interval considered. The basic idea depends upon Vasil'eva's method (see Section 3.1) but here we will show simpler results, following Hoppensteadt [Hp.71.1]. These results are equivalent to those derived by O'Malley [O.71.1].

We shall seek a solution of the form

$$x = x(t,\varepsilon) + X(\tau,\varepsilon),$$
$$y = y(t,\varepsilon) + Y(\tau,\varepsilon), \tag{1.4.4}$$

where $x(t,\varepsilon)$ and $y(t,\varepsilon)$ have an asymptotic expansion (1.4.2) and correspond to the outer solution which represents "slow mode". And $X(\tau,\varepsilon)$ and $Y(\tau,\varepsilon)$ have also an asymptotic expansion

$$X(\tau,\varepsilon) = X_0(\tau) + \varepsilon X_1(\tau) + \frac{\varepsilon^2}{2} X_2(\tau) + \ldots,$$
$$Y(\tau,\varepsilon) = Y_0(\tau) + \varepsilon Y_1(\tau) + \frac{\varepsilon^2}{2} Y_2(\tau) + \ldots, \qquad (1.4.5)$$

which is valid in the boundary layer and is required to tend to zero as $\tau \to \infty$. The quantities $X(\tau,\varepsilon)$ and $Y(\tau,\varepsilon)$ are called boundary layer correction terms and represent "fast mode". Substituting the expansions (1.4.2) and (1.4.5) into Eq.(1.4.4), we have an asymptotic solution uniformly valid

$$x = \sum_{r=0}^{\infty} [x_r(t) + X_r(\tau)](r!)^{-1}\varepsilon^r,$$
$$y = \sum_{r=0}^{\infty} [y_r(t) + Y_r(\tau)](r!)^{-1}\varepsilon^r. \qquad (1.4.6)$$

In Vasil'eva's method, we must construct series solutions of three kinds, outer, inner, and intermediate solution. The boundary layer correction term is equivalent to the difference between the inner and intermediate solution. Then we have recursive equations for $X_i(\tau)$ and $Y_i(\tau)$

$$\frac{d}{d\tau} X_0 = 0,$$
$$\frac{d}{d\tau} Y_0 = g(x_0(0) + X_0,\ y_0(0) + Y_0,\ 0,\ 0), \qquad (1.4.7)_0$$
$$\frac{d}{d\tau} X_r = P_r(\tau),$$
$$\frac{d}{d\tau} Y_r = g_x(x_0(0) + X_0,\ y_0(0) + Y_0,\ 0,\ 0)X_r \qquad (1.4.7)_r$$
$$+ g_y(x_0(0) + X_0,\ y_0(0) + Y_0,\ 0,\ 0)Y_r + Q_r(\tau).$$

Here the remainders P_r and Q_r are polynomials consisting only of t, $x_0(\tau)$, $y_0(\tau)$, ..., $x_{r-1}(\tau)$, $y_{r-1}(\tau)$, X_0, Y_0, ..., X_{r-1}, Y_{r-1}.

Initial conditions for (1.4.6) are determined by the prescribed conditions (1.4.1), i.e.,

$$x_r(0) + X_r(0) = \xi_r, \quad y_r(0) + Y_r(0) = \eta_r, \tag{1.4.8}$$

where ξ_r and η_r are the coefficients of ε^r of Taylor series of ξ and η respectively. It is required that

$$\lim_{\tau\to\infty} X_r(\tau) = 0, \qquad \lim_{\tau\to\infty} Y_r(\tau) = 0. \tag{1.4.9}$$

Now we can solve Eq.$(1.4.3)_0$ under the initial condition

$$x_0(0) = \xi_0, \tag{1.4.10}$$

then from Eq.$(1.4.7)_0$ and (1.4.8), we have

$$X_0(\tau) = 0. \tag{1.4.11}$$

Thus we must solve

$$\frac{d}{d\tau} Y_0 = g(x_0(0),\ y_0(0) + Y_0,\ 0,\ 0), \quad Y_0(0) = \eta_0 - y_0(0). \tag{1.4.12}$$

This problem can be written as

$$\frac{d}{d\tau} Y_0 = g_y\, Y_0 + q_0(Y_0) \tag{1.4.13}$$

where

$$q_0(Y_0) = o(|Y_0|), \quad \text{as} \quad |Y_0| \to 0.$$

If the initial data is given in the domain of influence of the asymptotically stable root $y = \phi(x)$ of

$$g(x,\ \phi(x),\ t,\ 0) = 0, \tag{1.4.14}$$

then we can determine the leading terms X_0, Y_0, from $(1.4.7)_0$, (1.4.8), and (1.4.9).

$X_1(\tau)$ can be determined from (1.4.7) as

$$X_1(\tau) = X_1(0) + \int_0^\tau P_1(\sigma)d\sigma. \tag{1.4.15}$$

Assertion (1.4.9) requires $X_1(0)$ to satisfy

$$X_1(0) = - \int_0^\infty P_1(\sigma)d\sigma. \tag{1.4.16}$$

This infinite integral exists since the integrand is dominated by functions of boundary layer type which decay exponentially. Then we obtain the initial data for the second coefficient of the outer expansion

$$x_1(0) = \xi_1 - X_1(0). \tag{1.4.17}$$

Thus we can solve the first correction equation $(1.4.3)_1$ in the outer region. Then $y_1(0)$ can be determined from $(1.4.3)_1$, and we can solve the boundary layer correction equation $(1.4.7)_1$ for $Y_1(\tau)$. In this way, recursive equations (1.4.3) and (1.4.7) are solved stepwise.

Several conditions are needed to establish main theorems. First we state the prerequisite conditions, and then basic results are offered. These conditions are denoted by C1, C2, ..., as follows:

C1. The reduced problem $(1.4.3)_0$ has a continuous solution $x = x_0(t)$, $y = y_0(t)$, in some interval $t \in [0,T]$.

C2. The functions f, g have continuous derivatives up to the order $n + 2$ with respect to their arguments (x, y, t) in some neighborhood of the points $(x_0(t), y_0(t), t)$, $t \in [0,T]$, and $\varepsilon \in [0,\varepsilon_0]$.

C3. The initial data $\xi(\varepsilon)$, $\eta(\varepsilon)$ are smooth functions of ε for $\varepsilon \in [0,\varepsilon_0]$.

The most important condition is from the assertion (1.4.9).

C4. The Jacobian matrix of (1.4.12) satisfies the condition

$$\frac{\partial g}{\partial y}(x_0(t), y_0(t), t, 0) \leqq -\kappa < 0 \tag{1.4.18}$$

for each $t \in [0,T]$.

We can replace C4 by the following equivalent condition.

C4'. All the eigenvalues of the Jacobian matrix $g_y(t)$ have negative real parts for each $t \in [0,T]$.

Lemma 1-1 If conditions C1 - C4 hold, then the reduced solution is uniquely determined.

Note that C4 implies that the solution of (1.4.12) is asymptotically stable with respect to the root $y = \phi(t_0,\xi)$ of

$$g(x_0, \phi(t_0,\xi), 0, 0) = 0. \tag{1.4.19}$$

C5. The initial data exist in the domain of influence of the root $y = \phi(t_0,\xi)$.

The following theorem is due to Tikhonov [Tk.52]

Theorem 1-1 If conditions C1 - C5 hold, then the following convergence relations between the reduced solution and the full solution are satisfied:

$$\lim_{\varepsilon\to 0} x(t,\varepsilon) = x_0(t), \qquad \text{for } t \in [0,T], \tag{1.4.20}$$

$$\lim_{\varepsilon\to 0} y(t,\varepsilon) = y_0(t), \qquad \text{for } t \in]0,T]. \tag{1.4.21}$$

Following Vasil'eva [Vs.63.1] we have an error estimation theorem:

Theorem 1-2 If conditions C1 - C5 hold, then there exist $\varepsilon_0 > 0$ and functions $R^n(t,\varepsilon)$ and $S^n(t,\varepsilon)$ uniformly bounded in the interval considered, such that

$$x(t,\varepsilon) = \sum_{r=0}^{n} [x_r(t) + X_r(\tau)](r!)^{-1}\varepsilon^r + R^n(t,\varepsilon)\varepsilon^{n+1}, \tag{1.4.22}$$

$$y(t,\varepsilon) = \sum_{r=0}^{n} [y_r(t) + Y_r(\tau)](r!)^{-1}\varepsilon^r + S^n(t,\varepsilon)\varepsilon^{n+1}, \tag{1.4.23}$$

for $\varepsilon \in [0,\varepsilon_0]$.

If we restrict our attention to the outer expansion, we have the following theorem as a corollary of Theorem 1-2.

Theorem 1-3 If conditions C1 - C5 hold, then there exist $\varepsilon_0 > 0$ and bounded functions $R(t,\varepsilon)$ and $S(t,\varepsilon)$ such that

$$x(t,\varepsilon) = \sum_{r=0}^{n} x_r(t)(r!)^{-1}\varepsilon^r + R^n(t,\varepsilon)\varepsilon^{n+1}, \quad \text{for } t \in [\delta,T], \tag{1.4.24}$$

$$y(t,\varepsilon) = \sum_{r=0}^{n} y_r(t)(r!)^{-1}\varepsilon^r + S^n(t,\varepsilon)\varepsilon^{n+1}, \quad \text{for } t \in [\delta,T], \tag{1.4.25}$$

where

$$\delta = -C\varepsilon\log\varepsilon; \qquad C \text{ is independent of } \varepsilon.$$

In Hoppensteadt [Hp.71.1] condition C4 or C4' is replaced by the more general condition, as follows:

C6. The matrix $g_y(t)$ has k eigenvalues, $1 \leqq k \leqq n$, with negative real parts satisfying $\mathrm{Re}(\lambda(t)) \leqq -\mu$, and n-k eigenvalues with positive real parts satisfying $\mathrm{Re}(\lambda(t)) \geqq \mu$ for $t \in [0,T]$.

1.5 A related topic

In this section, a relation between the Vasil'eva's method and the method of matched asymptotic expansions is considered with regard to the determination of the initial conditions of the recursive set of equations.

For simplicity, we consider the following set of equations:

$$\begin{aligned} \frac{d}{dt}x &= f(x, y, t, \varepsilon), \\ \varepsilon\frac{d}{dt}y &= g(x, y, t, \varepsilon), \end{aligned} \tag{1.5.1}$$

where x and y are scalars. The initial condition is given as follows:

$$x(0) = \xi, \qquad y(0) = \eta. \tag{1.5.2}$$

The reduced system is obtained by setting $\varepsilon = 0$ in Eq.(1.5.1), with the initial condition for x. The first order correction of the outer expansion can be represented as a solution of

$$\begin{aligned} \frac{d}{dt} x_1 &= f_x x_1 + f_y y_1 + p_1(x_0, y_0), \\ \frac{d}{dt} y_0 &= g_x x_1 + g_y y_1 + q_1(x_0, y_0), \end{aligned} \tag{1.5.3}$$

where f_x denotes the partial derivative of the function f with respect to x at (x_0, y_0), similar notations are adopted for the others, and x_0 and y_0 are solutions determined by the reduced system.

Following Vasil'eva [Vs.63.1], the initial condition for the first order correction equation (1.5.3) is obtained as follows:

$$x_1(0) = \int_0^\infty [f(x_0(0), \bar{y}(\tau) - f(x_0(0), y_0(0))]d\tau, \tag{1.5.4}$$

where $\bar{y}(\tau)$ is a solution of the boundary layer system associated with the full system (1.5.1) with (1.5.2),

$$\frac{d}{d\tau} \bar{y}(\tau) = g(x_0(0), \bar{y}(\tau), 0, 0) \tag{1.5.5}$$

with the initial condition

$$\bar{y}(0) = \eta. \tag{1.5.6}$$

The conditions needed are assumed to hold so that we can apply the singular perturbation theory to the system (1.5.1) with (1.5.2). Then the boundary layer system (1.5.5) is asymptotically stable, i.e., the condition C4 in Section 1.4 holds

$$\frac{\partial g}{\partial y}(x_0(t), y_0(t), t, 0) \leqq -\kappa < 0 \tag{1.5.7}$$

for each $t \in [0,T]$.

Since Lemma 1-1 gives the solvability of the reduced problem, we may represent formally y_0 in the form of a function of x_0,

$$y_0 = \phi(x_0) \tag{1.5.8}$$

which is a solution of

$$g(x_0(t),\ y_0(t),\ t,\ 0) = 0. \tag{1.5.9}$$

If the condition (1.5.7) holds, then we obtain

$$\lim_{\tau\to\infty} \frac{d\bar{y}}{d\tau} = \lim_{\tau\to\infty} g(x_0(0),\ \bar{y}(\tau),\ 0,\ 0) = 0. \tag{1.5.10}$$

Considering Eqs.(1.5.9) and (1.5.10) collectively, we have the following equality relation:

$$\lim_{\tau\to\infty} \bar{y}(\tau) = y_0(0), \tag{1.5.11}$$

if the function $g\colon x \to y$ is one-valued. In the case where g is a multi-valued function, Eq.(1.5.11) also holds by taking an appropriate branch $\phi(x)$ satisfying Eq.(1.5.9).

The matching principle of the method of matched asymptotic expansions is stated as below [Vn.64]:

$$\lim_{\tau\to\infty} [\text{the k-th order solution of the inner expansion}]$$
$$= \lim_{t\to 0} [\text{ the k-th order solution of the outer expansion}], \tag{1.5.12}$$

which is equivalent to Eq.(1.5.11) for $k = 0$. For the higher order, the similar argument is carried out. Extensions to the case where x and y are multi-dimensional vectors are easily obtained.

The outline of the above situation is shown in Fig.1.5.1

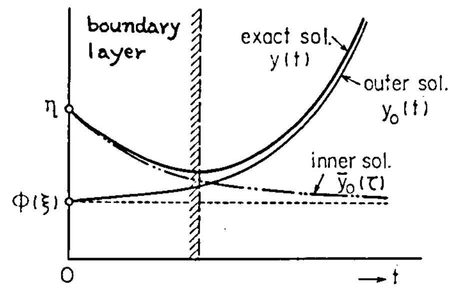


Fig. 1.5.1 A schematic figure of limit relations.

Chapter 2 Near-Optimum Control of Lumped Parameter System

2.1 Introduction

In this chapter, a finite dimensional model is treated, which is described by a set of ordinary differential equations. Such models may offer a description of a broad class of realistic physical systems.

In the optimization problem of such a system of the high order, it is urged from practical points of view to develop an effective synthesis method based on an approximate model, "low order model", obtainable by reducing the system order. Designers usually make the model order to be lower by neglecting some small parasitic parameters, e.g., time-constants, masses,moments of inertia, inductance, capacitance, and so forth, whose presence causes the model order higher than acceptable for practical treatment. For instance the numerical calculation of optimal control needs increased endeavor with the increase of the order of the control system, because the generated canonical system equation is "2n-dimensional" when the original system is "n-dimensional".

Such a conventional "low-order design" is made on the basis of the designer's intuition or his practical experiences. The "low-order design" has some drawbacks, e.g., that it may destroy original stable properties of more accurate "high-order design" (see for example [Ko. 68]), or may not maintain controllability and observability of the original system. Moreover no means are available to improve the low-order approximation because of the absence of systematic and theoretical basis.

The singular perturbation theory offers very useful and powerful tools to the problems. Kokotović and Sannuti [Ko.68] considered first the singular perturbations of optimal control problems of certain class. They analyzed the regular degeneration of control and state vector, and constructed the higher correction term in the outer region by using Vasil'eva's theory [Vs.63.1]. In singular perturbations, the stability of boundary layer system plays an important role, mentioned on occasion in the preceding chapter. In the study [Ko.68] of nonlinear regulator problem it is needed that the boundary layer system be asymptotically stable as $\tau \to \infty$. The result was improved in Sannuti and Kokotović [Sa.69.1], by removing the assumption on continuity and differentiability of control with respect to a singularly perturbing parameter (see also [Sa.69.1]).

In Sannuti and Kokotović [Sa.69.2], another approach is taken by utilizing the Riccati transformation. They studied linear regulator problems under the condition that the state matrix of the boundary layer system should be stable and the full system should satisfy a special condition. Kokotović and Yackel [Ko.72] also studied linear regulator problems by introducing new concepts of boundary layer controllablility and boundary layer observability; the theorem obtained in [So.69.2] was refined and generalized by using the new concepts, and was derived as a corollary of the main theorem in [Ko.72].

A quite different approach to the problem was made in O'Malley [O. 72.1], Hadlock et al. [Hd.70.1], [Hd.70.2], and Sannuti [Sa.71]. In these studies, the so-called boundary layer method proposed in O'Malley [O.71.1] or in Hoppensteadt [Hp.71.1] was adopted in order to obtain a temporally uniform approximate solution of the asymptotic expansion form. The results of O'Malley [O.72.1] and Hadlock [Hd.70.2] are more general than those of Sannuti [Sa.71] in that their results correspond to the conditionally stable case in Section 1.1., which all the eigenvalues of boundary layer system of the state equation concerned, have negative real part.

Boundary layer method is also applied to matrix Riccati differential equations in Yackel and Kokotović [S.73.1].

In this chapter, the author shall show the most generalized results of the singular perturbation theory of optimal control problems derived by him. The chapter treats three problems occuring typically in optimization problems, and covers many variations. In the following section, we derive basic results on tracking problems. The results can be easily extended to the linear regulator problem, yielding theorem under hypotheses different from those of Yackel and Kokotović [Y.73]. In Section 2.3, fixed-end-point, fixed-end-time, minimum energy problems will be treated. The generalized Riccati transformation is introduced to treat the ill-conditioned two-point boundary value problems generated by the fixed-terminal problems. Section 2.4 deals with the system involving two singularly perturbing parameters of different orders of smallness. The results of this section can be easily applied to systems involving more parameters. The last section is devoted to the study of the asymptotic behaviour of the performance index.

2.2 Tracking problem

2.2.1 Problem statement

Let the state equation be

$$\frac{d}{dt} x(t,\varepsilon) = A(t,\varepsilon)x(t,\varepsilon) + B(t,\varepsilon)u(t,\varepsilon), \tag{2.2.1}$$

$$A = \begin{bmatrix} A_1 & A_2 \\ \varepsilon^{-1}A_3 & \varepsilon^{-1}A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \varepsilon^{-1}B_2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where x_1 is an n-dimensional state vector, x_2 m-dimensional, control vector u r-dimensional, and A_i's, B_i's consistent dimensional matrices respectvely. A small positive parameter is expressed by ε.

The tracking problem is to minimize the performance index

$$J = \frac{1}{2} e(t_f)' \; Fe(t_f) + \frac{1}{2} \int_{t_0}^{t_f} [e(t)'Qe(t) + u(t)'Ru(t)]dt, \tag{2.2.2}$$

where

$$F = \begin{bmatrix} F_1 & \varepsilon F_2 \\ \varepsilon F_2' & \varepsilon F_3 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2' & Q_3 \end{bmatrix}, \tag{2.2.3}$$

under the prescribed initial conditions

$$x(t_0) = \xi, \tag{2.2.4}$$

or in the partitioned form

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}.$$

The partitioned form of F shall prove to be logical in the sequel. The vector e(t) is an error vector of l-dimensional

$$e(t) = z(t) - y(t) \tag{2.2.5}$$

where z(t) is a desired vector of l-dimensional and y(t) is an output vector of l-dimensional

$$\begin{aligned} y(t) &= Cx(t) = C_1x_1(t) + C_2x_2(t), \\ C &= [C_1, C_2]. \end{aligned} \tag{2.2.6}$$

As well-known, the optimal control u* is given, through the constructing Hamiltonian, as below (Athan and Falb [At.66]):

$$u^* = -R^{-1}B'p. \tag{2.2.7}$$

We have the canonical equation by eliminating costate vector p,

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B' \\ -Q & -A' \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \tag{2.2.8}$$

The equation (2.2.8) is to be solved under the boundary conditions

$$x(t_0) = \xi, \tag{2.2.9a}$$

$$p(t_f) = Fe(t_f). \tag{2.2.9b}$$

In the course of deriving the canonical equation (2.2.8), we can easily see that the costate vector p_2 corresponds to the quantity of x_2/ε. Then the form of terminal condition (2.2.9b) supports the partitioned form of F mentioned previously.

Several methods are proposed to handle such two-point boundary value problems, e.g., sweep method (Gelfand and Fomin [G1.63]) shooting method (Keller [K1.68]), invariant imbedding (Bailey and Wing [B1.65]), etc.

The Riccati transformation is known well in the field of optimal control as one of initial value methods which transform the original two-point boundary value problem into a single terminal value problem. The Riccati transformation is rigorously based upon the linear space theory. The validity of the transformation is proved in many articles (see for instance Meyer [My.73]).

The transformation is defined by

$$p(t) = K(t)x(t) + g(t). \qquad (2.2.10)$$

By eliminating p(t) in Eq.(2.2.8) by using Eq.(2.2.10), the following differential equation of matrix Riccati type is obtained:

$$\dot{K} = -KA - A'K + KBR^{-1}B'K - C'QC, \qquad (2.2.11)$$

with the associated differenatial equation

$$\dot{g} = -[A - BR^{-1}B'K]'g. \qquad (2.2.12)$$

The terminal conditions for K(t), and g(t) are given as follows from Eqs.(2.2.9b) and (2.2.10):

$$K(t_f) = C'(t_f)FC(t_f), \qquad (2.2.13)$$

$$g(t_f) = -C'(t_f)Fz(t_f). \qquad (2.2.14)$$

The optimum trajectory is a solution of

$$x = [A - BR^{-1}B'K]x + BR^{-1}B'g, \qquad (2.2.15)$$

under the initial value condition (2.2.9a)

It is to be noted that the partitioned form of F suggests that the effect of the state vector $x_2(t)$ upon the terminal cost should be little. In other words, if the order of each element of F is roughly equal to each other, the elements F_2 and F_3 should take a greater value than F_1.

2.2.2 Asymptotic expansions

The asymptotic property of the original system (2.2.1) as $\varepsilon \to 0$, indicates that K has the partitioned form

$$K = \begin{bmatrix} K_1 & \varepsilon K_2 \\ \varepsilon K_2' & \varepsilon K_3 \end{bmatrix}, \tag{2.2.16}$$

and for g

$$g = \begin{bmatrix} g_1 \\ \varepsilon g_2 \end{bmatrix}, \tag{2.2.17}$$

where K_1 is an nxn matrix, K_2 nxm, K_3 mxm, and g_1 is an n-dimensional vector, and g_2 m-dimensional. The partitioning is seen reasonable by simple manipulations of direct substitutions. Moreover the partitioned forms of K and g, and the terminal conditions (2.2.12), (2.2.13) for K and g also support the partitioned form of F.

Equation (2.2.11) can be written for each element K_i as

$$\dot{K}_1 = -K_1A_1 - A_1'K_1 - K_2A_3 - A_3'K_2 + K_1E_1K_1 + K_2E_2'K_1 + K_1E_2K_2' + K_2E_3K_2' - C_1'Q_1C_1, \tag{2.2.18a}$$

$$\varepsilon\dot{K}_2 = -K_1A_2 - K_2A_4 - A_3'K_3 + K_1E_2K_3 + K_2E_3K_3 - C_1'Q_2C_2 + \varepsilon(-A_1'K_2 + K_1E_1K_2 + K_2E_2'K_2), \tag{2.2.18b}$$

$$\varepsilon\dot{K}_3 = -K_3A_4 - A_4'K_3 + K_3E_3K_3 - C_2'Q_3C_2 + \varepsilon(-K_2'A_2 - A_2'K_2$$

$$+ \; K_3E_2'K_2 + K_2'E_2K_3) + \varepsilon^2K_2'E_1K_2. \tag{2.2.18c}$$

And for g_i's, we have

$$\dot{g}_1 = -(A_1 - E_1K_1 - E_2K_2')'g_1 - (A_3 - E_2'K_1 - E_3K_2')'g_2 - C_1Q_1z_1$$

$$-C_2Q_2z_2, \tag{2.2.19a}$$

$$\varepsilon\dot{g}_2 = -(A_2 - \varepsilon E_1K_2 - E_2K_3)'g_2 - (A_4 - \varepsilon E_2'K_2 - E_3K_3)'g_2 - C_1\bar{Q}_2'z_1$$

$$-C_2Q_3z_2, \tag{2.2.19b}$$

where

$$E_1 = B_1R^{-1}B_1', \quad E_2 = B_1R^{-1}B_2', \quad E_3 = B_2R^{-1}B_2'.$$

The reduced system for K_i^0's, is obtained by setting $\varepsilon = 0$

$$\dot{K}_1^0 = -K_1^0A_1 - A_1'K_1^0 - K_2^0A_3 - A_3'K_2^0 + K_1^0E_1K_1^0 + K_2^0E_2'K_1^0$$

$$+ K_1^0E_2K_2^{0\prime} + K_2^0E_3K_2^{0\prime} + C_1'Q_1C_1, \tag{2.2.20a}$$

$$0 = -K_1^0A_2 - K_2^0A_4 - A_3'K_3^0 + K_1^0E_2K_3^0 + C_1'Q_1C_2, \tag{2.2.20b}$$

$$0 = -K_3^0A_4 - A_4'K_3^0 + K_3^0E_3K_3^0 + C_2'Q_3C_2. \tag{2.2.20c}$$

For g_i^0's the reduced system is

$$\dot{g}_1^0 = -(A_1 - E_1K_1^0 - E_2K_2^{0\prime})'g_1^0 - (A_3 - E_2'K_1^0 - E_3K_2^{0\prime})'g_2^0$$

$$- C_1Q_1z_1 - C_2Q_2z_2, \tag{2.2.21a}$$

$$0 = -(A_2 - E_2'K_3^0)'g_1^0 - (A_4 - E_3K_3^0)'g_2^0 - C_1Q_2'z_1 - C_2Q_3z_2. \tag{2.2.21b}$$

The terminal conditions for these systems are given by, using the cancellation law of O'Malley et al. [0.68]

$$K_1^0(t_f) = C_1'(t_f)F_1C_1(t_f), \tag{2.2.22}$$

and

$$g_1^0(t_f) = -C_1'(t_f)F_1 z_1(t_f), \tag{2.2.23}$$

where z_1 corresponds to the vector C_1X_1.

The first correction system can be derived by differentiating Eq.(2.2.17) with respect to ε and setting $\varepsilon = 0$, as follows:

$$\dot{K}_1^1 = -K_1^1A_1 - A_1'K_1^1 - K_2^1A_3 - A_3'K_2^1 + K_1^1E_1K_1^0 + K_1^0E_1K_1^1 + K_2^1E_2'K_1^0$$
$$+ K_2^0E_2'K_1^1 + K_1^1E_2K_2^{0\prime} + K_1^0E_2K_2^{1\prime} + K_2^1E_3K_2^{0\prime} + K_2^0E_3K_2^{1\prime}, \tag{2.2.24a}$$

$$0 = -K_1^1A_2 - K_2^1A_4 - A_1'K_2^0 - A_3'K_3^1 + K_1^0E_1K_2^0 + K_2^0E_2'K_2^0 + K_1^0E_2K_3^1$$
$$+ K_1^1E_2K_3^0 + K_1^0E_3K_2^0 - \dot{K}_2^0, \tag{2.2.24b}$$

$$0 = -K_3^1A_4 - A_4'K_3^1 + K_3^1E_3K_3^0 + K_3^0E_3K_3^1 - K_2^{0\prime}A_2 - A_2'K_2^0 + K_3^0E_2'K_3^0$$
$$+ K_2^0E_2K_3^0 - \dot{K}_3^0. \tag{2.2.24c}$$

For g_i^j, we have

$$\dot{g}_1^1 = -(A_1 - E_1K_1^0 - E_2K_2^0)'g_1^1 - (A_3 - E_2'K_1^0 - E_3K_2^{0\prime})'g_2$$
$$+ (E_1K_1^1 + E_2K_2^{1\prime})g_1^0 + (E_2'K_1^1 + E_3K_2^{1\prime})'g_2^0, \tag{2.2.25a}$$

$$0 = -(A_2 - E_2K_3^0)'g_1^1 + (A_4 - E_3K_3^0)'g_2^1 + (E_1K_2^0 + E_2K_3^1)'g_1^0$$
$$+ (E_2'K_2^0 + E_3K_3^1)'g_2^0 - \dot{g}_2^0, \tag{2.2.25b}$$

where it is assumed that A_i's, B_i's, C_i's, Q_i's, etc. are constant with respect to ε in order to avoid introducing superfluous notations, but this assumption is not essential. The first correction system (2.2.24) - (2.2.25) can also be obtained in the following way.

Expanding K_i's and g_i's into Taylor series in ε, we get

$$K_i = K_i^0 + \varepsilon K_i^1 + \frac{\varepsilon^2}{2} K_i^2 + \ldots\ldots, \quad i = 1, 2, 3, \tag{2.2.26}$$

$$g_i = g_i^0 + \varepsilon g_i^1 + \frac{\varepsilon^2}{2} g_i^2 + \ldots\ldots, \quad i = 1,2. \tag{2.2.27}$$

Substituting (2.2.26) and (2.2.27) into Eqs.(2.2.17) and (2.2.18b), and equating the coefficients of like powers of ε, we have the reduced system as a coefficient of ε^0, and the first correction system as a coefficient of ε. The higher correction systems can be derived by using the same procedures.

The terminal conditions under which Eqs.(2.2.24) and (2.2.25) are solved are not generally set to be zero. This situation is peculiar to the singular perturbation theory. Following Vasil'eva's integral formula [Vs.63.1], we obtain the relations between the terminal conditions and the associate boundary layer systems.

The boundary layer system plays a crucial role in the singular perturbation theory; in this case, for K it results in

$$-\frac{d}{d\tau}\bar{K}_2 = -K_1^0A_2 - \bar{K}_2A_4 - A_3'\bar{K}_3 + K_1^0E_2\bar{K}_3 + \bar{K}_2E_3\bar{K}_3 - C_1'Q_2C_2, \tag{2.2.28}$$

$$-\frac{d}{d\tau}\bar{K}_3 = -\bar{K}_3A_4 - A_4'\bar{K}_3 + \bar{K}_3E_3\bar{K}_3 - C_2'Q_3C_2. \tag{2.2.29}$$

For g, we get

$$-\frac{d}{d\tau}\bar{g}_2 = -(A_2 - E_2\bar{K}_3)'g_1^0 - (A_4 - E_3\bar{K}_3)'\bar{g}_2 - C_1Q_2'z_1 - C_2Q_3z_2, \tag{2.2.30}$$

where the independent variable is τ, and t is regarded as a fixed parameter. The above system is considered as a kind of stretched system derived by the "stretching transformation" $\tau = [(t_f-t)/\varepsilon]$, $\tau \in [0,\infty]$. The terminal conditions for these equations are

$$\bar{K}_2(t_f) = C_1'(t_f)F_2C_2(t_f), \tag{2.2.31}$$

$$\bar{K}_3(t_f) = C_2'(t_f)F_3C_2(t_f), \tag{2.2.32}$$

and

$$\bar{g}_2(t_f) = -C_2'(t_f)F_3z_2(t_f). \tag{2.2.33}$$

Then the terminal conditions for the first correction equations are given as follows:

$$K_1(t_f) = \int_0^\infty [(-\bar{K}_2 A_3 - A_3'\bar{K}_2 + \bar{K}_2 E_2' K_1^0 + K_1^0 E_2 \bar{K}_2' + \bar{K}_2 E_3 \bar{K}_2') \\ - (-K_2^0 A_3 - A_3' K_2^0 + K_2^0 E_2' K_1^0 + K_1^0 E_2 K_2^0 + K_2^0 E_3 K_2^{0'})] d\tau, \quad (2.2.34)$$

$$g_1(t_f) = \int_0^\infty [E_2(\bar{K}_2' - K_2^{0'}) g_1^0 - (A_3 - E_2' K_1^0 - E_3 \bar{K}_2')\bar{g}_2 \\ - (A_3 - E_2' K_1^0 - E_3 K_2^{0'}) g_2^0] d\tau, \quad (2.2.35)$$

where $\bar{K}_2$, $\bar{K}_3$ and $\bar{g}_2$ are solutions of the boundary layer systems (2.2.28) - (2.2.30) with the boundary conditions (2.2.31) - (2.2.33).

A similar procedure is applied to solve the recursive equation for the higher order, and it is possible to construct an approximate solution whose accuracy is a desired one.

2.2.3 Basic theorems

We have derived a representation of the approximate solution in the form of asymptotic expansions. Here the basic theorems are given upon which the validity of the approximation depends.

In the regular optimization problem, the following conditions are usually assumed to hold for $t \in [t_0, t_f]$, and $\varepsilon \in [0, \varepsilon_0]$:

C1. A_i's and B_i's are holomorphic with respect to t and ε.

C2. R and Q are positive definite and holomorphic with respect to t and ε.

C3. F is time invariant, positive semidefinite and continuous with respect to ε.

C4. z is bounded, and holomorphic with respect to ε.

We state below the conditions C5 and C6, essential to the singular perturbation theory.

C5. The system (2.2.1) is boundary layer controllable.

In other words,

$$\text{rank}[B_2,\ A_4B_2,\ A_4^2B_2\ \ldots,\ A_4^{m-1}B_2] = m.$$

C6. A_4 is a stable matrix.

We first state a main theorem which gives asymptotic accuracy.

Theorem 2-1 If conditions C1 - C6 hold, then there exist functions $U_i^k(t,\varepsilon)$ and $V_i^k(t,\varepsilon)$ bounded in the outer region such that

$$K_1(t) = \sum_{j=0}^{k} K_1^{\,j}(t)\varepsilon^j (j!)^{-1} + U_1(t,\varepsilon)\varepsilon^{k+1}, \text{ for } t \in [t_0, t_f - \delta], \tag{2.2.36a}$$

$$K_i(t) = \sum_{j=0}^{k} K_i^{\,j}(t)\varepsilon^j (j!)^{-1} + U_i(t,\varepsilon)\varepsilon^{k+1}, \text{ for } t \in [t_0, t_f - \delta];\quad i = 2,3, \tag{2.2.36b,c}$$

$$g_1(t) = \sum_{j=0}^{k} g_1^{\,j}(t)\varepsilon^j (j!)^{-1} + V_1(t,\varepsilon)\varepsilon^{k+1}, \text{ for } t \in [t_0, t_f - \delta], \tag{2.2.37a}$$

$$g_2(t) = \sum_{j=0}^{k} g_2^{\,j}(t)\varepsilon^j (j!)^{-1} + V_2(t,\varepsilon)\varepsilon^{k+1}, \text{ for } t \in [t_0, t_f - \delta], \tag{2.2.37b}$$

where

$\delta = -C\varepsilon\log\varepsilon$; C is independent of ε.

The proof of this theorem is carried out by proving the following three lemmas. In Lemma 2-1 it is shown that the solution $K_3(\tau)$ of the boundary layer system (2.2.29) is asymptotically stable with respect to the positive definite root $K_3(t)$ of Eq.(2.2.20c) for $t = t_f$. The similar result for $K_2(\tau)$ is shown in Lemma 2-2. Lemma 2-3 establishes the existence and uniqueness of the solution of the reduced system (2.2.20). Lemma 2-2 corresponds to Lemma 1-1 in Chapter 1. These facts make Theorem 1-3 applicable to this problem.

Lemma 2-1 If conditions C1- C6 hold, then the boundary layer equation (2.2.29) has an isolated and asymptotically stable solution as $\tau \to \infty$, which is positive definite and is obtained by equating the right hand side of Eq.(2.2.29) to zero. Moreover, the matrix

$-(-A_4 + K_3Q_3)$ is stable.

proof The lemma is proved from the results obtained in proper theorems in Reid [R.63], [R.65] (see Appeendix A) dealing with two point boundary value problems associated with the state regulator problem. In the boundary layer system (2.2.24), t is considered as a fixed parameter, so that every coefficient matrix appearing in Eq.(2.2.29) is time invariant. Conditions C1 - C6 may involve every assumption in the Theorems A-2 and A-3 in Appendix A. Q.E.D.

Lemma 2-2 If conditions C1 - C6 hold, then the boundary layer equations (2.2.29) and (2.2.30) have, respectively, an isolated and asymptotically stable solution, which is given as in Lemma 2-1.

proof The Jacobian matrix of Eq.(2.2.28) is given by

$$-(-A_4 + K_3Q_3)$$

which is a stable matrix as $\tau \to \infty$ by Lemma 2-1. Hence the solution $K_2(t_f)$ of Eq.(2.2.28) is asymptotically stable with respect to the equilibrium $K_2(t_f)$ obtained in the same manner as in Lemma 2-1. Q.E.D.

Lemma 2-3 ([Ko.72]) If conditions C1 - C6 hold, then the solution of the reduced system (2.2.20) exists and is unique on the interval $[t_0,t_f]$.

proof We can derive the following matrix Riccati equation by manipulating Eq.(2.2.20).

$$\dot{K}_1^0 = -K_1^0A_0 - A_0'K_1^0 + K_1^0B_0R^{-1}B_0K_1^0 - Q_0,$$
$$K_1(t_f) = C_1'(t_f)F_1C_1(t_f), \tag{2.2.38}$$

where

$$A_0 = A_1 + H_1A_3 + E_2H_2' + H_1E_1H_2',$$

$$B_0 = B_1 + H_1 B_2,$$
$$Q_0 = -H_2 A_3 - A_3' H_2' - H_2 E_3 H_2' + Q_1,$$
$$H_1 = (E_2 K_3 - A_2)(A_4 - E_3 K_3)^{-1},$$
$$H_2 = (A_3' K_3 + Q_2)(A_4 - E_3 K_3)^{-1},$$

and K_3 is a positive definite solution of Eq.(2.2.20c).
The well-known theory of optimal control [At.66] ascertains that if R is positive definite, Q and F are positive semidefinite, then the solution K_1 exists and is unique in the interval $[t_0, t_f]$. In order to show the positive definiteness of Q_0, we derive

$$v'Q_0 v = w'Q_3 w + v'Q_2 w + w'Q_2' v + v'Q_1 v$$
$$= [v', w'] \begin{bmatrix} Q_1 & Q_2 \\ Q_2' & Q_3 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} > 0, \qquad (2.2.39)$$

where v is an arbitrary vector and w is given by

$$w = -(A_4 K_3 + Q_3)'^{-1}(Q_2 + A_3' K_3)' v. \qquad (2.2.40)$$

Hence Q_0 is positive definite, which completes the proof. Q.E.D.

Lemma 2-1 - 2-3 establish the conditions needed under which Theorem 1-3 holds. Therefore the proof of Theorem 2-1 is completed. It is remarked that the result of this section has been deduced under the conditions different from that Kokotović and Yackel [Ko.72] who adopted the following condition C7 in place of our condition C6.

C7. The system is boundary layer observable, i.e.,

$$\text{rank}[C_2', A_4' C_2', \ldots, (A_4')^{m-1} C_2'] = m.$$

The boundary layer correction should be added to the outer expansion in order to obtain the uniformly valid solution. But it is not considered here, and shall be investigated in Section 3.2. The result of Section 3.2.3 can be easily applied to this problem. If we obtain the uniformly valid expansion, the similar theorem to Theorem 1-2 holds.

2.3 Fixed-end-point, minimum energy problem

This section deals with fixed-end-point, fixed-end-time, minimum energy problems, which lead to the so-called ill-conditioned two-point boundary value problems.

Here is introduced the generalized Riccati transformation, and the situations peculiar to the fixed-end-point problems are shown. For the free-end-point problem treated in Section 2.2, it is needed that the state matrix of the boundary layer system should be stable, but for the fixed-end-point problem in this section our analysis shows that the assumption required becomes that it is either positively or negatively stable.

2.3.1 Problem statement

The state equation having the same form as shown in Section 2.2, is as follows:

$$\frac{d}{dt} x(t,\varepsilon) = A(t,\varepsilon)x(t,\varepsilon) + B(t,\varepsilon)u(t,\varepsilon), \tag{2.3.1}$$

$$A = \begin{bmatrix} A_1 & A_2 \\ \varepsilon^{-1}A_3 & \varepsilon^{-1}A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \varepsilon^{-1}B_2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The prescribed boundary conditions are

$$x(t_0,\varepsilon) = \xi(\varepsilon), \qquad x(t_f,\varepsilon) = \eta(\varepsilon), \tag{2.3.2}$$

or in the partitioned form

$$\begin{bmatrix} x_1(t_0,\varepsilon) \\ x_2(t_0,\varepsilon) \end{bmatrix} = \begin{bmatrix} \xi_1(\varepsilon) \\ \xi_2(\varepsilon) \end{bmatrix}, \quad \begin{bmatrix} x_1(t_f,\varepsilon) \\ x_2(t_f,\varepsilon) \end{bmatrix} = \begin{bmatrix} \eta_1(\varepsilon) \\ \eta_2(\varepsilon) \end{bmatrix}.$$

The problem is to minimize the performance index

$$J = \frac{1}{2} \int_{t_0}^{t_f} [x(t)'Qx(t) + u(t)'Ru(t)]dt. \qquad (2.3.3)$$

This problem, as easily seen, leads to the canonical equation

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B' \\ -Q & -A' \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}, \qquad (2.3.4)$$

where p is a costate vector of x. The canonical system (2.3.4) under the boundary conditions (2.3.2) is ill-conditioned in the sense of Mufti et al. [Mf.69], i.e., in our case the boundary condition of the costate vector p is not specified.

Now we define a Riccati transformation [Mf.69]

$$x(t) = M(t)p(t) + d(t), \qquad (2.3.5)$$

where M(t) is an (n + m) x (n + m)-matrix and d(t) is an (n + m)-dimendional vector. Notice that (2.3.5) is different from the conventional Riccati transformation (2.2.9) in control theory. By simple manipulation, we obtain the differential equation of Riccati type for M(t) and the associated differential equation for d(t) as below:

$$\dot{M} = AM + MA' + MQM - BR^{-1}B, \qquad (2.3.6)$$

$$\dot{d} = (A + MQ)d. \qquad (2.3.7)$$

Equations (2.3.6) and (2.3.7) are solved under the initial conditions

$$M(t_0) = 0, \qquad (2.3.8)$$

and

$$d(t_0) = \xi, \qquad (2.3.9)$$

or under the final conditions

$$M(t_f) = 0, \tag{2.3.8'}$$

$$d(t_f) = \eta. \tag{2.3.9'}$$

Partitioning M(t) and d(t) into the forms

$$M = \begin{bmatrix} M_1 & M_2 \\ M_2' & \varepsilon^{-1}M_3 \end{bmatrix}, \qquad d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \tag{2.3.10}$$

we get the equations for M_i, i = 1, 2, 3;

$$\begin{aligned} \dot{M}_1 &= A_1M_1 + M_1A_1 + A_2M_2' + M_2A_2' + M_1Q_1M_1 + M_2Q_2M_1 + M_1Q_2M_2 \\ &\quad + M_2Q_2M_2' - B_1R^{-1}B_1', \\ \varepsilon\dot{M}_2 &= M_2A_4' + M_2Q_3M_3 + M_1A_3 + M_1Q_2M_3 - B_1R^{-1}B_2' \\ &\quad + \varepsilon(A_1M_2 + A_2M_3 + M_1Q_1M_2 + M_2Q_2M_2), \\ \varepsilon\dot{M}_3 &= A_4M_3 + M_3A_4' + M_3Q_3M_3 - B_2R^{-1}B_2 + \varepsilon(A_3M_2 + M_2'A_3' \\ &\quad + M_3Q_2'M_2 + M_2'Q_2M_3) + \varepsilon^2M_2'Q_1M_2, \end{aligned} \tag{2.3.11}$$

and for d_i, i = 1, 2;

$$\begin{aligned} \dot{d}_1 &= (A_1 + M_1Q_1 + M_2Q_2)d_1 + (A_2 + M_1Q_2 + M_2Q_2)d_2, \\ \varepsilon\dot{d}_2 &= (A_3 + M_3Q_2' + \varepsilon M_2'Q_1)d_1 + (A_4 + M_3Q_3 + \varepsilon M_2'Q_2)d_2, \end{aligned} \tag{2.3.12}$$

where Q_i's are the elements of

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2' & Q_3 \end{bmatrix}.$$

We call the system (2.3.11), (2.3.12) with (2.3.8), (2.3.9) the "forward full system" and the system (2.3.11), (2.3.12) with (2.3.8'), (2.3.9') the "backward full system".

After M and d are determined under the terminal conditions, we get the optimal control u*

$$u^* = -R^{-1}B'M(t)^{-1}(x(t) - d(t)). \tag{2.3.13}$$

The optimum trajectory is then given by

$$\dot{x} = (A - BR^{-1}B'M^{-1})x + BR^{-1}B'M^{-1}d, \tag{2.3.14}$$

where it is assumed that M is invertible, so that Eqs.(2.3.13) and (2.3.14) hold for $t \in\ ]t_0, t_f]$ for the forward system, or for $t \in [t_0, t_f[$ for the backward system.

2.3.2 Asymptotic expansions and approximate solutions

The reduced equations for M_i's and d_i's are obtained by letting $\varepsilon = 0$ in Eqs.(2.3.11) and (2.3.12),

$$\begin{aligned}
\dot{M}_1^0 &= A_1M_1^0 + M_1^0A_1' + A_2M_2^{0\prime} + M_2^0A_2' + M_1^0Q_1M_1^0 + M_2^0Q_2M_1^0 \\
&\quad + M_1^0Q_2M_2^{0\prime} + M_2^0Q_2M_2^{0\prime} - B_1R^{-1}B_1, \\
0 &= M_2^0A_4' + M_2^0Q_3M_3^0 + M_1^0A_3 + M_1^0Q_2M_3^0 - B_1R^{-1}B_2', \\
0 &= A_4M_3^0 + M_3^0A_4' + M_3^0Q_3M_3^0 - B_2R^{-1}B_2',
\end{aligned} \tag{2.3.15}$$

and

$$\begin{aligned}
\dot{d}_1^0 &= (A_1 + M_1^0Q_1 + M_2^0Q_2)d_1^0 + (A_2 + M_1^0Q_2 + M_2^0Q_3)d_2^0, \\
0 &= (A_3 + M_3^0Q_2')d_1^0 + (A_4 + M_3^0Q_3)d_2^0,
\end{aligned} \tag{2.3.16}$$

with the initial conditions for the forward reduced system

$$M_1^0(t_0) = 0, \tag{2.3.17}$$

and

$$d_1^0(t_0) = \xi_1, \tag{2.3.18}$$

or with the final conditions for the backward reduced system

$$M_1^0(t_0) = 0, \tag{2.3.17'}$$

and

$$d_1^0(t_0) = \eta_1. \tag{2.3.18'}$$

The superscript zero means that the corresponding variables are of the reduced system.

Now expanding M_i's and d_i's into Taylor series in ε, substituting them into Eqs.(2.3.11) and (2.3.12), and comparing the coefficients of like powers of ε thereof, we have the first correction system, the second, and so forth. The deduction is easy to derive as in the preceding section.

In this case, for the forward system the boundary layer system is

$$\frac{d}{d\tau}\,\overline{M}_2(\tau) = \overline{M}_2(\tau)A_4' + \overline{M}_2(\tau)Q_3\overline{M}_2(\tau) + M_1^0(t)A_3 + M_1^0(t)Q_2\overline{M}_3(\tau), \tag{2.3.19}$$

$$\frac{d}{d\tau}\,\overline{M}_3(\tau) = A_4\overline{M}_3(\tau) + \overline{M}_3(\tau)A_4' + \overline{M}_3(\tau)Q_3\overline{M}_3(\tau) - B_2R^{-1}B_2', \tag{2.3.20}$$

$$\frac{d}{d\tau}\,\overline{d}_2(\tau) = (A_3 + \overline{M}_3(\tau)Q_2')d_1^0(t) + (A_4 + \overline{M}_3(\tau)Q_3)\overline{d}_2(\tau), \tag{2.3.21}$$

where the independent variable is τ, and t is considered as a fixed parameter. This system can be obtained by the "left stretching transformation" $\tau = [(t-t_0)/\varepsilon]$, $\varepsilon \in [0,\varepsilon_0]$. The terminal conditions for these equations are

$$\overline{M}_2(\tau = 0) = 0, \tag{2.3.22}$$

$$\overline{M}_3(\tau = 0) = 0, \tag{2.3.23}$$

$$\overline{d}_2(\tau = 0) = \xi_2. \tag{2.3.24}$$

For the backward system, we can derive easily the backward boundary layer system in place of Eqs.(2.3.19) - (2.3.21) by introducing the "right stretching transformation" $\tau' = [(t_f - t)/\varepsilon]$, $\varepsilon \in [0,\varepsilon_0]$, with terminal conditions,

$$\overline{M}_2(\tau' = 0) = 0, \qquad (2.3.22')$$

$$\overline{M}_3(\tau' = 0) = 0, \qquad (2.3.23')$$

$$\overline{d}_2(\tau' = 0) = \eta_2. \qquad (2.3.24')$$

Then the terminal conditions for the first correction system are given as follows for the forward system:

$$M_1^1(t_0) = \int_0^\infty [(A_2\overline{M}_2'(\tau) + \overline{M}_2(\tau)A_2' + \overline{M}_2(\tau)Q_2M_1 + M_1^0Q_2\overline{M}_2'(\tau)) - (A_2M_2^0 + M_2^0A_2' + M_2^0Q_2M_1^0 + M_1^0Q_2M_2^{0\prime})]d\tau, \qquad (2.3.25)$$

$$d_1^1(t_0) = \int_0^\infty [(\overline{M}_2(\tau) - M_2^0)Q_2d_1^0 + (A_2 + M_1^0Q_2)(\overline{d}_2(\tau) - d_2^0) + \overline{M}_2(\tau)Q_3\overline{d}_2(\tau) - M_2^0Q_3d_2^0]d\tau, \qquad (2.3.26)$$

where $\overline{M}_2$, $\overline{M}_3$, and $\overline{d}_2$ are solutions of the boundary layer system (2.3.19) - (2.3.21) with the boundary conditions (2.3.22) - (2.3.24). For the backward system, the variable τ in Eqs.(2.3.25) and (2.3.26) should be replaced by τ', and instead of the initial conditions (2.3.22) - (2.3.24), we adopt the final conditions (2.3.22') - (2.3.24') for the backward boundary layer system, then obtaining $M_i^1(t_f)$ and $d_i^1(t_f)$.

A similar procedure is applied to solve the recursive equation for the higher order. As discussed later, whether the forward or backward system should be selected depends upon the property of the boundary layer system.

2.3.3 Suboptimum trajectory

Between the variables M, d in Eq.(2.3.5) and K, g in Eq.(2.2.9), there are the following relations:

$$K(t) = M(t)^{-1}, \tag{2.3.27}$$

and

$$g(t) = -M(t)^{-1}d(t). \tag{2.3.28}$$

Expanding $M(t)^{-1}$ into Taylor series in ε, we have(see Appendix B)

$$K(t) = M(t)^{-1}$$

$$= \begin{bmatrix} M_1^{0^{-1}} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} M_1^{1^{-1}} + M_1^{-2}M_2M_3^{-1}M_2' & -M_1M_2M_3^{-1} \\ -M_3^{-1}M_2'M_1^{-1} & M_3^{-1} \end{bmatrix} + O(\varepsilon^2)$$

$$= K^0 + \varepsilon K^1 + O(\varepsilon^2)$$

$$= \begin{bmatrix} K_0 & 0 \\ 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} K_1 & K_2 \\ K_2' & K_3 \end{bmatrix} + O(\varepsilon^2), \tag{2.3.29}$$

where the superscript zero of M_i is omitted for simplicity.

The optimum trajectory is given by Eq.(2.3.14). Expanding each variable into Taylor series in ε and comparing coefficients of like powers of ε thereof, we have for the reduced system

$$\begin{aligned} \dot{x}_1^0 &= (A_1 - E_1K_0)x_1^0 + A_2x_2^0 + (E_1K_0 + E_2K_2')d_1^0 + E_2K_3'd_2^0, \\ 0 &= (A_3 - E_2K_0)x_1^0 + (A_4 - E_3K_3)x_2^0 + E_2K_0d_1^0 + E_3K_3'd_2^0, \end{aligned} \tag{2.3.30}$$

with

$$x_1(t_0) = \xi_1, \quad \text{or } x_1(t_f) = \eta_1,$$

where

$$E_1 = B_1R^{-1}B_1', \quad E_2 = B_1R^{-1}B_2', \quad E_3 = B_2R^{-1}B_2'.$$

For the first correction system

$$\begin{aligned} \dot{x}_1^1 &= (A_1 - E_1K_0)x_1^1 + A_2x_2^1 + (E_1K_0 + E_2K_2')d_1^1 + E_2K_3'd_2^1 \\ &\quad - E_1K_2x_1^0 + (E_1K_2' + E_1K_1)d_1^0, \qquad (2.3.31a) \\ 0 &= (A_3 - E_2K_0)x_1^1 + (A_4 - E_3K_3)x_2^1 + E_2K_0d_1^1 + E_3K_3'd_2^1 - E_3K_2'x_1^0 \end{aligned}$$

$$- E_2{}'K_2x_2^0 + (E_3K_2{}' + E_2K_1)d_1^0 + E_2{}'K_2d_2^0, \tag{2.3.31b}$$

with the terminal condition for the forward system

$$x_1^1(t_0) = \int_0^\infty [A_2(\bar{x}_2(\tau) - x_2^0) + E_2K_2{}'(\bar{d}_2(\tau) - d_2^0)]d\tau, \tag{2.3.32}$$

where $\bar{x}_2(\tau)$ in the integrand is a solution of the boundary layer system

$$\frac{d}{d\tau}\bar{x}_2(\tau) = (A_3 - E_2K_0)x_1^0 + (A_4 - E_3K_3)\bar{x}_2(\tau) + E_2{}'K_0d_1^0 + E_3K_2{}'\bar{d}_2(\tau), \tag{2.3.33}$$

with

$$\bar{x}_2(\tau = 0) = \xi_2. \tag{2.3.34}$$

For the backward system, similar conditions are easily obtained by the same procedure as in the preceding description. It is remarked that we need to solve the forward system for x_i's when we solve the forward system for M_i's and d_i's and vice versa. This situation is different from the case using the conventional Riccati transformation (2.2.10).

After determining M_i's, d_i's, and x_i's, the suboptimal control u_{sub} is given as follows:

$$u_{sub} = u^0 + \varepsilon u^1, \tag{2.3.35}$$

where

$$u^0 = -R^{-1}[(B_1{}'K_0 + B_2{}'K_2{}')(x_1^0 - d_1^0) + B_2{}'K_3(x_2^0 - d_2^0)],$$

$$u^1 = -R^{-1}[(B_1{}'K_0 + B_2{}'K_2{}')(x_1^1 - d_1^1) + B_2{}'K_3(x_2^1 - d_2^1) + B_1{}'K_1(x_1^0 - d_1^0) + B_1{}'K_2(x_2^0 - d_2^0)].$$

2.3.4 A simple example

We shall show the outline of constructing the terminal values of higher order system for

$$\frac{d}{dt} x_1 = x_1 - u,$$
$$\varepsilon \frac{d}{dt} x_2 = x_1 - x_2 + u, \tag{2.3.36}$$

where x_1, x_2 and u are scalars with $x_i(t_0) = \xi_i$ and $x_i(t_f) = 0$ prescribed. The performance index to be minimized is given as follows:

$$J = \frac{1}{2}\int_{t_0}^{t_f} (x'x + ru^2)dt. \tag{2.3.37}$$

The Riccati transformation $x(t) = M(t)p(t) + d(t)$ yields

$$\frac{d}{dt} m_1 = m_1{}^2 + 2m_2 + m_2{}^2 - r^{-1}, \tag{2.3.38a}$$

$$\varepsilon \frac{d}{dt} m_2 = m_1 - m_2 + m_3 + m_2 m_3 - r^{-1} + \varepsilon m_1 m_2, \tag{2.3.38b}$$

$$\varepsilon \frac{d}{dt} m_3 = -2m_3 + m_3{}^2 - r^{-1} + 2\varepsilon m_2 + \varepsilon^2 m_2{}^2, \tag{2.3.38c}$$

for each element of $M = \begin{bmatrix} m_1 & m_2 \\ m_2 & \varepsilon^{-1} m_3 \end{bmatrix}$, where m_1, m_2 and m_3 are scalars.

The boundary layer system of Eq.(2.3.38) is

$$\frac{d}{d\tau} \bar{m}_2 = m_1(t_0) - \bar{m}_2 + \bar{m}_3 + \bar{m}_2\bar{m}_3 - r^{-1}, \tag{2.3.39a}$$

and

$$\frac{d}{d\tau} \bar{m}_3 = -2\bar{m}_3 + \bar{m}_3{}^2 - r^{-1}. \tag{2.3.39b}$$

Now we have an asymptotically stable point as $\tau \to \infty$ for $\bar{m}_3$,

$$\bar{m}_3(\infty) = 1 - (1 + r^{-1})^{1/2} < 0, \tag{2.3.40}$$

by equating the right hand side of Eq.(2.3.39b) to zero, where

$$\bar{m}_3 = 1 + (1 + r^{-1})^{1/2} > 0$$

is excluded because of Lemma 2-4 in Section 2.3.5. For $\bar{m}_2$ we get from Eqs.(2.3.39a) and (2.3.40)

$$\bar{m}_2(\infty) = -1 - (1 + r^{-1})^{1/2} \tag{2.3.41}$$

The initial value of $m_1^1(t_0)$ is given from Eq.(2.3.25):

$$m_1(t_0) = \int_0^{\infty} [2\bar{m}_2(\tau) + \bar{m}_2{}^2(\tau) - 2m_2^0(t_0) - m_2^{02}(t_0)]d\tau. \tag{2.3.42}$$

Eliminating $m_2(\tau)$ in Eq.(2.3.42) by using Eqs.(2.3.39a), (2.3.39b), (2.3.40) and (2.3.41), we get

$$m\ (t_0) = \left| -\frac{1}{2(1 + r^{-1})^{1/2}}\,\bar{m}_2\ (\tau) - \frac{1 + (1 + r^{-1})^{1/2}}{1 + r^{-1}}\,\dot{\bar{m}}_2(\tau)\right|_0^{\infty}$$

$$= \frac{-2 + (r^{-1} - 2)(1 + r^{-1})^{1/2}}{2(1 + r^{-1})^{1/2}}\,, \tag{2.3.43}$$

which can be obtained by a simple manipulation under the asymptotic stability of Eqs.(2.3.39a) and (2.3.39b), considering $\bar{m}_2(0) = 0$. Now we can compute the reduced system for $m_i^0(t)$ and the first correction system for $m_i^1(t)$, and the similar procedure can be applied to higher order terms, $m_i^j(t)$, for $j > 1$ and to the recursive equations for $d_i^j(t)$, and $x_i^j(t)$.

2.3.5 Basic theorems

Some conditions are needed to establish main theorems and lemmas. We first state the prerequisite conditions and then the basic theorems are offered.

As in the preceding section, the following conditions are usually assumed to hold for $t \in [t_0, t_f]$ and $\varepsilon \in [0,\varepsilon_0]$:

C1. A_i's and B_i's are holomorphic with respect to t and ε.

C2. R and Q are positive definite and holomorphic with respect to t and ε.

C3. ξ and η are continuous with respect to ε.

The conditions C1 - C3 are equivalent to C1 - C3 in Section 2.2.

We introduce an important condition in an analogous form to the observability, as follows:

C4. $\text{rank}\ [Q_3,\ A_4Q_3,\ A_4^2Q_3,\ \ldots\ldots,\ A_4^{m-1}Q_3] = m.$

We add the following two conditions which are essential to the singular perturbation theory.

C5a. A_4 is a stable matrix.

C5b. $-A_4$ is a stable matrix.

These conditions C5a. and C5b. are exclusive and main theorems need either one.

<u>Lemma 2-4</u> If conditions C1 - C5a (C1 - C4 and C5b) hold, then the forward (backward) boundary layer equation (2.3.20) has an isolated and asymptotically stable solution as $\tau \to \infty$ ($\tau' \to \infty$), which is negative definite (positive definite) and is given by equating the right hand side of Eq.(2.3.20) to zero. Moreover a matrix $A_4 + M_3Q_3$ ($-A_4 - M_3Q_3$) is stable.

Lemma 2-4 plays a decisive role in the sequel and directly leads to Lemma 2-5.

<u>Lemma 2-5</u> If conditions C1 - C5a (C1 - C4 and C5b) hold, then the forward (backward) boundary layer equations (2.3.19) and (2.3.21) have respectively an isolated and asymptotically stable solution, which is given as in Lemma 2-4.

Lemmas 2-4 and 2-5 lead to the main theorems.

Theorem 2-2 If conditions C1 - C5a (C1 - C4 and C5b) hold for the forward system (backward system), then the following convergence relations between the reduced solutions and the full solutions are satisfied,

$$\lim_{\varepsilon\to 0} M_1(t) = M_1^0(t), \text{ for } t\in[t_0,t_f] \quad (t\in[t_0,t_f]),$$
$$\lim_{\varepsilon\to 0} M_i(t) = M_i^0(t), \text{ for } t\in]t_0,t_f] \quad (t\in[t_0,t_f[); \quad i = 2, 3, \tag{2.3.44}$$

$$\lim_{\varepsilon\to 0} d_1(t) = d_1^0(t), \text{ for } t\in[t_0,t_f] \quad (t\in[t_0,t_f]),$$
$$\lim_{\varepsilon\to 0} d_2(t) = d_2^0(t), \text{ for } t\in]t_0,t_f] \quad (t\in[t_0,t_f[). \tag{2.3.45}$$

Following Vasil'eva [Vs.63.1], we have Theorem 2-3 which gives error estimations.

Theorem 2-3 If conditions C1 - C5a (C1 - C4 and C5b) hold, then exist bounded functions $U_i(t,\varepsilon)$ and $V_i(t,\varepsilon)$ such that

$$M_1(t) = \sum_{j=0}^{k} M_1^{\,j}(t)(j!)^{-1}\varepsilon^j + U_1(t,\varepsilon)\varepsilon^{k+1},$$
$$M_i(t) = \sum_{j=0}^{k} M_i^{\,j}(t)(j!)^{-1}\varepsilon^j + U_i(t,\varepsilon)\varepsilon^{k+1},$$
$$; \text{ for } t\in[t_0+\delta,t_f](t\in[t_0,t_f-\delta]); \ i = 2,3, \tag{2.3.46}$$

$$d_1(t) = \sum_{j=0}^{k} d_1^{\,j}(t)(j!)^{-1}\varepsilon^j + V_1(t,\varepsilon)\varepsilon^{k+1},$$
$$d_2(t) = \sum_{j=0}^{k} d_2^{\,j}(t)(j!)^{-1}\varepsilon^j + V_2(t,\varepsilon)\varepsilon^{k+1},$$
$$; \text{ for } t\in[t_0+\delta,t_f](t\in[t_0,t_f-\delta]), \tag{2.3.47}$$

where

$$\delta = -C\varepsilon\log\varepsilon; \quad C \text{ is independent of } \varepsilon.$$

Now we have similar theorems for x and u, and these are obtained by simple manipulation directly from Theorems 2-2 and 2-3, so that the detail is omitted.

The proofs of these theorems and lemmas are shown directly or by simple modification from relevant theorems in Refs. [W.65, Vs.63.1, R63, R65, Lv.54]. Lemma 2-4 is proved from Theorems A-2 and A-3 in Reid [R.63, R65] (see Appendix A). Note that in the boundary layer system, t is regarded as a parameter and therefore the matrices in the matriceare constant, so that all assumptions of Theorem A in Reid are satisfied. Each Jacobian matrix of the right hand sides of the boundary layer systems (2.3.19) – (2.3.21) with respect to M_2, M_3 and d_2 respectively is $A_4 + M_3Q_3$, hence the proof of Lemma 2-5 is deduced straight from Lemma 2-4.

Theorem 2-2 is a modified one of the popular theorem [Lv.54] in the two-point boundary value problem of singular perturbation type. In this regard, Tikhonov's convergence theorem [Tk.52] in the initial value problem should be referred to. Theorem 2-3 is an extended one of Theorem 1-3 of Chapter 1.

2.4 Systems with two small parameters

The singular perturbations of differential equations with several parameters were studied in a few papers by Hoppensteadt [Hp.69.3], O'Malley [O.71.2], Vasil'eva [Vs.63.2], etc. In Hoppensteadt, the regular degeneration was considered as for the many parameter case. O'Malley [O.71.2], and Vasil'eva [Vs.63.2] obtained asymptotic expansion of solutions as to the two parameter case.

But no studies can be found on the subject of singular perturbation approach to optimal control problems of the system involving several parameters. This section treats state regulator problems of systems containing two parameters of different orders of smallness. The results of two parameter cases can be easily extended to many parameter cases with slight modifications.

2.4.1 Problem statement

Let the state equation be given as below:

$$\frac{d}{dt} x(t,\varepsilon) = A(t,\varepsilon)x(t,\varepsilon) + B(t,\varepsilon)u(t,\varepsilon), \tag{2.4.1}$$

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ \varepsilon_1^{-1}A_{21} & \varepsilon_1^{-1}A_{22} & \varepsilon_1^{-1}A_{23} \\ \varepsilon_2^{-1}A_{31} & \varepsilon_2^{-1}A_{32} & \varepsilon_2^{-1}A_{33} \end{bmatrix}; \quad B = \begin{bmatrix} B_1 \\ \varepsilon_1^{-1}B_2 \\ \varepsilon_2^{-1}B_3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

where x_1 is an n-dimensional state vector, x_2 m_1-dimensional, x_3 m_2-dimensional, and

$$0 << \varepsilon_2 << \varepsilon_1 << 1, \tag{2.4.2}$$

or

$$\lim_{\varepsilon\to 0} \varepsilon_2/\varepsilon_1 = 0. \tag{2.4.3}$$

Here we consider the state regulator problem to minimize the performance index

$$J = \frac{1}{2} x'(t_f)Fx(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [x'(t)Qx(t) + u'(t)Ru(t)]dt, \tag{2.4.4}$$

where

$$F = \begin{bmatrix} F_{11} & \varepsilon_1 F_{12} & \varepsilon_2 F_{13} \\ \varepsilon_1 F'_{12} & \varepsilon_1 F_{22} & \varepsilon_2 F_{23} \\ \varepsilon_2 F'_{13} & \varepsilon_2 F'_{23} & \varepsilon_2 F_{33} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q'_{12} & Q_{22} & Q_{23} \\ Q'_{13} & Q'_{23} & Q_{33} \end{bmatrix}.$$

The elements of a matrix F are expressed as in the above for the sake of convenience in our formulation. We assume $Q_{ij} = 0$ when $i = j$, for simplicity, though this is not essential. We can find feedback control u*, under the usual hypothesis of regulator problem:

$$u^* = -R^{-1}B'K(t)x(t), \tag{2.4.5}$$

where K(t) satisfies the following matrix differential equation of

Riccati type:

$$\dot{K} = -KA - A'K + KBR^{-1}B'K - Q, \quad K(t_f) = F, \qquad (2.4.6)$$

Now partitioning K into

$$K = \begin{bmatrix} K_{11} & \varepsilon_1 K_{12} & \varepsilon_2 K_{13} \\ \varepsilon_1 K_{12}' & \varepsilon_1 K_{22} & \varepsilon_2 K_{23} \\ \varepsilon_2 K_{13}' & \varepsilon_2 K_{23}' & \varepsilon_2 K_{33} \end{bmatrix},$$

we get the differential equations for each array

$$\frac{d}{dt} K_{11} = -K_{11}A_{11} - A_{11}'K_{11} + K_{11}E_{11}K_{11} - Q_{11} + f_1(K_{ij}); \ (i,j) \neq (1,1), \qquad (2.4.7a)$$

$$\mu_2 \frac{d}{dt} K_{12} = K_{12}(-A_{22} + E_{22}K_{22} + E_{23}K_{23}') + f_2(K_{ij}); \ (i,j) \neq (1,2), \qquad (2.4.7b)$$

$$\mu_2 \frac{d}{dt} K_{22} = K_{22}(-A_{22} + E_{23}K_{23}') + (-A_{22}' + K_{23}E_{23}')K_{22} + K_{22}E_{22}K_{22} - Q_{22} - f_3(K_{23}), \qquad (2.4.7c)$$

$$\mu_1\mu_2 \frac{d}{dt} K_{13} = K_{13}(-A_{33} + E_{33}K_{33}) + f_4(K_{ij}); \ (i,j) \neq (3,3), \qquad (2.4.7d)$$

$$\mu_1\mu_2 \frac{d}{dt} K_{23} = K_{23}(-A_{33} + E_{33}K_{33}) + f_5(K_{22}, K_{33}), \qquad (2.4.7e)$$

$$\mu_1\mu_2 \frac{d}{dt} K_{33} = -K_{33}A_{33} - A_{33}'K_{33} + K_{33}E_{33}K_{33} - Q_{33}, \qquad (2.4.7f)$$

where $E_{ij} = B_i R^{-1} B_j'$, $\mu_1 = \varepsilon_2/\varepsilon_1$, and $\mu_2 = \varepsilon_1$, and f_i's are polynomials of the first order of its argument with respect to the order of μ_i^0. For details, Appendix C may be referred to. The terminal conditions for the full system are

$$K_{ij}(t_f) = F_{ij}; \quad i, j = 1, 2, 3. \qquad (2.4.8)$$

2.4.2 Asymptotic expansions

We develop a method simpler than those presented by O'Malley

[O.71.2] or by Vasil'eva [Vs.63.2] in different ways. This method is an extended and modified one of Hoppensteadt's method [Hp.71.1] dealing with single parameter case.

For simplicity of notations, imbedding each element of K_{ij} into appropriate dimensional vectors u, v, and w, where u consists of elements of K_{13}, K_{23}, and K_{33}, we have

$$\frac{d}{dt} u = - f(u, v, w; \mu, t), \quad u(t_f) = u^0, \tag{2.4.9a}$$

$$\mu_2 \frac{d}{dt} v = - \phi(u, v, w; \mu, t), \quad v(t_f) = v^0, \tag{2.4.9b}$$

$$\mu_1\mu_2 \frac{d}{dt} w = - \psi(u, v, w; \mu, t), \quad w(t_f) = w^0, \tag{2.4.9c}$$

where $\mu = (\mu_1, \mu_2)'$, and minus signs of the right hand side of Eq.(2.4.9) is for the convenience's sake in the arguments below. It is to be noted that the system (2.4.9) has a hierarchy of boundary layers, here doubly structured.

Now introducing new independent variables by

$$\tau_1 = (t_f - t)/\mu_1\mu_2, \quad \tau_2 = (t_f - t)/\mu_2, \tag{2.4.10}$$

$$\tau_1 \in [0,\infty[, \quad \tau_2 \in [0,\infty[; \quad \text{as } |\mu|\to 0,$$

we have stretched systems of two kinds, the fundamental boundary layer system

$$\frac{d}{d\tau_1} u = \mu_1\mu_2 f(u, v, w; \mu, \mu_1\mu_2\tau_1), \tag{2.4.11a}$$

$$\frac{d}{d\tau_1} v = \mu_1 \phi(u, v, w; \mu, \mu_1\mu_2\tau_1), \tag{2.4.11b}$$

$$\frac{d}{d\tau_1} w = \psi(u, v, w; \mu, \mu_1\mu_2\tau_1), \tag{2.4.11c}$$

and the secondary boundary layer system

$$\frac{d}{d\tau_2} u = \mu_2 f(u, v, w; \mu,\mu_2\tau_2), \tag{2.4.12a}$$

$$\frac{d}{d\tau_2} v = \phi(u, v, w; \mu, \mu_2\tau_2), \tag{2.4.12b}$$

$$\mu_1 \frac{d}{d\tau_2} w = \psi(u, v, w; \mu, \mu_2\tau_2). \tag{2.4.12c}$$

We seek solutions of Eq.(2.4.9) of the form

$$u = \sum_{r,s=0}^{\infty} [u_{rs}(t) + U_{rs}(\tau_1) + \overline{U}_{rs}(\tau_2)](r!s!)^{-1}\mu_1{}^r\mu_2{}^s, \tag{2.4.13a}$$

$$\lim_{\tau_1\to\infty} U_{rs}(\tau_1) = 0, \quad \lim_{\tau_2\to\infty} \overline{U}_{rs}(\tau_2) = 0, \tag{2.4.14a}$$

and other variables v and w are also assumed to have the same form and referred to Eqs.(2.4.13b), (2.4.13c), (2.4.14b) and (2.4.14c).

In the outer region, $\Sigma u_{rs}(t)$, $\Sigma v_{rs}(t)$ and $\Sigma w_{rs}(t)$ satisfy Eq.(2.4.9), then we have a recursive system in the outer region by substituting the outer expansions into Eq.(2.4.9) and comparing the coefficent of μ_i^r, r = 0, 1, 2, .., thereof,

$$\begin{aligned} \frac{d}{dt} u_{00} &= -f(u_{00}, v_{00}, w_{00}; 0, t), \\ 0 &= -\phi(u_{00}, v_{00}, w_{00}; 0, t), \\ 0 &= -\psi(u_{00}, v_{00}, w_{00}; 0, t), \end{aligned} \tag*{(2.4.15)$_{00}$}$$

and

$$\begin{aligned} \frac{d}{dt} u_{rs} &= -f_u u_{rs} - f_v v_{rs} - f_w w_{rs} + p_{rs}(t), \\ \frac{d}{dt} v_{r,s-1} &= -\phi_u u_{rs} - \phi_v v_{rs} - \phi_w w_{rs} + q_{rs}(t), \\ \frac{d}{dt} w_{r-1,s-1} &= -\psi_u u_{rs} - \psi_v v_{rs} - \psi_w w_{rs} + r_{rs}(t). \end{aligned} \tag*{(2.4.15)$_{rs}$}$$

For the fundamental boundary layer system, we have the recursive set of equations by substituting Eq.(2.4.13) into Eq.(2.4.11) as follows:

$$\frac{d}{d\tau_1} U_{00} = 0,$$

$$\frac{d}{d\tau_1} V_{00} = 0, \qquad (2.4.16)_{00}$$

$$\frac{d}{d\tau_1} W_{00} = \psi(u_{00}(t_f)+U_{00}+\overline{U}_{00}(0),\ v_{00}(t_f)+V_{00}+\overline{V}_{00}(0),\ w_{00}(t_f)+W_{00}+\overline{W}_{00}(0);\ 0,\ 0),$$

and

$$\frac{d}{d\tau_1} U_{rs} = P_{rs}(\tau_1),$$

$$\frac{d}{d\tau_1} V_{r,s-1} = Q_{rs}(\tau_1), \qquad (2.4.16)_{rs}$$

$$\frac{d}{d\tau_1} W_{r-1,s-1} = \psi_u U_{rs} + \psi_v V_{rs} + \psi_w W_{rs} + R_{rs}(\tau_1).$$

For the secondary boundary layer system, we obtain

$$\frac{d}{d\tau_2} \overline{U}_{00} = 0,$$

$$\frac{d}{d\tau_2} \overline{V}_{00} = \phi(u_{00}(t_f)+U_{00}(0)+\overline{U}_{00}(\tau_2),\ v_{00}(t_f)+V_{00}(0)+\overline{V}_{00}(\tau_2),\ w_{00}(t_f)+W_{00}(0)+\overline{W}_{00}(\tau_2);\ 0,\ 0), \qquad (2.4.17)_{00}$$

$$0 = \psi(u_{00}(t_f)+U_{00}(0)+\overline{U}_{00}(\tau_2),\ \cdots;\ 0,\ 0),$$

and

$$\frac{d}{d\tau_2} \overline{U}_{rs} = \overline{P}_{rs},$$

$$\frac{d}{d\tau_2} \overline{V}_{rs} = \phi_u \overline{U}_{rs} + \phi_v \overline{V}_{rs} + \phi_w \overline{W}_{rs} + \overline{Q}_{rs}(\tau_2), \qquad (2.4.17)_{rs}$$

$$\frac{d}{d\tau_2} \overline{W}_{r,s-1} = \psi_u \overline{U}_{rs} + \psi_v \overline{V}_{rs} + \psi_w \overline{W}_{rs} + \overline{R}_{rs}(\tau_2),$$

where the remainders p_{rs}, q_{rs}, r_{rs}, P_{rs}, Q_{rs}, R_{rs}, etc. are polynomials involving only the terms known successively in the preceding steps.

The terminal conditions for these systems can be derived in the initial conditions for the terms with subscript $(r,s) = (1,0)$. From the secondary boundary layer system (2.4.12), we get

$$U_{10}(\tau_1) = U_{10}(0) + \int_0^{\tau_1} P_{10}(\sigma)d\sigma,$$
$$V_{10}(\tau_1) = V_{10}(0) + \int_0^{\tau_1} Q_{10}(\sigma)d\sigma. \tag{2.4.18}$$

The restriction on the boundary layer correction terms (2.4.14) makes us choose $U_{10}(0)$ and $V_{10}(0)$ as

$$U_{10}(0) = -\int_0^{\infty} P_{10}(\sigma)d\sigma, \tag{2.4.19}$$

$$V_{10}(0) = -\int_0^{\infty} Q_{10}(\sigma)d\sigma. \tag{2.4.20}$$

As for the fundamental boundary layer system (2.4.11), we can obtain

$$\overline{U}_{10}(0) = -\int_0^{\infty} \overline{P}_{10}(\sigma)d\sigma. \tag{2.4.21}$$

The integrals (2.4.19) - (2.4.21) may exist if the conditions

$$\phi_v \leqq -\kappa < 0, \tag{2.4.22}$$

$$\Psi_w \leqq -\kappa < 0, \tag{2.4.23}$$

hold. Then we can determine the terminal condition for the outer system of u_{10} by using Eqs.(2.4.13), (2.4.19) - (2.4.21),

$$u_{10}(t_f) = u_{10}^0 - U_{10}(0) - \overline{U}_{10}(0). \tag{2.4.24}$$

Under the condition (2.4.24) we can solve $u_{10}(t)$, $v_{10}(t)$, and $w_{10}(t)$, and we get $v_{10}(t_f)$ and $w_{10}(t_f)$.

From Eq.(2.4.20), $V_{10}(0)$ is known then $\overline{V}_{10}(0)$ is derived as follows:

$$\overline{V}_{10}(0) = v_{10}^0 - v_{10}(t_f) - V_{10}(0). \tag{2.4.25}$$

The secondary recursive system $(2.4.17)_{10}$ is solved with conditions (2.4.21) and (2.4.25). Thus we have $W_{10}(0)$, then $W_{10}(0)$ as follows:

$$W_{10}(0) = w_{10}^0 - w_{10}(t_f) - \overline{W}_{10}(0). \qquad (2.4.26)$$

Using Eq.(2.4.26), the fundamental recursive system $(2.4.16)_{10}$ is worked out. In this way, every system can be solved thoroughly.

A similar procedure may treat each higher order system satisfactorily. This method has the advantage that we need not solve any differential equation in order to get initial conditions (2.4.25) and (2.4.26).

2.4.3 Basic theorems

We first state prerequisite conditions for clarity, though they may be obvious from conditions stated in the preceding sections.

C1. A_i's, and B_i's are holomorphic with respect to t and ε_i.

C2. R and Q are positive definite and holomorphic with respect to t and ε_i.

C3. F is time invariant, positive semidefinite and continuous in ε_i.

The following conditions are proper to the several parameter case.

C4. The system (2.4.1) is fundamental boundary layer controllable, which is defined by

$$\text{rank } [B_3, A_{33}B_3, \ldots., A_{33}^{m_2-1}B_3] = m_2,$$

for fixed $t \in [t_0, t_f]$.

C5. A_{33} is a stable matrix.

C6. The system (2.4.1) is secondary boundary layer controllable, which is defined by

$$\text{rank } [B_2, A_2B_2, \ldots., A_2^{m_1-1}B_2] = m_1,$$

for fixed $t \in [t_0, t_f]$, where A_2 is given by

$$A_2 = A_{22} - E_{23}K_{23}. \qquad (2.4.27)$$

C7. A_2 is a stable matrix.

As discussed before, conditions C5 and C7 can be replaced by the following C5' and C7' respectively.

C5'. The system (2.4.1) is fundamental boundary layer observable, which is defined as,

$$\text{rank } [C_3', A_{33}'C_3', \ldots, (A_{33}')^{m_2-1}C_3'] = m_2,$$

for fixed $t \in [t_0, t_f]$, where C_3 is a solution of

$$C_3'C_3 = Q_{33}. \tag{2.4.28}$$

C7'. The system (2.4.1) is secondary boundary layer observable, which is defined as,

$$\text{rank } [C_2', A_2'C_2', \ldots, (A_2')^{m_1-1}C_2'] = m_1$$

for fixed $t \in [t_0, t_f]$, where C_2 is a solution of

$$C_2'C_2 = Q_2, \tag{2.4.29}$$

where

$$Q_2 = Q_{22} + K_{23}A_{23} + A_{32}K_{23}' - K_{23}E_{33}K_{23}'. \tag{2.4.30}$$

Under these conditions, we now give main theorems.

Theorem 2-4 If conditions C1 - C7 hold, then the following convergence relations between the reduced solutions and the full solutions are satisfied,

$$\lim_{\mu\to 0} K_{11}(t) = K_{11}^{00}(t), \quad \text{for } t \in [t_0, t_f], \tag{2.4.31}$$

$$\lim_{\mu\to 0} K_{ij}(t) = K_{ij}^{00}(t), \quad \text{for } t \in [t_0, t_f[;\ i,j=1,2,3, i\neq j. \tag{2.4.32}$$

As easily seen this theorem gives regular degeneration of the several parameter case. The succeeding theorem assures the asymptotic correctness.

Theorem 2-5 If conditions C1 - C7 hold, then there exist functions $S_{ij}(t,\mu)$ uniformly bounded in the interval considered, such that

$$K_{ij}(t) = \sum_{r+s\leq N} [K_{ij}^{rs}(t) + M_{ij}^{rs}(\tau_1) + N_{ij}^{rs}(\tau_2)](r!s!)^{-1}\mu_1^r\mu_2^s + \lambda^{N+1}S_{ij}(t,\mu), \qquad (2.4.33)$$

where $\lambda = \max(\mu_1, \mu_2)$, and $M_{ij}^{rs}(\tau_1)$ and $N_{ij}^{rs}(\tau_2)$ are respectively fundamental and secondary boundary layer correction terms corresponding to coefficients of the outer expansion $K_{ij}^{rs}(t)$. These matrices can be obtained through rearranging the elements of corresponding $U_{rs}(\tau_1)$, $U_{rs}(\tau_2)$, $V_{rs}(\tau_1)$, etc.

As in the single parameter case, if we restrict our attention to the outer expansion, it is easy to find the following theorem.

Theorem 2-6 If conditions C1 - C7 hold, then there exist functions $S_{ij}(t,\mu)$ uniformly bounded, such that

$$K_{11}(t) = \sum_{r+s\leq N} K_{11}^{rs}(t)(r!s!)^{-1}\mu_1^r\mu_2^s + \lambda^{N+1}S_{11}(t,\mu), \quad \text{for } t\in[t_0,t_f-\delta], \qquad (2.4.34)$$

$$K_{ij}(t) = \sum_{r+s\leq N} K_{ij}^{rs}(t)(r!s!)^{-1}\mu_1^r\mu_2^s + \lambda^{N+1}S_{ij}(t,\mu), \quad i,j=1,2,3;\ i\neq 1, \quad \text{for } t\in[t_0,t_f-\delta], \qquad (2.4.35)$$

where

$$\delta = -C\lambda\log\lambda; \quad C \text{ is independent of } \lambda.$$

Proofs of these theorems can be derived by inferences analogous to those of Section 2.2, and the outline is given here on the proof of Theorem 2-5. Theorem 2-4, and 2-6, will be proved as corollaries of Theorem 2-5.

Under the conditions C4 and C5, it is proved that the fundamental boundary layer equation for K_{33}

$$\frac{d}{d\tau_1}\overline{K}_{33} = \overline{K}_{33}A_{33} + A_{33}'\overline{K}_{33} - \overline{K}_{33}E_{33}\overline{K}_{33} + Q_{33} \qquad (2.4.36)$$

is asymptotically stable as $\tau_1 \to \infty$, which is equivalent to the system (2.2.29). Thus the following lemma is obtained in the same form as Lemma 2-1 of the single parameter systems.

Lemma 2-6 If conditions C1 - C5 hold, then the fundamental boundary layer equation (2.4.36) has an isolated and asymptotically stable solution as $\tau_1 \to \infty$, which is positive definite and is given by equating the right hand side of Eq.(2.4.36) to zero. Furthermore, $(A_{33} - E_{33}K_{33})$ is a stable matrix.

This lemma as usual plays a crucial role in the singular perturbations of matrix Riccati equations, and is directly followed by Lemma 2-7.

Lemma 2-7 If conditions C1 - C5 hold, then the fundamental boundary layer equations for K_{13} and K_{23} have respectively an isolated and asymptotically stable solution, which is derived as in Lemma 2-6.

The proof of this lemma is obtained easily, since each Jacobian matrix of the right hand side of the boundary layer equations for K_{13} and K_{23} with respect to K_{13} and K_{23} respectively is $A_{33} - E_{33}K_{33}$, and asymptotic behaviors of K_{13} and K_{23} are dominated by that of K_{33}.

The above argument is not seen peculiar to the several parameter case, if we refer to the results of Section 2.2. But the following discussions are characteristic of this case.

As for the secondary boundary layer system, we can have the following lemma.

Lemma 2-8 If conditions C1 - C3, C6, and C7 hold, then the secondary boundary layer equation for K_{22}

$$\frac{d}{d\tau_2}K_{22} = K_{22}A_2 + A_2'K_{22} - K_{22}E_{22}K_{22} + Q_2 \qquad (2.4.37)$$

has an isolated and asymptotically stable solution as $\tau_2 \to \infty$, which is positive definite and is given as in Lemma 2-6. Furthermore, the matrix $(A_{22} - E_{23}K_{23}' - E_{22}K_{22})$ is a stable matrix.

Lemma 2-8 leads to a lemma similar to Lemma 2-7, which assures the asymptotic stablity of the boundary layer equation for K_{12}, and is ommited in order to avoid repeating the same number over again.

Lemmas 2-6 - 2-8 show that every boundary layer equation associated with the original full system (2.4.7) has an isolated and asymptotically stable solution respectively, and therefore the conditions that assure the regular degeneration stated in Theorem 2-4 are all satisfied, the proof of which can be obtained by using a theorem of Hoppensteadt [Hp.69.3].

As for Theorem 2-5, we can prove it following O'Malley's result [O.71.2], with simple modifications and manipulations.

The results derived above are easily extended to the several parameters system of the form

$$\begin{aligned} \frac{d}{dt} x &= A_{00}x + \sum_{i=1}^{m} A_{0i}y_i + B_0u, \\ \mu_m \frac{d}{dt} y_1 &= A_{10}x + \sum_{i=1}^{m} A_{1i}y_i + B_1u, \\ &\cdots\cdots\cdots\cdot \\ \mu_1\mu_2\cdots\mu_m \frac{d}{dt} y_m &= A_{m0}x + \sum_{i=1}^{m} A_{mi}y_i + B_mu, \end{aligned} \qquad (2.4.38)$$

where $\mu_1, \mu_2, \ldots, \mu_m$ are positive small parameters. It is to be noted that the system (2.4.38) has a hierarchy of m-fold structured boundary layers.

It can easily be seen that there is a close relation between the several parameter case and the time invariant case with $t_f = \infty$ of the single parameter case (see Appendix D).

2.5 Asymptotic behaviour of performance index

It is natural to investigate the asymptotic behaviour of the performance index, once we have obtained the asymptotic expansion of the control and the trajectory.

In this section, we are concerned with the performance index of the tracking problem treated in Section 2.2, since the problem involves the regulator problem as a special case. And it is easy to extend the result to the other optimization problems, with performance index of non-quadratic type.

Concerning the terminal free, state regulator problem, the following theorem was proved in case of excluding the terminal cost by Werner and Cruz [Wr.68].

<u>Theorem 2-7</u> It is assumed that the performance index J has a quadratic form of

$$J = \frac{1}{2} e'(t_f)Fe(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [e'(t)Qe(t) + u'(t)Ru(t)]dt, \quad (2.5.1)$$

Let the exact optimal control be denoted by $u_e(t)$ and the approximate solution of the control represented in the form of the truncated series up to the n-th term be denoted by $u_a{}^n(t)$ as

$$u_a{}^n(t) = \sum_{r=0}^{n} u^r \varepsilon^r (r!)^{-1}. \quad (2.5.2)$$

Expanding the performance index J into Taylor series in

$$J(u) = \sum_{r=0}^{\infty} \frac{d^r}{d\varepsilon^r} J\bigg|_{\varepsilon=0} \varepsilon^r (r!)^{-1}, \quad (2.5.3)$$

we obtain the following approximate relations

$$\frac{d^k}{d\varepsilon^k} J(u_e(t))\bigg|_{\varepsilon=0} = \frac{d^k}{d\varepsilon^k} J(u_a{}^n(t))\bigg|_{\varepsilon=0}, \quad (2.5.4)$$

for k = 0, 1, 2, ..., 2n+1.

The proof of this theorem was originally made for the case that the terminal cost is not included in the performance index, and the system is of the regular perturbation type by Werner and Cruz [Wr.68]. Sannuti and Kokotović mentioned first the theorem above as to the singular perturbations of the regulator problem [Sa.69.1].

It is easy to see that, if we obtain the approximate solutions temporally uniform, there is little difference between the singular perturbations and the regular perturbations. The proof can be made in line with Werner and Cruz, and is omitted here.

As to the case that the performance index is of nonquadratic type, as for instance

$$J = \int_{t_0}^{t_f} (|x| + r|u|)dt, \tag{2.5.5}$$

we can easily obtain the following theorem instead of Theorem 2-7. If the problem is restricted to the fixed terminal systems, then the following theorem also holds.

<u>Theorem 2-8</u> If we obtain the approximate solution u_a^n as given in Theorem 2-7, and expand the performance index into Taylor series in ε, then we have the following approximate relations

$$\frac{d^k}{d\varepsilon^k} J(u_e(t))\Big|_{\varepsilon=0} = \frac{d^k}{d\varepsilon^k} J(u_a^n(t))\Big|_{\varepsilon=0}, \tag{2.5.6}$$

for $k = 0, 1, 2, \ldots, n$.

This theorem is intuitive and it may be proved when the performance index has no singularity with respect to its arguments. Such a condition holds usually in the ordinary optimization problems.

It is to be noted that if we restrict our attention to the outer expansion, which is rational from the practical motivation, only the slight changes are needed in the theorems described above as to the interval in which the validity of the asymptotic accuracy holds. The effect of the boundary layer upon the performance index is the same order of ε width of the boundary layer, unless the size of the boundary layer jump is the same order of ε^{-1}.

O'Malley [0.72.1] also stated the subject in the same form as Theorem 2-8 in case of the state regulator problem with quadratic cost.

A remarkable result is Theorem 2-7 particularly when the quadratic cost function is considered. The extension to the several parameter case is easy to be handled.

Chapter 3 Nuclear Reactor Control with Lumped Parameters

In this chapter, the nuclear reactor control as a lumped parameter system is considered. Such a lumped parameter model is well known in the field of reactor engineering as a "point reactor" or "one point approximation". Small reactors can be treated by the one point approximation with relatively small errors, if we restrict ourself to study the case in which the neutron flux changes slowly. Studies of dynamics and control of a point reactor have been made by many authors (see for instance Weaver[We.68], or Mohler and Shen [Mh. 70]).

The results in the previous chapters are successfully applied to the analysis and syntheses of suboptimal control of the nuclear reactor as a lumped parameter system.

This chapter consists of two sections. The early section is concerned with the error estimation of prompt jump approximation with the aid of singular perturbation theory. The prompt jump approximation is intuitively done by engineers and the singular perturbation theory can establish the validity of the approximation. The latter section treats the regulator problem of the reactor. Both sections have some numerical examples which enable us to understand the importance and validity of the theory presented.

3.1 Error estimation of prompt jump approximation

3.1.1 Introduction

In reactor dynamics, many ideas have been developed to give an error estimation of prompt jump approximation and to improve its accuracy. Goldstein and Shotkin [Gs.69] proposed the method of using integration by parts after transforming a system equation into an integral equation of Volterra type. Their method is effective in the outer region, but is not successful in giving uniformly valid solutions in the interval considered. The singular perturbation theory, an outline of which is described in Chapter 1, has been applied to this aim with satisfactory results.

In this section, we obtain uniformly valid solutions with use made of Vasil'eva's theory. The basic idea of the theory is different from that of boundary layer method given in Section 1.4. But the resulting series are equivalent to each other. And the singular perturbation theory has a principal innovation in the treatment of initial conditions of the generated recursive equations, quite defferent from that of the classical perturbation theory, as is mentioned on occasions.

Our object in this section is to give a refined error estimation of prompt jump approximation providing a method of improving its accuracy in preparation for the synthesis of suboptimal control of a point reactor system.

3.1.2 Prompt jump approximation

In the development of the point reactor model, spatial independence and one energy grouping are assumed. The kinetic equation of the model without source term and temperature-dependent reactivity feedback is given by

$$\frac{d}{dt} n = \frac{\delta k - \beta}{\ell} n + \lambda C, \tag{3.1.1}$$

$$\frac{d}{dt} C = \frac{\beta}{\ell} n - \lambda C, \tag{3.1.2}$$

where $n(t)$ and $C(t)$ are respectively neutron density and precursor concentration. Normalization of $n(t)$ and $C(t)$ with respect to their equilibrium values n_0, C_0 makes the system (3.1.1), (3.1.2) into the form of

$$\frac{\ell}{\beta}\frac{d}{dt} n(t) = (\rho(t) - 1)n(t) + C(t), \tag{3.1.3}$$

$$\frac{d}{dt} C(t) = \lambda n(t) - \lambda C(t), \tag{3.1.4}$$

where $\rho(t)$ is the reactivity in dollar units, i.e.,

$$\rho(t) = k(t)/\beta. \tag{3.1.5}$$

The initial condition is given by

$$n(0) = 1, \quad C(0) = 1. \tag{3.1.6}$$

The derivation of the above model can be found in many of the texts, for example in Weaver [Wv.68]

If the relative rate of the change of the reactor power in the mean prompt generation time is sufficiently small so that

$$\left|\frac{\ell}{\beta}\frac{\dot{n}}{n}\right| << |1 - \rho(t)| \tag{3.1.7}$$

holds, then the term $(\ell/\beta)\dot{n}(t)$ in the kinetic equation (3.1.3), (3.1.4) can be neglected as compared to $(\rho(t) - 1)n(t)$. The resulting approximation is written as follows:

$$\begin{aligned} 0 &= (\rho(t) - 1)n(t) + C(t), \\ \frac{d}{dt} C &= \lambda n(t) - \lambda C(t). \end{aligned} \tag{3.1.8}$$

This approximation is called the prompt jump approximation. Figure 3.1.1 is a sketch of the exact kinetic equation and the prompt jump approximation to it in case of step reactivity insertion where the prompt jump approximation is shown by the dashed line.

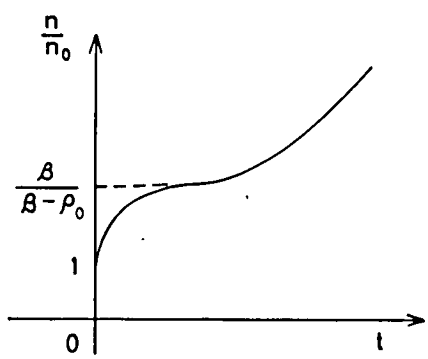


Fig. 3.1.1 Prompt jump approximation and the exact solution.

It is clear that the prompt jump approximation fails near prompt-critical, as seen in the condition (3.1.7), in which the right hand side of Eq.(3.1.7), $1 - \rho(t) \fallingdotseq 0$. The condition that gives the validity of the prompt jump approximation is given in various forms (see for instance Akcasu, Lellouche, and Shotkin [Ac.71]).

3.1.3 Analysis via singular perturbation theory

In Eq.(3.1.1), if we set $\varepsilon = \ell/\beta$, then ε has a relatively small value, for example $\varepsilon = 0(10^{-2})$ for the thermal reactor and $\varepsilon = 0(10^{-5})$ for the fast reactor, therefore equations (3.1.1) and (3.1.2) are of singular perturbation type.

Let solutions in the outer region be in the form of asymptotic expansions

$$
\begin{aligned}
n(t,\varepsilon) &= \sum_{r=0}^{\infty} n_r(t)\varepsilon^r(r!)^{-1}, \\
C(t,\varepsilon) &= \sum_{r=0}^{\infty} C_r(t)\varepsilon^r(r!)^{-1}.
\end{aligned}
\qquad (3.1.9)
$$

Substituting the above into Eqs.(3.1.1) and (3.1.2) and comparing coefficients of like powers of ε thereof, we obtain a recursion set of differential equations

$$
\varepsilon^0: \qquad
\begin{aligned}
0 &= (\rho(t) - 1)n_0(t) + C_0(t), \\
\dot{C}_0(t) &= \lambda(n_0(t) - C_0(t)),
\end{aligned}
\qquad (3.1.10)
$$

$$
\varepsilon^r: \qquad
\begin{aligned}
0 &= (\rho(t) - 1)n_r(t) + C_r(t) - \dot{n}_{r-1}(t), \\
\dot{C}_r(t) &= \lambda(n_r(t) - C_r(t)).
\end{aligned}
\qquad (3.1.11)
$$

Equation (3.1.8) for the leading terms $n_0(t)$ and $C_0(t)$ is equivalent to the prompt jump approximation (PJA).

The classical perturbation theory cannot be applied to this system, which may allow initial conditions of the form

$$C_0(0) = C(0), \quad C_r(0) = 0; \quad r > 0. \tag{3.1.12}$$

and leads to a wrong result. Therefore we must treat initial conditions by the singular perturbation theory.

Adopting the current procedure, we first obtain solutions in the inner region (the boundary layer) and the outer region, and then we use the singular perturbation theory by Vasil'eva to connect these solutions [Vs.63.1].

In order to obtain a solution in the boundary layer, we consider a stretched system based on the stretching transformation($\tau = t/\varepsilon$),

$$\begin{aligned} \frac{d}{d\tau}\bar{n}(\tau) &= (\rho(\tau) - 1)\bar{n}(\tau) + \bar{C}(\tau), \\ \frac{d}{d\tau}\bar{C}(\tau) &= \varepsilon\lambda(\bar{n}(\tau) - \bar{C}(\tau)), \end{aligned} \tag{3.1.13}$$

which admits given initial conditions (3.1.6). The solutions of this problem have convergent expansions in powers of ε,

$$\begin{aligned} \bar{n}(\tau,\varepsilon) &= \sum_{r=0}^{\infty} \bar{n}_r(\tau)\varepsilon^r(r!)^{-1}, \\ \bar{C}(\tau,\varepsilon) &= \sum_{r=0}^{\infty} \bar{C}_r(\tau)\varepsilon^r(r!)^{-1}. \end{aligned} \tag{3.1.14}$$

As the stretched system (3.1.13) has no singularity at $\varepsilon = 0$, through the usual procedure of the regular perturbation method, we have a recursive set of differential equations

$$\varepsilon^0: \quad \begin{aligned} \frac{d}{d\tau}\bar{n}_0(\tau) &= (\rho(\tau) - 1)\bar{n}_0(\tau) + \bar{C}_0(\tau), \\ \frac{d}{d\tau}\bar{C}_0(\tau) &= 0, \end{aligned} \tag{3.1.15}$$

$$\varepsilon^r: \quad \begin{aligned} \frac{d}{d\tau}\bar{n}_r(\tau) &= (\rho(\tau) - 1)\bar{n}_r(\tau) + \bar{C}_r(\tau), \\ \frac{d}{d\tau}\bar{C}_r(\tau) &= \lambda(\bar{n}_{r-1}(\tau) - \bar{C}_{r-1}(\tau)), \end{aligned} \tag{3.1.16}$$

with initial conditions

$$\bar{n}_0(0) = n(0), \quad \bar{C}_0(0) = C(0),$$
$$\bar{n}_r(0) = 0, \qquad \bar{C}_r(0) = 0; \quad r > 0. \tag{3.1.17}$$

While many authors have proposed various methods in order to relate the two types of expansions, (3.1.9) and (3.1.14), we expand the coefficients $n_r(t)$ and $C_r(t)$ of series (3.1.9) into formal Taylor series involving τ and ε; possibly divergent,

$$n_r(t) = \sum_{s=0}^{\infty} n_{rs} t^s = \sum_{s=0}^{\infty} n_{rs} \tau^s \varepsilon^s,$$
$$C_r(t) = \sum_{s=0}^{\infty} C_{rs} t^s = \sum_{s=0}^{\infty} C_{rs} \tau^s \varepsilon^s. \tag{3.1.18}$$

Substituting Eq.(3.1.18) into Eq.(3.1.9) and rearranging the summations according to powers of ε, we have the coupling series in the form

$$n(t,\varepsilon) = \sum_{r=0}^{\infty} \hat{n}_r(\tau)\varepsilon^r, \quad \hat{n}_r(\tau) = \sum_{s=0}^{r} n_{r-s,s}\tau^s,$$
$$C(t,\varepsilon) = \sum_{r=0}^{\infty} \hat{C}_r(\tau)\varepsilon^r, \quad \hat{C}_r(\tau) = \sum_{s=0}^{r} C_{r-s,s}\tau^s. \tag{3.1.19}$$

The series (3.1.19) are also formal solutions of the stretched system (3.1.13), but the difference between these two series lies in the initial values at $\tau = 0$.

Thus $\hat{n}_r(\tau)$ and $\hat{C}_r(\tau)$ satisfy the differential equations in the form

$$\hat{n}_0(\tau) = \frac{1}{1 - \rho(\tau)} C_0(0),$$
$$\hat{C}_0(\tau) = C(0), \tag{3.1.20}$$

and

$$\frac{d}{d\tau} \hat{n}_r(\tau) = (\rho(\tau) - 1)\hat{n}_r(\tau) + \hat{C}_r(\tau),$$
$$\frac{d}{d\tau} \hat{C}_r(\tau) = \lambda(\hat{n}_{r-1}(\tau) - \hat{C}_{r-1}(\tau)). \tag{3.1.21}$$

Several papers on the related subjects have been published [Am.69, Am.71], which have been based on the method of matched asymptotic expansions [Ku.67, Vd.64, Cl.68], but one of these [Am.71] leaves some doubt on the author's choice of initial conditions of the outer region.*

Here, we make use of Vasil'eva's method to determine initial conditions of outer region equations in such a manner that solutions of inner region equations (3.1.15) and (3.1.16) and those of Eqs.(3.1.20) and (3.1.21) coincide at $\tau = \infty$. Thus we have the initial conditions of outer region equations (3.1.10) and (3.1.11) and those of Eqs.(3.1.20) and (3.1.21) as follows:

$$\begin{aligned} C_0(0) &= \hat{C}_0(0) = C(0), \\ C_r(0) &= \hat{C}_r(0) = \int_0^\infty [\lambda\bar{n}_{r-1}(\sigma) - \lambda n_{r-1}(0)]d\sigma; \; r > 0, \end{aligned} \tag{3.1.22}$$

where $n_{r-1}(\sigma)$ is a solution of the boundary layer equation belonging to the full system (3.1.1) and (3.1.2):

$$\frac{d}{d\sigma}\bar{n}_{r-1}(\sigma) = (\rho(0) - 1)\bar{n}_{r-1}(\sigma) + \bar{C}_{r-1}(0), \tag{3.1.23}$$

with the initial conditions

$$\bar{n}_0(0) = n(0), \quad \bar{n}_r(0) = 0; \quad r > 0. \tag{3.1.24}$$

---

* The method adopts the initial conditions $n_0(0) = n(0)$, $n_1 = 0$. But it is not logical in the light of the theory of matched asymptotic expansion, which requires the limit process to decide initial values for outer expansion.

It is to be noted that in Eq.(3.1.23) $\overline{C}_{r-1}$ is regarded as a fixed paremeter, while in Eq.(3.1.15) $\overline{C}_{r-1}$ is a variable.

After determining successively n_r, $\overline{n}_r$, $\hat{n}_r$, C_r, $\overline{C}_r$, and $\hat{C}_r$, we define the functions R(t,ε) and S(t,ε) by

$$n(t,\varepsilon) = \sum_{r=0}^{m} [n_r(t) + \overline{n}_r(\tau) - \hat{n}_r(\tau)]\varepsilon^r(r!)^{-1} + R(t,\varepsilon)\varepsilon^{m+1}, \quad (3.1.25)$$

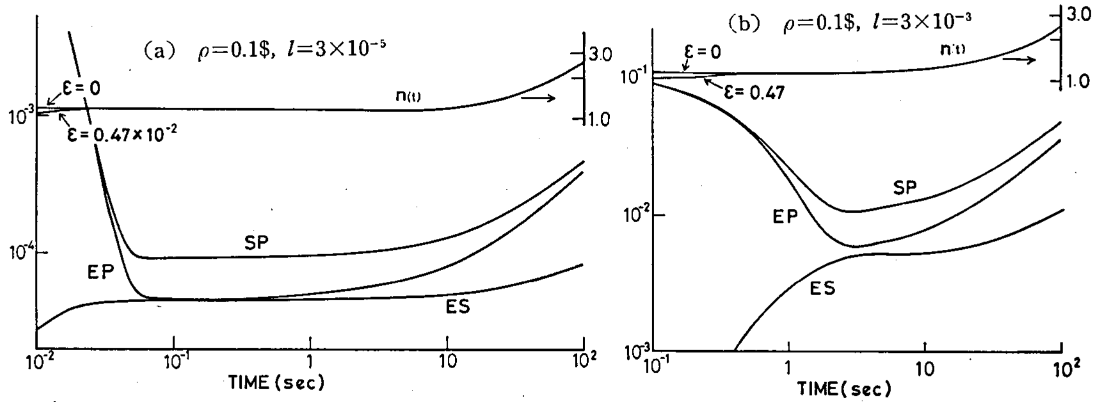
$$C(t,\varepsilon) = \sum_{r=0}^{m} [C_r(t) + \overline{C}_r(\tau) - \hat{C}_r(\tau)]\varepsilon^r(r!)^{-1} + S(t,\varepsilon)\varepsilon^{m+1}, \quad (3.1.26)$$

where $\tau = t/\varepsilon$. Assuming that $\rho(t) - 1 < 0$, the functions R(t,ε) and S(t,ε) are bounded for $0 \leq t \leq \infty$; $0 \leq \varepsilon << 1$. This means that an error of truncated series up to the (m + 1)-th term is of the order of ε^{m+1}.

3.1.4 Numerical examples

For simplicity we restrict ourselves to a step input of reactivity ($\rho(t) = \rho_0$, time independent).

In Fig.3.1.2 we can see PJA has a boundary layer at t = 0 and SPA has uniform accuracy with an error of the order of ε^2 as shown in Eq.(3.1.25). Figure 3.1.2 also shows that EP is less than expected in some time interval, and outside it, SP well Approximates EP. Our study by numerical calculations shows that, as far as step insertion of reactivity is concerned, SPA has a good accuracy when $\rho < 0.5\$$. In Fig.3.1.2 we used constants $\beta = 0.0064$, $\lambda = 0.078$.



$SP = \frac{|SPA-PJA|}{EXACT}$, $EP = \frac{|EXACT-PJA|}{EXACT}$ and $ES = \frac{|EXACT-SPA|}{EXACT}$, where EXACT is an exact solution.

Fig. 3.1.2 Accuracy of PJA and SPA.

The method developed here can treat arbitrary reactivity inputs so far as the asymptotic stability of the system for the neutron density n is preserved, by a simple modification and has close relation to the matched asymptotic expansion method proposed by Kaplun et al. independently of the method by Vasil'eva. Namely Eq.(3.1.9) corresponds to the "outer expansion" of the matched asymptotic expansions and Eq.(3.1.14) to the "inner expansion".

This method permits us to deal with more complex systems such as expressed by kinetic equations with multi-group of delayed neutrons, and furthermore with nonlinear systems.

3.2 Regulator problem of a point reactor

This section deals with the state regulator problem of a point reactor, which occurs when deviation from the equlibrium occurs. The same formulation may treat the power level change control in the sense of tracking problems.

3.2.1 Formulation of the problem

We consider here the same reactor model discussed in Section 3.1. Linearizing the equations (3.1.1) and (3.1.2) by considering small perturbations about some equilibrium neutron and precursor concentration level n_0 and C_0, we obtain

$$\frac{d}{dt}\,\delta n(t) = -\frac{\beta}{\ell}\,\delta n(t) + \lambda\delta C(t) + \frac{n_0}{\ell}\delta k(t), \qquad (3.2.1)$$

$$\frac{d}{dt}\,\delta C(t) = \frac{\beta}{\ell}\,\delta n(t) - \lambda\delta C(t), \qquad (3.2.2)$$

where δn, δC, and δk are small deviations in n, C, and k respectively.

Normalizing quantities $\delta n(t)$ and $\delta C(t)$ with respect to each steady state values n_0, C_0 respectively, the reactivity being represented in dollar units, we obtain the state equation to begin with

$$\frac{\ell}{\beta}\frac{d}{dt}\tilde{n}(t) = -\tilde{n}(t) + \tilde{C}(t) + \rho(t), \tag{3.2.3}$$

$$\frac{d}{dt}\tilde{C}(t) = \lambda\tilde{n}(t) - \lambda\tilde{C}(t), \tag{3.2.4}$$

where

$$\tilde{n}(t) = \frac{\delta n}{n_0}(t), \quad \tilde{C}(t) = \frac{\delta C}{C_0}(t), \tag{3.2.5}$$

and $\rho(t)$ is the externally applied reactivity. When the system deviates from the equilibrium, we meet the problem of regulating the deviation of the power level, or neutron density in a fixed time so that the performance index is a minimum, which is given here as below:

$$J = \frac{1}{2}\int_{t_0}^{t_f}(\tilde{n}^2 + r\rho^2)dt, \tag{3.2.6}$$

where $r > 0$ is a scalar weight. Therefore we assume that the value of the deviation can be measured at the initial time t_0, and should be represented in the form of initial conditions, as follows:

$$\tilde{n}(t_0) = \tilde{n}_0, \quad \tilde{C}(t_0) = \tilde{C}_0. \tag{3.2.7}$$

Further we assume for simplicity the end-terminal is free and the control may be obtained without constraints.

Letting $\tilde{C} = x_1$, $\tilde{n} = x_2$, $\rho = u$ and $\lambda = a$ for the sake of convenience, Eqs.(3.2.3) and (3.2.4) become

$$\frac{d}{dt}x_1 = -ax_1 + ax_2, \tag{3.2.8}$$

$$\varepsilon\frac{d}{dt}x_2 = x_1 - x_2 + u, \tag{3.2.9}$$

with initial conditions

$$x_1(t_0) = \xi_1, \quad x_2(t_0) = \xi_2. \tag{3.2.10}$$

Defining the Hamiltonian function H by

$$H = -\frac{1}{2}(x_1{}^2 + ru^2) + p_1(-ax_1 + ax_2) + p_2(x_1 - x_2 + u), \qquad (3.2.11)$$

where p_1 and p_2 are costate variables of x_1 and x_2, we derive the canonical equation

$$\dot{x}_1 = -ax_1 + ax_2, \qquad (3.2.12a)$$

$$\varepsilon\,\dot{x}_2 = x_1 - x_2 + r^{-1}p_2, \qquad (3.2.12b)$$

$$\dot{p}_1 = ap_1 + p_2, \qquad (3.2.12c)$$

$$\varepsilon\,\dot{p}_2 = x_2 - ap_1 + p_2. \qquad (3.2.12d)$$

The optimal control u* is given by

$$u^* = r^{-1}p_2. \qquad (3.2.13)$$

The terminal conditions at both ends of the interval $[t_0, t_f]$ can be derived by considering condition (3.2.10) and the transversality condition as follows:

$$x_1(t_0) = \xi_1, \qquad x_2(t_0) = \xi_2, \qquad (3.2.14)$$

and

$$p_1(t_f) = 0, \qquad p_2(t_f) = 0. \qquad (3.2.15)$$

3.2.2 Asymptotic expansions

Expanding each variable into Taylor series in ε, and substituting the resulting series into Eqs.(3.2.12) and (3.2.13), and comparing coefficients of like powers of ε, we obtain the recursive set of equations, i.e., the outer recursive system,

$$\dot{x}_1^0 = -ax_1^0 + ax_2^0,$$

$$0 = x_1^0 - x_2^0 + r^{-1}p_2^0,$$

$$\dot{p}_1^0 = ap_1^0 - p_2^0,$$
$$0 = x_2^0 - ap_1^0 - p_2^0, \tag{3.2.16}$$

and

$$\begin{aligned}
\dot{x}_1{}^k &= - ax_1{}^k + ax_2{}^k, \\
0 &= x_1{}^k - x_2{}^k + r^{-1}p_2{}^k - \dot{x}_2{}^{k-1}, \\
\dot{p}_1{}^k &= ap_1{}^k - p_2{}^k, \\
0 = &= x_2{}^k - ap_1{}^k - p_2{}^k - \dot{p}_2{}^{k-1},
\end{aligned} \tag{3.2.17}$$

for $k > 0$.

Following the results obtained in Section 2.2, we can treat the two point boundary value problem (3.2.12) with (3.2.14) and (3.2.15) with use made of the Riccati transformation. The resulting matrix Riccati differential equation is

$$\begin{aligned}
\dot{k}_1 &= 2ak_1 - 2k_2 + r^{-1}k_2{}^2, \\
\varepsilon\, \dot{k}_2 &= - ak_1 + k_2 - k_3 + r^{-1}k_2k_3 + \varepsilon ak_2, \\
\varepsilon\, \dot{k}_3 &= 2k_3 + r^{-1}k_3^2 - 1 - 2\varepsilon ak_2,
\end{aligned} \tag{3.2.18}$$

with the final terminal conditions

$$k_i(t_f) = 0; \quad i = 1, 2, 3. \tag{3.2.19}$$

Then we have the reduced system as follows:

$$\begin{aligned}
\dot{k}_1^0 &= 2ak_1^0 - 2k_2^0 + r^{-1}k_2^{0\,2}, \\
0 &= - ak_1^0 + k_2^0 - k_3^0 + r^{-1}k_2^0k_3^0, \\
0 &= 2k_3^0 + r^{-1}k_3^{0\,2} - 1,
\end{aligned} \tag{3.2.20}$$

with the terminal condition

$$k_1(t_f) = 0, \tag{3.2.21}$$

and the first correction system in the outer region

$$\dot{k}_1^1 = 2ak_1^1 - 2k_2^1 + 2r^{-1}k_2^0k_2^1,$$
$$\dot{k}_2^0 = -ak_1^1 + k_2^1 - k_3^1 + r^{-1}(k_2^0k_3^1 + k_2^1k_3^0) + ak_2^0, \qquad (3.2.22)$$
$$\dot{k}_3^0 = 2k_3^1 + r^{-1}2k_3^0k_3^1 - 2ak_2^0.$$

The final terminal condition for the first correction system is given as follows:

$$k_1^1(t_f) = \int_0^\infty [(-2\bar{k}_2 + r^{-1}\bar{k}_2^{\,2}) - (-2k_2^0 + r^{-1}k_2^{02})]d\tau, \qquad (3.2.23)$$

where $\bar{k}_2$ in the integrand is a solution of the boundary layer system

$$-\frac{d}{d\tau}\bar{k}_2(\tau) = -ak_1^0(t_f) + \bar{k}_2(\tau) - \bar{k}_3(\tau) + r^{-1}\bar{k}_2(\tau)\bar{k}_3(\tau), \qquad (3.2.24)$$

$$-\frac{d}{d\tau}\bar{k}_3(\tau) = 2\bar{k}_3(\tau) + r^{-1}\bar{k}_3^{\,2}(\tau) - 1, \qquad (3.2.25)$$

with

$$\bar{k}_2(0) = 0, \qquad \bar{k}_3(0) = 0. \qquad (3.2.26)$$

After solving the Riccati equations (3.2.20) and (3.2.21), we have a suboptimum trajectory, as follows:
for the reduced trajectory

$$\dot{x}_1 = -ax_1^0 + ax_2^0,$$
$$0 = x_1^0 - x_2^0 + r^{-1}(k_2^0x_1^0 + k_3^0x_2^0), \qquad (3.2.27)$$

with

$$x_1(t_0) = \xi_1. \qquad (3.2.28)$$

For the first correction trajectory

$$\dot{x}_1^1 = -ax_1^1 + ax_2^1,$$
$$\dot{x}_2^0 = x_1^1 - x_2^1 + r^{-1}(k_2^0x_1^1 + k_3^0x_2^1 + k_2^1x_1^0 + k_3^1x_2^0), \qquad (3.2.29)$$

with the initial condition

$$x_1^1(t_0) = \int_0^{\infty} a[\overline{x}_2(\tau) - x_2^0(t_0)]d\tau, \qquad (3.2.30)$$

where $x_2(\tau)$ is a solution of the following boundary layer system associated with the trajectory equation (3.2.12):

$$\frac{d}{d\tau}\overline{x}_2(\tau) = x_1^0(t_0) - \overline{x}_2(\tau) + r^{-1}(k_2^0(t_0)x_1(t_0) + k_3^0(t_0)\overline{x}_2(\tau)), \qquad (3.2.31)$$

with

$$\overline{x}_2(t_0) = \xi_2. \qquad (3.2.32)$$

At last we obtain the suboptimum control as follows:

$$u_{sub} = r^{-1}[(k_2^0 + \varepsilon k_2^1)(x_1^0 + \varepsilon x_1^1) + (k_3^0 + \varepsilon k_3^1)(x_2^0 + \varepsilon x_2^1)]. \qquad (3.2.33)$$

3.2.3 Boundary layer corrections

In order to obtain the uniformly valid expansion throughout the interval $[t_o, t_f]$, we seek a solution of the form

$$\tilde{K}(t,\varepsilon) = K(t,\varepsilon) + H(\tau,\varepsilon), \qquad (3.2.34)$$

where $\tau = (t_f - t)/\varepsilon$. Here the boundary layer correction $H(\tau,\varepsilon)$ vanishes as τ tends to infinity, i.e.,

$$\lim_{\tau\to\infty} H(\tau,\varepsilon) = 0. \qquad (3.2.35)$$

Further $H(\tau,\varepsilon)$ has an asymptotic expansion in ε as $\varepsilon \to 0$ of the form

$$H(\tau,\varepsilon) = \sum_{r=0}^{\infty} H^r(\tau)\varepsilon^r(r!)^{-1}. \qquad (3.2.36)$$

To determine each coefficient $H^r(\tau)$, as in the usual way, we substitute the asymptotic power series

$$\tilde{K}(t,\varepsilon) = \sum_{r=0}^{\infty} [K^r(t) + H^r(\tau)]\varepsilon^r (r!)^{-1}, \tag{3.2.37}$$

into the stretched system

$$\begin{aligned}
-\frac{d}{d\tau}\tilde{k}_1 &= \varepsilon(2a\tilde{k}_1 - 2\tilde{k}_2 + r^{-1}\tilde{k}_2{}^2), \\
-\frac{d}{d\tau}\tilde{k}_2 &= -a\tilde{k}_1 + \tilde{k}_2 - \tilde{k}_3 + r^{-1}\tilde{k}_2\tilde{k}_3, \qquad (3.2.38) \\
-\frac{d}{d\tau}\tilde{k}_3 &= 2\tilde{k}_3 + r^{-1}\tilde{k}_3{}^2 - 1,
\end{aligned}$$

and compare coefficients of like powers of ε thereof, then the following recursive equations for the boundary layer coefficients are obtained:

$$\begin{aligned}
&-\frac{d}{d\tau} h_1^0 = 0, \qquad -\frac{d}{d\tau} h_2^0 = -ah_1^0(\tau) + h_2^0(\tau) - h_3^0(\tau) \\
&\quad + r^{-1}[k_2^0(t_f)h_3^0(\tau) + k_3^0(t_f)h_2^0(\tau) + h_2^0(\tau)h_3^0(\tau)], \qquad (3.2.39)_0 \\
&-\frac{d}{d\tau} h_3^0 = 2h_3^0(\tau) + r^{-1}[2k_3^0(t_f)h_3^0(\tau) + (h_3^0(\tau))^2] - 1,
\end{aligned}$$

$$\begin{aligned}
&-\frac{d}{d\tau} h_1^1 = -2ah_1^0(\tau) - 2h_2^0(\tau) + r^{-1}[k_2^0(t_f)h_2^0(\tau) + (h_2^0(\tau))^2], \\
&-\frac{d}{d\tau} h_2^1 = -ah_1^1(\tau) + h_2^1(\tau) - h_3^1(\tau) + ah_2^0(\tau) \qquad (3.2.39)_1 \\
&\qquad + r^{-1}[k_1^0(t_f)h_2^0(\tau) + k_2^0(t_f)h_1^0(\tau) + h_1^0(\tau)h_2^0(\tau)], \\
&-\frac{d}{d\tau} h_3^1 = 2h_3^1(\tau) + r^{-1}[2k_3^0(t_f)h_3^1(\tau) + 2k_3^1(t_f)h_3^0(\tau) + \\
&\qquad + h_3^0(\tau)h_3^1(\tau)] - 2ah_2^0(\tau).
\end{aligned}$$

Each recursive equations can be solved under the terminal conditions derived from the requirement

$$K^r(t_f) + H^r(0) = 0. \tag{3.2.40}$$

3.2.4 Numerical example

We shall present the results obtained by numerical calculations according to the method described in the preceding section. The data used in the example are the same as in Section 3.1.4, namely $\beta = 0.0064$, $a = \lambda = 0.078$; mean life time of the prompt neutron $\ell = 3\text{x}10^{-5}$, and therefore $\varepsilon = 0.47\text{x}10^{-2}$.

In Figs. 3.2.1 and 3.2.2, the time behaviours of the solution of the Riccati equation (3.2.18) and those of the approximate solution derived via the singular perturbation theory are illustrated for two values of the weight r. The approximation is taken of the form of truncated series solution at the second term. The boundary layer appears in the case of the reduced solution.

In general, it can be seen that the feedback coefficients determined by solving the Riccati equation change rapidly in the neighbourhood of the terminal time, and computational difficulties should be encountered such as overflow of computers or increased size of the memory needed. Moreover, in this case the existence of the boundary layer makes the rate of the change larger and the difficulties should become more serious. The method adopted can make us avoid the difficulties, by using the idea of separation of the time scale.

Figures 3.2.3 and 3.2.4 show the time behaviours of the optimum and suboptimum trajectories of neutrons for $r = 10$ and $r = 0.4$ respectively. Table 3.2.1 represents values of the performance index calculated by using the optimum and suboptimum control.

Our numerical study shows that the reduced solution with boundary layer corrections (0-th order approximation) gives a satisfactory result. The error of the 0-th order approximation is, except the terminal time, of the order of 0.1 percent for trajectories of neutrons and 1 percent for feedback coefficients. The first correction term improves the approximation uniformly in the time coordinate, especially does in the boundary layers. The memory can be saved about 60 percent when we adopt our 0-th order approximation. This saving is a great advantage of the present method.

These consequences show that the method gives satisfactory results from view points of computaional labour, computing time, accuracy of the approximation.

It is to be noted that when the original state equation is stiff, then the generated matrix Riccati equation will be stiffer because of the nonlinearity of the Riccati equation. So the method of separation of the time scale or the method of multiple time scale may be effective to such a problem. Above all the singular perturbation theory plays a very important role as seen in this example.

Table 3.2.1 Performance index

	r = 10	r = 0.4
exact	0.4173×10^{-1}	0.8435×10^{-2}
approximate	0.4170×10^{-1}	0.8437×10^{-2}

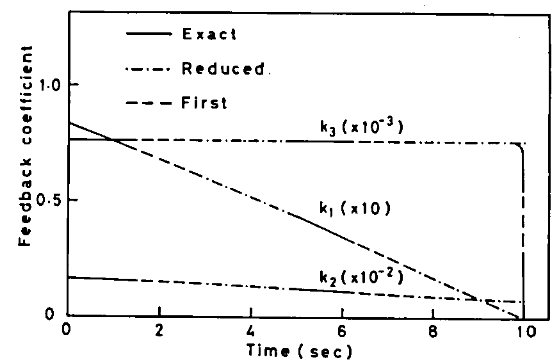


Fig. 3.2.1 Feedback coefficients for r = 10.

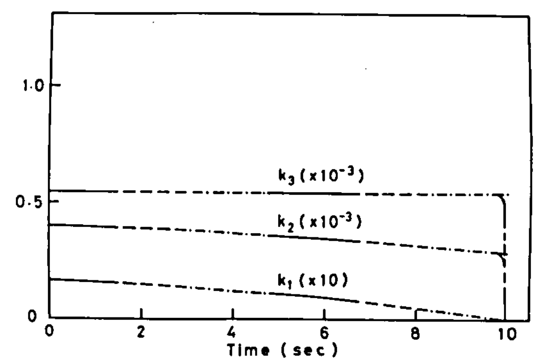


Fig. 3.2.2 Feedback coefficients for r = 0.4.

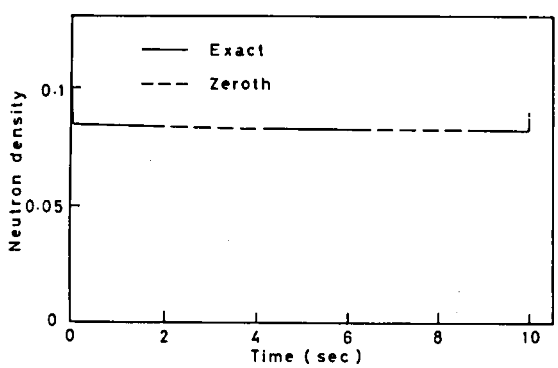


Fig. 3.2.3 Suboptimum and optimum trajectories of the neutron density for r = 10.

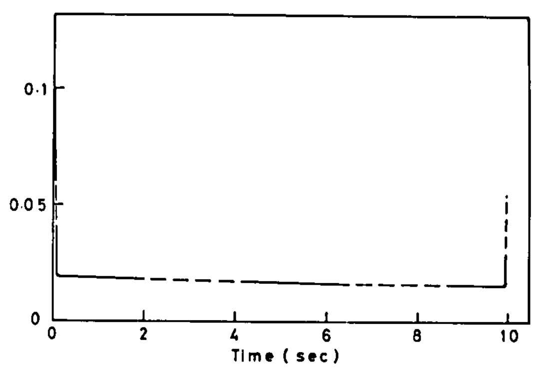


Fig. 3.2.4 Suboptimum and optimum trajectories of the neutron density for r = 0.4.

Chaper 4 Near-Optimum Control of Distributed Parameter System

4.1 Introduction

It is true that every realistic physical system is intrinsically distributed in nature. From practical motivations, however, the system's spatial distributed parameters are usually assumed to be sufficiently concentrated or invariant in form during the course of motion so that an approximate lumped parameter model may provide an adequate description. Although such a treatment is feasible in many physical systems, it is obvious that the effects of spatial distribution of practical systems are lost. In the precise consideration of the effects it is required to investigate the system maintaining the spatial distribution of physical parameters, so that a study of the theory of partial differential equations becomes indispensable. Typical examples of this case are nuclear reactors, many metallurgical plants, chemical reactors, and so forth.

Optimal control theory of lumped parameter systems has been developed powerfully and successfully by many mathematicians and control engineers, and is almost complete in theory. A maximum principle (and minimum principle) was obtained by Pontryagin and his students, which is a generalization of the classical variational theory in a sense [Pn.62]. Dynamic programming was found by Bellman independently of the maximum principle [B.57]. Kalman showed that the maximum principle, dynamic programming, and Hamilton-Jacobi theory are equivalent to each other, and constructed general solutions of linear state regulator problem by introducing the Riccati transformation and established the basic results [K.60]. The treatment adopted by Kalman seems to be one of the main streams in control theory. Theories above mentioned are systematic and complete in theory, and provide us powerful gu des.

On the other hand, control theory of distributed parameter system is not so sufficiently developed, since it is by far complex and difficult to handle or to analyze partial differential equations. Modern treatments of partial differential equations have recently made remark able progress with the development of functional analysis; see for instance Dunford and Schwartz [Df.58], Lions and Magenes [L.72], or Mizohata [M.65]. On the basis of this development, optimal control of distributed parameter systems has become one of the most productive areas.

One of the earliest investigators of the subject was Butkovskii (see references listed in [Bt.69]). Butkovskii extended Pontryagin's maximum principle to a certain class of distributed systems, and studied the approximation method with the aid of method of moments.

Another approach was made by Wang [W.64]. His main theme seems to lie in applying the results of the theory of lumped parameter systems to distributed ones in line with Kalman, i.e., he obtained Hamilton-Jacobi formalism, a maximum principle, and Riccati-like partial differential integral equations. He also devoted himself to the study of stability, controllability, observability, and approximation methods.

Some other studies on the subject were made by Axelband [Ax.69], Brogan [Br.68], Lions [L.71], etc. with use made of functional analysis. Since it is not our object to review the theory comprehensively, for general surveys, Brogan [Br.68] or Lions's detailed and extensive work [L.71] may be referred to.

It is to be noted here that the practical treatment of partial differential equations cannot be made without involving an approximation, spatial discretization, modal expansion, etc., since the equations can be regarded to be infinite dimensional. Considering such a situation, it is easy to see the importance of the reduction of the number of independent variables if possible.

As mentioned in Section 1.3, the singular perturbations of abstract evolution equations are not fully developed, this chapter presents materials to contribute to the theory concerned: the chapter consists of two parts, the former one is concerned with the singular perturbations of a certain class of a set of evolution equations. The latter deals with near-optimum control problems on the basis of the results derived in the former one. So far as the author knows, the singular perturbations of a set of partial differential equations involving construction of asymptotic expansions have never been treated. The regular degeneration of such systems was studied recently by Lions [L.73].

4.2 Evolution equations containing a small parameter

In this section, an investigation is made of the singular perturbation theory of a certain class of abstract evolution equations of the form

$$\frac{\partial}{\partial t} z = Az + f,$$
$$z = \begin{bmatrix} x \\ y \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & A_2 \\ \varepsilon^{-1} A_3 & \varepsilon^{-1} A_4 \end{bmatrix},$$

where z is a state vector, operators A_1 and A_4 are parabolic, A_2, A_3 are bounded, and ε is a small positive parameter.

This kind of system does not seem to have been treated with the closed form.

4.2.1 Mathematical notations and preliminaries

We shall consider functions defined on the sets as follows:

Ω : a simply connected, open set in R^r.

Γ : the boundary of Ω.

$\Sigma = \Gamma \times]0,T]$.

$Q = \Omega \times]0,T]$.

Points of Ω are denoted by $\omega_r = (\omega_1, \omega_2, \ldots, \omega_r)$.

Further some definitions are needed regarding the equivalence class of functions considered.

$H = L^2(\Omega)$: space of functions square integrable on Ω.

$L^2(S;E)$: space of functions defined on S with values in a Hilbert space E, and whose second powers are integrable with respect to the Lebesgue measure of S, $d\mu(t)$, i.e.

$$\int_S \|f(t)\|_E^2 \, d\mu(t) < \infty.$$

$H^1(\Omega)$: Sobolev space of order 1, i.e., space of functions ϕ such that

$$\phi, \frac{\partial \phi}{\partial \omega_1}, \frac{\partial \phi}{\partial \omega_2}, \ldots, \frac{\partial \phi}{\partial \omega_r} \in L^2(\Omega).$$

$H_0^1(\Omega)$: subspace of $H^1(\Omega)$, i.e., space of functions ϕ such that

$$\phi \in H^1(\Omega), \quad \phi|_\Gamma = 0.$$

The inner products and norm of elements f, g ε H are defined:

Inner product: $(f,g)_H = \int_\Omega f'(\omega) g(\omega) d\omega.$

Norm: $\| f \|_H = [\int_\Omega |f(\omega)|^2 d\omega]^{1/2}.$

For elements f, g ε $L^2(0,T;H)$,

Inner product: $(f,g)_{L^2(0,T;H)} = \int_0^T (f,g)_H dt.$

Norm: $\| f \|_{L^2(0,T;H)} = [\int_0^T \| f \|_H^2 \, dt]^{1/2}.$

In order to distinguish functions as elements of a function space and values taken by them, the following notations are adopted:

$f(\omega,t)$ is a point in R^1, where $(\omega,t) \in Q$.

$f(\cdot,t)$ is an element of the Hilbert space H.

$f(\cdot,\cdot)$ is an element of the Hilbert space $L^2(0,T;H)$.

$f(\cdot, t, \varepsilon)$ is an element of the Hilbert space $L^2(0,T;H)$ parameterized by a small parameter ε.

Hence the function $f(\cdot)$ is considered to be an element of some Hilbert space H. If the function f considered is m-dimensional vector valued, the minute notation

$$f \in L^2(\Omega)\times L^2(\Omega)\times\ldots\times L^2(\Omega) = L^{2,m}(\Omega)$$

should be adopted. In order to avoid notational trouble, it is noted that

$$f \in L^2(\Omega),$$

when no misunderstanding may occur.

For practical reasons, we consider here only separable Hilbert spaces; such a treatment is rather general. In a separable Hilbert space H, there exists a set of known "elementary" functions, countable and everywhere dense in H. The complete orthonormal system $\{\phi_i\}$, i = 1, 2, ... is adopted as such a set, and we have the following classical theorems [Km.71]:

<u>Theorem 4-1</u> In a separable Hilbert space, every complete orthonormal system is closed and conversely.

For L^2 the above theorem becomes more useful.

<u>Theorem 4-2</u>(Riesz-Fischer) Given a numerical sequence $\{c_k\}$ for which the series $\sum_{k=1}^{\infty}|c_k|^2$ is convergent, there exists a unique element $f \in H$ such that its Fourier coefficients a_k are equal to c_k (k = 1, 2, ...) and

$$\sum_{k=1}^{\infty}|c_k|^2 = (f,f)_H = \| f \|_H^2. \tag{4.2.1}$$

Thus for any $f \in H$,

$$\lim_{m\to\infty} \| f - \sum_{k=1}^{m} c_k\phi_k \|_H^2 \to 0, \tag{4.2.2}$$

where

$$c_k = (f,\phi_k) = \int_\Omega f(\omega)\phi_k(\omega)d\omega.$$

We use in the sequel mainly a system of complete orthonormal functions generated by the Strum-Liouville equation:

$$A[\phi] = \lambda\phi, \qquad \text{in } \Omega \tag{4.2.3}$$

with any one of the boundary conditions

$$\text{i) } \phi(\Gamma) = 0, \tag{4.2.4}$$

$$\text{ii) } \frac{\partial\phi}{\partial\nu_A}(\Gamma) = 0, \tag{4.2.4'}$$

$$\text{iii) } \frac{\partial\phi}{\partial\nu_A}(\Gamma) + \sigma(\Gamma)\phi(\Gamma) = 0; \quad \sigma(\Gamma)>0, \tag{4.2.4''}$$

where

$$\frac{\partial}{\partial\nu_A}(\cdot) = \sum_{j=1}^{r} a_{ij}\frac{\partial}{\partial x_j}(\cdot)\cos(n,x), \qquad \text{on } \Gamma .$$

It is assumed that

$$A[\phi] = -\sum_{i,j=1}^{r}\frac{\partial}{\partial\omega_i}[a_{ij}(\omega)\frac{\partial\phi}{\partial\omega_j}] + a_0(\omega)\phi(\omega), \tag{4.2.5}$$

$$\sum a_{ij}(\omega)\xi_i\xi_j \geqq \alpha(\xi_1^2 + \xi_2^2 + \cdots + \xi_r^2) \tag{4.2.6}$$

holds almost everywhere in Ω for all $\xi = (\xi_1, \xi_2, \ldots, \xi_r)$ in R^r.

The theorem described below is also called the Riesz-Fischer's theorem, and shall play an important role in Section 4.2.3.

Theorem 4-3(Riesz-Fischer) A necessary and sufficient condition for the given sequence of functions $\{f_n\}$ to be convergent to an element $f \varepsilon H$ is that

$$\| f_m - f_n \|_H \to 0 \quad \text{as } m,\ n \to \infty.$$

4.2.2 Formal construction of asymptotic expansion

We describe the formal procedure of constructing the asymptotic expansion of the solution of the Cauchy problem of the form

$$\frac{\partial}{\partial t}\tilde{x}(\omega,t,\varepsilon) + A_1\tilde{x}(\omega,t,\varepsilon) + A_2\tilde{y}(\omega,t,\varepsilon) = f(\omega,t,\varepsilon), \quad (4.2.7)$$

$$\varepsilon\frac{\partial}{\partial t}\tilde{y}(\omega,t,\varepsilon) + A_3\tilde{x}(\omega,t,\varepsilon) + A_4\tilde{y}(\omega,t,\varepsilon) = g(\omega,t,\varepsilon), \quad (4.2.8)$$

with initial data

$$\left.\begin{aligned}\tilde{x}(\omega,0,\varepsilon) &= \alpha(\omega,\varepsilon),\\ \tilde{y}(\omega,0,\varepsilon) &= \beta(\omega,\varepsilon),\end{aligned}\right\} \quad \text{on } \omega \in \Omega, \quad (4.2.9)$$

and boundary condition

$$B(\tilde{x}, \tilde{y}) = 0, \quad \text{on } \omega \in \Omega, \quad (4.2.10)$$

where A_1, A_2, A_3, and A_4 are linear operators whose properties are specified in the succeeding Section 4.2.3.

On the analogy of the lumped parameter case, we seek the solution of (4.2.7) and (4.2.8) of the form,

$$\begin{aligned}\tilde{x}(\omega,t,\varepsilon) &= x(\omega,t,\varepsilon) + X(\omega,t/\varepsilon,\varepsilon),\\ \tilde{y}(\omega,t,\varepsilon) &= y(\omega,t,\varepsilon) + Y(\omega,t/\varepsilon,\varepsilon),\end{aligned} \quad (4.2.11)$$

where $x(\omega,t,\varepsilon)$, $y(\omega,t,\varepsilon)$, $X(\omega,t/\varepsilon,\varepsilon)$ and $Y(\omega,t/\varepsilon,\varepsilon)$ are assumed to have asymptotic expansions with respect to ε, as follows:

$$\begin{aligned}x(\omega,t,\varepsilon) &= \sum_{r=0}^{\infty} x^r(\omega,t)\varepsilon^r(r!)^{-1},\\ y(\omega,t,\varepsilon) &= \sum_{r=0}^{\infty} y^r(\omega,t)\varepsilon^r(r!)^{-1},\end{aligned} \quad (4.2.12)$$

and

$$\begin{aligned}X(\omega,t/\varepsilon,\varepsilon) &= \sum_{r=0}^{\infty} X^r(\omega,t/\varepsilon)\varepsilon^r(r!)^{-1},\\ Y(\omega,t/\varepsilon,\varepsilon) &= \sum_{r=0}^{\infty} Y^r(\omega,t/\varepsilon)\varepsilon^r(r!)^{-1}.\end{aligned} \quad (4.2.13)$$

It is easily seen that $x(\omega, t, \varepsilon)$ and $y(\omega, t, \varepsilon)$ represent the outer solution, and $X(\omega, \tau, \varepsilon)$ and $Y(\omega, \tau, \varepsilon)$ the boundary layer correction. Since Eq.(4.2.11) satisfies the given initial data (4.2.9), the following relations hold:

$$\begin{aligned}x^r(\omega,0) + X^r(\omega,0) &= \alpha^r(\omega),\\ y^r(\omega,0) + Y^r(\omega,0) &= \beta^r(\omega),\end{aligned} \quad (4.2.14)$$

where $\alpha^r(\omega)$ and $\beta^r(\omega)$ are coefficients of the Taylor series of the given initial data (4.2.9). Further the boundary layer corrections $X^r(\omega,\ t/\varepsilon)$ and $Y^r(\omega,\ t/\varepsilon)$ arerequired to vanish as the stretched coordinate $\tau = t/\varepsilon$ tends to infinity,

$$\lim_{\tau\to\infty} X^r(\omega,\tau) = 0, \tag{4.2.15}$$

$$\lim_{\tau\to\infty} Y^r(\omega,\tau) = 0. \tag{4.2.16}$$

In the outer region, Eqs.(4.2.7) and (4.2.8) are valid for the outer expansion. We substitute the expansion (4.2.12) into Eqs.(4.2.7) and (4.2.8), and compare the coefficients of like powers of ε, obtaining the following recursive set of equations.

$$\frac{\partial}{\partial t} x^0 + A_1x^0 + A_2y^0 = f, \qquad (4.2.17)_0$$

$$A_3x^0 + A_4y^0 = g, \qquad (4.2.18)_0$$

$$\frac{\partial}{\partial t} x^r + A_1x^r + A_2y^r = 0, \qquad (4.2.17)_r$$

$$\frac{\partial}{\partial t} y^{r-1} + A_3x^r + A_4y^r = 0. \qquad (4.2.18)_r$$

For simplicity, we assume here that A_1, A_2, A_3, A_4, f and g are independent of ε. This assumption is not essential.

In order to make the boundary layer correction, the boundary layer method is adopted. The stretching transformation

$$\tau = t/\varepsilon, \tag{4.2.19}$$

makes Eqs.(4.2.7) and (4.2.8) to be of the stretched form

$$\frac{\partial}{\partial\tau} \tilde{x}(\omega,\tau,\varepsilon) + A_1\tilde{x}(\omega,\tau,\varepsilon) + A_2\tilde{y}(\omega,\tau,\varepsilon) = f(\omega,\tau), \tag{4.2.20}$$

$$\frac{\partial}{\partial\tau} \tilde{y}(\omega,\tau,\varepsilon) + A_3\tilde{x}(\omega,\tau,\varepsilon) + A_4\tilde{y}(\omega,\tau,\varepsilon) = g(\omega,\tau). \tag{4.2.21}$$

Substituting the solution (4.2.11) into Eqs.(4.2.20) and (4.2.21), we have

$$\frac{\partial}{\partial\tau} X(\omega,\tau,\varepsilon) + \varepsilon A_1X(\omega,\tau,\varepsilon) + \varepsilon A_2Y(\omega,\tau,\varepsilon) = 0, \tag{4.2.22}$$

$$\frac{\partial}{\partial\tau} Y(\omega,\tau,\varepsilon) + A_3X(\omega,\tau,\varepsilon) + A_4Y(\omega,\tau,\varepsilon) = 0, \qquad (4.2.23)$$

where it is considered that outer solution x(t) and y(t) satisfy Eq. (4.2.7) and (4.2.8) with $t = \varepsilon\tau$. Putting each expansion (4.2.12) and (4.2.13) in place of $x(\omega, t, \varepsilon)$, $y(\omega, t, \varepsilon)$, $X(\omega, \tau, \varepsilon)$ and $Y(\omega, \tau, \varepsilon)$ in Eqs.(4.2.22) and (4.2.23), we have the following recursive equations for the boundary layer corrections,

$$\frac{\partial}{\partial\tau} X^0(\omega,\tau) = 0, \qquad (4.2.24)_0$$

$$\frac{\partial}{\partial\tau} Y^0(\omega,\tau) + A_3X^0(\omega,\tau) + A_4Y^0(\omega,\tau) = 0, \qquad (4.2.25)_0$$

$$\frac{\partial}{\partial\tau} X^r(\omega,\tau) = P_r(\omega,\tau), \qquad (4.2.24)_r$$

$$\frac{\partial}{\partial\tau} Y^r(\omega,\tau) + A_3X^r(\omega,\tau) + A_4Y^r(\omega,\tau) = 0. \qquad (4.2.25)_r$$

The remainders $P_r(\omega,\tau)$ are known successively in the preceding steps.

The initial data for the reduced outer equation is given by using the cancellation law, as

$$x^0(\omega,0) = \alpha^0(\omega), \quad \omega \in \Omega, \qquad (4.2.26)$$

hence the initial data for the boundary layer correction of the 0-th order(reduced), obtained considering Eq.(4.2.13), is as follows:

$$X^0(\omega,0) = 0, \quad \omega \in \Omega, \qquad (4.2.27)$$

$$Y^0(\omega,0) = \beta^0(\omega) - y^0(\omega,0); \quad \omega \in \Omega. \qquad (4.2.28)$$

For the higher order recursive equations, the initial data is determined as follows. From Eq.$(4.2.17)_1$, $X^1(\omega,0)$ can be determined as

$$X^1(\omega,\tau) = X^1(\omega,0) + \int_0^\tau P_1(\omega,\tau)d\tau$$

$$= X^1(\omega,0) - \int_0^\tau A_2Y^0(\omega,\tau)d\tau. \qquad (4.2.29)$$

Considering the property (4.2.15), we choose $X^1(\omega,0)$ as

$$X^1(\omega,0) = \int_0^\infty A_2 Y^0(\omega,\tau)d\tau. \tag{4.2.30}$$

This infinite integral may exist if the integrand is dominated by functions of the boundary layer type which should be ascertained if Eq.(4.2.23) is asymptotically stable with respect to the root $Y = 0$. The conditions needed can be found in the follwing section. Then the initial data for the outer second coefficient of the outer expansion is derived by using Eq.(4.2.14) as

$$x^1(\omega,0) = \alpha^1(\omega) - \int_0^\infty A_2 Y^0(\omega,\tau)d\tau. \tag{4.2.31}$$

A similar procedure may treat successfully the higher order equations.

If we obtain the Green's function $G_4(\omega,\omega')$ for $(4.2.18)_0$, then we can solve Eq.$(4.2.18)_0$ for $y^0(\omega,t)$ as

$$y^0(\omega,t) = \int_\Omega G_4(\omega,\omega')[g(\omega',t) - A_3 x^0(\omega',t)]d\omega', \tag{4.2.32}$$

or in the notation of the operator theory

$$y^0(\omega,t) = A_4^{-1}[g(\omega,t) - A_3 x^0(\omega,t)], \tag{4.2.33}$$

where A_4^{-1} is defined by

$$A_4^{-1}[\cdot] = \int_\Omega G_4(\omega,\omega')[\cdot]d\omega'. \tag{4.2.34}$$

The above procedure eliminates $y^0(\omega,t)$ in Eq.(4.2.7) formally, and we can solve Eq.(4.2.7) for $x^0(\omega,t)$ with the initial data (4.2.26).

The same procedure can be applied to the higher order equation.

If the right hand sides of Eqs.(4.2.7) and (4.2.8) depend upon $\tilde{x}$ and $\tilde{y}$ regularly in the sense that

$$f,\ g = o(|\tilde{x}|,\ |\tilde{y}|) \quad \text{as } |\tilde{x}|,\ |\tilde{y}| \to 0$$

holds, the resulting recursive set of equations can be solved stepwise similarly.

4.2.3 Main theorems

The regular degeneration of the system (4.2.1) and (4.2.2), and the asymptotic accuracy of the result obtained formally above shall be studied.

Some prerequisite assumptions are offered, first we state considering Theorems 4-1 and 4-2 in Section 4.2.1:

<u>Assumption 1</u> A_4 is a linear operator which generates an orthonormal set of eigenfunctions, i.e., there exist a sequence of eigenfunctions $\{\phi_i\}$ and a sequence of eigenvalues $\{\lambda_i\}$ satisfying

$$- A_4[\phi] = \lambda\phi, \tag{4.2.35}$$

with the boundary condition that is any one of the conditions given above (4.2.4) - (4.2.4") and denoted here by

$$B_2[\phi] = 0. \tag{4.2.36}$$

And the sequence of eigenfunctions is orthonormal in the space $L^2(\Omega)$, that is

$$\int_\Omega \phi_n \phi_m d\omega = \delta_{nm},$$

$$\delta_{nm} = \begin{cases} 1 & n = m \\ 0, & n \neq m. \end{cases}$$

<u>Assumption 2</u> A_1 admits the same eigenfunction as A_4, e.g., A_1 can be expressed as,

$$A_1 = -kA_4 + h \tag{4.2.37}$$

where k and h are constant with respect to the spatial coordinates.

<u>Assumption 3</u> A_2 and A_3 are spatially homogeneous.

If the Assumptions 1 and 2 hold, we can expand each variable into an eigenfunction expansion by using Theorems 4-1 and 4-2:

$$\begin{aligned} x(\omega,t,\varepsilon) &= \sum_{i=1}^{\infty} \xi_i(t,\varepsilon)\phi_i(\omega), \\ y(\omega,t,\varepsilon) &= \sum_{i=1}^{\infty} \eta_i(t,\varepsilon)\phi_i(\omega). \end{aligned} \tag{4.2.38}$$

Substituting the expansion (4.2.38) into Eqs.(4.2.7) and (4.2.8), we derive the following set of equations by using the orthonormality of eigenfunctions ϕ_i,

$$\frac{d}{dt}\xi_i(t,\varepsilon) + (k\lambda_i + h)\xi_i(t,\varepsilon) + A_2\eta_i(t,\varepsilon) = f_i(t), \tag{4.2.39}$$

$$\varepsilon\frac{d}{dt}\eta_i(t,\varepsilon) + A_3\xi_i(t,\varepsilon) - \lambda_i\eta_i(t,\varepsilon) = g_i(t), \tag{4.2.40}$$

where

$$\begin{aligned} f_i(t) &= \int_\Omega \phi_i'(\omega)f(t,\omega)d\omega, \\ g_i(t) &= \int_\Omega \phi_i'(\omega)g(t,\omega)d\omega. \end{aligned} \tag{4.2.41}$$

For the boundary layer correction terms, the same expansion by the eigenfunctions is adopted. The resulting set of equations is as follows:

$$\frac{d}{d\tau}\Xi_i(\varepsilon\tau,\varepsilon) + \varepsilon(k\lambda_i + h)\Xi_i(\varepsilon\tau,\varepsilon) + \varepsilon A_2 H_i(\varepsilon\tau,\varepsilon) = \varepsilon f_i(\varepsilon\tau), \tag{4.2.42}$$

$$\frac{d}{d\tau}H_i(\varepsilon\tau,\varepsilon) + A_3\Xi_i(\varepsilon\tau,\varepsilon) - \lambda_i H_i(\varepsilon\tau,\varepsilon) = g_i(\varepsilon\tau). \tag{4.2.43}$$

Now we introduce an important assumption which ascertains the asymptotic stability of the boundary layer system, as mentioned on occasion in the lumped parameter case.

<u>Assumption 4</u> The Cauchy problem

$$\frac{\partial u}{\partial t} + A_4 u = 0 \tag{4.2.44}$$

is uniformly well-posed in the sense of Hadamard, and the semigroup

$U(t,\tau)$ of this problem satisfies

$$\| U(t,\tau) \| \leqq C\exp(-\delta t), \quad \text{for} \quad \delta > 0. \tag{4.2.45}$$

Now consider the eigenfunction expansion of the reduced system using the same eigenfunction as used for the full system (4.2.39), (4.2.40):

$$\frac{d}{dt}\xi_i(t,0) + (k\lambda_i + h)\xi_i(t,0) + A_2\eta_i(t,0) = f_i(t), \tag{4.2.46}$$

$$A_3\xi_i(t,0) - \lambda_i\eta_i(t,0) = g_i(t), \tag{4.2.47}$$

where $f_i(t)$, $g_i(t)$ are the same as those given in Eq.(4.2.41).

We have the following lemmas required in the sequel.

<u>Lemma 4-1</u> If Assumptions 1 - 4 hold, then

$$\lim_{\varepsilon\to 0} \xi_i(t) = \xi_i^0(t) \equiv \xi_i(t,0), \quad \text{for } t \in [0,T], \tag{4.2.48}$$

$$\lim_{\varepsilon\to 0} \eta_i(t) = \eta_i^0(t) \equiv \eta_i(t,0), \quad \text{for } t \in]0,T]. \tag{4.2.49}$$

<u>Proof</u> Assumption 2 leads to

$$\begin{aligned} &- \int_\Omega \phi_i'(\omega)A_4\phi_i(\omega)d\omega \\ &= \int_\Omega \phi_i'(\omega)\lambda_i\phi_i(\omega)d\omega \\ &= \lambda_i < 0. \end{aligned}$$

Hence the boundary layer system for each mode η_i is asymptotically stable, which assures the regular degeneration of the full expanded i-th mode equations (4.2.39) and (4.2.40) from Theorem 1-1. This completes the proof. Q.E.D.

<u>Lemma 4-2</u> If Assumptions 1 - 4 hold, then

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\sum_{j=1}^{n} \| \xi_j(t) - \xi_j(t)\|^2 = 0, \tag{4.2.50}$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\sum_{j=1}^{n} \| \eta_j(t) - \eta_j(t)\|^2 = 0. \qquad (4.2.51)$$

Proof Assumptions 1 - 3 assure that the eigenvalues of the problem

$$- A_4\phi(\omega) = \lambda\phi(\omega), \quad B_2[\phi] = 0,$$

constitute a sequence of monotonously decreasing

$$0 > \lambda_1 \geqq \lambda_2 \geqq \lambda_3 \geqq \cdots\cdots \to -\infty,$$

furthermore the series $\sum_{i=1}^{\infty} (\frac{1}{\lambda_i})^2$ converges to a finite value, i.e.

$$\sum_{i=1}^{\infty} (\frac{1}{\lambda_i})^2 < \infty .$$

Hence we have the following estimate for λ_i:

$$\lambda_i = O(i^r), \quad r \geqq 1 \quad \text{as } i \to \infty.$$

Transforming λ_i into μ_i by the relation

$$\lambda_i = \mu_i / i^r,$$

Eqs.(4.2.39) and (4.2.40) become

$$\frac{d}{dt}\xi_i(t,\varepsilon) + (ki^r\mu_i + h)\xi_i(t,\varepsilon) + A_2\eta_i(t,\varepsilon) = f_i(t), \qquad (4.2.52)$$

$$\frac{\varepsilon}{i^r}\frac{d}{dt}\eta_i(t,\varepsilon) + A_3\xi_i(t,\varepsilon) - \mu_i\eta_i(t,\varepsilon) = g_i(t). \qquad (4.2.53)$$

Therefore Theorem 1-2 can be applied to this case. The result is as follows:

$$\| \xi_i(t) - \xi_i^0(t) \| = O(\frac{\varepsilon}{i^r}), \qquad (4.2.54)$$

$$\| \eta_i(t) - \eta_i^0(t) \| = O(\frac{\varepsilon}{i^r}). \qquad (4.2.55)$$

Then the limit relation

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \sum_{j=1}^{\infty} \left(\frac{\varepsilon}{ir} \right)^2 = 0$$

completes the proof. Q.E.D.

For example, the most typical eigenfunctions we often meet are those given by

$$\frac{\partial^2}{\partial\omega^2} \phi + \lambda\phi = 0 \quad \text{on }]0,1[, \tag{4.2.56}$$

$$\phi(0) = 0, \quad \phi(1) = 0, \tag{4.2.57}$$

namely

$$\{\sqrt{2} \sin \sqrt{\lambda_i}\, \omega\}, \; i = 1, 2, \ldots\ldots, \quad \lambda_i = (i\pi)^2.$$

The sequence of the eigenvalues $\{\lambda_i\}$ satisfies

$$\sum_{i=1}^{\infty} \left(\frac{1}{\lambda_i} \right)^2 < \infty.$$

By using Lemmas 4-1 and 4-2, we obtain the desired theorem.

<u>Theorem 4-4</u> If Assumptions 1 - 4 hold, then the regular degeneration of the system (4.2.7), (4.2.8) is concluded, as follows:

$$\lim_{\varepsilon\to 0} x(\omega,t,\varepsilon) = x^0(\omega,t), \quad \text{for } t \in [0,T], \tag{4.2.58}$$

$$\lim_{\varepsilon\to 0} y(\omega,t,\varepsilon) = y^0(\omega,t), \quad \text{for } t \in]0,T]. \tag{4.2.59}$$

<u>Proof</u> Lemma 4-2 assures that, as ε tends to zero

$$\| \xi_i(t,\varepsilon) - \xi_i^0(t) \| \to 0; \text{ uniformly,}$$

$$\| \eta_i(t,\varepsilon) - \eta_i^0(t) \| \to 0; \text{ uniformly.}$$

Hence the series

$$x(\omega,t,\varepsilon) - x^0(\omega,t) = \sum_{i=1}^{\infty} [\xi_i(t,\varepsilon) - \xi_i^0(t)]\phi_i(\omega), \qquad (4.2.60)$$

$$y(\omega,t,\varepsilon) - y^0(\omega,t) = \sum_{i=1}^{\infty} [\eta_i(t,\varepsilon) - \eta_i^0(t)]\phi_i(\omega), \qquad (4.2.61)$$

converge uniformly to zero respectively, which complete the proof. Q.E.D.

We expand each variable ξ_i and η_i into Taylor series in ε, as

$$\begin{aligned} \xi_i(t,\varepsilon) &= \sum_{r=0}^{\infty} \xi_i{}^r(t)\varepsilon^r(r!)^{-1}, \\ \eta_i(t,\varepsilon) &= \sum_{r=0}^{\infty} \eta_i{}^r(t)\varepsilon^r(r!)^{-1}. \end{aligned} \qquad (4.2.62)$$

Substituting the resulting series (4.2.62) into Eqs.(4.2.39) and (4.2.40), then comparing the coefficients of like powers of ε, we derive the following set of equations:

$$\frac{d}{dt}\xi_i{}^r(t) + (k\lambda_i + h)\xi_i{}^r(t) + A_2\eta_i{}^r(t) = p_r(t), \qquad (4.2.63)$$

$$\frac{d}{dt}\eta_i{}^{r-1}(t) + A_3\xi_i{}^r(t) - \lambda_i\eta_i{}^r(t) = q_r(t), \qquad (4.2.64)$$

where $r > 0$ and the remainders P_r, Q_r are known stepwise in the preceding steps.

By the analogous arguments made in the proof of Theorem 4-4, the following theorem, giving the asymptotic accuracy, is derived.

Theorem 4-5 If Assumptions 1 - 4 hold, then there exist bounded functions $R_n(\omega, t; \varepsilon)$ and $S_n(\omega, t; \varepsilon)$ such that

$$x(\omega,t,\varepsilon) = \sum_{r=0}^{n} \varepsilon^r [\sum_{i=1}^{\infty} \xi_i{}^r(t)\phi_i(\omega)](r!)^{-1} + \varepsilon^{n+1}R_n(\omega,t,\varepsilon), \quad \text{for } t \in [\delta,T], \qquad (4.2.65)$$

$$y(\omega,t,\varepsilon) = \sum_{r=0}^{n} \varepsilon^r [\sum_{i=1}^{\infty} \eta_i{}^r(t)\phi_i(\omega)](r!)^{-1} + \varepsilon^{n+1}S_n(\omega,t,\varepsilon), \quad \text{for } t \in [\delta,T], \qquad (4.2.66)$$

where $\delta = - C\varepsilon\log\varepsilon$; C is independent of ε.

The proof of this theorem can be made by using the following lemma.

Lemma 4-3 If Assumptions 1 - 4 hold, then

$$x^r(\omega,t) = \sum_{i=1}^{\infty} \xi_i^r(t)\phi_i(\omega), \quad (4.2.67)$$

$$y^r(\omega,t) = \sum_{i=1}^{\infty} \eta_i^r(t)\phi_i(\omega), \quad (4.2.68)$$

where x^r and y^r are coefficients of ε^r of their Taylor series. This lemma is a well-known idea of separation of variables.

As to the boundary layer corrections, we can proceed our argument in line with the lumped parameter case and derive the results similar to Theorem 1-2.

The proofs given in this section are constructive when the eigenfunctions are determined.

4.2.4 Further generalization

The results obtained in the preceding section are applicable to the restricted class of systems satisfying Assumptions 1 - 4 described therein. Such systems may provide a broad class in view point of applications, considering the complexity of computational treatment of distributed parameter systems. For instance, very few numerical examples can be seen in literatures published, which treat exceptional systems violating Assumptions 1 - 4.

This section shows the outline of the approach to the more general systems, time variant systems, systems involving spatially inhomogeneous coefficients, and so forth.

Consider the Cauchy problem

$$\frac{d}{dt} x + A_1(t,\varepsilon)x + A_2(t,\varepsilon)y = f(x, y, t, \varepsilon), \; x(0,\varepsilon) = \xi(\varepsilon), \quad (4.2.69)$$

$$\varepsilon \frac{d}{dt} y + A_3(t,\varepsilon)x + A_4(t,\varepsilon)y = g(x, y, t, \varepsilon), \; y(0,\varepsilon) = \eta(\varepsilon). \tag{4.2.70}$$

Systems of this type are intractable, for showing the existence and uniqueness of the solution. So several hypotheses are made for it.

H1. The operators $A_4(t,\varepsilon)$ are closed (possibly unbounded) linear operators acting in H with the domain of definition $D = D(A_4(t,\varepsilon))$ everywhere dense in H and independent of (t,ε) for $t \in [0,T]$, and $\varepsilon \in [0,\varepsilon_0]$.

H2. The resolvent of $-A_4(t,\varepsilon)$, $R(A_4(t,\varepsilon); \lambda) = [A_4(t,\varepsilon) + \lambda I]^{-1}$, exists as a bounded operator in H for each λ, $\mathrm{Re}\lambda \geqq 0$, and

$$\| R(A_4(t,\varepsilon);\lambda) \| \leqq C'(1 + |\lambda|)^{-1}, \tag{4.2.71}$$

for $\mathrm{Re}\lambda \geqq 0$ and some positive constant c' independent of (t,ε). The identity operator in H is denoted by I, and the operator norm over H by $\| \cdot \|$.

Thus we have:

For each $\sigma \in [0,T]$, $- A_4(t,\varepsilon)$ generates an analytic semigroup $\{\exp[-tA_4(\sigma,\varepsilon)]\}$, $t \geqq 0$.

There exist positive numbers δ and C independent of t, σ, and ε such that for each $\sigma \in [0,T]$,

$$\frac{d}{dt}\{\exp[-tA_4(\sigma,\varepsilon)]x\} = - A_4(\sigma,\varepsilon)\exp[-tA_4(\sigma,\varepsilon)]x, \quad x \in \Omega, \; t > 0,$$

$$\| \exp[-tA_4(\sigma,\varepsilon)] \| \leqq C\exp(- t), \quad t > 0,$$

$$\| A_4(\sigma,\varepsilon)\exp[-tA_4(\sigma,\varepsilon)] \| \leqq Ct^{-1}\exp(- t), \quad t > 0.$$

Hence Assumption 4 in Section 4.2.3 holds. Also it is assumed that

H3. There is a positive constant c_0 such that

$$\| [A_4(t,\varepsilon) - A_4(\tau,\varepsilon)]A_4^{-1}(s,\varepsilon) \| \leqq C_0|t - \tau|, \tag{4.2.72}$$

for all $0 \leqq t, \tau, s \leqq T$ and $0 \leqq \varepsilon \leqq \varepsilon_0$.

In order to specify the dependence of A_4 on (t,ε) we assume

H4. The operator function $A_4(t,\varepsilon)A_4^{-1}(0,0)$ has strongly continuous derivatives of all orders with respect to (t,ε), for $0 \leqq t \leqq T$, $0 \leqq \varepsilon \leqq \varepsilon_0$.

H5. The functions f and g have continuous derivatives of all orders with respect to $(t, x, y, \varepsilon) \in [0,T] \times S \times [0,\varepsilon_0]$ where S is some sphere about the origin in HxH (see Dieudonne [D.60, P186]). With hypothesis H5, f and g can be expanded into the formal Taylor series

$$f = \sum_{i,j,k,l=0}^{\infty} f_{ijkl}\varepsilon^i t^j x^k y^l, \qquad (4.2.73)$$

$$g = \sum_{i,j,k,l=0}^{\infty} g_{ijkl}\varepsilon^i t^j x^k y^l, \qquad (4.2.74)$$

where f_{ijkl} and g_{ijkl} are kl-linear bounded operators in H, while the operator A_4 is linear but unbounded. The above power series converges uniformly in the Fréchet sense, in the sphere $\|x_0\|^2 + \|y_0\|^2 + \varepsilon^2 \leq \rho$.

The following conditions give a specification of the certain solution of the reduced problem.

H6. There are functions $x_0(t)$ and $y_0(t)$ which satisfy the reduced Cauchy problem

$$\frac{d}{dt} x^0 + A_1(t,0)x^0 + A_2(t,0)y^0 = f(x^0, y^0, t, 0), \qquad (4.2.75)$$

$$A_3(t,0)x^0 + A_4(t,0)y^0 = g(x^0, y^0, t, 0), \qquad (4.2.76)$$

with the initial data

$$x^0(0) = \xi(0), \qquad (4.2.77)$$

and which has continuous derivatives of all orders.

H7. There is a function $y^*(t)$ which satisfies

$$A_3(t,0)x + A_4(t,0)y^* = g(x, y^*, t, 0), \qquad (4.2.78)$$

and which has continous derivatives of all orders. Further $y^*(t)$ is isolated in the sense that for each $t \varepsilon [0,T]$, there is a sphere about $y^*(t)$ which contains no other solutions of this equation.

H8.
$$f_x(x^0, y^0, t, 0) = 0, \quad f_y(x^0, y^0, t, 0) = 0,$$
$$g_x(x^0, y^0, t, 0) = 0, \quad g_y(x^0, y^0, t, 0) = 0, \qquad (4.2.79)$$

for $t \varepsilon [0,T]$. Here f_x denotes the Fréchet derivative of f with respect to x and similar for f_y, g_x, g_y.

The boundary layer system

$$\frac{d}{d\tau} Y + A_3(0,0)X + A_4(0,0)Y = g(X, Y, 0, 0), \; Y(0) = \eta(0), \qquad (4.2.80)$$

is derived through stretching transformation $\tau = t/\varepsilon$ and setting $\varepsilon = 0$.

H9. The boundary layer Cauchy problem (4.2.80) has a unique solution, $Y = Y_0(\tau)$, which exists for $\tau \varepsilon [0,\infty]$.

H10. The full problem has a unique solution, $(x(t), y(t))$ which exists for $t \varepsilon [0,T]$.

H11. The linear operator $A_4^{-1}(t,0)$ takes infinitely differentiable functions into infinitely differentiable functions.

Under these hypotheses, the following theorem similar to Theorem 1-2 holds:

Theorem 4-6 If hypotheses H1 - H11 hold, there exist a constant $\varepsilon_0 > 0$ and bounded functions temporally uniform, $R_n(\omega, t, \varepsilon)$ and $S_n(\omega, t, \varepsilon)$, such that

$$x(\omega,t,\varepsilon) = \sum_{r=0}^{n} [x^r(\omega,t) + X^r(\omega,\tau)]\varepsilon^r(r!)^{-1} + R_n(\omega,t,\varepsilon)\varepsilon^{n+1}, \qquad (4.2.81)$$
$$y(\omega,t,\varepsilon) = \sum_{r=0}^{n} [y^r(\omega,t) + Y^r(\omega,\tau)]\varepsilon^r(r!)^{-1} + S_n(\omega,t,\varepsilon)\varepsilon^{n+1}, \qquad (4.2.82)$$

for $\varepsilon \varepsilon [0,\varepsilon_0]$, $t \varepsilon [0,T]$.

The proof of this theorem will be possible by using the lemmas similar to those of lumped parameter cases, and by rephrasing the proof given in the lumped parameter cases, e.g., in Wasow [Ws.65] or in Hoppensteadt [Hp.71.1].

<u>Lemma 4-4</u> If H1 and H2 hold, then there exist positive constants ε_0, δ and C independent of t, ε, and such that

$$\| \exp[-tA_4(s,\varepsilon)/\varepsilon] \| \leq C\exp[-\delta(t - s)/\varepsilon], \qquad (4.2.83)$$

for $0 \leq s \leq t \leq T$ and $\varepsilon \in [0,\varepsilon_0]$.

<u>Lemma 4-5</u> If H1 - H11 hold, then there exists a set of positive constants C_r and δ_r such that

$$\begin{aligned} &\| X^r(\tau) \| + \| Y^r(\tau) \| \leq C_r \exp(-\delta_r \tau), \\ &\left\| \frac{d}{d\tau} X^r(\tau) \right\| + \left\| \frac{d}{d\tau} Y^r(\tau) \right\| \leq C_r \exp(-\delta_r \tau), \end{aligned} \qquad (4.2.84)$$

where $X^r(\tau)$ and $Y^r(\tau)$ are solutions of the boundary layer equations corresponding to Eqs.$(4.2.24)_r$ and $(4.2.25)_r$.

These lemmas can be shown in the similar way to those given in Hoppensteadt [Hp.69.1]. But our present object being the investigation of the singular perturbations of optimal control problems, with regard to proofs of these lemmas, Hoppensteadt [Hp.69.1] may be referred to.

4.3 Tracking problem

4.3.1 Problem statement

We are given Hilbert spaces V and H. The dual of V is denoted by V' and the dual of H is here assumed to be identified with H, and hence

$$V \subset H \subset V'.$$

For simplicity we are concerned with the system described by the following set of equations:

$$\frac{\partial}{\partial t} x - (a_1\Delta - a_2)x + a_3 y = b_1 u_1, \tag{4.3.1}$$

$$\varepsilon \frac{\partial}{\partial t} y - (a_4\Delta - a_5)y + a_6 x = b_2 u_2, \tag{4.3.2}$$

where $x \varepsilon H_1$ and $y \varepsilon H_2$ are scalars and a_i's are spatially homogeneous and time invariant parameters, Adopting the matrix representation, we express the system, in place of Eqs.(4.3.1) and (4.3.2), as

$$I_{\varepsilon}\frac{\partial}{\partial t} z + Az = Bu, \tag{4.3.3}$$

where

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} -a_1\Delta + a_2 & a_3 \\ a_6 & -a_4\Delta + a_5 \end{pmatrix},$$
$$B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{4.3.4}$$

Let us denote the Hilbert space of control by $\mathcal{U}$, and we suppose that

$$u = \{u_1, u_2\} \varepsilon \mathcal{U} = L^2(Q) \times L^2(Q). \tag{4.3.5}$$

Then the operator B is given by

$$B \varepsilon \mathcal{L}(\mathcal{U}; V'). \tag{4.3.6}$$

We define $\hat{\mathcal{U}}$ by

$$\hat{\mathcal{U}} = L^2(\Omega) \times L^2(\Omega). \tag{4.3.7}$$

The initial and boundary conditions are given as below:

$$z(\omega,0) = z_0(\omega) \varepsilon H, \quad \text{in } \Omega, \tag{4.3.8}$$

$$z = 0, \quad \text{on } \Sigma. \tag{4.3.9}$$

We also assume that

$$z \varepsilon L^2(0,T;V), \quad \frac{\partial}{\partial t} z \varepsilon L^2(0,T;V'). \tag{4.3.10}$$

The notations $L^2(0, T; V), L^2(0, T; V')$ are defined in Section 4.2.1.

The problem concerned is the tracking problem to transport the given initial state $z_0(\omega)$ to the desired state $z_d(\omega)$ given as

$$z_d(\omega) = \begin{bmatrix} x_d(\omega) \\ y_d(\omega) \end{bmatrix} ; \quad z_d \in L^2(\Omega), \tag{4.3.11}$$

with minimizing the following performance index:

$$J = \int_0^T \| z(t;u) - z_d(t) \|_{\hat{\mathcal{H}}} dt + (u, Ru)_{\mathcal{U}} , \tag{4.3.12}$$

where

$$\mathcal{H} = L^2(0,T;\hat{\mathcal{H}}), \tag{4.3.13}$$

and $\hat{\mathcal{H}}$ is defined similarly as $\hat{\mathcal{U}}$ in Eq.(4.3.7).

In this case, we take

$$V = H_0^1(\Omega) \times H_0^1(\Omega), \tag{4.3.14}$$

$$H = L^2(\Omega) \times L^2(\Omega). \tag{4.3.15}$$

The adjoint equations satisfied by the adjoint variables p and q corresponding to the state variables x and y respectively, are introduced as follows:

$$- \varepsilon \frac{\partial}{\partial t} p - (a_1 \Delta - a_2)p + a_6 q = w_1 x - x_d, \tag{4.3.16}$$

$$- \frac{\partial}{\partial t} q - (a_4 \Delta - a_5)q + a_3 p = w_2 y - y_d, \tag{4.3.17}$$

or

$$- I_\varepsilon \frac{\partial}{\partial t} s + A^* s = Wz - z_d. \tag{4.3.18}$$

In the case where there are no constraints, that is the optimal control u* minimizing J(u) when the control U ranges over the whole space $\mathcal{U}$, is contained in the space of admissible control $\mathcal{U}_{ad}$, we can derive the optimal control u* as

$$u^* = R^{-1}B's, \tag{4.3.19}$$

or

$$u_1^* = r_1^{-1}b_1p, \quad u_2^* = r_2^{-1}b_2q. \tag{4.3.19'}$$

Eliminating u(t) in Eqs.(4.3.1) and (4.3.2) by using Eq.(4.3.18), the following canonical equation is obtained:

$$I_\varepsilon \frac{\partial}{\partial t} z + Az = BR^{-1}B's, \tag{4.3.20}$$

$$- I_\varepsilon \frac{\partial}{\partial t} s + A^*s = Wz - z_d, \tag{4.3.21}$$

with initial and final conditions

$$z(\omega,0) = z, \quad \omega \varepsilon \Omega, \tag{4.3.22}$$

$$s(\omega,T) = 0, \quad \omega \varepsilon \Omega. \tag{4.3.23}$$

We decouple the canonical system (4.3.20), (4.3.21) by using the Riccati-like transformation: there exist an operator P(t) and a function r(t) such that

$$s(t) = P(t)z(t) + r(t), \tag{4.3.24}$$

where P(t) is given by

$$- I_\varepsilon \frac{\partial}{\partial t} P(t) + P(t)A + A^*P(t) + P(t)BR^{-1}B'P(t) = W, \tag{4.3.25}$$

$$P(t) \varepsilon \mathcal{L}(H;H). \tag{4.3.26}$$

An operator P(t) satisfies Eq.(4.3.25) in the sense that

$$[- I_\varepsilon \frac{\partial}{\partial t} P(t)]\phi + P(t)A\phi + A^*P(t)\phi + P(t)BR^{-1}B'P(t)\phi = W\phi, \tag{4.3.27}$$

for any $\phi \varepsilon V$.

The final terminal data are given by

$$P(T) = 0. \tag{4.3.28}$$

which holds in the same sense as in Eq.(4.3.27).

The vector function r(t) satisfies the following linear parabolic equation

$$- I\varepsilon\frac{\partial}{\partial t} r(t) + A^* r(t) + P(t)BR^{-1}B' r(t) = z_d(t), \quad (4.3.29)$$

$$r(t) \in H, \quad (4.3.30)$$

with the terminal condition

$$r(T) = 0. \quad (4.3.31)$$

The validity of the above derivation of P(t) and r(t) was ascertained in Lions [L.71], whose approach is a straight extension of the results in lumped parameter case obtained by Kalman [K.60]. Meyer [My.73] may also be referred to for Riccati equations in abstract space.

By using the Kernel Theorem due to Schwartz (cf [L.71]), P(t) has the representation

$$P(t)\varphi = \int_{\Omega} P(\omega, \omega', t)\varphi(\omega')d\omega' \quad (4.3.32)$$

where $P(\omega, \omega', t)$ is a kernel of P(t), which is a distribution on $\Omega_\omega \times \Omega_{\omega'}$, defined uniquely by P(t). The generated kernel $P(\omega, \omega', t)$ can be expressed as

$$P(\omega, \omega', t) = \begin{bmatrix} \varepsilon P_1(\omega, \omega', t) & \varepsilon P_2(\omega, \omega', t) \\ \varepsilon P_2'(\omega, \omega', t) & P_3(\omega, \omega', t) \end{bmatrix}, \quad (4.3.33)$$

with

$$P_1(\omega, \omega', t) = P_1(\omega', \omega, t), \quad (4.3.34)$$

$$P_3(\omega, \omega', t) = P_3(\omega', \omega, t). \quad (4.3.35)$$

Then the kernel corresponding to the operator P(t) characterized by Eq.(4.3.25) satisfies the following integro-differential equation of Riccati type

$$-\varepsilon\frac{\partial}{\partial t}P_1 = (A_{1\omega} + A^*_{1\omega'})P_1(\omega, \omega', t) + r^{-1}b_1{}^2P_1 o P_1 - w_1\delta(\omega - \omega')$$
$$- 2\varepsilon a_6P_2 + \varepsilon^2 r^{-1}b_2{}^2P_2 o P_2, \tag{4.3.36}$$

$$-\varepsilon\frac{\partial}{\partial t}P_2 = A^*_{1\omega}P_2 - a_6P_3 + r^{-1}b_1{}^2P_2 o P_1 - \varepsilon A_{4\omega}P_2 + \varepsilon r^{-1}b_2{}^2P_3 o P_2, \tag{4.3.37}$$

$$-\frac{\partial}{\partial t}P_3 = (A_{4\omega} + A^*_{4\omega'})P_3 - 2a_3P_2 + r^{-1}b_1{}^2P_2 o P_2 - w_2\delta(\omega - \omega'), \tag{4.3.38}$$

where

$$A_{1\omega} = a_1\Delta_\omega - a_2, \qquad A_{4\omega} = a_4\Delta_\omega - a_5$$

$$PoQ(\omega, \omega', t) = \int_\Omega P(\omega, \sigma, t)Q(\sigma, \omega', t)d\sigma, \tag{4.3.39}$$

with the terminal and boundary conditions

$$P_i = 0, \quad \text{on } \partial(\Omega\times\Omega), \tag{4.3.40}$$

$$P_i(\omega, \omega', T) = 0. \tag{4.3.41}$$

It is to be noted that we obtained Eqs.(4.3.36) - (4.3.38) by using the relation

$$PA\phi(\omega) = \int_\Omega P(\omega, \omega', t)A_\omega\phi(\omega)d\omega$$
$$= \int_\Omega A^*_{\omega'}P(\omega, \omega', t)\phi(\omega)d\omega,$$

hence the kernel of PA is $A^*_{\omega'}P(\omega, \omega', t)$. It is added that the kernel of the identity is $\delta(\omega - \omega')$.

4.3.2 Eigenfunction expansions

By using Theorem 4-2, there exists an orthonormal system of eigenfunctions satisfying

$$- A_{1\omega}\phi_j(\omega) = \lambda_j\phi_j(\omega), \qquad (4.3.42)$$

with

$$\phi_j|_\Gamma = 0, \qquad (4.3.43)$$

where

$$A_{1\omega} = a_1\Delta_\omega - a_2.$$

Then it can be shown that approximations to the state vector z and the costate vector s may be represented to be of the form,

$$_m z = \sum_{j=1}^{m} z_j(t)\phi_j, \qquad _m s = \sum_{j=1}^{m} s_j(t)\phi_j, \qquad (4.3.44)$$

$$z_j \in L^2(0,T), \qquad s_j \in L^2(0,T), \qquad (4.3.45)$$

$$\sum_{j=1}^{\infty} |\lambda_j| \int_0^T \| z_j(t)\|^2 dt < \infty, \qquad (4.3.46)$$

$$\sum_{j=1}^{\infty} |\lambda_j| \int_0^T \| s_j(t)\|^2 dt < \infty. \qquad (4.3.47)$$

These approximations $_m z$ and $_m s$ have the properties

$$\lim_{m\to\infty} {_m z} = z, \quad \text{in } L^2(\Omega), \qquad (4.3.48)$$

$$\lim_{m\to\infty} {_m s} = s, \quad \text{in } L^2(\Omega). \qquad (4.3.49)$$

An approximation to the kernels $P_i(\omega, \omega', t)$ may also be represented as

$$mP_i(\omega,\omega', t) = \sum_{k,l=1}^{m} P_i^{kl}\phi_k(\omega)\phi_l(\omega'), \qquad (4.3.50)$$

$$P_i^{kl} \in L^2(0,T). \qquad (4.3.51)$$

It is to be noted that the sequence $\{\phi_k(\cdot)\phi_l(\cdot)\}_{k,l=1,2,\ldots}$ constitutes an orthonormal basis in $L^2(\Omega \times \Omega)$ and $_mP_i$ has also the property

$$\lim_{m\to\infty} {}_mP_i = P_i, \quad \text{in } L^2(\Omega \times \Omega). \tag{4.3.52}$$

If we set

$$u_j = (u, \phi_j), \tag{4.3.53}$$

$$z_{0j} = (z_0, \phi_j), \tag{4.3.54}$$

and

$$z_{dj} = (z_d, \phi_j), \tag{4.3.55}$$

then, we obtain, substituting Eq.(4.3.44) into Eqs.(4.3.3) and (4.3.18):

$$I_\varepsilon \frac{d}{dt} z_j + A_j z_j = Bu_j, \tag{4.3.56}$$

$$- I_\varepsilon \frac{d}{dt} s_j + A_j' s_j = Wz_j - z_{dj}, \tag{4.3.57}$$

where

$$A_j = \begin{bmatrix} \lambda_j & a_3 \\ a_6 & \alpha_1 \lambda_j + \alpha_2 \end{bmatrix}, \quad \alpha_1 = \frac{a_2 a_4}{a_1}, \quad \alpha_2 = - a_5 + \frac{a_2 a_4}{a_1}.$$

The performance index (4.3.12) can also be decoupled into each mode by using the orthonormality of the basis $\{\phi_j(\omega)\}$, such that

$$J = \sum_{j=1}^{\infty} J_j, \quad J_j = \int_0^T \| z_j - z_{dj} \|^2 dt + \int_0^T (u_j, Ru_j) dt. \tag{4.3.58}$$

It is to be noted that J_j includes only variables with suffix j. Then we can synthesize the decoupled modal control. The property of the decoupled modal control ascertains that the time coefficients $P_i^{kl}(t)$ in Eq.(4.3.50) are represented by the form

$$P_i^{kl}(t) = \delta_{kl} P_i^{kl}(t), \tag{4.3.59}$$

where δ_{kl} denotes Kronecker's delta, i.e., we derive the following expression in place of Eq.(4.3.50):

$$ {}_mP_i(\omega, \omega', t) = \sum_{k=1}^{m} P_i^{kk}\phi_k(\omega)\phi_k(\omega'). \tag{4.3.60} $$

It is clear that $P_i^{kl}(t)$ is equivalent to the solution $P_i^{k}(t)$, $i = 1, 2, 3$, of the Riccati equation derived for the k-th mode canonical equation with the k-th mode performance index J_k. Considering the situations described above, we give the matrix representation for $\{P_i^{kl}(t)\}$ $i = 1, 2, 3$, as

$$ -\varepsilon\frac{d}{dt} P_1 = 2A_1P_1 + r^{-1}b_1{}^2P_1P_1 - w_1 - 2\varepsilon a_6P_2 + \varepsilon^2r^{-1}b_2{}^2P_2P_2, \tag{4.3.61} $$

$$ -\varepsilon\frac{d}{dt} P_2 = A_1P_2 - a_6P_3 + r^{-1}b_1{}^2P_2P_1 - \varepsilon A_4P_2 + \varepsilon r^{-1}b_2{}^2P_3P_2, \tag{4.3.62} $$

$$ -\frac{d}{dt} P_3 = 2A_4P_3 - 2a_3P_2 + r^{-1}b_2{}^2P_3P_3 + r^{-1}b_1{}^2P_2P_2 - w_2, \tag{4.3.63} $$

where A_i, $i = 1, 4$ and P_i, $i = 1, 2, 3$, are of the diagonal form

$$ A_1 = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_m \end{bmatrix}, \quad A_4 = \begin{bmatrix} \alpha_1\lambda_1+\alpha_2 & & & 0 \\ & \alpha_1\lambda_2+\alpha_2 & \ddots & \\ 0 & & & \alpha_1\lambda_m+\alpha_2 \end{bmatrix}, $$

$$ P_i = \begin{bmatrix} P_i^{11} & & 0 \\ & P_i^{22} \ddots & \\ 0 & & P_i^{mm} \end{bmatrix} $$

Thus the Riccati equation for the mode time coefficients $P_i^{kl}(t)$ can be separated into each mode Riccati equation and the generated Riccati equation of the individual mode can be treated independently of those of other modes. The Riccati equation of the k-th mode is represented as

$$ -\varepsilon\frac{d}{dt} P_1^{kk} = 2\lambda_kP_1^{kk} + r^{-1}b_1{}^2(P_1^{kk})^2 - w_1 - 2\varepsilon a_6P_2^{kk} $$

$$+ \varepsilon^2 r^{-1} b_2^2 (P_2^{kk})^2, \tag{4.3.64}$$

$$-\varepsilon \frac{d}{dt} P_2^{kk} = \lambda_k P_2^{kk} - a_6 P_3^{kk} + r^{-1} b_1^2 P_2^{kk} P_1^{kk} - \varepsilon(\alpha_1 \lambda_k + \alpha_2) P_2^{kk}$$

$$+ \varepsilon r^{-1} b_2^2 P_3^{kk} P_2^{kk}, \tag{4.3.65}$$

$$-\frac{d}{dt} P_3^{kk} = 2(\alpha_1 \lambda_k + \alpha_2) P_3^{kk} - 2a_3 P_2^{kk} + r^{-1} b_2^2 (P_3^{kk})^2$$

$$+ r^{-1} b_1^2 (P_2^{kk})^2 - w_2, \tag{4.3.66}$$

with the terminal conditions

$$P_i^{kk}(T) = 0, \quad i = 1, 2, 3. \tag{4.3.67}$$

4.3.3 Asymptotic expansions via boundary layer method

As in the lumped parameter case, we are naturally led to constructing solutions of Eqs.(4.3.36) - (4.3.38) of the form

$$P(\omega, \omega', t) = K(\omega, \omega', t) + H(\omega, \omega', \tau), \tag{4.3.68}$$

where $\tau=(T-t)/\varepsilon$, and $K(\omega, \omega', t)$ and $H(\omega, \omega', \tau)$ admit the following asymptotic expansions in ε as ε tends to zero:

$$K(\omega, \omega', t) = \sum_{r=0}^{\infty} K^r(\omega, \omega', t)\varepsilon^r (r!)^{-1}, \tag{4.3.69}$$

$$H(\omega, \omega', \tau) = \sum_{r=0}^{\infty} H^r(\omega, \omega', \tau)\varepsilon^r (r!)^{-1}, \tag{4.3.70}$$

and

$$K^r(\omega, \omega', t) = K^r(\omega', \omega, t), \tag{4.3.71}$$

$$H^r(\omega, \omega', \tau) = H^r(\omega', \omega, \tau). \tag{4.3.72}$$

The results of the preceding section, i.e., availability of the decoupled modal control, the kernel corresponding to the "slow mode" (see Section 1.4), $K(\omega, \omega', t)$ and the "fast mode" kernel $H(\omega, \omega', \tau)$, can be expanded into the following eigenfunction expansions:

$$K(\omega, \omega', t) = \sum_{k,l=1}^{\infty} \delta_{kl} K^{kl}(t)\phi_k(\omega)\phi_l(\omega'), \tag{4.3.73}$$

$$H(\omega, \omega', \tau) = \sum_{k,l=1}^{\infty} \delta_{kl} H^{kl}(\tau)\phi_k(\omega)\phi_l(\omega'). \tag{4.3.74}$$

As to the coefficient kernel of ε^r in Eqs.(4.3.69) and (4.3.70), the results analogous to Eqs.(4.3.73) and (4.3.74) hold. Thus the coefficient kernels of the outer expansion $K^r(\omega, \omega', t)$ and those of the boundary layer corrections $H^r(\omega, \omega', \tau)$ can be approximated by

$${}_mK^r(\omega, \omega', t) = \sum_{k=1}^{m} K^{r,kk}(t)\phi_k(\omega)\phi_k(\omega'), \tag{4.3.75}$$

$${}_mH^r(\omega, \omega', \tau) = \sum_{k=1}^{m} H^{r,kk}(\tau)\phi_k(\omega)\phi_k(\omega'), \tag{4.3.76}$$

where ${}_mK^r(\omega, \omega', t)$ and ${}_mH^r(\omega, \omega', \tau)$ have the property

$$\lim_{m\to\infty} {}_mK^r(\cdot, \cdot, t) = K^r(\cdot, \cdot, t), \quad \text{in } L^2(\Omega \times \Omega), \tag{4.3.77}$$

$$\lim_{m\to\infty} {}_mH^r(\cdot, \cdot, \tau) = H^r(\cdot, \cdot, \tau), \quad \text{in } L^2(\Omega \times \Omega). \tag{4.3.78}$$

The above derivations are founded on the construction of the solutions of Eqs.(4.3.64) - (4.3.66) with use made of the boundary layer method developed in the lumped parameter case. We seek the solutions of Eqs.(4.3.64) - (4.3.66) with the terminal conditions (4.3.67) of the form

$$P_i^{kk}(t) = K_i^{kk}(t) + H_i^{kk}(\tau), \tag{4.3.79}$$

where $K_i^{kk}(t)$ and $H_i^{kk}(\tau)$ admit the asymptotic expansions in ε as ε tends to zero,

$$K_i^{kk}(t) = \sum_{r=0}^{\infty} K_i^{r,kk}(t)\varepsilon^r(r!)^{-1}, \tag{4.3.80}$$

$$H_i^{kk}(\tau) = \sum_{r=0}^{\infty} H_i^{r,kk}(\tau)\varepsilon^r(r!)^{-1}, \tag{4.3.81}$$

the suffix i indicates the position of the element in the original matrices such as

$$K = \begin{bmatrix} \varepsilon K_1 & \varepsilon K_2 \\ \varepsilon K_2 & K_3 \end{bmatrix}, \quad H = \begin{bmatrix} \varepsilon H_1 & \varepsilon H_2 \\ \varepsilon H_2 & H_3 \end{bmatrix}. \tag{4.3.82}$$

In order to determine the coefficients $K_i^{r,kk}(t)$ and $H_i^{r,kk}(\tau)$, the same procedure can be adopted as in the lumped parameter case. The Riccati equation (4.3.64) - (4.3.66) gives the recursive set of equations of the outer expansion, when we substitute the outer expansion (4.3.80) into Eqs.(4.3.64) - (4.3.66). The construction of the boundary layer corrections is carried out by stretching the Riccati equation (4.3.64) - (4.3.66) and by substituting Eq.(4.3.79) into them with consideration given to Eqs.(4.3.80) and (4.3.81). And the generated recursive set of equations determines each coefficient of the boundary layer corrections $H_i^{r,kk}(\tau)$. We have shown here the complete applicability of the method derived in Chaper 2.

4.3.4 Main theorem

The object of this section is to establish the theorem giving the asymptotic accuracy of the resulting series solutions derived in the preceding section. First we state the following important condition:

C1. The Cauchy problem

$$\frac{\partial u}{\partial t} - A_{1\omega}u = 0, \quad A_{1\omega} = a_1\Delta_\omega - a_2 \tag{4.3.83}$$

is uniformly well-posed in the sense of Hadamard, and the semigroup exp $[tA(\sigma)]$ of this problem satisfies

$$\| \exp[tA_1(\sigma)] \| \leq C\exp(-\delta t), \quad \text{for } \delta > 0. \tag{4.3.84}$$

In this case the inequality(4.3.84) means that

$$a_1 > 0, \quad a_2 - a_1 > 0 \tag{4.3.85}$$

holds for $t \in [0,T]$.

We are led to the next condition, which considers the boundary layer controllability in the lumped parameter case.

C2. The boundary layer system associated with Eqs.(4.3.1) and (4.3.2)

$$\frac{\partial}{\partial\tau} x - (a_1\Delta - a_2)x + a_3 y = b_1 u_1 \tag{4.3.86}$$

is controllable, i.e.

$$b_1 \neq 0. \tag{4.3.87}$$

This definition of the boundary layer controllability can be extended easily to the case where x is a state vector of multi-dimensional. Then we can offer the following main theorem.

<u>Theorem 4-6</u> If conditions C1 and C2 and other specified conditions in Sections 4.3.2. and 4.3.3 hold, then there exist $\varepsilon_0 > 0$ and functions $R_i^n(t,\varepsilon)$ bounded uniformly in the t-interval [0,T] such that

$$\begin{gathered} {}_nP_i(\omega,\omega',t) = \sum_{r=0}^{n} [K_i^r(\omega,\omega',t) + H_i^r(\omega,\omega',\tau)]\varepsilon^r(r!)^{-1} + R_i^n(t,\varepsilon)\varepsilon^{n+1}, \\ K_i^r(\omega,\omega',t) = \sum_{k=1}^{\infty} K_i^{r,kk}\phi_k(\omega)\phi_k(\omega'), \quad H_i^r(\omega,\omega',\tau) = \sum_{k=1}^{\infty} H_i^{r,kk}\phi_k(\omega)\phi_k(\omega'), \end{gathered} \tag{4.3.88}$$

for $\varepsilon \in [0,\varepsilon_0]$ where $K_i^{r,kk}(t)$ and $H_i^{r,kk}(\tau)$ are determined stepwise by solving the recursive set of equations derived by using the procedure described in Section 4.3.3.

The proof of this theorem can be carried out in line with the proofs made in Theorem 4-5. We state the following lemmas verified in the course of the derivation of the eigenfunction expansion of the kernels $K(\omega, \omega', t)$ and $H(\omega, \omega', \tau)$, as shown in Eqs.(4.3.73) and (4.3.74).

<u>Lemma 4-6</u> Under all the conditions specified in the preceding

Sections 4.3.2 and 4.3.3, the kernels $K(\omega, \omega', t)$ and $H(\omega, \omega', \tau)$ can be expanded into

$$K(\omega, \omega', t) = \sum_{k=1}^{\infty} K^{kk}(t)\phi_k(\omega)\phi_k(\omega'), \tag{4.3.89}$$

$$H(\omega, \omega', \tau) = \sum_{k=1}^{\infty} H^{kk}(\tau)\phi_k(\omega)\phi_k(\omega'), \tag{4.3.90}$$

where $K^{kk}(t)$ and $H^{kk}(\tau)$ are derived by

$$K^{kk}(t) = \int_{\Omega\times\Omega} \phi_k(\omega)K(\omega, \omega', t)\phi_K(\omega')d\omega d\omega' \tag{4.3.91}$$

$$H^{kk}(\tau) = \int_{\Omega\times\Omega} \phi_k(\omega)H(\omega, \omega', \tau)\phi_k(\omega')d\omega d\omega' \tag{4.3.92}$$

$$K^{kk} \in L^2(0,T), \quad H^{kk} \in L^2(0,\infty).$$

Proof Theorem 4-2 also holds in the case of $L^2(\Omega \times \Omega)$ instead of $L^2(\Omega)$ with adequate modifications. (see Phillipson [Ph.71]) Q.E.D.

The truncation series of (4.3.75) and (4.3.76) up to the finite terms gives the so-called Galerkin approximation scheme.

Lemma 4-7 If the conditions C1 and C2 and other specified conditions in Section 4.3.2 and 4.3.3 hold, then there exist $\varepsilon_0 > 0$ and functions $S^n(t,\varepsilon)$ bounded uniformly in t-interval considered such that

$$\begin{aligned} {}^{n}P^{kk}(t) &= {}^{n}K^{kk}(t) + {}^{n}H^{kk}(\tau) + S^n(t,\varepsilon)\varepsilon_k^{\,n+1} \\ &= \sum_{r=0}^{n} K^{r,kk}(t)\varepsilon_k^{\,r}(r!)^{-1} + \sum_{r=0}^{n} H^{r,kk}(\tau)\varepsilon_k^{\,r}(r!)^{-1} \\ &\quad + S^n(t,\varepsilon)\varepsilon_k^{\,n+1} \\ &= \sum_{r=0}^{n} [K^{r,kk}(t) + H^{r,kk}(\tau)]\varepsilon_k^{\,r}(r!)^{-1} + S^n(t,\varepsilon)\varepsilon_k^{\,n+1}, \end{aligned} \tag{4.3.93}$$

where

$$\varepsilon_k = \varepsilon/k^2, \quad \varepsilon\in[0,\varepsilon_0].$$

Proof The singular perturbation theory of the lumped parameter case can be applied to the ordinary differential equations of the

Riccati type (4.3.64) - (4.3.66). As in Chapter 2, we can construct solutions of Eqs.(4.3.64) - (4.3.66) of the form

$$P_i^{kk}(t) = \tilde{K}_i^{kk}(t) + \tilde{H}_i^{kk}(\tau), \tag{4.3.94}$$

where $\tilde{K}_i^{kk}(t)$ and $\tilde{H}_i^{kk}(\tau)$ admit asymptotic expansions in ε as ε tends to zero (see the proof of Lemma 4.2)

$$\tilde{K}_i^{kk}(t) = \sum_{r=0}^{\infty} \tilde{K}_i^{r,kk}(t)\varepsilon^r(r!)^{-1}, \tag{4.3.95}$$

$$\tilde{H}_i^{kk}(\tau) = \sum_{r=0}^{\infty} \tilde{H}_i^{r,kk}(\tau)\varepsilon^r(r!)^{-1}. \tag{4.3.96}$$

The coefficients $\tilde{K}_i^{kk}(t)$ and $\tilde{H}_i^{kk}(\tau)$ are identical with those derived in Section 4.3.3, which completes the proof. Q.E.D.

The connection of Lemma 4-6 with Lemma 4-7 completes the proof of Theorem 4-6.

We restrict ourself to the investigation of the Riccati equation (4.3.36) - (4.3.38), and the same procedure adopted in the proof of the main theorem can be applied to the associated parabolic equation (4.3.29) by using the eigenfunction expansions as

$$r(\omega,t) = \sum_{j=1}^{\infty} r_j(t)\phi_j(\omega). \tag{4.3.97}$$

With regard to the construction of coefficients $r_j(t)$, the result of Section 4.2 may be referred to.

Chapter 5 Nuclear Reactor Control with Distributed Parameters

5.1 Introduction

This chapter treats the nuclear reactor control involving spatially distributed parameters. The importance of this treatment has been increasing, since the size of reactors is becoming larger and the spatial effects cannot be neglected. For instance, control of the neutron flux distribution, burnup control with consideration of hot-spot-factor, Xenon oscillation, etc. can be treated by considering distributed parameters.

Many studies have been made on the subject via various approaches, i.e., modal expansion method with use made of various types of mode; clean reactor mode, Kaplan mode(natural mode), lambda mode, etc., nodal expansion method, function space method, dynamic programming, and so forth. For a more precise survey, Iwazumi and Koga [I.73], or references listed in it, may be referred to.

The method proposed in this chapter is the singular perturbation theory connected with one of the modal expansion method, by using the Helmholtz mode. The Helmholtz mode gives the complete base in the space L^2. The theoretical foundation, upon which the validity of the method presented relies, is given in the preceding chapter.

It is to be noted that the criticality of the reactor considered violates the asymptotic stability of the equation for the prompt neutron. Hence we can not neglect time derivative term of the prompt neutron in the original partial differential equation. But the resulting equation using the modal expansion can be reduced in dimensionality except for the fundamental mode. Such a situation shows that the dimension of the dynamics governing the spatial control can be reduced and that of the power control cannot, since the power control is dominated by the fundamental mode and the spatial control is by the higher mode (see Wiberg [Wi.67]). Thus the singular perturbation theory is effectively applied to the spatial control system.

The system described by multi-group diffusion equations with multi-group of delayed neutrons has a different aspect. In this case, it is possible to reduce the dimensionality of this system in both power control and spatial control. Considerations of the situation are found in Sections 5.2.3 and 5.3.2.

5.2 Formulation of the problem

5.2.1 Derivation of the state equation

We consider here an infinite slab reactor described by the following set of equations, the one-group diffusion equation with one delayed neutron group:

$$\frac{1}{v}\frac{\partial}{\partial t}\psi(\omega,t) = \frac{\partial}{\partial\omega}[D(\omega)\frac{\partial}{\partial\omega}\psi(\omega,t)] - \Sigma_a(\omega)\psi(\omega,t) + (1-\beta)\nu\Sigma_f(\omega)\psi(\omega,t) + \lambda C - \Sigma_c(\omega,t)\psi(\omega,t), \tag{5.2.1}$$

$$\frac{\partial}{\partial t}C(\omega,t) = \beta\nu\Sigma_f(\omega)\psi(\omega,t) - \lambda C(\omega,t), \tag{5.2.2}$$

where ω is the spatial coordinate, $\psi(\omega,t)$ is neutron flux, $C(\omega,t)$ is precursor density, and $\Sigma_c(\omega,t)$ is the absorption cross section which is used for control variable, and assumed to be distributed in this case for simplicity. This system is bilinear form as in the lumped parameter model dealt with in Chapter 3. Hence linearization is made as in Chapter 3 by expanding each variable about the initial steady state.

The spatially dependent coefficient $D(\omega)$ prevents usfrom constructing analytical eigenfunctions. The approximation made to avoid this trouble, is the separation of the coefficient $D(\omega)$ into two parts, spatially variant and invariant parts. To ensure the finality of the expansion series (see Kaplan [Ka.61]), the other coefficients are also separated in the same way. The spatially independent parts should be chosen so that the fundamental mode may sustain the initial steady state and the effects of the spatially variant parts are taken into the control as the initial distribution of the control. This approximation was proposed by Iwazumi and Koga [I.73]. If the system behaves well in the course of motion, the approximation gives satisfactory results. The validity depends upon the property of the problem considered. In this chapter, we consider the tracking problem which shall give the validity, i.e., the desired state is given not so far from the initial state.

The linearized state equation can be obtained by considering small deviations from the steady fundamental mode, which is determined by the system at the initial time involving only spatially invariant parameters, and by normalizing with respect to the fundamental mode of neutron flux ψ_{10}, as

$$I_\varepsilon \frac{\partial}{\partial t} \Phi(\omega,t) = S_\omega \Phi(\omega,t) + Bu(\omega,t), \qquad (5.2.3)$$

where

$$\Phi(\omega,t) = \begin{bmatrix} \delta\psi(\omega,t)/\psi_{10} \\ \delta C(\omega,t)/\psi_{10} \end{bmatrix}, \quad u = -\psi(\omega,t)\Sigma_c(\omega,t)/\psi_{10},$$

$$S_\omega = \begin{bmatrix} a_1\partial^2/\partial\omega^2 + a_2 & a_3 \\ a_4 & a_5 \end{bmatrix}, \quad I_\varepsilon = \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad (5.2.4)$$

$$a_1 = D, \quad a_2 = (1-\beta)\nu\Sigma_f - \Sigma_a, \quad a_3 = \lambda,$$

$$a_4 = \beta\nu\Sigma_f, \quad a_5 = -\lambda, \quad \varepsilon = v^{-1}.$$

The resulting equation (5.2.3) is of the same form that Kuroda and Makino [Kd.69] treated. The difference lies in that Kuroda and Makino first assumed the spatial homogenity of the system parameters, but the treatment presented takes the effects of spatially varying parameters into consideration.

The problem to be considered is the terminal cost problem to minimize the following performance index:

$$J = \int_0^{\ell} [\Phi(\omega,T) - \Phi_d(\omega)]' \, Q[\Phi(\omega,T) - \Phi_d(\omega)]d\omega + \int_0^T\int_0^{\ell} r[u(\omega,t) - u_0(\omega)]^2 d\omega dt, \qquad (5.2.5)$$

where

$$Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_3 \end{bmatrix}; \quad q_1,\ q_3,\ r > 0,$$

$\Phi_d(\omega)$ is the desired state, and $u_0(\omega)$ is the initial value of the control determined beforehand so that the spatial homogenity of the system parameters holds. As to the construction of this initial value, Iwazumi and Koga [I.73] may be referred to.

5.2.2 Modal expansion

In the case of one dimensional space, i.e., an infinite slab reactor, the Helmholtz mode is defined by

$$\frac{d^2}{d\omega^2}\phi + \lambda\phi = 0, \qquad (5.2.6)$$

with

$$\phi(0) = 0, \quad \phi(\ell) = 0, \qquad (5.2.7)$$

where ℓ is an extrapolated thickness, λ_i eigenvalue, and ϕ_i eigenfunction;

$$\phi_i = \sqrt{2/\ell}\ \sin\sqrt{\lambda_i}\,\omega, \qquad (5.2.8)$$

$$\lambda_i = \left(\frac{i\pi}{\ell}\right)^2. \qquad (5.2.9)$$

The sequence $\{\phi_i\}$ constitutes an orthonormal set in L^2, i.e.,

$$\int_0^{\ell} \phi_i(\omega)\phi_j(\omega)d\omega = \delta_{ij}, \qquad (5.2.10)$$

where δ_{ij} is the Kronecker's delta.

If the existence of the solution has been proved, it can be shown that the system $\{\sqrt{2/\ell}\ \sin\sqrt{\lambda i}\ \omega\}$ can be replaced by other suitable orthonormal systems chosen in infinite ways (see Green [Gr.53]).

The state vector Φ can be expanded into infinite series with respect to the eigenfunctions generated above,

$$\Phi(\omega,t) = \sum_{i=1}^{\infty} \Phi_i(t)\sqrt{2/\ell}\ \sin\frac{i\pi}{\ell}\omega, \tag{5.2.11}$$

or

$$\begin{bmatrix}\psi(\omega,t)\\ \\ C(\omega,t)\end{bmatrix} = \sum_{i=1}^{\infty}\begin{bmatrix}\xi_i(t)\\ \\ \eta_i(t)\end{bmatrix}\sqrt{2/\ell}\ \sin\frac{i\pi}{\ell}\omega\ .$$

The control rods are physically concentrated, which should be represented mathematically by the Dirac measure δ. Then the control term is given by

$$u(\omega,t) = \sum_{k=1}^{M} u_k(t)\delta(\omega - \omega_k), \tag{5.2.12}$$

where M is the number of control rods and ω_k is the position of the k-th control rod. In the case where there are lots of control rods, however, the spatially concentrated control can be represented approximately by the spatially distributed control, which is simpler to analyze. The spatially concentrated case will be considered later. Here we restrict ourself to the spatially distributed case for simplicity.

The control term $u(\omega,t)$ can be expanded with respect to the eigenfunction of the Helmholtz mode, if the control $u(\cdot,t)$ is given in $L^2(\Omega)$.

$$u(\omega,t) = \sum_{i=1}^{\infty} u_i(t)\sqrt{2/\ell}\ \sin\frac{i\pi}{\ell}\omega\ . \tag{5.2.13}$$

In order to obtain a set of equations for the modal coefficients, $\xi_i(t)$, $\eta_i(t)$ and $u_i(t)$ which are functions of time, the expansions (5.2.11) and (5.2.13) are substituted into Eq.(5.2.3) and the orthonormality (5.2.10) is used, yielding

$$\varepsilon \frac{d}{dt} \xi_i(t) = [-a_1(\frac{i\pi}{\ell})^2 + a_2]\xi_i(t) + a_3\eta_i(t) + u_i(t), \quad (5.2.14)$$

$$\frac{d}{dt} \eta_i(t) = a_4\xi_i(t) + a_5\eta_i(t). \quad (5.2.15)$$

It is to be noted that the modal equations (5.2.14) and (5.2.15) are decoupled with each of the other modes, which results from the approximation made beforehand about the spatial homogenity of parameters.

To synthesize the optimal feedback control in the modal space, the performance index should be expanded into modal form. Thus the desired terminal distribution $\Phi_d(\omega)$ and the initial control $u_0(\omega)$ are expanded by using the same spatial eigenfunctions, as follows:

$$\Phi_d(\omega) = \sum_{i=1}^{\infty} \Phi_{di}\sqrt{2/\ell}\, \sin\frac{i\pi}{\ell}\omega = \sum_{i=1}^{\infty} \begin{bmatrix} \xi_{di} \\ \eta_{di} \end{bmatrix} \sqrt{2/\ell}\, \sin\frac{i\pi}{\ell}\omega,$$

$$u_0(\omega) = \sum_{i=1}^{\infty} u_{0i}\sqrt{2/\ell}\, \sin\frac{i\pi}{\ell}\omega \quad . \quad (5.2.16)$$

Then we obtain the modal expansion of the performance index, as

$$J = \sum_{i=1}^{\infty} \Psi_i'(T)Q\Psi_i(T) + r\int_0^T \sum_{i=1}^{\infty} \tilde{u}_i^2(t)dt \quad (5.2.17)$$

where

$$\Psi_i(t) = \Phi_i(t) - \Phi_{di}(t),$$

$$\tilde{u}_i(t) = u_i(t) - u_{0i}(t).$$

Hence the performance index for the i-th mode J_i becomes

$$J_i = \frac{1}{2} \Psi_i'(T)Q\Psi_i(T) + \frac{1}{2} r \int_0^T \tilde{u}_i^2(t)dt. \quad (5.2.18)$$

It is easily seen that optimal control theory for lumped parameter systems can be applied to each mode, since coupling between the other modes does not occur in total system including the performance index.

The state equation of the i-th mode and the corresponding i-th performance index are given in Eqs.(5.2.14), (5.2.15) and (5.2.18) respectively. Applying the well known results in optimal control theory of lumped parameter systems, the following control law is derived in the form of feedback type:

$$u_i^*(t) = -r^{-1}B'[K_i(t)\Phi_i(t) + g_i(t)] + u_{0i}, \qquad (5.2.19)$$

where $K_i(t)$ is a solution of the matrix Riccati differential equation of the i-th mode

$$\dot{K}_i(t) = -K_i(t)A_i - A_i'K_i(t) + r^{-1}K_i(t)BB'K_i(t), \qquad (5.2.20)$$

and $g_i(t)$ is a solution of the following linear differential equation associated with the Riccati equation derived above:

$$\dot{g}_i(t) = -(A_i - r^{-1}BB'K_i(t))'g_i(t) - K_i'(t)Bu_{0i}, \qquad (5.2.21)$$

where

$$A_i = \begin{bmatrix} a_1(i\pi/\ell)^2 + a_2 & a_3 \\ a_4 & a_5 \end{bmatrix}.$$

As the partitioning similar to Eqs.(2.2.16) and (2.2.17) holds also, then we obtain the following Riccati equations for each element,

$$\begin{aligned} \varepsilon\dot{k}_1 &= -2[-a_1(i\pi/\ell)^2 + a_2]k_1 + r^{-1}k_1^2 - 2\varepsilon a_4k_2, \\ \varepsilon\dot{k}_2 &= -a_3k_1 - [-a_1(i\pi/\ell)^2 + a_2]k_2 - a_4k_3 + r^{-1}k_1k_2 \\ &\quad - \varepsilon a_5k_2, \\ \dot{k}_3 &= -2a_3k_2 - 2a_5k_3 + r^{-1}k_2^2, \end{aligned} \qquad (5.2.22)$$

and for $g_i(t)$

$$\varepsilon\dot{g}_1 = -[a_1(i\pi/\ell)^2 + a_2 - r^{-1}k_1]g_1 - a_4g_2 - k_1u_0,$$
$$\dot{g}_2 = -a_3g_1 - a_5g_2 + r^{-1}k_2g_1 - k_2u_0. \tag{5.2.23}$$

These equations should be solved under the terminal conditions

$$K(T) = Q, \tag{5.2.24}$$

$$g(T) = -Q\Phi_d. \tag{5.2.25}$$

In the above expressions of Eqs.(5.2.22) - (5.2.25) the subscript of each variable representing the modal order is omitted, and the same expressions are adopted in what follows unless confusions may occur.

5.2.3 Consideration on the criticality condition

From the fact that the decoupled modal control is applicable to the system (5.2.3), the resulting infinite set of differential equations can be treated separately. Hence the technique of constructing asymptotic expansions of solutions developed in the lumped parameter systems can be applied to the i-th mode system of ordinary differential equations respectively, within the i-th mode system.

However, the distributed parameter reactor described by the kinetic equation of one-group diffusion with one delayed neutron or multiple-delayed neutron, has a distinct situation from the one-point reactor. The fundamental mode (the first mode) should be considered separately from the other higher modes, since the criticality of the reactor requires that the eigenvalue corresponding to the first mode of the clean reactor mode should be set to zero, which sustains the steady state of the initial state under consideration. The critical condition is thus obtained in the Helmholtz mode as

$$a_2a_5 - a_3a_4 - a_1a_5(\pi/\ell)^2 = 0 \tag{5.2.26}$$

The criticality condition is assumed to hold at the initial time, so the asymptotic stability of the boundary layer system of the first time coefficient of the neutron flux is not concluded [Hn. 69]. So the technique of the singular perturbations cannot be applied to the first mode system. The separation of the control system into two systems, the power control system and the spatial control system, gives a physical meaning of the above situation. The contributions of the higher modes to the power control are minor, and the reactor can be controlled by one feedback system represented by the fundamental mode. And the other feedback systems detect and control the other higher modes which determine the spatial shape of the neutron flux (see Wiberg [Wi.67, P322]). Hence we can say that the power control system should be treated in the original form, and the spatial control system can be dealt with by using the singular perturbation theory. In other words, singular perturbation theory is effectively applied to the spatial control system with reduction of the dimensionality.

The stability theory of the reactor with distributed parameters are found in literatures [S.68], [Ks.69], etc. The situations considered above is consistent with the result described in Kastenberg [Ks.69], which gives an extensive survey on the asymptotic stability of distributed systems adopting several approaches, Lyapunov method, semigroup theory, etc. The theory is a flourishing field.

5.2.4 Construction of asymptotic expansions

Applying the boundary layer method developed in the lumped parameter systems, we seek solutions of the form

$$\tilde{k}_j(t,\varepsilon) = k_j(t,\varepsilon) + h_j(\tau,\varepsilon); \quad j = 1, 2, 3, \tag{5.2.27}$$

where $\tau = (T - t)/\varepsilon$, and $k_j(t,\varepsilon)$, $h_j(\tau,\varepsilon)$ admit asymptotic expansions in ε, as ε tends to zero of the form

$$\begin{aligned} k_j(t,\varepsilon) &= \sum_{r=0}^{\infty} k_j^{\,r}(t)\varepsilon^r (r!)^{-1}, \\ h_j(\tau,\varepsilon) &= \sum_{r=0}^{\infty} h_j^{\,r}(\tau)\varepsilon^r (r!)^{-1}. \end{aligned} \tag{5.2.28}$$

The subscript j represents the position of the element such that

$$K = \begin{bmatrix} \varepsilon k_1 & \varepsilon k_2 \\ \varepsilon k_2 & k_3 \end{bmatrix}, \quad H = \begin{bmatrix} \varepsilon h_1 & \varepsilon h_2 \\ \varepsilon h_2 & h_3 \end{bmatrix}. \tag{5.2.29}$$

It can be easily seen that the sequences $\sum_i \sum_r K_{i,j}^{\ r}(t)$ and $\sum_i \sum_r H_{i,j}^{\ r}(\tau)$ converge to the solution of the Riccati integro-differential equation and that of the stretched Riccati equation associated with it, respectively, which are obtained directly by using the Riccati-like decoupling shown in the preceding chapter.

Substituting the outer expansion $\Sigma k_j^r \varepsilon^r (r!)^{-1}$ into (5.2.22), and comparing the coefficients of like powers of ε, we have the following set of equations:

for the higher mode $i > 1$

$$\begin{aligned} 0 &= -2[-a_1(i\pi/\ell)^2 + a_2]k_1^0 + r^{-1}k_1^{02}, \\ 0 &= -a_4k_3^0 - [-a_1(i\pi/\ell)^2 + a_2]k_2^0 - a_3k_1^0 + r^{-1}k_1^0k_2^0, \\ \dot{k}_3^0 &= -2a_5k_3^0 - 2a_3k_2^0 + r^{-1}(k_2^0)^2, \end{aligned} \tag{5.2.30$_0$}$$

and

$$\begin{aligned} \dot{k}_1^{\ r-1} &= -2[-a_1(i\pi/\ell)^2 + a_2]k_1^{\ r} + 2r^{-1}k_1^0k_1^{\ r} + p_1^{\ r}(t), \\ \dot{k}_2^{\ r-1} &= -a_3k_1^{\ r} - [-a_1(i\pi/\ell)^2 + a_2]k_2^{\ r} - a_4k_3^{\ r} + r^{-1}k_1^0k_2^{\ r} \\ &\quad + r^{-1}k_1^{\ r}k_2^0 + p_2^{\ r}(t), \\ \dot{k}_3^{\ r} &= -2a_3k_2^{\ r} - 2a_5k_3^{\ r} + 2r^{-1}k_2^0k_2^r + p_3^r(t), \end{aligned} \tag{5.2.30$_r$}$$

where the remainders $p_i^r(t)$, $i = 1 \sim 3$ are polynomials involving the known terms in the preceding steps.

In order to construct a uniformly valid expansion, the boundary layer corrections are considered. The recursive set of equations satisfied by the boundary layer correctors is obtained by using the stretched system as in the lumped parameter case in Section 3.2.3. The resulting recursive set of equations thus becomes

$$\begin{aligned}
-\frac{d}{d\tau} h_1^0 &= -2[-a_1(i\pi/\ell)^2 + a_2]h_1^0 + r^{-1}h_1^0[h_1^0 + k_1^0(T)], \\
-\frac{d}{d\tau} h_2^0 &= -a_3 h_1^0 - [-a_1(i\pi/\ell)^2 + a_2]h_2^0 - a_4 h_3^0 \\
&\quad + r^{-1}[h_1^0 h_2^0 + h_1^0 k_2^0(T) + k_1^0(T)h_2^0], \\
-\frac{d}{d\tau} h_3^0 &= 0.
\end{aligned} \tag{5.2.31$_0$}$$

For the higher order

$$\begin{aligned}
-\frac{d}{d\tau} h_1{}^r &= -2[-a_1(i\pi/\ell)^2 + a_2]h_1{}^r + r^{-1}h_1^0[h_1{}^r + k_1{}^r(T)] + p_1{}^r(\tau), \\
-\frac{d}{d\tau} h_2{}^r &= -a_3 h_1{}^r - [-a_1(i\pi/\ell)^2 + a_2]h_2{}^r - a_4 h_3{}^r \\
&\quad + r^{-1}h_1^0[h_2{}^r + k_2{}^r(T)] + r^{-1}h_2^0[h_1{}^r + k_1{}^r(T)] + p_2{}^r(\tau), \\
-\frac{d}{d\tau} h_3{}^r &= p_3{}^r(\tau),
\end{aligned} \tag{5.2.31$_r$}$$

where the remainders p_i^r consist only of functions decaying exponentially. Hence the terminal condition of the first boundary layer correction $h_3^1(0)$ can be determined by the following semi-infinite integral formula:

$$h_3{}^1(0) = \int_0^\infty (-2a_3h_2 - 2a_5h_3)d\tau. \tag{5.2.32}$$

The integrand involves only the terms determined in the 0-order boundary layer equation $(5.2.31)_0$ with the terminal condition derived by using the given data by setting ε to zero. Thus the terminal condition of the first coefficient of the outer expansion $k_3^r(t_f)$ can be derived by using the followng relation:

$$k_i(T) + h_i(0) = q_i; \; i = 1, \; 2, \; 3, \tag{5.2.33}$$

where q_i are given in Eq.(5.2.24). Hence we obtain

$$k_3{}^1(T) = q_3 - \int_0^{\infty} (-2a_3h_2 - 2a_5h_3)d\tau, \tag{5.2.34}$$

under which the first correction system of the outer expansion can be solved thoroughly. Then we calculate the value of each coefficient of the first outer expansion at the terminal time $t = t_f$. The values of the boundary layer correctors at the terminal time $\tau = 0$ directly follow by using the relation (5.2.33),

$$h_1{}^1(0) = q_1 - k_1{}^1(T), \quad h_2{}^1(0) = - k_2{}^1(T). \tag{5.2.35}$$

Thus we can carry out the computation of the first boundary layer correctors given in Eq.$(5.2.31)_1$.

The above procedure determines the total system of the reduced and first order completely. A similar result can be derived by using the same algorithm, for the higher order systems.

As to the linear differential equation for $g(t)$, the same procedure can be applied as in Section 2.2.

The practical computation can not be carried out with regard to the infinite terms of the modal expansion, hence the truncated series is adopted at finite N-th mode. The suboptimum distributed control is given by

$$u_{sub}(\omega,t) = \sum_{i=1}^{N} u_i{}^*(t)\sqrt{2/\ell}\ \sin\frac{i\pi}{\ell}\omega\ , \tag{5.2.36}$$

and the suboptimum trajectory, spatially dependent, is obtained by

$$\Phi_{sub}(\omega,t) = \sum_{i=1}^{N} \Phi_i(t)\ \sqrt{2/\ell}\ \sin\frac{i\pi}{\ell}\omega\ , \tag{5.2.37}$$

where the optimal trajectory Φ_i of the i-th mode can be derived as a solution of the i-th mode trajectory equation

$$\varepsilon\dot{\xi}_i = [-a_1(i\pi/\ell)^2 + a_2 - r^{-1}k_{i1}]\xi_i + a_3\eta_i - r^{-1}g_{i1} + u_{0i}, \tag{5.2.38}$$

$$\dot{\eta}_i = a_4\xi_i + a_5\eta_i, \qquad (5.2.39)$$

with the initial conditions

$$\xi_i(0) = [-a_1(i\pi/\ell)^2 + a_2 - a_4/a_5]^{-1} u_{0_i},$$
$$\eta_i(0) = -a_4/a_5 \cdot \xi_i(0), \qquad (5.2.40)$$

which can be obtained by equating the differential terms in Eqs.(5.2.14) and (5.2.15) to zero. The above procedure to compute the initial condition (5.2.40) is reasonable from the requirement of the steady state sustenance at the initial time. Equations (5.2.38) and (5.2.39) can be treated approximately by the singular perturbation theory as shown in Section 2.2.3.

5.2.5 Numerical example

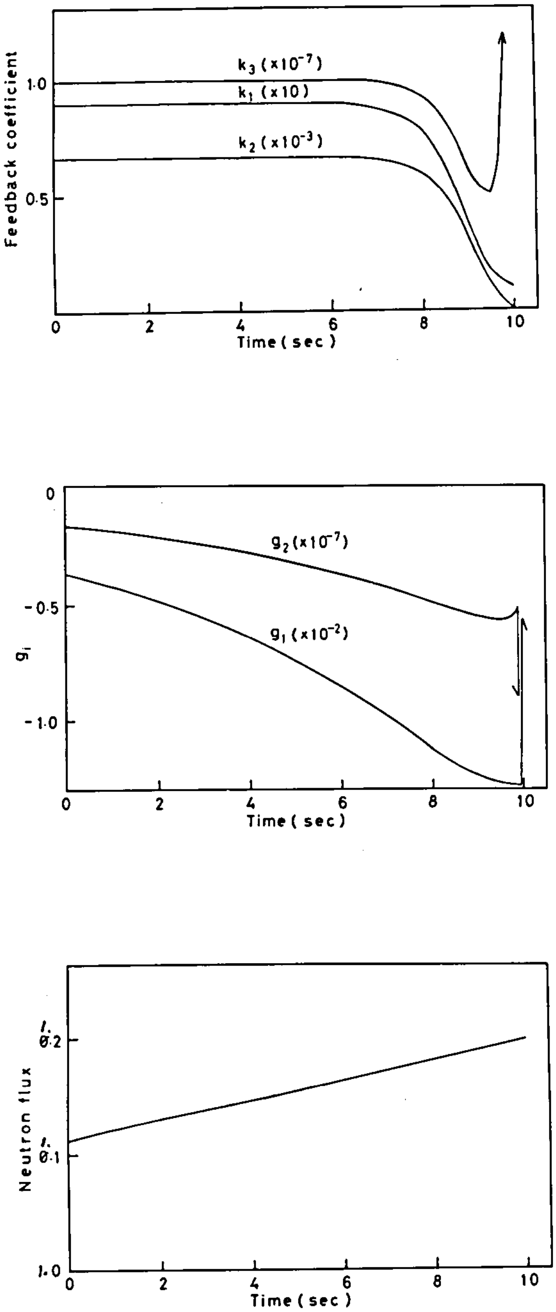
The results obtained by numerical computations are presented. The data used in this example are listed in Table 5.2.1.

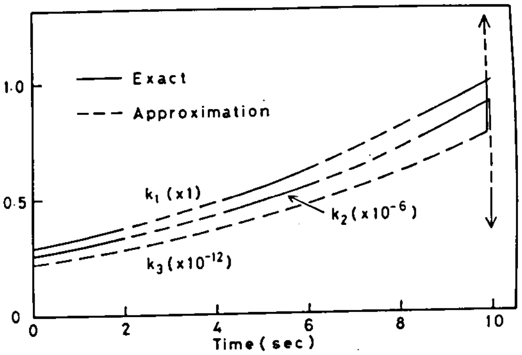
Table 5.2.1 Data used in the example.

λ	0.078	sec^{-1}	ℓ	100	cm
Σ_f	0.0202	cm^{-1}	T	10	sec
Σ_a	0.05	cm^{-1}	r	1000	
D	0.5071	cm	q_1	1	
v	2.2×10^5	cm/sec	q_3	78	

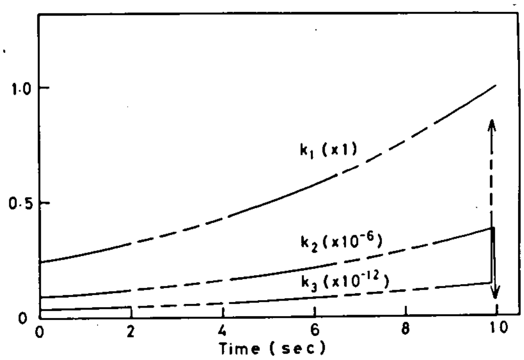
Figure 5.2.1 shows various time behaviours of the first mode: the elements of the feedback coefficient matrix, g_i, viz. the solution of the equation associated with the Riccati equation, and the amplitude of the neutron flux. This figure presents the exact solutions derived by using Runge-Kutta-Gill method.

Figure 5.2.2 presents the time behaviours of the elements of the

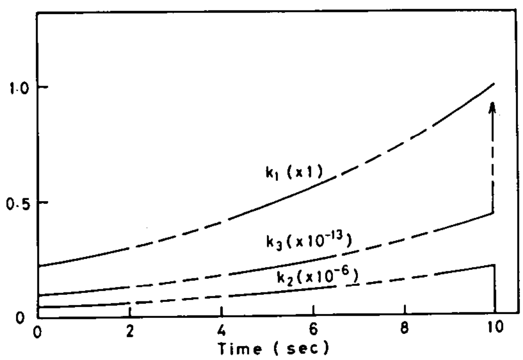




a) second mode



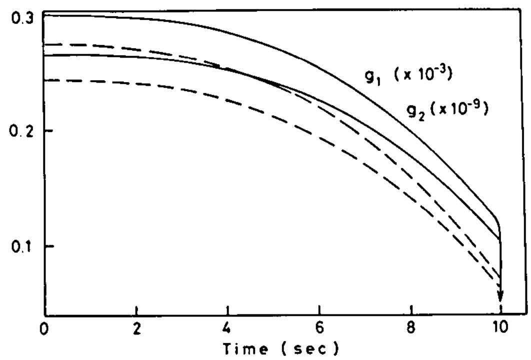
b) third mode



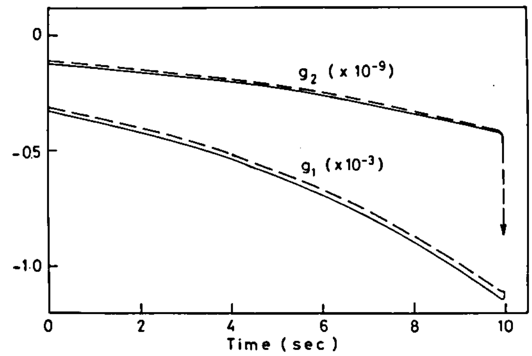
c) fourth mode

Fig. 5.2.1 First mode.

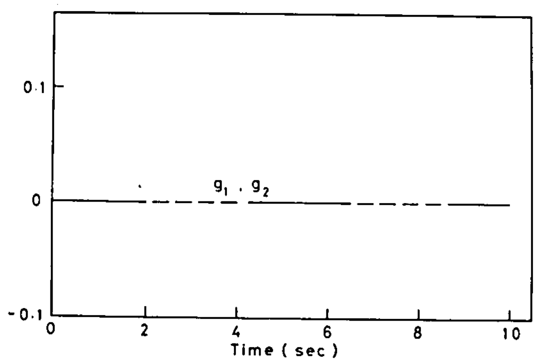
Fig. 5.2.2 Feedback coefficients.



a) second mode

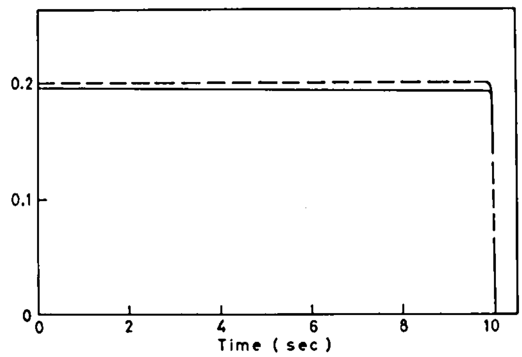


b) third mode

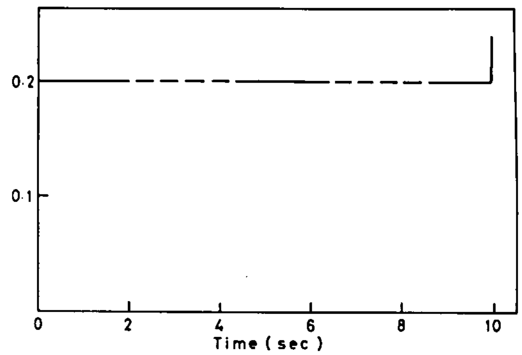


c) fourth mode

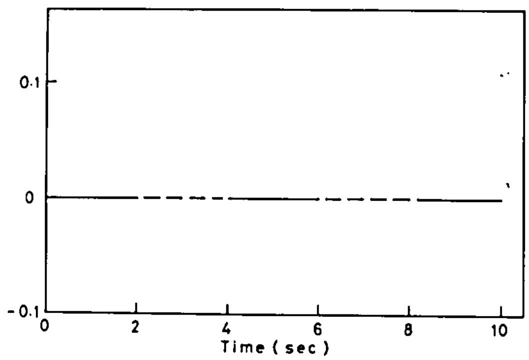
Fig. 5.2.3 g_1 and g_2.



a) second mode



b) third mode



c) fourth mode

Fig. 5.2.4 Amplitude of the neutron flux.

feedback coefficients matrix of the second to fourth mode; each figure shows both the exact and approximate solutions. The approximate solutions are derived of the form of truncated series at the second term.

As mentioned in the lumped parameter case, the feedback coefficients of each mode change rapidly in the neighbourhood of the terminal time, and computational difficulties should arise. The difficulties are much more serious in distributed parameter cases, because many modes equations must be treated and the existence of the boundary layers appearing in every mode makes the change more rapid. The present method can make us avoid the difficulties in the same way as in Section 3.2 except the fundamental mode.

The approximation to each feedback coefficient has as good accuracy as expected except k_3 of the second mode. But the error of k_3 of the second mode is of the order of 10^{-14} which is consistent with the theoretical results in the preceding sections. The order of k_3 itself is small relatively. Such situations show that the approximations except k_3 of the second mode is particularly good in this example.

Figure 5.2.3 shows the time behaviours of g_i of the second to fourth mode. These results for g_i do not give so good approximation as those for k_i. The cause of this situation is that the order of g_i itself is not sufficiently large compared with the order of the perturbing parameter ε. The truncated series approximation at the second term seems to be insufficient in appearance; but it is good enough from the practical point of view, considering the small effects of the higher modes to the motion considered.

In Fig. 5.2.4, the time behaviours of the amplitudes of the neutron flux of the higher modes are illustrated. It can be seen in this figure that the error of g_i of each higher mode is compensated.

Figure 5.2.5 shows the exact and approximate transients of the neutron flux shape.

Our numerical study shows that the computing time can be reduced to 21 percent of that in the case of the exact solution (the original modal solution). These savings prove that the method presented is very efficient in various points.

In this example, we have treated the terminal cost problem in order to compare with the results obtained by Iwazumi and Koga[I.73]. The method can be extended to the regulator problem, in which the rapid change of the feedback coefficients in the neighbourhood of the terminal time is relaxed more than this example. But as mentioned in Section 3.2, the existence of the boundary layer and the nonlinearity of the Riccati equation of each mode also cause the rapid change which leads computational difficulties. Of course, the singular perturbation theory is effective in the regulator problem.

It is to be noted that Iwazumi and Koga[I.73] and Asatani, Iwazumi and Hattori[A.73.1] used the Kaplan mode. The results of [A.73.1] are incorrect in that the modal expansion via the Kaplan mode is not of the singular perturbation type, and the method presented is not applicable. The modal expansion equation via the Kaplan mode approaches asymptotically to that via the Helmholtz mode.

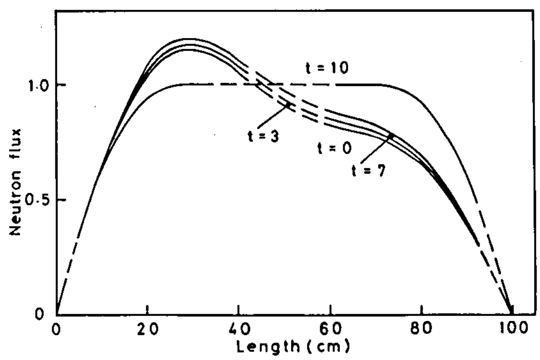


Fig. 5.2.5 Transient shape of the neutron flux.

After the completion of the theses, it has been found that the fundamental mode can also be solved approximately by utilizing the singular pertrubation theory, since the parameters adopted make the boundary layer system asymptotically stable even under the criticality condition. But in the case of reactor kinetic equation of one-group diffusion without delayed neutrons, the description of the text is valid.

5.3 Additional remarks

5.3.1 Spatially concentrated control

In the case of the spatially concentrated control the representation (5.2.12) by the Dirac measure gives the modal expansion form of the original system coupled with each other mode in place of Eqs.(5.2. 14) and (5.2.15), as

$$\frac{d}{dt}\begin{bmatrix} \varepsilon\xi_1 \\ \eta_1 \\ \varepsilon\xi_2 \\ \eta_2 \\ \vdots \end{bmatrix} = \left[\begin{array}{cc|cc|c} -a_1(\pi/\ell)^2 + a_2 & a_3 & \multicolumn{2}{c|}{0} & 0 \\ a_4 & a_5 & & & \\ \hline \multicolumn{2}{c|}{0} & -a_1(2\pi/\ell)^2 + a_2 & a_3 & 0 \\ & & a_4 & a_5 & \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & \ddots \end{array}\right] \begin{bmatrix} \xi_1 \\ \eta_1 \\ \xi_2 \\ \eta_2 \\ \vdots \end{bmatrix}$$

$$+ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \begin{bmatrix} \sin(\pi/\ell)\omega_1, & \cdots\cdots, & \sin(\pi/\ell)\omega_M \\ \sin(2\pi/\ell)\omega_1, & \cdots\cdots, & \sin(2\pi/\ell)\omega_M \\ \vdots & & \vdots \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_M \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad (5.3.1)$$

or in the matrix notation

$$I_\varepsilon \frac{d}{dt}\Phi_i = A_i\Phi_i + B\sum_{r=0}^{M} \sin\left(\frac{i\pi}{\ell}\omega_r\right)u_r(t); \quad i = 1, 2, \ldots\ldots, \infty ,$$

where

$$I = \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}.$$

If we truncate at the N-th finite mode, the matrix T becomes

$$T = \begin{bmatrix} \sin(\pi/\ell)\omega_1, \dots , & \sin(\pi/\ell)\omega_M \\ \vdots & \vdots \\ \sin(N\pi/\ell)\omega_1, & \dots, \sin(N\pi/\ell)\omega_M \end{bmatrix}. \tag{5.3.2}$$

The matrix T violates the modal decoupling in the case of spatially distributed control, hence the individual mode cannot be treated separately. The transformation

$$v(t) = T\cdot u(t), \tag{5.3.3}$$

gives the decoupled modal control formally, where v(t) is a new control vector [I.73]. The controllability in the modal space requires that

$$N \geqq M \tag{5.3.4}$$

where N is the number of the modes considered, and M is the dimension of the original control rods. The existence of the original M-dimensional control vector u(t) is ascertained through Eq.(5.3.3) by the following requirement :

$$\text{rank}(T) = M, \tag{5.3.5}$$

which imposes a restriction upon the positioning of the control rods. The decoupled modal control can be carried out if the following performance index is given:

$$J = \int_0^\ell [\Phi(\omega,T) - \Phi_d(\omega)]'Q[\Phi(\omega,T) - \Phi_d(\omega)]d\omega + \int_0^T\int_0^\ell r[v(\omega,t) - v_0(\omega)]^2 d\omega dt, \tag{5.3.6}$$

where $v(\omega,t)$ is a distributed new control vector determined from the vector v(t) of Eq.(5.3.3). Then the technique of the preceding section is applicable to the spatially concentrated control case.

5.3.2 Multi-group diffusion equation

The results obtained in Section 5.2 can be extended to the more general systems than that described by Eqs.(5.2.1) and (5.2.2), such as multi-group diffusion equations, which approximate better the accurate Boltzmann equation describing the balance of the neutrons in the reactor.

The form of the equation for one fast and one slow groups of neutrons and one group of delayed neutrons is given as follows:

$$
\begin{aligned}
\frac{1}{v_f}\frac{\partial}{\partial t}\psi_F &= [\nabla D_F \nabla - (\Sigma_{aF} + \Sigma_R) + (1-\beta)\nu\Sigma_{fF}]\psi_F \\
&\quad + (1-\beta)\nu\Sigma_{fs}\psi_s + \lambda_c C, \\
\frac{1}{v_s}\frac{\partial}{\partial t}\psi_s &= \Sigma_R\psi_F + (\nabla D_s \nabla - \Sigma_{as})\psi_s, \\
\frac{\partial}{\partial t} C &= \beta\nu\Sigma_{fF}\psi_F + \beta\nu\Sigma_{fs}\psi_s - \lambda_c C.
\end{aligned}
\tag{5.3.7}
$$

In this case the criticality condition can be sustained with consideration to both the fast group and slow group neutrons. This situation should affect the result of the criticality condition in the case of one group diffusion equation (5.2.1), derived in Section 5.2.3.

The fundamental modes of the fast and slow neutrons cannot sustain the steady state without the other, which ascertains that the eigenvalue corresponding to the fundamental mode should be negative. Hence the asymptotic stability of the boundary layer system belonging to Eq.(5.3.7) is established. We conclude that the singular perturbation theory can be satisfactorily applied to the multi-group diffusion equation with multi-group delayed neutrons, in both the power control systems and the spatial control systems with reduction of the dimensionalities.

Chapter 6 Conclusions

Near-optimum control of the singularly perturbed systems has been investigated. The results obtained are summarized in this chapter. Additional remarks involving suggestions to the future works remaining to be done, are also made.

Chapter 1 is concerned with the singular perturbation theory with the view point of historical survey. Further, a supplementary relation between the Vasil'eva's method(the method of inner and outer expansions) and the method of matched asymptotic expansions has been shown.

Chapter 2 deals with the near-optimum synthesis of controls arising in lumped parameter systems. The results obtained in each section are similar to each other in a formal sense. Indeed, the results of Section 2.2 and those of Section 2.4 are related closely in that, the results of Section 2.4(two small parameters case) is a generalization of those of Section 2.2(single parameter case). The extension to more parameters case can be made easily in the same way as in Section 2.4.

The situation in Section 2.3 is rather different from that in Sections 2.2 and 2.4. It is to be noted that the full system expressed by Eqs.(2.3.6) and (2.3.7) can be solved consistently with the initial conditions (2.3.8) and (2.3.9) or with the final conditions (2.3.8') and (2.3.9'). The recursive system, however, can be solved not in the forward direction but only in the backward direction and vice versa, according to the properties of the boundary layer systems (2.3.28) - (2.3.30).

Without loss of generality, we can fix the end-value to be zero, and then we have the advantage that we need not solve the equations for g_i's of the backward system because these have only trivial solutions.

Recently, O'Malley and others(e.g. [0.72.1.]) developed another technique to treat the singularly perturbed linear regulator problem from a differnt point of view via Turrittin's works [Tu.52] and Harris's [Hr.60, Hr.62] on two-point boundary value problems of the singular perturbation type. Their method may be applied to the fixed-terminal optimization problem and allows a more general property of the boundary layer system, viz. that its matrix of canonical system

$$G = \begin{bmatrix} A_4 & -B_2R^{-1}B_2' \\ -Q_3 & -A_4' \end{bmatrix}$$

should have m-eigenvalues with negative real part and m-eigenvalues with positive real part, which includes obviously our condition C5 of Section 2.3 as a special case. However, the generated recursive system must be solved under two-point boundary value conditions, and since the boundary layer occurs at both ends of the interval considered, it must treat both of them in order to make the "boundary layer corrections".

The results obtained in Section 2.3, fixed terminal optimization problems, can be extended to the several parameters case in the same manner as made in Section 2.4.

Chapter 3 is concerned with the application of the results derived in the preceding chapter to nuclear reactor control. The numerical examples show that the approximation by the singular perturbation theory is uniformly satisfactory in the interval considered. We restrict ourselves to the problem of the state regulation.

The method presented can be applied to the more general control problems such as power level change with minimum control energy, regulator problem of reactors involving temperature feedback, etc. (Weaver [Wv.68]).

Chapter 4 deals with near-optimum control of systems described by a set of partial differential equations of parabolic type. To the author's knowledge, the theory of singular perturbations of the systems described by evolution equations has not been developed sufficiently.

In Section 4.2 the singular perturbation theory for the restricted class of evolution equation is considered in respect of the regular degeneration and the construction of asymptotic expansions. The conditions assumed here are rather more rigid than necessary.
But from the practical point of view, the derived results offer us with powerful tools to attack a broad class of realistic distributed systems. Further extensions to the more general systems can be made in line with the method of Section 4.2.4.

In Section 4.3, asymptotic expansions of controls by extending the technique for lumped parameter systems was obtained. The advantage of this technique lies in that the result obtained is directly construc tive. As suggested in Section 4.2.4, the results derived in Section 4.3 can be generalized to the more general systems. Lions [L.73] studied also the regular degeneration of control systems of singular perturbation type with construction of the boundary layer corrections only of the 0-th order. He also studied the Riccati equation of singular perturbation type, but his result is not consistent with the author's, and there remains some doubt in the partitioning of the Riccati equation. While the author adopted Eq.(4.3.33), Lions's partitioning is as follows:
(Eq.(3.36) P563 of Lions [L.73])

$$P(\omega, \omega', t) = \begin{bmatrix} P_{11}(\omega, \omega', t) & P_{12}(\omega, \omega', t) \\ P_{21}(\omega, \omega', t) & P_{22}(\omega, \omega', t) \end{bmatrix}$$

which is not logical seeing that the reduction of dimensionality of the original state equation causes the reduction of that of the Riccati equation generated.

The theory presented is seen to be applicable to the problems considered in Chapter 2, such as regulator problems, fixed-end-point problems, and optimization problems of systems involving several parameters.

Chapter 5 is concerned with the regulator problems with terminal cost of the reactor with distributed parameters. The state equations (5.2.1) and (5.2.2) are not the same as treated in Chapter 4. Equation (5.2.2) does not involve any spatially differential operator, but this situation does not violate the applicability of the results obtained in Chapter 4, looking at it from the point of functional analysis (cf. Riesz-Fischer's Theorem, Theorem 4-2). Some numerical examples show the efficiency of the theory presented. The error estimation (Theorem 4-6) should be made with the consideration of the error estimation of the modal truncation. The error due to the truncation is estimated by Phillipson [Ph.71], or referred by Green [Gr.53].

Throughout this thesis, the case where the terminal time is fixed is treated, but the extension to the free-terminal time problem can be made in line with Kao and Bankoff [Kao. '73]. The system with small delay can be treated by the singular perturbation theory, constructing asymptotic expansions. Other interesting subjects are differential games of singular perturbation type, filtering theory of singularly pertubed system, etc. The theory is very widely applicable since the real physical system is often described by stiff differential equations, if the system is large in dimensionality.

It should be noted that the singular perturbation theory can be adopted in modeling, in order to select state variables essential to the motion considered, and this may offer us tools to avoid the knotty problem of exceeding dimensionality.

We add that the interesting field remains to be studied in respect of the deduction of the "canonical system of singular perturbation type" from the system of arbitrarily given form.

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Appendix A Theorems Related to Matrix Riccati Equations

The fundamental theorems due to Reid [R.63, R.65] are offered here.

The linear vector differential systems to be treated are of the form

$$\frac{d}{dt} u = Au + Bv, \quad \frac{d}{dt} v = Cu + Dv, \tag{A.1}$$

where u(t) and v(t) are n-dimensional vector functions, and A, B, C, and D are $n \times n$ matrices with time-invariant elements which are Lebesgue integrable on the interval considered. The theorems due to Reid are stated for the corresponding matrix differential equations

$$\frac{d}{dt} U = AU + BV, \quad \frac{d}{dt} V = CU + DV, \tag{A.2}$$

where in general U(t) and V(t) are matrices of $r \times n$, and here we consider the case where they are $n \times n$.

We adopt the following matrix notations in the sequel; the symbols E and 0 are used for the identity and zero matrices respectively of any dimensions; the conjugate transpose of a matrix M is denoted by M^*. The notation $M \geqq N$ or $N \leqq M$, ($M > N$ or $N < M$), is used to signify that M and N are Hermitian matrices of the same dimensions and $M - N$ is non-negative, (positive), definite.

By definition, a system (A.1) is identically normal if the only solution (u(t), v(t)) of this system with $u(t) \equiv 0$ on a nondegenerate interval is the identically vanishing solution $u(t) \equiv 0$, $v(t) \equiv 0$.

<u>Theorem A-1</u> ([R.63]) A system (A.2), identically normal and non-oscillatory on $]-\infty,\infty[$, has a principal solution at ∞ (at $-\infty$) if and only if the solution $(U_0(t), V_0(t))$ of (A.2) for which $U_0(0) = 0$, $V_0(0) = E$ is such that $W_0(t) = V_0(t)U_0(t)^{-1}$ converges to a limit W_∞ ($W_{-\infty}$) as $t \to -\infty (t \to \infty)$; the corresponding distinguished solution of the associated Riccati equation

$$- \frac{d}{dt} W = WA + DW + WBW - C, \qquad (A.3)$$

at $t = \infty (t = -\infty)$ is $W_\infty(t) = W_\infty (W_{-\infty}(t) = W_{-\infty})$.

Theorem A-2 ([R.63]) A system (A.2) with coefficient matrices satisfying $A^* = D$, $C^* = C$, $B^* = B \geqq 0$, and which is identically normal, is non-oscillatory on $]-\infty,\infty[$ if and only if there exists an Hermitian constant matrix W satisfying the algebraic matrix equation

$$WA + A^*W + WBW - C = 0; \qquad (A.4)$$

moreover, if such a system is non-oscillatory on $]-\infty,\infty[$ then there exist Hermitian matrices W_∞ and $W_{-\infty}$ which are individually solutions of Eq.(A.4), and are extreme solutions for Eq.(A.3) in the sense that if $W(t)$ is any Hermitian solution of Eq.(A.3) on $]-\infty,\infty[$ then $W_\infty \leqq W(t) \leqq W_{-\infty}$; in particular, if W is any Hermitian solution of Eq.(A.4) then $W_\infty \leqq W \leqq W_{-\infty}$.

Theorem A-3 ([R.65]) If A and B are constant n×n matrices with B Hermitian and $B \geqq 0$, while the $n \times n^2$ matrix

$$[B, AB, A^2B, \ldots., A^{n-1}B] \qquad (A.5)$$

has rank n, then there exist Hermitian matrices $W_\infty \leqq 0$ and $W_{-\infty} \geqq 0$ that are extreme solutions of the matrix equation

$$WA + A^*W + WBW = 0 \qquad (A.6)$$

in the sense that $W = W_\infty$ and $W = W_{-\infty}$ are individually solutions of Eq.(A.6), while if W is any Hermitian matrix satisfying Eq.(A.6) then $W_\infty \leqq W \leqq W_{-\infty}$. Moreover, $W_{-\infty} > 0$, $(W_\infty < 0)$, if and only if all eigenvalues λ of A, (-A), have $Re(\lambda) < 0$.

Theorem A-3 are related to the system (A.1) through the following lemma.

Lemma A-1 ([R.65]) The system (A.1) with constant coefficient matrices is identically normal if and only if the matrix (A.5) is of rank n.

Appendix B A Matrix Inversion Formula

The well-known inversion formula for a partitioned matrix leads to

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & \varepsilon^{-1}M_3 \end{bmatrix}^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{B.1}$$

where

$$A = [M_1 - \varepsilon M_2 M_3^{-1} M_2']^{-1} = M_1^{-1} + \varepsilon M_1^{-2} M_2 M_3^{-1} M_2' + O(\varepsilon^2),$$

$$B = -[M_1 - \varepsilon M_2 M_3^{-1} M_2']^{-1}[\varepsilon M_2 M_3^{-1}] = -\varepsilon M_1^{-1} M_2 M_3^{-1} + O(\varepsilon^2),$$

$$C = -\varepsilon M_3^{-1} M_2' [M_1 - \varepsilon M_2 M_3^{-1} M_2']^{-1} = -\varepsilon M_3^{-1} M_2' M_1^{-1} + O(\varepsilon^2),$$

$$D = [\varepsilon^{-1} M_3 - M_2' M_1^{-1} M_2]^{-1} = \varepsilon M_3^{-1} + O(\varepsilon^2).$$

Then we have

$$M^{-1} = \begin{bmatrix} M_1^{-1} + \varepsilon M_1^{-2} M_2 M_3^{-1} M_2' & -\varepsilon M_1^{-1} M_2 M_3^{-1} \\ -\varepsilon M_3^{-1} M_2' M_1^{-1} & \varepsilon M_3^{-1} \end{bmatrix} + O(\varepsilon^2). \tag{B.2}$$

Substituting Taylor expansions for M_i's (see Section 2.3.2) into Eq.(B.2), we derive Eq.(2.3.29).

Appendix C The Detailed Expression of Eq.(2.4.7)

$$\begin{aligned}
\frac{d}{dt} K_{11} = &- K_{11}A_{11} - A_{11}'K_{11} - K_{12}A_{21} - A_{21}'K_{12}' - K_{13}A_{31} - A_{31}'K_{13} \\
&+ K_{11}E_{11}K_{11} + K_{12}E_{12}'K_{11} + K_{13}E_{13}'K_{11} + K_{11}E_{12}K_{12}' \\
&+ K_{12}E_{22}K_{12}' + K_{13}E_{23}K_{12}' + K_{11}E_{13}K_{13}' + K_{12}E_{23}'K_{13}' \\
&+ K_{13}E_{33}K_{13}' - Q_1,
\end{aligned}$$

$$\begin{aligned}
\mu_1 \frac{d}{dt} K_{12} = &- K_{11}A_{12} - K_{12}A_{22} - K_{13}A_{32} - A_{21}'K_{22} - A_{31}'K_{23}' \\
&+ K_{11}E_{12}K_{22} + K_{12}E_{22}K_{22} + K_{13}E_{23}K_{22} + K_{11}E_{13}K_{23}' \\
&+ K_{12}E_{23}K_{23}' + K_{13}E_{33}K_{23}' + \mu_1(A_{11}'K_{12} + K_{11}E_{11}K_{12} \\
&+ K_{12}E_{12}'K_{12} + K_{13}E_{13}'K_{12}),
\end{aligned}$$

$$\begin{aligned}
\mu_1 \frac{d}{dt} K_{22} = &- K_{22}A_{22} - A_{22}'K_{22} - K_{23}A_{23} - A_{32}K_{32}' + K_{22}E_{22}K_{22} \\
&+ K_{23}E_{23}'K_{22} + K_{22}E_{23}K_{23}' + K_{23}E_{33}K_{23}' - Q_2 \\
&+ \mu_1(-K_{12}'A_{12} - A_{12}'K_{12} + K_{22}E_{12}'K_{12} + K_{23}E_{13}'K_{12} \\
&+ K_{12}'E_{12}K_{22} + K_{12}'E_{13}K_{23}) + \mu_1^2 K_{12}'E_{11}K_{12},
\end{aligned}$$

$$\begin{aligned}
\mu_1\mu_2 \frac{d}{dt} K_{13} = &- K_{11}A_{13} - K_{12}A_{23} - K_{13}A_{33} - A_{31}'K_{33} + K_{11}E_{13}K_{33} \\
&+ K_{12}E_{23}K_{33} + K_{13}E_{33}K_{33} + \mu_2(-A_{21}'K_{23} + K_{11}E_{12}K_{23} \\
&+ K_{12}E_{22}K_{23} + K_{13}E_{23}K_{23}) + \mu_1\mu_2(-A_{11}'K_{13} + K_{11}E_{11}K_{13} \\
&+ K_{12}E_{12}'K_{13} + K_{13}E_{13}'K_{13}),
\end{aligned}$$

$$\begin{aligned}
\mu_1\mu_2 \frac{d}{dt} K_{23} = &- K_{22}A_{23} - K_{23}A_{33} - A_{32}'K_{33} + K_{22}E_{23}K_{33} + K_{23}E_{33}K_{33} \\
&+ \mu_1(-K_{12}'A_{13} + K_{12}'E_{13}K_{33}) + \mu_2(-A_{22}'K_{23} + K_{22}E_{22}K_{23} \\
&+ K_{23}E_{23}'K_{23}) + \mu_1\mu_2(-A_{12}'K_{13} + K_{22}E_{12}'K_{13} + K_{23}E_{13}'K_{13} \\
&+ K_{12}'E_{12}K_{23}) + \mu_1^2\mu_2 K_{12}'E_{11}K_{13},
\end{aligned}$$

$$\mu_1\mu_2 \frac{d}{dt} K_{33} = - K_{33}A_{33} - A_{33}'K_{33} + K_{33}E_{33}K_{33} - Q_3 + \mu_2(-K_{23}'A_{23}$$

$$- A_{23}'K_{23} + K_{23}'E_{23}K_{33} + K_{33}E_{23}'K_{23}) + \mu_1\mu_2(-K_{13}A_{13}$$

$$- A_{13}'K_{13} + K_{33}E_{13}'K_{13} + K_{13}'E_{13}K_{33}) + \mu_2{}^2(K_{23}'E_{22}K_{23})$$

$$+ \mu_1\mu_2{}^2(K_{23}'E_{12}'K_{13} + K_{13}'E_{12}K_{23}) + \mu_1{}^2\mu_2{}^2K_{13}'E_{11}K_{13}.$$

Appendix D Time-Invariant Regulator Problem [Y.73]

Consider the regulator problem when $t_f \to \infty$ and all the system and weighting matrices are time-invariant. The solution of this problem is

$$u = -R^{-1}[(B_1H_1 + B_2'H_2')x_1 + (\varepsilon B_1'H_2 + B_2'H_3)x_2]. \qquad \text{(D.1)}$$

The matrix

$$H = \begin{bmatrix} H_1 & \varepsilon H_2 \\ \varepsilon H_2' & \varepsilon H_3 \end{bmatrix}$$

is the positive definite root of the following algebraic equation (cf. Eq.(2.2.18):

$$\begin{aligned}
0 &= -K_1A_1 - A_1'K_1 - K_2A_3 - A_3'K_2 + K_1E_1K_1 + K_2E_2'K_1 + K_1E_2K_2' \\
&\quad + K_2E_3K_2' - C_1'Q_1C_1, \\
0 &= -K_1A_2 - K_2A_4 - A_3'K_3 + K_1E_2K_3 + K_2E_3K_3 - C_1'Q_2C_2 \\
&\quad - \varepsilon(A_1'K_2 - K_1E_1K_2 - K_2E_2'K_2), \qquad \text{(D.2)} \\
0 &= -K_3A_4 - A_4'K_3 + K_3E_3K_3 - C_2'Q_3C_2 - \varepsilon(K_2'A_2 + A_2'K_2 \\
&\quad - K_3E_2'K_2 + K_2'E_2K_3) + \varepsilon^2K_2'E_1K_2.
\end{aligned}$$

This algebraic equation can be written compactly in the vector form as

$$F(H;\varepsilon) = 0. \qquad \text{(D.3)}$$

It defines the vector H as an implicit function of ε. If the Jacobian matrix $\partial F/\partial H$ at $\varepsilon = 0$ is nonsingular, the differentiability of the coefficient matrices in Eq.(D.2) is concluded. It was shown that the nonsingularity of $\partial F/\partial H$ at $\varepsilon = 0$ is guaranteed by the controllability of $\hat{A}$, $\hat{B}$ and the observability of $\hat{A}$, $\hat{C}$, where

$$S_1 = [E_2K_3 - A_2][A_4 - E_3K_3]^{-1},$$

$$S_2 = [A_3'K_3 + Q_2][A_4 - E_3K_3]^{-1},$$

$$\hat{A} = A_1 + S_1A_3 + E_2S_2' + S_1E_1S_2,$$

$$\hat{B} = B_1 + S_1B_2,$$

$$\hat{C} = \text{a solution of } \hat{C}'\hat{C} = \hat{Q},$$

$$\hat{Q} = - S_2A_3 - A_3'S_2' - S_2E_3S_2' + Q_1,$$

(cf. Section 2.3.4). Hence the following theorem.

<u>Theorem D-1</u> In addition to letting conditions C1 - C5 and C7 of Section 2.2.3 be satisfied for the time-invariant system considered, assume the following:

CD1. $\text{rank}[\hat{C}', \hat{A}'\hat{C}', \ldots, (\hat{A}')^{n-1}\hat{C}'] = n,$ (D.4)

CD2. $\text{rank}[\hat{B}, \hat{A}\hat{B}, \ldots, (\hat{A})^{n-1}\hat{B}] = n.$ (D.5)

Then there exists an $\varepsilon_0 > 0$ such that, for $\varepsilon \in [0, \varepsilon_0]$,

$$H_i(\varepsilon) = \sum_{j=0}^{n} H_i^{j}(0)(j!)^{-1}\varepsilon^j + H_i^{n+1}(\theta\varepsilon)(n+1)^{-1}\varepsilon^{n+1}, \quad i=1-3, \tag{D.6}$$

where $0 \leqq \theta \leqq 1$. The coefficients H_i^{0} are the solutions of Eq.(D.2) and H_3^{0} is the positive definite root of Eq.(D.2c) at $\varepsilon = 0$. Higher order coefficients satisfy the linear equations resulting from the successive differentiation of Eq.(D.2) with respect to ε.

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