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<th>State variable approaches to linear electrical network analysis (Dissertation_全文)</th>
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<tr>
<td>Author(s)</td>
<td>Nitta, Tanzo</td>
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<tr>
<td>Citation</td>
<td>Kyoto University (京都大学)</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1978-01-23</td>
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<tr>
<td>URL</td>
<td><a href="https://doi.org/10.14989/doctor.r3490">https://doi.org/10.14989/doctor.r3490</a></td>
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Kyoto University
STATE VARIABLE APPROACHES

TO

LINEAR ELECTRICAL NETWORK ANALYSIS

BY

TANZO NITTA

SEPTEMBER 1977

FACULTY OF ENGINEERING
KYOTO UNIVERSITY
JAPAN
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ACKNOWLEDGMENT

The author wishes to express his sincere gratitude to Dr. Akira Kishima, Professor of Kyoto University, for his constant guidance and encouragement during the course of this work and for his careful reading of the manuscript.

In the preparation of the present paper, the author was greatly aided by Dr. Takao Okada, Professor of Kyoto University, who gave him valuable suggestions.

The author would like to thank Dr. Takao Ozawa, Assistant Professor of Kyoto University, for his helpful advices and his careful reading of the manuscript.

The author is also grateful to Dr. Chikasa Uenosono, Professor of Kyoto University, for his constant advices and encouragement.

The author is also indebted to Dr. Kohshi Okumura, Assistant of Kyoto University, for his valuable discussions.

Acknowledgment must be made to the staffs and graduate students of Professor Kishima's, Professor Okada's and Professor Uenosono's researching groups for their excellent cooperations.
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CHAPTER 1  INTRODUCTION

In analyzing electro-magnetic phenomena, there are three steps to carry out:

First step: Finding an equivalent network for a electrical apparatus.

Second step: Finding equations describing the network.

Third step: Finding the solutions of the equations.

This paper is mainly concerned with the second step. The first and the third steps are considered to a small extent.

For networks which contain linear passive resistors, inductors and capacitors (RCL networks), Emeritus Professor S. Hayashi showed a method of formulating a set of equations, which be called the fundamental equations. The variables in the fundamental equations are the voltages across the capacitors and the currents through the inductors. The fundamental equations are simultaneous differential equations of the first order. However, they are not always independent.

In 1950's, the description of a network by means of the state equations,

\[ \dot{x} = Ax + Bu, \]

is proposed for RCL networks by T.R. Bashkow. After that, a great many studies on systematic formulations of the state equations for various networks are published. It is called state variable approach to network analysis. The state variables are closely relative to the network topology. The state variable approach is convenient for computer-aided network analysis.

In Chapter 2 - 5, the state variable approach to various
networks is studied. In Chapter 2, the state variable approach to RLC networks is described for later use.

In Chapter 3, is studied systematic formulations of state equations for linear, reciprocal, passive, time-invariant networks with mutual coupling elements. These networks are equivalent to those which contain resistors, inductors, capacitors, ideal-transformers and independent sources (RCLT networks). The properties of the state equations are also studied.

In Chapter 4, are studied the solvability and a systematic formulation of the state equations for linear, nonreciprocal passive time-invariant networks. These networks are equivalent to those which contain resistors, capacitors, gyrators and independent sources (RCG networks).

In Chapter 5, the solvability and a systematic formulation of state equations for linear, active, time-invariant networks (active networks) is presented.

In Chapter 6, networks containing periodically operated switches are considered by use of the results of Chapter 2 - 5. The networks have important applications. These networks are studied from a point of view different from that by S. Hayashi.
CHAPTER 2  STATE VARIABLE APPROACH

2-1 INTRODUCTION

In this chapter, graph theory and its terminology necessary to the state variable approach of lumped networks are described. A formulation of explicit state equations of linear passive lumped networks containing resistors, inductors, capacitors and independent sources as network-elements (so called RLC networks) is described, as a preparation for formulating explicit state equations of more general linear lumped networks.

2-2 NETWORK AND GRAPH

Consider a lumped network, then the graph of the network is obtained by replacing each lumped two terminal element with an oriented branch. The graph obtained from a graph $G$ by contracting a branch $b$"a set of branches $S"$ is denoted by $G[b]$"$G[S]$". The graph obtained from $G$ by deleting a branch $b$"a set of branches $S$" is denoted by $G\{b\}$"$G\{S\}$". The graph obtained from $G$ by contracting a branch $b_1$"a set of branches $S_1$" and deleting a branch $b_2$"a set of branches $S_2$" is denoted by $G\{b_1;b_2\}$"$G\{S_1;S_2\}$".

A branch of a graph corresponding to a capacitor"resistor, inductor, voltage source, current source" is called a capacitor"resistor-, inductor- voltage source-, current source-" branch.

A network and its graph are, respectively, denoted by $N^*$ and $G^*$. The graph[voltage source-branches;current source-branches] is denoted by $G$, and its corresponding network is denoted by $N$.

\[\text{\(\theta\)}\text{ The repetition of the (dual) sentence is avoided. The word in "" are to replace their preceding words in the (dual) sentence.}\]
The rank"nullity" of \( G \) is denoted by \( r(G) \) or \( \mu(G) \). A tree and its corresponding cotree are denoted by \( T \) and \( \overline{T} \), respectively.

A branch of \( T"\overline{T}" \) is called a twig"a link". A fundamental loop"cut-set" uniquely determined by a link \( l \) and some twigs"a twig t and some links" is denoted by \( \text{loop}(l)"\text{cut-set}(t)" \)

An \((i,j)\)element of a matrix \( A \) is denoted by \( a(i,j) \). The cofactor of \( a(i,j) \) is denoted by \( \text{Cof}.A(i,j) \). The determinant of \( A \) is denoted by \( |A| \).

Let \( B \) and \( Q \) denote basic loop and basic cut-set matrices, respectively, then nonsingular major submatrices of \( B"Q" \) are in one to one correspondence with cotrees"trees".

A graph is assumed to posses \( n_n \) nodes and \( n_b \) branches. A fundamental loop matrix of \( G \) for a tree \( T \) is written as

\[
B=[1,F],
\]

\[ (2-2.1) \]

where the matrix is a \( \mu(G) \times n_b \) matrix, \( 1 \) indicates the unit matrix of the order \( \mu(G) \) and \( F \) is known to be a unimodular matrix. Similarly, a fundamental cut-set matrix of \( G \) for \( T \) is written as

\[
Q=[-F',1],
\]

\[ (2-2.2) \]

where the matrix \( Q \) is an \( r(G) \times n_b \) matrix and \( 1 \) indicates the unit matrix of the order \( r(G) \).

The matrix \( F \) is called the characteristic part of a fundamental matrix for a tree \( T \). Every row"column" of \( F \) corresponds to a link "a twig".

The \( i \)-th row"the \( j \)-th column" of a matrix \( A \) is denoted by \( a(i) "a(j)" \).

If by exchanging a link \( l_1 \) with a twig \( t_1 \) for \( T \), another tree,
Lemma 2-2.1

Assume that the fundamental loop matrix $B$ the fundamental cut-set matrix $Q$ for a tree $T$ is given by $B = [1, F]' Q = [-F', 1]'$. If a tree $T_1$ is obtained by $m$ tree-transformations from $T$ such as $T_1 = T - (t_1 + \cdots + t_m) + (l_1 + \cdots + l_m)$, and the matrix $[B_1, B_2]' [Q_1, Q_2]'$ is obtained by permutating columns corresponding to $t_i$ and $l_i \ (i = 1, \cdots, m)$ of $B'Q'$, then the next equality holds,

$$|Q_2| \cdot |B_1| = (-1)^m. \quad (2-2.3)$$

Proof: If the matrix $B$ is written as

$$B = \begin{bmatrix} 1 & a_{11} & \cdots & a_{1r} \\ \cdot & 0 & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & \cdot & \cdots & \cdot \\ 1, a_{u1}, \cdots, a_{ur} \end{bmatrix}, \quad (2-2.4)$$

then $Q$ is written as

$$Q = \begin{bmatrix} -a_{11}, \cdots, -a_{u1}, 1, \\ \cdot & \cdots & 0 \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & 0 \\ -a_{1r}, \cdots, -a_{ur}, 1 \end{bmatrix}, \quad (2-2.5)$$

where $a_{i,j} = \pm 1, 0$.

Then the matrices $B_1$ and $Q_2$ may be written as
\[ B_1 = \begin{bmatrix}
1, & a_{11}, & \ldots, & a_{1t}, & \ldots, & a_{1T_m} \\
& \ddots & & \ddots & & \vdots \\
& & \ddots & & \ddots & \ddots \\
& & & \ddots & & \ddots \\
& & & & \ddots & & \ddots \\
& & & & & \ddots & \ddots \\
\end{bmatrix} \]
\[ Q_2 = \begin{bmatrix}
1, & -a_{11}, & \ldots, & -a_{11}, & \ldots, & -a_{11}, \\
& \ddots & & \ddots & & \vdots \\
& & \ddots & & \ddots & \ddots \\
& & & \ddots & & \ddots \\
& & & & \ddots & & \ddots \\
& & & & & \ddots & \ddots \\
\end{bmatrix} \]

Then the following equality holds
\[ |Q_2|_{B_1} = \begin{vmatrix}
-a_{11}, & \ldots, & -a_{11}, & \ldots, & -a_{11}, \\
& \ddots & & \ddots & & \vdots \\
& & \ddots & & \ddots & \ddots \\
& & & \ddots & & \ddots \\
& & & & \ddots & & \ddots \\
& & & & & \ddots & \ddots \\
\end{vmatrix} \]
The number of elements which are contained in a set $S$ is denoted by $|S|$.

The set of twig-"link-"capacitor-branches, twig-"link-"resistor-branches and twig-"link-"inductor-branches are, respectively, denoted by $T(C)^{\Gamma(C)}$, $T(R)^{\Gamma(R)}$ and $T(L)^{\Gamma(L)}$.

Definition 2-2.1

A normal tree (as denoted by $T_n$) for an RLC network is a tree of $G$ such that $|T(C)| + |\Gamma(L)|$ is a maximum.

2-3 STATE EQUATION

Assumption A

In $G^*$, there exists neither loop composed of only independent voltage source-branches nor cut-set composed of only independent current source-branches.

In a network which does not satisfy Assumption A, the Kirchhoff's voltage"current" law may not holds. In the Kirchhoff's voltage and current laws hold in this network, the voltages"currents" of independent voltage"current" sources in such a loop"cut-set" are
linearly dependent. The network obtained by deleting "contracting" any one independent voltage "current" source in the loop "cut-set" is equivalent to the original network. Its graph satisfies Assumption A. Then consider an RLC network which satisfies Assumption A.

Let \( v \) and \( i \) denote the branch voltage and the branch current respectively. The \( i \)-th components of \( v \) and \( i \) are voltage across the \( i \)-th branch and current through the \( i \)-th branch. Let a fundamental loop and a fundamental cut-set matrices for a tree \( T \) be \( B \) and \( Q \), then the Kirchhoff's voltage and current laws state

\[
BV = e, \quad \text{(2-3.1)}
\]
\[
Qi = j, \quad \text{(2-3.2)}
\]

where \( e \) is a loop source voltage vector, the \( i \)-th component of which is the algebraic sum of source voltages appearing in the \( i \)-th fundamental loop, and \( j \) is a cut-set source current vector, the \( k \)-th component of which is the algebraic sum of source currents appearing in the \( k \)-th fundamental cut-set.

The relation between branch voltages and branch currents may be represented by

\[
Yv = Zi, \quad \text{(2-3.3)}
\]

where \( Y \) and \( Z \) are diagonal matrices. The each \((k,k)\) component of \( Y \) and \( Z \) is specified as follows.

\[
y(k,k) = 1, \quad z(k,k) = r_k \quad \text{for a resistor-branch}
\]
\[
y(k,k) = C_k p, \quad z(k,k) = 1 \quad \text{for a capacitor-branch}
\]
\[
y(k,k) = 1, \quad z(k,k) = L_k p \quad \text{for an inductor-branch},
\]

where \( p \equiv d/dt \).

Eq. (2-3.1), (2-3.2) and (2-3.3) may be written as a matrix equation:
which is called a network equation. The coefficient matrix is denoted by \( \Theta(p) \).

It is known that if and only if the matrix \( \Theta(p) \) is nonsingular, Eq. (2-3.4) has a unique solution. The unique solvability of network means that Eq. (2-3.4) has a unique solution. Whether the network has a unique solution or not is the problem on network solvability.

The next theorem has already obtained.

**Theorem 2-3.1**

An RLC network which satisfies Assumption A has a unique solution.

**Definition 2-3.1**

The natural angular frequencies of a network are defined to be the roots of the characteristic polynomial of \( \Theta(p) \). The number of these natural angular frequencies (denoted by \( \sigma \)) is called the order of complexity of the network.

**Theorem 2-3.2**

The order of complexity \( \sigma \) of an RLC network is given by

\[
\sigma = |T_n(C)| + |T_n(L)|. \tag{2-3.5}
\]

A method of formulating the state equations of RLC networks been already given\(^2\)[3]. Let us describe it briefly for the
studies in latter chapters.

The branch voltages and currents may be, conveniently, partitioned as

\[
\begin{align*}
\mathbf{v} &= \begin{pmatrix} 
\mathbf{v}_S \\
\mathbf{v}_R \\
\mathbf{v}_L \\
\mathbf{v}_C \\
\mathbf{v}_G \\
\mathbf{v}_\Gamma 
\end{pmatrix}, \\
\mathbf{i} &= \begin{pmatrix} 
\mathbf{i}_S \\
\mathbf{i}_R \\
\mathbf{i}_L \\
\mathbf{i}_C \\
\mathbf{i}_G \\
\mathbf{i}_\Gamma 
\end{pmatrix},
\end{align*}
\]

(2-3.6)

where the subscripts \( S, R \) and \( L \) denote link-capacitors, link-resistors and link-inductors, respectively. The subscript \( C, G \) and \( \Gamma \) denote twig-capacitors, twig-resistors and twig-inductors, respectively.

The characteristic part of a fundamental loop matrix for \( T_N \) may be represented by

\[
\begin{pmatrix}
F_{SC}, & 0, & 0 \\
F_{RC}, & F_{RG}, & 0 \\
F_{LC}, & F_{LG}, & F_{LT}
\end{pmatrix},
\]

(2-3.7)

The relations between branch voltages and currents can be expressed as

\[
\begin{pmatrix}
C_1, & 0, & \frac{d}{dt} \mathbf{v}_S = \mathbf{i}_S, \\
0, & C_2, & \mathbf{v}_C = \mathbf{i}_C \\
R_1, & 0, & \mathbf{i}_R = \mathbf{v}_R, \\
0, & G_2, & \mathbf{v}_G = \mathbf{i}_G \\
L_1, & 0, & \frac{d}{dt} \mathbf{i}_L = \mathbf{v}_L, \\
0, & L_2, & \mathbf{i}_\Gamma = \mathbf{v}_\Gamma
\end{pmatrix},
\]

for capacitor-branches,

for resistor-branches,

for inductor-branches,

where the matrices \( C_1, C_2, R_1, G_2, L_1 \) and \( L_2 \) are diagonal.

From the Kirchhoff's voltage and current laws, we obtain
where the loop source voltage vector \( e \) and the cut-set source current vector \( j \) are partitioned (for example, the \( k \)-th component of \( e_S \) is the algebraic sum of source voltages appearing in the loop (the \( k \)-th link-capacitor branch)).

Eliminating voltage and current vectors except \( v_C \) and \( i_L \) from Eq. (2-3.8), (2-3.9) and (2-3.10), we obtain a differential state equation as

\[
\dot{x} = Ax + Bu,
\]

where

\[
x = \begin{bmatrix} v_C \\ i_L \end{bmatrix}, \quad u = \begin{bmatrix} j_C \\ j_G \\ j_I \end{bmatrix}, \quad A = \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix}, \quad B = \begin{bmatrix} 1, F_{RC} R^{-1} \end{bmatrix}, \quad Y = \begin{bmatrix} 0, F_{RG} G^{-1} \end{bmatrix},
\]

\[
H = \begin{bmatrix} 0, F_{RG} G^{-1} \end{bmatrix}, \quad C = C_2 + F_{SC} C_1 F_{SC}, \quad L = L_1 + F_{LT} L_2 F_{LT},
\]

\[
R = R_1 + F_{RG} G^{-2} F_{RG}, \quad g = G_2 + F_{RG} R^{-1} F_{RG}
\]
Property 2-3.1

The matrices $C$, $L$, $R$ and $g$ are positive definite.

Proof: Consider the matrix $C$. From Eq. (2-3.12), it is rewritten as

$$C = \begin{vmatrix} -F'_{SC}, 1 & C_1, 0 & -F_{SC} \\ 0, C_2 & 1 \end{vmatrix}. \quad (2-3.13)$$

Then, the matrix $C$ may be positive semi-definite. The first matrix of the right hand side of Eq. (2-3.13) is a fundamental cut-set matrix of $G(R,L)$ and the last matrix is a transposed matrix of the fundamental cut-set matrix. Therefore the determinant of $C$ is given by

$$|C| = \prod_{\text{all tree}} \text{(tree-capacitance product in } G(R,L)) .$$

Since $|C| \neq 0$ from this expansion, the matrix $C$ is positive definite. Since the matrices $L$, $R$ and $g$ are of the same form as $C$, they are positive definite in similar way.

Q.E.D

Nonsingular $R$ and $g$ guarantee the existence of a hybrid matrix, whose rows correspond to the currents through the capacitors and the voltages across the inductors, and whose columns correspond to the voltages across the capacitors and the currents through the inductors. Nonsingular $C$ and $L$ guarantee that the state equation (2-3.11) has a unique solution. The above discussions are essential to deriving state equations.

Property 2-3.2

Let $h(i,j)$ denote an $(i,j)$ element of $H$, then the absolute value $|h(i,j)|$ is not more than 1.
Property 2-3.2 is a result of the non-amplification property of a resistive network. Here another proof is given.

Proof: In order to obtain the value of \( h(i,j) \), consider the graph \( G_{i,j} = G \{ \text{all capacitor-branches except } C_i; \text{ all inductor-branches except } L_j \} \), where \( C_i \) and \( L_j \) are twig-capacitor and link-inductor corresponding to \( h(i,j) \). The graph \( G_{i,j} \) may be separable, that is, it may have more than one non-separable component. When \( C_i \) is not in the component where \( L_j \) is, \( h(i,j) = 0 \). When \( C_i \) and \( L_j \) are in one component, consider the possibility of a normal tree in \( G_{i,j} \) such that \( C_i \) is not in the loop \( L_j \). But such a normal tree can or cannot be chosen. Consider the case where such a tree does not exist. Then, if both \( f_{RC}(i) \) and \( f_{LG}(j) \) are nonzero, the elements of \( F_{RG} \) corresponding to nonzero \( f_{RC}(i) \) and \( f_{LG}(j) \) are zero. For if they were nonzero, a normal tree with the above-mentioned property could have existed. Then we obtain

\[
f'_{RC}(i)R^{-1}f_{RG}G^{-1}f'_{LG}(j) = 0.
\] (2-3.14)

Eq. (2-3.14) holds evidently in the case where \( f_{RC}(i) \) or \( f_{LG}(j) \) is zero. Since \( C_i \) is in the loop \( L_j \), \( |h(i,j)| = 1 \) in this case.

Consider the case where there exists a normal tree for which \( C_i \) is not in the loop \( L_j \). Let \( N_{i,j} \) denote the corresponding 2-ports of \( G_{i,j} \) as shown in Fig.2-3.1. Let \( T_{2i,j,i,j} \), denote a 2-tree, where the nodes \( i \) and \( j \) are in one connected part and the nodes \( i' \) and \( j' \) are in the other connected part. Let \( N_{i,j,i',j'} \) denote the sum of 2-tree admittance products corresponding to 2-trees \( T_{2i,j,i,j} \). The relation between voltages and currents of the 2-ports \( N_{i,j} \) is assumed to be

\[
\begin{align*}
|v_i| &= |r_{ii}^* r_{ij}^*| i_i^* \\
|v_j| &= |r_{ji}^* r_{jj}^*| i_j^*
\end{align*}
\] (2-3.15)
From Eq. (2-3.15), the element $h(i,j)$ is described as

$$h(i,j) = \frac{r_{i,j}}{r_{j,j}}$$  \hspace{1cm} (2-3.16)

The topological formula [4] leads

$$r_{i,j} = \{w_{i,j}, i,j', w_{i,j'}, i,j\}/V(Y)$$

$$r_{j,j} = \{w_{j,j}, j,j', w_{j,j'}, j,j\}/V(Y),$$

where $V(Y)$ denote the sum of tree admittance products and $w_{j,j'}$, the sum of tree admittance products corresponding to trees $T_{2j,j'}$, where the node $j$ is in one connected part and the node $j'$ in the other connected part. Since

$$w_{j,j'} = w_{i,j}, i,j', +w_{i,j'}, i', j', +w_{i',i,j',j'}, i', j',$$

we obtain

$$|r_{i,j}| < |r_{j,j}|$$

$$|h(i,j)| < 1.$$  \hspace{1cm} (2-3.17)

Consequently, we obtain

$$|h(i,j)| \leq 1,$$

where the equal sign holds if there is no normal tree for which the branch $C_{i}$ is not in the loop($L_{j}$).

Q.E.D
Property 2-3.2 does not hold in RLCT, RCG and active networks.

Property 2-3.3\(^1\)

If the derivatives of input functions appear in the state equations, there exists at least one loop which consists of voltage source branches and capacitor branches only, or at least one cut-set which consists of current source branches and inductor branches only in \(G^*\).

Property 2-3.4\(^5\)

If there exists at least one cut-set which consists of capacitor-branches only, or at least one loop which consists of inductor-branches only in \(G\), the matrix \(A\) of the state equation is singular.
CHAPTER 3  RLCT NETWORK(1)

3-1 INTRODUCTION

A network considering in this chapter contains, as the network-elements, only resistors, inductors, capacitors, ideal-transformers and independent sources. Furthermore it is assumed to satisfy Assumption A in Chapter 2. The network is called an RLCT network $N^*$. Several papers on the solvability of RLCT networks have been published[6][7][8]. A necessary condition for the solvability in a graphical sense has been given by [6]. Different from the case of RLC networks, however, solvability conditions depend not only on network-topology, but also on network-element values. A necessary and sufficient condition for the unique solvability is that the coefficient matrix of the network equation is nonsingular. In this chapter, a necessary and sufficient condition which depends on network-topology and turn-ratios of the ideal-transformers is given.

An explicit form of state equations of RLCT networks is given in [6], however, the process of deriving them is not discussed.

In Section 3-2, the magnetic and the electric properties of the multiwinding ideal-transformers are studied.

In Section 3-3, the problems on solvability and the order of complexity are studied.

In Section 3-4, the explicit forms of state equations are formulated.
Consider an ideal-transformer five-port as shown in Fig.3-2.1, where two of the ports are connected to two inductors and the rest are free. Let \( v \) and \( i \) denote the voltage- and current-vectors of free ports. Then we obtain

\[
v = K'A \frac{d}{dt} i + A \frac{d}{dt} i,
\]

where

\[
K = \begin{bmatrix} k_{11}, k_{12}, k_{13} \\ k_{21}, k_{22}, k_{23} \end{bmatrix}, \quad A = \begin{bmatrix} L_1, 0 \\ 0, L_2 \end{bmatrix}.
\]

The matrix \( M \) may be considered to be an inductance matrix of a mutual coupling inductor.

Consider a mutual coupling inductor whose inductance matrix is represented by an \( n \times n \) matrix of the rank \( r \), which is denoted by \( M \). The matrix \( M \) is known to be symmetric and positive semi-definite, therefore it is represented by

\[
M = K'AK,
\]

where \( A \) is an \( r \times r \) positive diagonal matrix. The matrix \( K \) is an \( r \times n \) matrix whose elements \( k(i,j) = 0 \), if \( i > j \) and \( k(i,i) = 1 \).
Consequently a mutual coupling inductor is equivalent to an ideal-transformer $(n+r)$-port, where $r$ port are connected to $r$ inductors and $n$ ports are free. For example a mutual coupling inductor which is characterized by $3 \times 3$ inductance matrix of rank 2 is equivalent to a network shown in Fig. 3-3.2.

Therefore an RLCT network may be studied in stead of a general passive reciprocal network with mutual coupling inductors.

Consider the electric and magnetic properties of multiwinding ideal-transformers. The magnetic property of multiwinding ideal-transformers can be characterized by magnetic graphs. For example the magnetic graph of the multiwinding ideal-transformer shown in Fig. 3-2.3(a) is that in Fig. 3-2.3(b).

The magnetic graphs represent the magnetic circuits. Since ideal-transformers can be regarded as unity coupled inductors of infinite inductance, we have

$$Q_M \psi = 0$$
$$B_M \xi = 0,$$

where $\psi$ and $\xi$ are the branch-flux and the branch-magnetomotive force vectors, respectively. The matrices $Q_M$ and $B_M$ are fundamental cut-set- and the fundamental loop matrices derived from the
magnetic graph $M$. Let $v_I$ and $i_I$ denote the port-voltage- and the
port-current vectors of ideal transformers. Let $K$ denote the
diagonal matrix that specifies turn-ratios of ideal-transformers.
Then we obtain

$$v_I = K \frac{d}{dt} \psi \quad \text{or} \quad \frac{d}{dt} \psi = K^{-1} v_I$$

$$\xi = K i_I$$

$$K = \text{diag.}\{k_1, \ldots, k_n\}.$$  \hspace{1cm} (3-2.4)

Eq. (3-2.3) and (3-2.4) lead us to

$$Q_W v_I = 0 \quad Q_W = Q_M K^{-1}$$

$$B_W i_I = 0 \quad B_W = B_M K$$  \hspace{1cm} (3-2.5)

Since the matrices $B_M$ and $Q_M$ are ,respectively, written as

$$B_M = [1, F_M] \quad Q_M = [-F_M, 1],$$

where $F_M$ is the characteristic part of the fundamental loop
matrix derived from the magnetic graph $M$. Eq.(3-2.5) becomes

$$v_I = K F_M K^{-1} v_I \Rightarrow M^M v_I$$
\[ i_L = -K_L^M V_M I_I K_{t}^{M} t = M_i t, \quad (3-2.6) \]

where \[ \begin{matrix} v_L & = v_I \\ i_L & = i_I \end{matrix} \quad K_L, 0 = K \quad \begin{matrix} v_M & 0 \end{matrix} \quad i_M \]

The subscripts \( L \) and \( t \) denote links and twigs, respectively, of the magnetic graph \( M \).

Consider the electric property of an RLCT network. It can be characterized by an electric graph \( G \) which is obtained from an RLCT network by replacing each of two terminal elements and each port of ideal-transformers with a branch. For example, the electric and magnetic graphs of the RLCT network shown in Fig.3-2.4(a) are in Fig.3-2.4(b) and (c), respectively.

![RLCT network](a)

![Electric graph](b)

![Magnetic graph](c)

Fig.3-2.4
(a) RLCT network
(b) its electric graph
(c) its magnetic graph.
In electric and magnetic graphs (G and M), every branch in M corresponds to a branch in G. However only every transformer branch in G has one-to-one correspondence with that of M.

we define

Definition[5] 3-2.1
A proper tree $T_p$ of an RLCT network is a set of branches which forms a tree in its electric graph $G$. Besides the transformer branches in the set form a tree in its magnetic graph $M$.

Consider an algorithm of selecting a proper tree of RLCT networks. Let $S_{G_1}$" $S_{G_2}$" denote a set of all branches in $G_1$"$G_2$".

Definition 3-2.2
A common tree is a tree in two graphs $G_1$ and $G_2$ where each branch in $G_1$ has a one-to-one correspondence with a branch in $G_2$, i.e. $S_{G_1}$=$S_{G_2}$<br>$S_G$.

Selecting common trees have been given in [10] and [11]. They may be summerized as follows. Let $T_1$"$T_2$" denote a tree of $G_1$"$G_2$". Then we obtain

$$T_1 \cup \bar{T}_2 = \bar{T}_2 \cup (T_1 \cap T_2).$$

Since $\bar{T}_2 \cap (T_1 \cap T_2) = \emptyset$, we obtain

$$|T_1 \cup \bar{T}_2| = |\bar{T}_2| + |T_1 \cap T_2|.$$  

Since $|T_1 \cap T_2| \leq |T_1| (or |T_2|)$ and $|\bar{T}_2|$ is constant, if and only if $|T_1 \cap T_2|$ is maximized, $|T_1 \cup \bar{T}_2|$ is maximized. Since $T_1$ is identical with $T_2$ if and only if $|T_1 \cup \bar{T}_2| = |S_G|$, the algorithm is one of maximizing $|T_1 \cup \bar{T}_2|$, instead of $|T_1 \cap T_2|$.
Modifying the above common tree algorithm, we obtain an algorithm of a proper tree in RLCT networks.

Let \( S_G, S_M \) denote a set of all branches in \( G"M" \). The set \( S_M \) is a subset of \( S_G \), that is, \( S_G \supseteq S_M \). Let \( T_G, T_M \) denote a tree of \( G"M" \). Corresponding to Eq.(3-2.7) and (3-2.8), the following equations hold.

\[
\begin{align*}
\bar{T}_G \cup T_M &= \bar{T}_G \cup (T_G \cap T_M) \\
|\bar{T}_G \cup T_M| &= |\bar{T}_G| + |T_G \cap T_M| \\
&\leq \mu(G) + r(M).
\end{align*}
\]  

(3-2.9)

By use of the common tree algorithm, trees \( T_G \) and \( T_M \) such that \( |\bar{T}_G \cup T_M| \) is maximum can be chosen. If \( T_G \) is a proper tree, it is necessary that \( T_G \cap T_M = \emptyset \) and \( T_G \bar{T}_M = \emptyset \) as shown in Fig.3-2.5(a). In general, \( T_G \cap T_M \neq \emptyset \), even if \( |T_G \cup T_M| \) is maximum and \( \bar{T}_G \cap T_M = \emptyset \). (as shown in Fig.3-2.5(b).) Therefore \( |\bar{T}_G \cup T_M| \) is maximized by the common tree algorithm, and then if \( \bar{T}_G \cap T_M = \emptyset \), \( |T_G \cup T_M| \) is maximized by its dual algorithm. If \( T_G \bar{T}_M = \emptyset \), \( T_G \) is a proper tree.

![Venn's diagrams](a) T<sub>G</sub> is a proper tree. (b) \( |T_G \cup T_M| \) is maximum and \( \bar{T}_G \cap T_M = \emptyset \).

Let us obtain a proper tree of an RLCT network whose electric and magnetic graphs are given in Fig.3-2.6. Assume that \( T_G \) and \( T_M \)
are initially as $T_G = \{3, 4, 5, 6\}$ and $T_M = \{1, 3\}$. By the common tree algorithm, $T_G$ and $T_M$ such that $|T_G \cup T_M|$ is a maximum are obtained as $T_G = \{2, 3, 4, 5\}$ and $T_M = \{1, 3\}$. From these, $|T_G \cup T_M|$ is maximized, and then a proper tree is obtained as $T_P = \{2, 3, 6, 7\}$, where $T_G = T_P$ and $T_M = \{2, 3\}$.

Let us define a proper tree giving the order of complexity.

Definition[6] 3-2.3

A maximum proper tree $T_{pmax}$ of an RLCT network is defined to be a proper tree having maximum $(|T_P(C)| + |T_P(L)|)$.

3-3 NETWORK EQUATION AND SOLVABILITY

A network equation of an RLCT network is given by
(3-3.1)

\[
\begin{vmatrix}
\Delta & Q_\Sigma, Q_M, 0, 0 \\
v & 0, 0, B_\Sigma, B_M \\
& Z, 0, -Y, 0 \\
& 0, B_W, 0, 0 \\
& 0, 0, 0, Q_W \\
i_\Sigma & j \\
i_M & e \\
v_\Sigma & 0 \\
v_M & 0
\end{vmatrix}
\]

where the subscript \(M\) and \(\Sigma\) denote the ideal-transformers and the network-elements except the ideal-transformers, respectively.

The matrix \(Q_\Delta|Q_\Sigma, Q_M|B_\Delta|B_\Sigma, B_M\) is a basic cut-set matrix"a basic loop matrix" in the electric graph.

The next theorem is given in [6].

Theorem 3-3.1[6]

A necessary condition for an RLCT network to have a unique solution is that there exists at least one proper tree in its electric graph.

The proof of theorem 3-3.1 is not given in [6] and therefore we give one here.

(proof) By the Laplace expansion of \(\det|\Theta(p)|\), we obtain

\[
\det|\Theta(p)| = \Sigma \text{sgn}(S_\Sigma) \cdot \text{sgn}(S_I) \cdot (Q_\Sigma, Q_M)_{S_\Sigma \cup S_I} \cdot (B_\Sigma, B_M)_{\bar{S}_\Sigma \cup \bar{S}_I} \\
\times Z; -Y |S_\Sigma, S_I| B_W |\bar{S}_I| Q_W |S_I\]

(3-3.2)

where

\[
\begin{align*}
S_\Sigma & : \text{A subset of } S_G - S_M \\
S_I & : \text{A subset of } S_M \\
\bar{S}_\Sigma & : S_G - S_M - S_I \\
\bar{S}_I & : S_M - S_I
\end{align*}
\]
The determinant of the submatrix which is composed of the columns of \([Q_L, Q_M] [B_L, B_M] \) corresponding to \(S^L \cup S^I\) is equal to \(\text{det} [Q_L, Q_M] [B_L, B_M] \).

\[ |Z; -Y| \]

The determinant of the diagonal matrix whose elements of \(Z\) corresponding to \(S^L\) and those of \(-Y\) to \(S^L\).

Since \(\text{det} [\theta (p)] \neq 0\) if the network has a unique solution, there exist \(S^L\) and \(S^I\) such that

\[ |[Q_L, Q_M] [B_L, B_M] \| S^L \cup S^I\| B_W \| S^I\| Q_W \| S^I\| \neq 0, \]

that is, \(S^L \cup S^I\) is a tree in \(G\) and \(S^I\) is a tree in \(M\). Consequently \(S^L \cup S^I\) is a proper tree.

Q.E.D.

Theorem 3-3.2 [6]

The order of complexity of an RLCT network is not greater than the sum of the number of capacitors in any maximum proper tree and the number of inductors in its corresponding cuttree.

Above two theorems are obtained in considerations of network topology. However, solvability of an RLCT network, that is, whether it has a unique solution, depends upon not only network topology but also network-element-values.

Consider how the network element-values affect network solvability. Assuming that a proper tree \(T_p\) can be obtained, we have the fundamental loop and cut-set matrices as

\[
B = \begin{vmatrix}
F_{LT}, 1, F_{Lt}, 0 \\
F_{IT}, 0, F_{It}, 1
\end{vmatrix}
\]

\[
Q = \begin{vmatrix}
1, -F'_{LT}, 0, -F'_{It} \\
0, -F'_{Lt}, 1, -F'_{It}
\end{vmatrix}
\]
where the subscript $l^\prime_T$ denotes the links "twigs" in $\Sigma$. The subscript $l^\prime_t$ denotes the links "twigs" in $M$.

Let $T_P(\Sigma) T_P(M)$ denote the subset of $T_P$ excluding the ideal transformer-branches $T_P - T_P(\Sigma)$. Then the coefficient matrix may be written as

$$
\begin{bmatrix}
T_P(\Sigma) & T_P(\Sigma) & T_P(M) & T_P(M) & T_P(\Sigma) & T_P(M) & T_P(M) \\
1, -F_L & 0, -F_L & 0 & F_L & 1, F_L & 0 & 0 \\
0, -F_L & 1, -F_L & 0 & F_L & 0 & F_L & 1 \\
Z_T, 0, & 0, -Y_T, 0 & 0 & 0 & 0 \\
0, Z_L, & 0, & -Y_L & 0 & 0 \\
0, 0, F_MK_t & K_l, 0, 0, 0, 0 & 0, K_t^{-1} & -F_MK_l^{-1} \\
0, 0, 0, 0, 0, 0 & 0, K_t^{-1} & -F_MK_l^{-1}
\end{bmatrix}
$$

where the matrix $Z_L"Z_T$ is a diagonal matrix whose diagonal elements are those of $Z$ corresponding to the links "twigs" of $T_P$, and the matrix $Y_L"Y_T$ is a diagonal matrix whose diagonal elements are those of $Y$ corresponding to the links "twigs" of $T_P$.

Consider the term in the Laplace expansion of $\det|\theta_{os}(p)|$ corresponding to $T_P(\Sigma)$. It is rewritten by

$$
|\theta_{os}(p)|_{TP(\Sigma)} = a |Z_L| |Y_T| |K_1 + F_MK_tF_{l_t}| |K_t^{-1} + F_MK_l^{-1} F_{l_t}| \\
= a |Z_L| |Y_T| a_{TP(\Sigma)},
$$

where $a = i$.

Consider the term in $|\theta_{os}(p)|$ corresponding to another proper tree $T_p$. By exchanging the rows and columns of $\theta_{os}(p)$, it becomes
\[
\theta_1(p) = \begin{pmatrix}
I_{Q_1} & I_{Q_2} & I_{Q_3} & I_{Q_4} & 0 & & & & & & & & & & & & & & & \\
I_{Q_5} & I_{Q_6} & I_{Q_7} & I_{Q_8} & & & & & & & & & & & & & & & & & \\
& & I_{B_1} & I_{B_2} & I_{B_3} & I_{B_4} & 0 & & & & & & & & & & & & & & \\
& & I_{B_5} & I_{B_6} & I_{B_7} & I_{B_8} & 0 & -I_{Y_T} & 0 & & & & & & & & & & & & & & \\
0 & I^{B_W_t} & I^{B_W_l} & 0 & 0 & I^{Q_W_t} & I^{Q_W_l} & 0 & & & & & & & & & & & & & & \\
\end{pmatrix}
\]

where the matrix \(I^Z_T I^Z_L\) is a diagonal matrix whose diagonal elements are those of \(Z\) corresponding to the links"twigs" of \(T_{p1}\). The matrix \(I^Y_T I^Y_L\) is a diagonal matrix whose diagonal elements are those of \(Y\) corresponding to the links"twigs" of \(T_{p1}\). The matrix \(I^{Q_1} I^{B_1}\) is obtained from \(Q^1 B^1\) by exchanging columns, and

\[
I^Q = \begin{vmatrix}
I_{Q_1} & I_{Q_2} & I_{Q_3} & I_{Q_4} \\
I_{Q_5} & I_{Q_6} & I_{Q_7} & I_{Q_8} \\
\end{vmatrix}
\]

\[
I^B = \begin{vmatrix}
I_{B_1} & I_{B_2} & I_{B_3} & I_{B_4} \\
I_{B_5} & I_{B_6} & I_{B_7} & I_{B_8} \\
\end{vmatrix}
\]

Assume that \(T_{p1}\) is obtained from \(T_{p}\) by \(m_1\) tree-transformations among the branches of \(\Sigma\) and \(n_1\) tree-transformations among the branches of \(M\). The matrix \(\theta_1(p)\) is obtained from \(\theta_0 p(p)\) by \(m_1 + n_2\) row-exchanges and \(2m_1 + n_2\) column-exchanges.

By the elementary transformation, \(\theta_1(p)\) is transformed to \(\theta_{1s}(p)\) as
\[
\begin{align*}
\theta_{1s}(p) &= \begin{vmatrix}
1 & -I_L^T & 0 & -I_L^T \\
0 & -I_L^T & 1 & -I_L^T \\
0 & I_L & 1 & -I_L^T \\
0 & I_L & 0 & -I_L^T \\
0 & I_L & 0 & -I_L \\
0 & I_L & 0 & -I_L \\
0 & I_L & 0 & I_L^{-1} \\
0 & 0 & 0 & I_L^{-1}
\end{vmatrix} \\
&= \begin{vmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{vmatrix}
\end{align*}
\]

(3-3.7)

where \( I_L \) denote the submatrices of the characteristic part of the fundamental loop matrix for the tree \( T_{p1} \) (associated with the subscripts). The matrix \( I_L^{-1} I_L^{-1} \) is a diagonal matrix whose diagonal elements are those of \( K \) corresponding to the links "twigs" of \( T_{p1}(M) \).

The term in the Laplace expansion of \( \det \theta_{1s}(p) \) corresponding to \( T_{p1}(\Sigma) \) is written as

\[
\begin{align*}
|\theta_{1s}(p)|_{T_{p1}(\Sigma)} &= a |I_L||I_T|\left| I_L^{-1} + I_M^{-1} \right| I_L^{-1} F_{1L}^T \\
&\times \left| I_L^{-1} + I_M^{-1} \right| I_L^{-1} F_{1L}^T \\
&= \Delta a |I_L||I_T|a_{T_{p1}(\Sigma)}^2.
\end{align*}
\]

(3-3.8)

Let us compare \( |\theta_{1s}(p)|_{T_{p1}(\Sigma)} \) with \( |\theta_{0s}(p)|_{T_{p1}(\Sigma)} \) which is the term in the Laplace expansion of \( \det \theta_{1s}(p) \) corresponding to \( T_{p1}(\Sigma) \). From Lemma 2-2.1, we obtain

\[
|\theta_{0s}(p)|_{T_{p1}(\Sigma)} = (-1)^{3m_1+3n_1}(-1)^{m_1+n_1}\theta_{1s}(p)|_{T_{p1}(\Sigma)}
\]

(3-3.9)

The term \( a_{T_{p1}(\Sigma)}^2 \) is written as
\[
\begin{aligned}
\alpha^J_{p1}(\Sigma) &= |1K_1 + 1F_M \cdot K_1 \cdot F_{lt}^1||1K_t^{-1} + 1F_M \cdot K_1^{-1} \cdot F_{lt}^1| \\
&= |1K_1||1K_t|^{-1}|1K_1 + 1F_M \cdot K_1 \cdot F_{lt}^1| \\
&\times |1 + |1K_t^{-1} + 1F_M \cdot K_1^{-1} \cdot F_{lt}^1|
\end{aligned}
\] (3-3.10)

From Lemma A-3.1, we obtain
\[
\alpha^J_{p1}(\Sigma) = |K|^{-1} g^2_I(k_1, k_2, \ldots, k_M),
\] (3-3.11)
where \( g_I(k_1, k_2, \ldots, k_M) = |1K_1 + 1F_M \cdot K_1 \cdot F_{lt}^1| \), which is a polynomial of turn ratios \( k_i \) (\( i = 1, 2, \ldots, M \)).

Only one proper tree \( T_{p1} \) obtained from \( T_p \) by tree-transformation is considered above, however, without loss of generality, Eq. (3-3.9) and (3-3.11) hold for any proper tree. Then we obtain

Lemma 3-3.1

If an RLCT network has at least one proper tree, the determinant of the coefficient matrix \( \Theta(p) \) in the network equations is given by
\[
\det |\Theta(p)| = \prod_{i=1}^{M} K_i^{-1} \{ \Sigma \} \prod_{j} C_j^{T_{pm}(\Sigma)} R_k^{T_{pm}(\Sigma)} L_l^{T_{pm}(\Sigma)}
\]
\[
\times g^2_m(k_1, k_2, \ldots, k_M),
\] (3-3.12)
where \( g_m(k_1, k_2, \ldots, k_M) \) is a polynomial of \( k_i \), which is specified by the \( m \)-th proper tree \( T_{pm} \).

From Lemma 3-3.1, two theorems are obtained

Theorem 3-3.3

A necessary and sufficient condition for an RLCT network to have a unique solution is that there exists at least one proper tree such that \( \det |K_1 + F_M \cdot K_1 \cdot F_{lt}^1| \neq 0 \).
Proof: From Eq. (3-3.12), \( \det |0(p)| \neq 0 \) if and only if \( g_m^2(k_1, k_2, \cdots, k_{|M|}) \neq 0 \). Since \( g_m = |mK_1 + mF_M \cdot K \cdot F_{1t}| \), then the theorem holds.

Q.E.D.

We have a sufficient condition depending on network topology only, that is,

Theorem 3-3.4

A sufficient condition for an RLCT network to have a unique solution is that there exists at least one proper tree \( T_p \) such that there exists no other proper tree which contains \( T_p (\Sigma) \).

Proof: For a proper tree satisfying the above condition, \( g_m \) is a monomial (not a polynomial). Then, \( g_m \neq 0 \).

Q.E.D.

Next, the theorems concerning the order of complexity are obtained.

Theorem 3-3.5

If there exists a maximum proper tree \( T_{p_{\max}} \) for which \( \det |K_1 + F_{M1}K_{1t}| \neq 0 \), the order of complexity of an RLCT network is equal to \( |T_{p_{\max}}(C) + \overline{T_{p_{\max}}(L)}| \).

Theorem 3-3.6

If there exists no other proper tree which contains a maximum proper tree \( T_{p_{\max}}(\Sigma) \), the order of complexity of an RLCT network is equal to \( |T_{p_{\max}}(C) + \overline{T_{p_{\max}}(L)}| \).
Some examples of networks for Theorem 3-3.4 - 3-3.6 are shown in Section 3-5.

3-4 STATE EQUATION

In this section, a method of formulating state equations of RLCT networks for maximum proper trees is studied.

The characteristic part of the fundamental loop matrix for a maximum proper tree \( T_{p_{\text{max}}} \) of an RLCT network is assumed to be given as

\[
\begin{bmatrix}
F_{SC}, 0, 0, F_{St} \\
F_{RC}, F_{RG}, 0, F_{Rt} \\
F_{LC}, F_{LG}, F_{Lt}, F_{Lt}
\end{bmatrix}
\]

(3-4.1)

From the matrix (3-3.1), Kirchhoff's voltage and current laws state,

\[
\begin{align*}
-v_s + e_s &= F_{SC}v_C + F_{St}v_t \\
-v_R + e_R &= F_{RC}v_C + F_{RG}v_G + F_{Rt}v_t \\
-v_L + e_L &= F_{LC}v_C + F_{LG}v_G + F_{Lt}v_t \\
-v_i + e_I &= F_{IC}v_C + F_{Ig}v_G + F_{Igt}v_t
\end{align*}
\]

(3-4.2)

\[
\begin{align*}
i_{C} - j_C &= F'_{SC}i + F'_{RC}i + F'_{LC}i + F'_{Ic}i \\
i_{G} - j_G &= F'_{RG}i + F'_{LG}i + F'_{Ig}i \\
i_{I} - j_I &= F'_{Lt}i + F'_{Ilt}i \\
i_{t} - j_t &= F'_{St}i + F'_{Rt}i + F'_{Lt}i + F'_{Ilt}i
\end{align*}
\]

(3-4.3)

where \( e_s \) is a loop source voltage vector, "a cut-set source current vector" associated with the subscripts.

From Eq. (3-2.6), (3-4.2d) and (3-4.3d), we obtain
\[-(K_1^t + F_{1t} K_1 F') K_1^{-1} v_1 = F_{1c} v_C + F_{1g} v_G + F_{1t} v_t - e_1 \quad (3-4.4)\]
\[-(K_1^t + F_{1t} K_1 F') K_1^{-1} i = F'_{1t} i + F_{1t} i + F'_{1t} i + j \quad (3-4.5)\]

If \( T_{pmax} \) satisfies the conditions of Theorem 3-3.5, that is, if the matrix \((K_1 + F_{1t} K_1 F')\) is nonsingular, then the matrix \((K_1^t + F_{1t} K_1 F')\) is nonsingular from Lemma A-3.1. Then the vectors \(v_1\), \(v_t\), \(i\), and \(i_t\) can be eliminated from Eq. (3-4.2) and (3-4.3), and we obtain

\[-v_S = (F_{SC} F_{St} \tilde{M}_G I_C) v_C + F_{St} \tilde{M}_G v_G - F_{St} \tilde{M}_G v_t - e_0 + F_{St} \tilde{M}_e_1 \quad (a)\]
\[-v_R = (F_{RC} F_{Rt} \tilde{M}_G I_C) v_C + (F_{RC} F_{Rt} \tilde{M}_G I_G) v_G - F_{Rt} \tilde{M}_G v_t - e_0 + F_{Rt} \tilde{M}_e_1 \quad (b)\]
\[-v_L = (F_{LC} F_{Lt} \tilde{M}_G I_C) v_C + (F_{LG} F_{Lt} \tilde{M}_G I_G) v_G + (F_{Lt} F_{Lt} \tilde{M}_G I_G) v_t - e_0 + F_{Lt} \tilde{M}_e_1 \quad (c)\]

\[i_c = (F'_{SC} - F'_{1c} \tilde{M}_G) i + (F'_{RC} - F'_{1c} \tilde{M}_G) i l + j + F'_{1c} \tilde{M}_j \quad (a)\]
\[i_c = -F'_{1g} \tilde{M}_G i + (F'_{RG} - F'_{1g} \tilde{M}_G) i l + (F'_{LG} - F'_{1g} \tilde{M}_G) i l + j + F'_{1g} \tilde{M}_j \quad (b)\]
\[i_l = -F'_{1l} \tilde{M}_G i + (F'_{RG} - F'_{1l} \tilde{M}_G) i l + (F'_{LG} - F'_{1l} \tilde{M}_G) i l + j l + F'_{1l} \tilde{M}_j \quad (c)\]

where \( \tilde{M} = M' (1 + F'_{1t} M')^{-1} \).

If the matrix-products \( F_{St} \tilde{M}_G \), \( F_{St} \tilde{M}_G I' \), \( F_{Rt} \tilde{M}_G I' \), \( F_{1g} \tilde{M}_G \), \( F'_{1g} \tilde{M}_G \), \( F'_{1l} \tilde{M}_G F'_{1t} \), and \( F'_{1l} \tilde{M}_G F'_{1t} \) in Eq. (3-4.6) and (3-4.7) are zero-matrices, Eq. (3-4.6) and (3-4.7) are of the same form as Eq. (2-3.9) and (2-3.10) in Chapter 2. Then a state equation of an RLCT network can be obtained by the same procedure as in Section 2-3.

**Lemma 3-4.1**

The matrix-products \( F_{St} \tilde{M}_G \), \( F_{St} \tilde{M}_G I' \), \( F_{Rt} \tilde{M}_G I' \), \( F_{1g} \tilde{M}_G \), \( F'_{1g} \tilde{M}_G \), \( F'_{1l} \tilde{M}_G F'_{1t} \), and \( F'_{1l} \tilde{M}_G F'_{1t} \) are all zero-matrices for any maximum
proof tree.

**Proof:** Assume that a \((p,q)\) element of \(F_{St} M_{I}\) is nonzero. Then there exist \(m\) and \(i\) such that

\[
(3-4.8)
\]

\[f_{St}(p,m) M(m,i) f_{I}(i,q) \neq 0.\]

If \(f_{St}(p,m) \neq 0\), there exist a link-capacitor-branch (denoted by \(S_p\)) corresponding to the \(p\)-th row of \(F_{St}\), \(f_{St}(p)\), and a twig-transformer-branch (denoted by \(t_m\)) in the loop(\(S_p\)) in the electric graph \(G\). If \(f_{I}(i,q) \neq 0\), there exist a link-transformer-branch (denoted by \(l_i\)) corresponding to \(f_{I}(i)\) and a twig-inductor-branch (denoted by \(t_q\)) in the loop(\(l_i\)) in \(G\).

Examine the relation between \(m(m,i) \neq 0\) and the network-topology.

From Eq. (3-2.6), the matrix \(\tilde{M}\) is written as

\[
\tilde{M}=M'(1+F_{It'} M')^{-1}
=K_{t'} F_{I} K_{I}^{-1} (1+F_{It'} K_{I}^{-1} F_{I} K_{I}^{-1})^{-1}
=K_{t'} F_{I} K_{I}^{-1} [1,F_{It'}] K_{I}, 0 | 1 |
| 0 , K_{t'} | F_{I} |

\Delta K_{t'} F_{I} K_{I}^{-1},
\]

where

\[
K=[1,F_{It'}] K_{I}, 0 | 1 |
| 0 , K_{t'} | F_{I} |
\tag{3-4.10}
\]

The matrices \([1,F_{It'}](\Delta B_I)\) and \([1,F_{M}](\Delta B_M)\) are the fundamental loop matrices of the electric graph \(G[I_{p_{max}(\Sigma)},\tilde{T}_{p_{max}(\Sigma)}]\) (denoted by \(G_I\)) and the magnetic graph \(M\).

If \(\tilde{m}(m,i) \neq 0\), there exist \(j\) and \(k\) such that

\[
k_{t'}(m,j) f_{M}(k,j) K_{I}^{-1}(k,j) \neq 0
\tag{3-4.11}
\]

If \(k_{t'}(m,j) \neq 0\), \(m=j\), since \(K_{t'}\) is a diagonal matrix. If \(f_{M}(k,j) \neq 0\), there exist a twig-transformer-branch \(t_m\) in \(M\), and a link-
transformer-branch (denoted by $l_k$) in the $\text{cut-set}(t_m)$ in $M$.

Examine the network topology when $k^{-1}(k,i)\neq 0$. The value $k^{-1}(k,i)$ is given by

$$k^{-1}(k,i) = \text{Cof}_{K(i,k)} / |K|.$$  \hfill (3-4.12)

Therefore it is rewritten as

$$k^{-1}(k,i) = (-1)^{k+i} |B_G(i)KB_M(k)| / |K|,$$  \hfill (3-4.13)

where $B_G(i)$ is the matrix obtained from $B_I$ by deleting the $i$-th row. The matrix $B_G(i)$ is the fundamental loop matrix of the graph $G_I\{l_i\}$. The matrix $B_M(k)$ is the fundamental loop matrix of the graph $M\{l_k\}$.

From the electro-magnetic point of view, the magnetic-motive force of $l_i$ is zero since $l_i$ is deleted in $G_I\{l_i\}$. Therefore the branch $l_i$ in $M$ may be considered to be contracted. The induced electro-motive force of $l_k$ is zero since $l_k$ is deleted in $M\{l_k\}$. Therefore the branch $l_k$ in $G_I$ may be considered to be contracted. Consequently, two graphs $G_I\{l_k;l_i\}$ and $M\{l_i;l_k\}$ may be considered for Eq. (3-4.13), which may be rewritten as

$$k^{-1}(k,i) = (-1)^{k+i} |B_G(l_k;l_i)KB_M(k)| / |K|,$$  \hfill (3-4.14)

where $B_G(l_k;l_i)$ is the fundamental loop matrix of $G_I\{l_k;l_i\}$, $B_M(l_i;l_k)$ is the fundamental loop matrix of $M\{l_k;l_i\}$. The matrix $K(i,k)$ is a matrix obtained by removing the $i$-th row and the $k$-th column from $K$. That $k^{-1}(k,i) \neq 0$ leads to that there exists at least one proper tree in $G_I\{l_k;l_i\}$ whose magnetic graph is $M\{l_i;l_k\}$.

Consequently, if $f_{St}(p,m|m;i)f_{\Gamma}(i,q) \neq 0$, there exist (1) a twig-transformer-branch $t_m$ in $\text{loop}(S_p)$ and a link-transformer-branch $l_i$ in $\text{cut-set}(T_q)$ in $G[T_{p_{\text{max}}}(\Sigma);T_{q_{\text{max}}}(\Sigma)-S_p]$, (2) a link-transformer-branch $l_k$ in $\text{cut-set}(t_m)$ in $M$, and (3) a proper
tree $T_{pI}$ in $G[T_{p_{max}^{max}}(\mathcal{L})+l_k;l_{p_{max}}^{max}]$ whose magnetic graph is $M[l_k;l_k]$.  

From (3), there exists a proper tree represented by $T_{pI+q}$ in $G[T_{p_{max}^{max}}(\mathcal{L})-q+l_k;l_{p_{max}}^{max}-q+l_i]$ whose magnetic graph is $M[l_k;l_k]$. Therefore from (1) and (3), there exists a proper tree $T_{pI+t_m+l_i}$ in $G[T_{p_{max}^{max}}(\mathcal{L})-q;l_{p_{max}}^{max}-q+l_i]$ whose magnetic graph is $M$.  

Let $T_{p_{new}}$ denote

$$T_{p_{new}} = T_{pI} + l_i + T_{p_{max}^{max}} - q$$  

Then, $T_{p_{new}}$ is a proper tree in $G$. Furthermore,

$$|T_{p_{new}}(C)+T_{p_{new}}(L)| > |T_{p_{max}^{max}}(C)+T_{p_{max}^{max}}(L)|$$

which contradicts that $T_{p_{max}^{max}}$ is a maximum proper tree. Therefore

$$F_{St} \tilde{M}_I = 0.$$  

In the similar manner, it can be proved that the matrix-products $F_{St} \tilde{M}_I G$, $F_{St} \tilde{M}_I G$, $F_{St} \tilde{M}_I G$, $F_{St} \tilde{M}_I G$, $F_{St} \tilde{M}_I G$, $F_{St} \tilde{M}_I G$ and $F_{St} \tilde{M}_I G$ are all zero-matrices.  

Q.E.D. 

From Lemma 3-4.1, Eq. (3-4.6) and (3-4.7) become

$$v_S \tilde{e}_S = -F_{SC} v_C$$  

$$v_R \tilde{e}_R = -F_{RC} v_C - F_{RG} v_G$$  

$$v_L \tilde{e}_L = -F_{LC} v_C - F_{LG} v_G - F_{LT} v_T$$  

$$i_C \tilde{c}_C = F_{SC} i_S + F_{RC} i_R + F_{LC} i_L$$  

$$i_R \tilde{c}_R = F_{RG} i_R + F_{LG} i_L$$  

$$i_L \tilde{c}_L = F_{LT} i_L$$

where

$$F_{SC} = F_{SC}^2 = F_{St} \tilde{M}_I G, \quad F_{RC} = F_{RC}^2 = F_{Rg} \tilde{M}_I G, \quad F_{RG} = F_{RG}^2 = F_{Rg} \tilde{M}_I G$$

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\[ F_{LC} = F_{LC} - F_{Lt} \tilde{M}_{1C}, \quad F_{LG} = F_{LG} - F_{Lt} \tilde{M}_{1G}, \quad F_{L \Gamma} = F_{L \Gamma} - F_{Lt} \tilde{M}_{1 \Gamma} \]
\[ \dot{e}_S = e_S - F_{S} \tilde{M}_{1 \Gamma}, \quad \dot{e}_R = e_R - F_{R} \tilde{M}_{1 \Gamma}, \quad \tilde{e}_L = e_L - F_{L} \tilde{M}_{1 \Gamma} \]
\[ j_C = j_C - F_{C} \tilde{M}_{1 \Gamma}, \quad \tilde{j}_C = j_C - F_{C} \tilde{M}_{1 \Gamma}, \quad \tilde{j}_\Gamma = j_\Gamma - F_{C} \tilde{M}_{1 \Gamma}. \]

Eq. (3-4.18) and (3-4.19) are ones where \( F_{..} \) are replaced with \( F_{..} \) in Eq. (2-3.9) and (2-3.10).

Since the voltage vs current relations of the network-elements are specified by Eq. (2-3.8), a matric form of state equations of an RLCT network may be written as
\[ \dot{x} = Ax + Bu, \quad (3-4.20) \]
where
\[ x = \begin{bmatrix} v_C \\ i_L \end{bmatrix}, \quad A = \begin{bmatrix} C^{-1}, & 0 \\ 0, & -H, -z \end{bmatrix} \]
\[ B = \begin{bmatrix} C^{-1}, & 0 \\ 0, & -F'_{LC}R^{-1} \tilde{F}'_{RC}, \end{bmatrix} \]
\[ C = C_C + F'_{SC} \tilde{F}_C, \quad L = L_L + L'_{L \Gamma} \tilde{F}'_{L \Gamma} \]
\[ \tilde{j}_C = F'_{C} \tilde{M}_{1 \Gamma}, \quad \tilde{j}_\Gamma = F'_{C} \tilde{M}_{1 \Gamma} \]
\[ u = \begin{bmatrix} \tilde{j}_C \\ \tilde{j}_\Gamma \end{bmatrix}, \quad R = R_1 + F'_{RG} \tilde{F}'_{RG}, \quad g = G_2 + F'_{RG} R^{-1} \tilde{F}'_{RG} \]
\[ \tilde{e}_S = e_S, \quad \tilde{e}_R = e_R, \quad \tilde{e}_L \]

In an RLC network, Property 2-3.1 - 2-3.4 hold. In an RLCT network, consider properties of state equations.

Property 3-4.1

The matrices \( C \), \( L \), \( R \) and \( g \) are positive definite unless there are no special relations among the network-element-values.

Proof: Since the matrix \( C \) is rewritten as
\[ C = \begin{bmatrix} 1, & -F'_{SC} \end{bmatrix} \begin{bmatrix} C_2, & 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 0, & C_1 \end{bmatrix} - F'_{SC} \]  \hspace{1cm} (3-4.21)

which is positive semi-definite. Since there are no special relations among the network-element-values, \(|C| \neq 0\). Then, \(C\) is positive definite. Similarly the matrix \(L\) is positive definite. Similarly the matrices \(R\) and \(g\) can be proved to be positive definite.

Q.E.D.

A property similar to Property 2-3.2 does not hold since there may exist elements of \( \tilde{L}_{LC} \) whose absolute values are greater than 1.

Property 3-4.2

If the state equations contain derivatives of input functions, there exists at least one set of branches which consists of capacitor-branches"inductor-branches" and voltage-source-branches "current-source-branches" only, which is the ring-sum of some loops"cut-sets" in B-S-graphs\(\oplus\) of \(G\{all\ resistor-\, inductor-\ and current-source-branches\}\) and \(M\{S-B-graphs\ \oplus\ of \ G\{all\ capacitor-\, resistor-\ and voltage-source-branches\}\ and \ M\}\).

Proof: See Appendix IV.

Q.E.D.

Property 3-4.3

If there exists at least one set of branches containing capacitor-branches"inductor-branches" which is the ring-sum of

\(\oplus\) See Appendix IV.

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some cut-sets "loops" in S-B-graphs of $G$ [all resistor- and inductor-branches] and $M$ "B-S-graphs of $G$ [all capacitor- and resistor-branches] and $M", the matrix $A$ of the state equation is singular.

Proof: See Appendix IV.

Q.E.D.

3-5 CONCLUDING REMARKS

Based on proper trees, the problems on the solvability-conditions and the formulation of state equations are discussed in the previous sections. The networks without proper trees are ones which do not satisfy the Kirchhoff's laws, or which are physically meaningless. Consider the matrices,

\[
\begin{bmatrix}
Q_A & Q \Sigma & Q_M \\
0 & B_W & B_M \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
B_A & B \Sigma & B_M \\
0 & Q_W & Q_M \\
\end{bmatrix}
\]

(noting that $B_W Q_W' = 0$.)

The row-rank of $Q_A B_A^T$, derived from networks without proper trees, is not equal to the number of rows. Then there are cut-sets "loop" containing only transformer-branches in the electric graphs. A simple example is shown in Fig.3-5.1(a), where the network has no proper tree. It has no unique solution.

The network shown in Fig.3-5.1(b) has proper trees, however, if $k_1 = k_2$, it has no unique solution.

Consider whether there exists a network which is equivalent to a given network $N$ and contains less ideal-transformers than $N$. If there exists a graph (denoted by $G_A$) whose basic cut-set matrix is $Q_A$, then the network corresponding to $G_A$ may have
no ideal transformers and is equivalent to $N$. If there exists a graph whose basic cut-set matrix is a proper submatrix of $Q_A$, and cannot be $Q_A$, then the remaining submatrix of $Q_A$ corresponds to ideal transformers which is contained in a network which is equivalent to $N$ and contains less ideal-transformers than $N$.  

Fig. 3-5.1
(a) Network without proper trees.
(b) Network without unique solution if $k_I = k_2$. 

CHAPTER 4  RCG NETWORK(2)(3)

4-1 INTRODUCTION

The network considered in this chapter contains resistors, capacitors, two-port-ideal gyrators and independent sources. Furthermore it is assumed to satisfy Assumption A in Chapter 2.

Several papers[6][12][13] on the solvability have been published. E.S.Kuh and et.al.[6] gave two basic conjectures on the solvability and the order of complexity. K.Abdullah and et.al.[12] proved that general passive networks are terminal solvable, but did not touch completely solvability. M.M.Milic[13][14] has proved the conjectures given by [6] using the theorem that the determinant of an even order skew-symmetric is a 2nd order polynomial in its elements.

Several papers[6][15] on the formulations of explicit form of state equations, have been published. They formulate the state equations of the networks under certain special assumptions.

In Section 4-2, the properties of multi-port-ideal gyrators are studied. It is pointed out that instead of a linear passive network, its equivalent RCG network may be analyzed.

In Section 4-3, conditions on the solvability and the order of complexity are studied.

In Section 4-4, an explicit form of state equations based on the solvability condition obtained in Section 4-3 is formulated.

@That is, the complementary variables of the sources associated with two-terminal elements can be determined uniquely but the other variables may not be determined uniquely.
A multi-port gyrator is a passive element. Let $v_g$ and $i_g$ denote port-voltage- and current-vectors, respectively, of a multi-port gyrator, whose port-characteristic is given by

$$v_g = R_g i_g \quad \text{or} \quad i_g = G_G v_g,$$


It is well-known that for a skew-symmetric matrix $R_G$, there exists a nonsingular matrix $P$ such that \[^16\]

$$R_G = P(E^0)P^T,$$  \hspace{1cm} \((4-2.2)\)

where $E = \begin{bmatrix} E_0 & E_1 & \cdots & E_1 \end{bmatrix}$, $E_1 \triangleq \begin{bmatrix} 0,1 \end{bmatrix}$.

The matrix $E_1$ represents a port-characteristic of a two-port gyrator, the symbol of which is shown in Fig.4-2.1(a).

As mentioned in Section 3-2, any multiwinding transformer whose port characteristic is represent is specified by a symmetric

---

\[(a)\]

Fig.4-2.1 (a) Symbol of two-port gyrator. (b) Equivalent network of multi-port gyrator.
positive semi-definite matrix, is equivalent to a network which consists only of ideal transformers and inductors without mutual coupling. It can be seen from Eq. (4-2.2) that any multi-port gyrator is equivalent to a network which consists only of ideal transformers and two-port gyrators (Fig. 4-2.1(b)). Furthermore, any ideal transformer is equivalent to a network which consists only of two-port gyrators. The ideal transformer in Fig. 3-2.2 is equivalent to the network shown in Fig. 4-2.2. It is well known that any inductor is equivalent to a cascade connection of a two-port gyrator and a capacitor as shown in Fig. 4-2.3. Consequently, a linear passive nonreciprocal network can be converted to an RCG network, and the RCG network obtained may be analyzed instead of the linear passive nonreciprocal network.

The voltage-current-relation of two-port gyrators in a network
is given by
\[ V_G = G_i G^* \]  \hspace{1cm} (4-2.4)
where the matrix given by
\[ G = E G_D, \]  \hspace{1cm} (4-2.5)
where
\[ G_D = G_1 \cdots G_{n_G} \]
\[ G_i = \begin{bmatrix} r_i & 0 \\ 0 & r_i \end{bmatrix} \]

\( n_G \): number of two-port gyrators.

(Note that the subscript \( G \) denotes two-port gyrators and matrix \( G \) denotes the resistance matrix of two-port gyrators, although in Chapter 2 and 3, the subscript \( G \) denotes twig-resistors and the matrix \( G \) denotes the conductance matrix of twig-resistors.)

**AN RCG NETWORK AND ITS GRAPH**

A graph for a two-port gyrator is obtained by replacing each port of the gyrator with a branch which is oriented from a node \( j \) to \( j' \). A graph \( G \) corresponding to an RCG network \( N \) is obtained as shown in Fig. 4-2.4. If one branch corresponding to a port of gyrator is labeled with \( i \), the other is assumed to be labeled with \( i^+ \). Two branches \( i \) and \( i^+ \) for a two-port gyrator are called a pair of gyrator-branches.

**Definition**[6] 4-2.1

A proper tree \( T_p \) in an RCG network is a tree of the graph \( G \) where every pair of gyrator branches are either in the tree or in its cotree.
A maximum proper tree $T_{p_{\text{max}}}$ for an RCG network is defined to be a proper tree having maximum $|T_p(C)|$.

Let us introduce another graph denoted by $G^+$ induced from the graph $G$ for an RCG network.

Definition 4-2.3
A graph $G^+$ induced from the graph $G$ for an RCG network is defined as
1) Two graphs $G$ and $G^+$ except labelings and orientations are isomorphic.
2) For non-gyrator-branches, the labeling and the orientations of every branch in $G^+$ is the same as that in $G$.
3) For gyrator branches, a gyrator-branch in $G^+$ corresponding to
a gyrator-branch $i$ in $G$ is labeled with $i^+$, and it is oriented in the same direction as the branch $i$ in $G$. A gyrator-branch in $G^+$ corresponding to a gyrator branch $i^+$ in $G$ is labeled with $i$, and it is oriented in the opposite direction as the branch $i^+$ and $G$.

![Diagram](image)

Fig.4-2.5 Example for (a) $G$ and (b) $G^+$.

The graph $G$ and $G^+$ for the network shown in Fig.4-2.4(a) are shown in Fig.4-2.5 (a) and (b), respectively.

Definition 4-2.4

A common tree $T_c$ of $G$ and $G^+$ is a tree in both $G$ and $G^+$.

The mark $^+$ may be regarded as an operator for a branch $i$ or a set of branches, that is,

i) for gyrator branches, $(i)^+ = i^+$, $(i^+)^+ = i$

ii) for non-gyrator branches, $(i)^+ = i$

iii) for a set of branches $S$, $S^+$ is a set of branches which contains exactly every $(i)^+$ for branch $i$ in $S$.

For a common tree $T_c$, define a set $S_g$ as

$$S_g = T_c \setminus T_c^+$$

(4-2.6)
The set $S_g$ consists only of gyrator branches, and if a branch is in $S_g$, its pair-branch is in $S_g$.

Define a set $S_{gm}$ which includes either of every pair of gyrator branches in $S_g$, then

$$S_g = S_{gm} + S_{gm}^+ \quad S_{gm} \cap S_{gm}^+ = \emptyset \quad (4-2.7)$$

Definition 4-2.5

For a common tree $T_c$, there may exist a common tree $T_{cm} = (T_c \setminus T_c^+)$ $\cup S_{gm}$, which is called a common tree of the same class of $T_c$.

Lemma 4-2.1

If $T_c$ is a common tree of $G$ and $G^+$, then $T_c^+$ is a common tree of $G$ and $G^+$.

Proof: Every branch $i$ in $G$ corresponds to a branch $(i)^+$ in $G^+$, and every branch $i$ in $G^+$ corresponds to a branch $(i)^+$ in $G$. From Definition 4-2.3, if $T$ is a tree of $G$, then $T^+$ is a tree of $G^+$, and vice versa. If $T_c$ is a common tree of $G$ and $G^+$, $T_c^+$ is a common tree of $G^+$ and $G$.

Q.E.D.

From Definition 4-2.2 and 4-2.3, we obtain

Lemma 4-2.2

A proper tree $T_p$ for an RCG network is a common tree of its graphes $G$ and $G^+$.

Proof: Since $T_p$ is a tree of $G$, $(T_p)^+$ is a tree of $G^+$. Since $(T_p)^+ = T_p$, $T_p$ is a common tree of $G$ and $G^+$.

Q.E.D.
4-3 NETWORK EQUATION AND UNIQUE SOLVABILITY

In this section, unique solvability of an RCG network is studied. Solvability of the network may depend not only on the network-topology but also on the network-element's values.

In Section 4-3.1, solvability-conditions depending only on the network-topology (so called topological condition on solvability) are studied.

In Section 4-3.2, solvability-conditions depending on the network-element's values are studied. (overall solvability)

4-3.1 TOPOLOGICAL CONDITIONS ON SOLVABILITY

Network equations of an RCG network containing $n_C$ capacitors, $n_R$ resistors and $n_G$ two-port gyrators are given as

\[
\begin{vmatrix}
\sigma(p) & | & \Delta & | & \begin{bmatrix}
Q_C, Q_R, Q_G, 0, 0, 0 \\
0, 0, 0, B_C, B_R, B_G \\
-1, 0, 0, C_p, 0, 0 \\
0, R, 0, 0, -1, 0 \\
0, 0, G, 0, 0, -1
\end{bmatrix} \\
\end{vmatrix}
\]

where the subscript $G$ denotes gyrators. The matrix $\sigma(p)$ is nonsingular. The matrix $\sigma(p)$ is transformed into

\[
\begin{vmatrix}
\sigma & | & \Delta & | & \begin{bmatrix}
i_C & = & j \\
& & & i_R & = & e \\
& & & i_G & = & 0 \\
& & & v_C & = & 0 \\
& & & v_R & = & 0 \\
& & & v_G & = & 0
\end{bmatrix}
\end{vmatrix}
\]
\[
\begin{pmatrix}
Q_C, Q_R, Q_G | 0, 0, 0 \\
B_C^{-1}p^{-1}, B_R, B_G | B_C, B_R, B_G \\
0 | 0, -1, 0 \\
0, 0, -1
\end{pmatrix}, \quad (4-3.2)
\]

or

\[
\begin{pmatrix}
Q_C, Q_R, Q_G | Q_C C^{-1}, Q_R R^{-1}, Q_G G^{-1}, 0, 0, 0 \\
0, 0, 0 | B_C, B_R, B_G \\
-1, 0, 0 | \\
0, R, 0 | 0, 0, 0 \\
0, 0, G |
\end{pmatrix}, \quad (4-3.3)
\]

by multiplying the matrices,

\[
\begin{pmatrix}
1, 0, 0 | \\
0, 1, 0 | 0 \\
0, 0, 1 | \\
1, 0, 0 | 0, 0, 0
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
1, 0, 0 | C, 0, 0 \\
0, 1, 0 | 0, R^{-1}, 0 \\
0, 0, 1 | 0, 0, G^{-1} \\
1, 0, 0 | 0, 1, 0 \\
0, 0, 1 | 0, 0, 1
\end{pmatrix}
\]

from the right, respectively. Here the dotted lines show the partitioning of the matrices.

Since the lower right partitioned submatrix of (4-3.2) is a nonsingular diagonal matrix, the upper left submatrix (denoted by \(A_1\)) is nonsingular if \(\theta(p)\) is nonsingular and vice versa. Similarly the upper right submatrix of (4-3.3) (denoted by \(A_2\)) is nonsingular if \(\theta(p)\) is nonsingular and vice versa. Since the matrices \(A_1\) and \(A_2\) are of the dimension \(n_C + n_R + 2n_G\), the product of \(A_1 A_2\) exists and
\[
A_1 A_2^T = \begin{vmatrix} A_{11}, A_{12} \\ A_{21}, A_{22} \end{vmatrix},
\]

where

\[
A_{11} = Q_C C P Q_C^T + Q_R R^{-1} Q_R^T + Q_G (G^{-1})^T Q_G^T,
\]
\[
A_{12} = Q_C B_C^T + Q_R B_R^T + Q_G B_G^T,
\]
\[
A_{21} = B_C^T Q_C + B_R^T Q_R + B_G (G^{-1})^T Q_G^T,
\]
\[
A_{22} = B_C^T C^{-1} P^{-1} B_C^T + B_R^T R B_R^T + B_G^T G B_G^T.
\]

The matrix \( A_{11} A_{22} \) is known as the cut-set-admittance"the loop-impedance" matrix. The matrix \( A_{12} \) is a zero matrix because of the orthogonality of the cut-set- and the loop matrices of a graph. Therefore, the matrices \( A_{11} \) and \( A_{22} \) are nonsingular if the matrix \( A_1 A_2^T \) is nonsingular and \textit{vice versa}.

Using the matrix \( G_D \) defined in (4-2.5), the matrices \( A_{11} \) and \( A_{22} \) may be rewritten as

\[
A_{11} = \begin{bmatrix} Q_C, Q_R, Q_G \end{bmatrix} \begin{bmatrix} C_P, 0, 0 \\ 0, R^{-1}, 0 \\ 0, 0, G_D^{-1} \end{bmatrix} \begin{bmatrix} Q_C^T \\ Q_R^T \\ Q_G^T \end{bmatrix},
\]

\[
A_{22} = \begin{bmatrix} B_C, B_R, B_G \end{bmatrix} \begin{bmatrix} C^{-1} P^{-1}, 0, 0 \\ 0, R, 0 \\ 0, 0, G_D \end{bmatrix} \begin{bmatrix} B_C^T \\ B_R^T \\ B_G^T \end{bmatrix}.
\]

The last matrix on the right hand-side of Eq.(4-3.5)"Eq.(4-3.6)" is the transposed matrix of the basic cut-set"loop" matrix. The second matrix on the right hand-side of Eq.(4-3.5)"Eq.(4-3.6)" is a diagonal matrix.

The first matrix on the right hand-side of Eq.(4-3.5)"Eq.(4-3.6)" is the matrix whose submatrix corresponding to the non-gyrator elements is identical to the submatrix of the basic cut-set"loop" matrix corresponding to non-gyrator-elements, and whose submatrix
corresponding to the gyrators is the matrix obtained from the sub-matrix of the basic cut-set"loop" matrix corresponding to the gyrators by multiplying the matrix \( E \) from the right.

The multiplication of the matrix \( E \) from the right means that every pair of columns corresponding to every pair of gyrator-branches are interchanged and every one column of the pair is multiplied by \(-1\). From a topological point of view, the first matrix on the right hand-side of Eq.\((4-3.5)\)"Eq.\((4-3.6)\)" is a basic cut-set"loop" matrix of the graph \( G^+ \) defined in Definition 4-2.3. Then the matrices \( A_{11} \) and \( A_{22} \) may be written as

\[
\begin{align*}
A_{11} &= Q_+ Y_D Q', \\
A_{22} &= B_+ Z_D B',
\end{align*}
\]

where \( Q_+ \) and \( B_+ \) is a basic cut-set"loop" matrix of \( G^+ \). The matrix \( Y_D \) and \( Z_D \) are, respectively,

\[
Y_D = \begin{bmatrix} C_p & 0 & 0 \\ 0 & r^{-1} & 0 \\ 0 & 0 & G_D^{-1} \end{bmatrix}, \quad Z_D = \begin{bmatrix} C^{-1} r^{-1} & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & G \end{bmatrix}.
\]

The columns for a nonzero major determinant of \( Q_+ B_+ \) correspond to a tree"cotree" of \( G^+ \). The columns for a nonzero major determinant of \( Q B \) correspond to a tree"cotree" of \( G \). There are some relations between the determinant \(|A_{11}||A_{22}|\) and trees "cotree" in \( G \) and \( G^+ \).

Property 4-3.1

By Binet-Cauchy theorem, the determinant \(|A_{11}||A_{22}|\) may be written as

\[
|A_{11}| |A_{22}| = \sum \{ \text{Products of corresponding major determinants of } Q_+ Y_D B_+ Z_D \text{ and } Q'B' \},
\]

\[
(4-3.10)
\]
where the summation is over all such major determinants.

A nonzero product of the major determinants of $Q + Y_D^\times B_0 + Z_D^\times$ and $Q'^\times B'^\times$ corresponds to a tree"a cotree" of $G^+$ and a tree"a cotree" of $G$, that is, to a common tree"a common cotree" of $G$ and $G^+$.

Note that the products of the major determinants are all positive in case of an RCL network, but they may not be all positive in Eq.(4-3.10).

From Definition 4-2.5 and Property 4-3.1, we obtain

Property 4-3.2

Let $a^\times b^\times$ be the product of major determinants corresponding to a common tree $T_c^\times T_{om}$, a common tree of the same class of $T_c$.

If $|K|$ is even, $a=b$,

If $|K|$ is odd, $a=-b$, \hspace{1cm} (4-3.11)

where $K=|T_c^\times T_{om}|$.

Proof: From Lemma 4-2.1, $T_c^\times$ and $T_{om}$ common trees of $G$ and $G^+$.

Since $T_{om}$ is a common tree of the same class of $T_c$, $T_c \cap T_{om} = T_{om} \cap T_{om}$.

Let $U$ and $W$ denote $T_{om} \cap T_{om}$ and $T_c - T_{om} \cap T_{om}$, respectively, then the following equalities hold;

$U \cap W = \phi$, $U \cap K = \phi$, $W \cap K = \phi$.

Then we obtain

$T_c = U + W + K$, $T_{om} = U + W^+ + K^+$, $T_{om} = U + W + K$, $T_{om} = U + W + K$.

Let $\gamma_W^\times \gamma_K^\times$ and $\gamma_U^\times$ denote the major determinants for $W^\times W^+$, $K^\times K^+$ and $U$ of the submatrices of $Q$ corresponding to $G[U, K]$, $G[U, W]$ and $G[K, W]$, respectively. Let $\gamma_{Q T_c^\times}^\times \gamma_{Q T_{om}}^\times$ and $\gamma_{Q + T_c^\times}^\times \gamma_{Q + T_{om}}^\times$ denote the major determinants of $Q$ and $Q_+$ for $T_c^\times T_{om}$, respectively. Then the following equalities hold;

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Let \( n_K \) denote the number of the elements with the mark '+\' in \( \hat{W} ^{K} \), and \( m_{K|m|} = |K| - n_K \). Then we obtain

\[
\gamma_{Q^+_T} = \gamma_{U} (-1)^{n_H} \gamma_{W} (-1)^{n_K} \gamma_{K}^{+}, \quad \gamma_{Q^+_T} = \gamma_{U} (-1)^{n_H} \gamma_{W} (-1)^{m_K} \gamma_{K}^{-}
\]

Therefore we obtain:

\[
\alpha = (-1)^{n_H} (-1)^{n_K} \gamma_{K}^{+}, \quad \beta = (-1)^{n_H} (-1)^{m_K} \gamma_{K}^{-}
\]

where \( Y \) is the admittance-product for \( T_c \) or \( T_{cm} \).

Then if \(|K|\) is even, \( \alpha = \beta \), and if \(|K|\) is odd, \( \alpha = -\beta \).

Q.E.D.

Property 4-3.1 states that a necessary condition\(^{[15]}\) that an RCG network has a unique solution is that there exists a common tree in \( G \) and \( G^+ \). Property 4-3.2 states that even if there exists a common tree in \( G \) and \( G^+ \), the network may not have a unique solution. Since even if there exists a common tree of the same class of \( T_c \), where \(|K|\) is even, there may exist another common tree of the same class of \( T_c \), where \(|K|\) is odd, we introduce a special common tree of \( G \) and \( G^+ \).

Definition 4-3.1

A common tree"cotree" corresponding to a product of major determinants of \( Q_{+} Y_{0} B_{+} Z_{0} \) and \( Q' B' \) which remains in the expansion of the determinant \(|A_{11}| |A_{22}|\) without cancellation is called an independent common tree"cotree".

From the above definition, if \(|A_{11}| \neq 0\) and \(|A_{22}| \neq 0\), there exists at least one independent common tree"cotree" in \( G \) and \( G^+ \).
Lemma 4-3.1

If there exists an independent common tree in $G$ and $G^+$, there exists at least one proper tree in $G$.

Proof: Let $T_c$ be an independent common tree in $G$ and $G^+$. If $S_g = \emptyset$, $T_c$ is a proper tree. If $S_g \neq \emptyset$, consider graphes $G[T_c \cap T_c^+; T_c \cap T_c^+]$ and $G^+(T_c \cap T_c^+; T_c \cap T_c^+)$ (denoted by $G_g$ and $G_g^+$, respectively). The basic cut-set matrix of $G_g$ and $G_g^+$ as denoted by $Q_g$ and $Q_g^+$ exist such that

$$Q_g = Q_g^+$$

where the dimension of $E$ is equal to $|S_g|$. The submatrix of $A_{11}$ for $S_g$ is $Q_g E_g^{-1} Q_g'$, where $E_g$ is the submatrix of $G_D$ for $S_g$. This matrix-product is a skew-symmetric matrix. Its determinant is written as

$$|Q_g E_g^{-1} Q_g'| = p(\ldots, x_{i,j}, \ldots)^2$$

where $p(\ldots, x_{i,j}, \ldots)$ is a pfaffian and $x_{i,j}$ is an $(i,j)$ element of the matrix $Q_g E_g^{-1} Q_g'$. The element $x_{i,j}$ is a sum of $r_{k}^{-1}$, where $r_{k}$ is defined in Eq. (4-3.5). Therefore there exists one term represented by $(r_{k_1}^{-1}, r_{k_2}^{-1}, \ldots, r_{k_i}^{-1}, \ldots)^2$ in the determinant $|Q_g E_g^{-1} Q_g'|$, which corresponds to a proper tree in $G_g$. Consequently, there exists a proper tree in $G$ if there exists an independent common tree in $G$ and $G^+$.

Similarly it can be proved that there exists a proper tree if there exists an independent common cotree in $G$ and $G^+$.

Q.E.D.

From the proof of Lemma 4-3.1, we obtain

\@ See Eq. (4-2.6)
Corollary

There exists a proper tree which contains any one branch of an independent common tree. And there exists a proper tree which contains $T_c \cap T'_c$, where $T_c$ is an independent common tree.

From Lemma 4-3.1, we obtain

Theorem 4-3.1

A necessary condition for an RCG network to have a unique solution is that there exists at least one proper tree.

we obtain the theorem on the order of complexity as

Theorem 4-3.2

The order of complexity of an RCG network is not greater than $|T_{p_{\text{max}}}(C)|$, where $T_{p_{\text{max}}}$ is a maximum proper tree.

Proof: Let $D_1$ and $D_2$ denote the right lower submatrix of (4-3.2) and the left lower submatrix of (4-3.3), respectively, that is,

$$D_1 = \begin{bmatrix} C_p, 0, 0 \\ 0, -1, 0 \\ 0, 0, -1 \end{bmatrix}, \quad (4-3.13)$$

$$D_2 = \begin{bmatrix} -1, 0, 0 \\ 0, R, 0 \\ 0, 0, G \end{bmatrix}, \quad (4-3.14)$$

We obtain

$$|\Theta(p)|^2 = |D_1| |D_2| |A_{11}| |A_{22}|. \quad (4-3.15)$$

Since the maximum degree of $p$ in $|A_{11}|$ is $|T_{p_{\text{max}}}(C)|$ and the minimum degree of $p^{-1}$ in $|A_{22}|$ is $n_C - |T_{p_{\text{max}}}(C)|$, the maximum degree in $|A_{11}||A_{22}|$ is $2|T_{p_{\text{max}}}(C)| - n_C$. From Eq. (4-3.15), the maximum degree of $p$ in $|\Theta(p)|^2$ is $2|T_{p_{\text{max}}}(C)|$. Consequently, the maximum degree of $p$ in $|\Theta(p)|$ is $|T_{p_{\text{max}}}(C)|$.

Q.E.D.
We have studied a topological solvability-condition in Section 4-3-1. If an RCG network has a unique solution, there exists at least one proper tree, but the converse is not always true. It depends on network-element-values. In this section, it is studied how network-element-values affect solvability.

Assume that there exists at least one proper tree in the graph corresponding to a given RCG network. A network equation of the network is written as

\[ \theta(p) | i = j, \]
\[ v | e \]
\[ 0 \]

where

\[ \theta(p) = \begin{bmatrix} -F', 1, 0 \end{bmatrix}, \]
\[ F = \begin{bmatrix} F_{CC}, F_{CR}, F_{CG} \\ F_{RC}, F_{RR}, F_{RG} \\ F_{GC}, F_{GR}, F_{GG} \end{bmatrix}, \]
\[ Z = \begin{bmatrix} 1, R_1, 0 \end{bmatrix}, \]
\[ Y = \begin{bmatrix} -C_1 p, -1, 0 \end{bmatrix}, \]
\[ i_1 = i_{C1}, \]
\[ i_2 = i_{R1}, \]
\[ i_3 = i_{C2}, \]
\[ v_1 = v_{C1}, \]
\[ v_2 = v_{C2}, \]

\[ (4-3.16) \]

\[ (4-3.17) \]
The subscripts 1 and 2 represent links and twigs, respectively.

Lemma 4.3.2

If an RCG network has at least one proper tree, the determinant of \( \Theta(p) \) is written as

\[
|\Theta(p)| = \alpha \Pi \frac{1}{T_P(C,R)} \left\{ \Pi \frac{C_I^P}{T_P(C)} \frac{R_{1}^{2}}{T_P(R)^2} \right\}, \quad (4-3.18)
\]

where \( (P_{T_P(C,R)})^2 = \text{det} \left[ G_2^{-1} + F_{\text{GG}}^{-1} F \right] \). The matrix \( F_{\text{GG}} \) is the fundamental loop matrix in \( G[T_P(C,R); T_P(C,R)] \). The value \( P_{T_P(C,R)} \) is a sum of tree-gyrator admittance products corresponding to independent common trees which contain \( T_P(C,R) \). \( \alpha = 1 \) or \( -1 \).

Proof: By multiplying by the Gauss matrix,

\[
\begin{bmatrix}
1 & -F' & 1 \\
0 & 0 & Z^{-1} \\
0 & 1 \\
\end{bmatrix}
\]

on the left, the matrix \( \Theta(p) \) is transformed into

\[
\begin{bmatrix}
1 & -F' & C_{1P} & 0 & 0 & | & C_{2P} & 0 \\
0 & 0 & R_1^{-1} & 0 & | & R_2^{-1} & 0 \\
0 & 0 & G_1^{-1} & 0 & | & G_2^{-1} & 0 \\
0 & 0 & 1 & -1 & | & F & -1 \\
1 & -C_{2P} & 0 & 0 & | & 0 & 0 \\
R_1 & 0 & -1 & 0 & | & 0 & 0 \\
1 & | & C_{2P} & 0 & 0 & | & 0 & 0 \\
R_2 & 0 & -1 & 0 & | & 0 & 0 \\
G_1 & 0 & -1 & 0 & | & -1 & 0 \\
G_2 & 0 & 0 & -1 & | & 0 & 0 \\
\end{bmatrix}
\]"
Then, by the Laplace expansion, the determinant of $0(p)$ is represented by the products of the determinants of the left lower submatrices and the right upper submatrices of \((4-3.19)\) as

$$
\text{det} | 0(p) | = \prod_{R} R_{j} n_{R}^{2} \text{det} | QDQ^{+} |, \quad (4-3.20)
$$

where

\[
Q = \begin{vmatrix}
1,0,0,-F_{CC}^{r},-F_{RC}^{r},-F_{GC}^{r} \\
0,1,0,-F_{CR}^{r},-F_{RR}^{r},-F_{GR}^{r} \\
0,0,1,-F_{CG}^{r},-F_{RG}^{r},-F_{GG}^{r}
\end{vmatrix}
\]

\[
D = \text{Diag.}\{C_{2}^{p}, R_{2}^{-1}, G_{D2}^{-1}, C_{1}^{p}, R_{1}^{-1}, G_{D1}^{-1}\}
\]

\[
Q^{+} = \begin{vmatrix}
1,0,0,-F_{CC}^{l},-F_{RC}^{l},-F_{GC}^{l} \\
0,1,0,-F_{CR}^{l},-F_{RR}^{l},-F_{GR}^{l} \\
0,0,1,-F_{CG}^{l},-F_{RG}^{l},-F_{GG}^{l}
\end{vmatrix} \quad (4-3.21)
\]

The dimension of $E$ is $2|T_{p}(G)|$. The determinant $|QDQ^{+}|$ is given by a sum of independent common tree-admittance products, therefore

\[
\text{det} | 0(p) | \text{ is written as}
\]

\[
\text{det} | 0(p) | = a \sum_{T_{cI}} \text{sgn}(T_{cI}) \prod_{T_{cI}(C)} C_{p} \prod_{T_{cI}(R)} R_{i} \prod_{T_{cI}(G)} r_{i}, \quad (4-3.22)
\]

where $T_{cI}$ is an independent common tree.

\[
\text{sgn}(T_{cI}) = \frac{|Q|}{|Q_{+}|}
\]

From Corollary of Lemma 4-3.1, there exists a proper tree $T_{p}$ which contains $T_{c}(C,R)$. Let $T_{g}$ denote an independent common tree of $G[T_{p}(C,R); T_{p}(C,R)]$ and $G^{+}[T_{p}(C,R); T_{p}(C,R)]$. Then Eq.\((4-3.22)\) is rewritten as

\[
\text{det} | 0(p) | = a \sum_{T_{p}(C,R)} \text{sgn}(T_{p}(C,R)) \prod_{T_{p}(C)} C_{p} \prod_{T_{p}(R)} R_{i} \cdot f(T_{p}), \quad (4-3.23)
\]

where $f(T_{p}) = \sum_{T_{g}} \text{sgn}(T_{g}) \prod_{T_{g}} r_{i}$.

Since $f(T_{p})$ is represented by the sum of independent common cotree impedance products, then
\[ f(T_p) = \text{sgn}(T_p(G)) \prod_{i} r_{i}^{2} |Q_{T_p(G)}G^{-1}Q_{+T_p(G)}| \]

\[ \Delta \text{sgn}(T_p(G)) \prod_{i} r_{i}^{2} P_{T_p(C,R)}^{*} \]

where \( P_{T_p(C,R)} \) is the pfaffian of the matrix \( G^{-1}_{2} + F_{G} G^{-1}_{1} F_{G} \).

The matrix \( Q_{T_p(G)} \) and \( Q_{+T_p(G)} \) are basic cut-set matrices of the graphs \( G(T_p(C,R); \overline{T_p(C,R)}) \) and \( G^{+}(T_p(C,R); \overline{T_p(C,R)}) \), respectively.

Then Eq. (4-3.23) is rewritten as

\[ \det |\theta(p)| = a \prod_{T_p(C,R)} \text{sgn}(T_p) \prod_{T_p(C)} C_{i}^{p} \prod_{T_p(R)} R_{i}^{2} P_{T_p(C,R)}^{*} \]

where \( \text{sgn}(T_p) = |Q_{T_p}|_{|Q_{+T_p}|} \).

Let us prove that \( \text{sgn}(T_p) = 1 \). From Eq. (4-3.21), we obtain

\[ |Q_{+T_p}|_{T_p} = \det |1,0| |Q_{T_p}|_{T_p} \]

\[ = 1. \]

Consequently, we obtain

\[ \det |\theta(p)| = a \prod_{T_p(C,R)} r_{i}^{2} \prod_{T_p(C)} C_{i}^{p} \prod_{T_p(R)} R_{i}^{2} (P_{T_p(C,R)})^{2}. \]

Q.E.D.

From Lemma 4-3.2, we obtain

Theorem 4-3.3

A necessary and sufficient condition that an RCG network has a unique solution is that there exists at least one proper tree such that \( P_{T_p(C,R)}^{2} = |G^{-1}_{2} + F_{G} G^{-1}_{1} F_{G}| \neq 0. \)

We have a sufficient condition depending only on network
topology, that is,

Theorem 4-3.4

A sufficient condition that an RCG network has a unique solution is that there exists at least one proper tree $T_p$ such that there exists no other proper tree which contains $T_p(C,R)$. 

Proof: Since there exists no other proper tree which contains $T_p(C,R)$, $P_{T_p(C,R)} = |G_2^{-1}|$. Since $|G_2^{-1}| \neq 0$, $|\theta(p)| \neq 0$. Q.E.D.

4-4 STATE EQUATION

In this section, a method of formulating state equations of RCG networks for maximum proper trees is studied.

The characteristic part of the fundamental loop matrix for a maximum proper tree $T_{p_{max}}$ of an RCG network is assumed to be given as

$$
\begin{vmatrix}
F_{CC} & F_{CL} \\
F_{SC} & F_{SL}
\end{vmatrix}
$$

(4-4.1)

where the subscript $\Sigma$ represents resistor- and gyrator-branches, that is,

$$
F_{CC} = [F_{CR}, F_{CG}], \quad F_{SC} = [F_{RC}], \quad F_{SL} = [F_{RR}, F_{RG}] \quad (4-4.2)
$$

From the matrix (4-4.1), Kirchhoff's voltage and current laws state,

$$
-V_{C1} + e_{C1} = F_{CC} V_{C2} + F_{CL} V_{L2}
$$

$$
-V_{\Sigma 1} + e_{\Sigma 1} = F_{\Sigma C} V_{C2} + F_{\Sigma L} V_{L2}
$$

and
\[ i_{C2} - j_{C2} = F'_C i_{C1} + F'_E i_{E1} \]  (4-4.4)
\[ i_{E2} - j_{E2} = F'_C i_{C1} + F'_E i_{E1} \]

where the subscripts 1 and 2 represent links and twigs, respectively. The vector \( e \) is a loop source voltage vector associated with the subscripts, and \( j \) is a cut-set source current vector associated with the subscripts.

The voltage and current relations of network-elements are given as

\[
\begin{pmatrix}
i_{C1} \\
i_{C2}
\end{pmatrix} = \frac{d}{dt} \begin{pmatrix} C_{1}, & 0 \\ 0, & C_{2} \end{pmatrix} \begin{pmatrix} v_{C1} \\
v_{C2} \end{pmatrix} \]  (4-4.5)
\[
\begin{pmatrix} v_{E1} \\
i_{E2}
\end{pmatrix} = \begin{pmatrix} R_{E1}, & 0 \\ 0, & G_{E2} \end{pmatrix} \begin{pmatrix} v_{E1} \\
v_{E2} \end{pmatrix} \]  (4-4.6)

where

\[
R_{E1} = \begin{pmatrix} R_{1}, & 0 \\ 0, & G_{1} \end{pmatrix}, \quad G_{E2} = \begin{pmatrix} R_{2}^{-1}, & 0 \\ 0, & G_{2}^{-1} \end{pmatrix}
\]

Eliminating \( v_{E1}, v_{E2}, i_{E1}, i_{E2}, i_{C1} \) and \( i_{C2} \) from Eq. (4-4.3)-(4-4.6), we obtain a matrix network first-order differential equation as

\[
M \frac{d}{dt} \begin{pmatrix} v_{C2} \\
v_{C1} \end{pmatrix} = -N \begin{pmatrix} v_{C2} \\
v_{C1} \end{pmatrix} + L \begin{pmatrix} e_{C1} \\
e_{E2} \end{pmatrix} \]  (4-4.7)

where

\[
M = \begin{pmatrix}
C_{2} , & (-F'_C R^{-1}_E F'_E G^{-1}_E F'_E C_{1} ) \\
0 , & F'_C G^{-1}_E F'_E C_{1}
\end{pmatrix}
\]
\[
N = \begin{pmatrix}
F'_E R^{-1}_E F'_E C_{1}, & 0 \\
F'_C R^{-1}_E F'_E C_{1}, & F'_C G^{-1}_E F'_E C_{1}
\end{pmatrix}
\]

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\[ L = \begin{vmatrix}
0,1, & -F_{\Sigma L}^{-1} & -F_{\Sigma L}^{-1} F_{\Sigma L} C L G_{L L}^{-1} \\
1,0, & F_{L L} G_{L L}^{-1} F_{\Sigma L}^{-1} C L & -F_{CL} G_{L L}^{-1}
\end{vmatrix} \]

\[ R_0 = F_{\Sigma L}^{-1} F_{\Sigma L}^{-1} F_{\Sigma L} G_{L L} + F_{\Sigma L}^{-1} F_{\Sigma L} C L G_{L L}^{-1} F_{\Sigma L} \]

If the submatrix of \( M, F_{CL} G_{L L}^{-1} F_{\Sigma L} C L \), is a zero matrix, the vector \( v_{CL} \) may be eliminated from Eq. (4-4.7), and a state equation may be obtained whose order is equal to the order of complexity given by Theorem 4-3.2. Here let us prove that the matrix 
\[ F_{CL} G_{L L}^{-1} F_{\Sigma L} C L \]
is a zero matrix.

Lemma 4-4.1

For a maximum proper tree \( T_{p max} \), the matrix \( F_{CL} G_{L L}^{-1} F_{\Sigma L} C L \) is a zero matrix.

Proof: Assume that the matrix \( F_{CL} G_{L L}^{-1} F_{\Sigma L} C L \) is a nonzero matrix. There exist \( i, j, k \) and \( l \) such that

\[ f_{CL}(i,j)g_0^{-1}(j,k)f_{\Sigma L}(l,k) \neq 0 \]  

(4-4.9)

If \( f_{CL}(i,j) \neq 0 \), there exists a link-capacitor-branch (denoted by \( S_i \)) corresponding to the row, \( f_{\Sigma L}(i) \), and there exists a twig-\( \Sigma \)-branch (denoted by \( g_j \)) corresponding to the column, \( f_{\Sigma L}(j) \). The branch \( g_j \) is in the loop \( S_i \). If \( f_{\Sigma L}(l,k) \neq 0 \), there exists a link-capacitor-branch (denoted by \( S_l \)) corresponding to \( F_{\Sigma L}(l) \), and there exists a twig-\( \Sigma \)-branch (denoted by \( g_k \)) corresponding to \( F_{\Sigma L}(k) \). The branch \( g_k \) is in the loop \( S_l \). Then \( g_j \) and \( g_k \) are gyrator-branch, otherwise, there exists a proper tree which contains more capacitor-branches than \( T_{p max} \).

Examine the relation between \( g_0^{-1}(j,k) \neq 0 \) and the network-topology. From the matrices (4-4.2) and (4-4.8), the matrix \( G_0 \) is written as

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\[
\begin{bmatrix}
G_0 &=& & \begin{bmatrix} 1,0,-F'_{RR},-F'_{GR} \end{bmatrix} & \begin{bmatrix} R_2^{-1}, 0, 0, 0 \end{bmatrix} & 1, 0 \\
0,1,-F'_{RG},-F'_{GG} & & 0, C_{D2}^{-1}, 0, 0 & 0, E_{D2} \\
& & 0, 0, R_1^{-1}, 0 & -F_{RR}, -F_{RG} \\
& & 0, 0, 0, C_{D1}^{-1} & -E_{D1}F_{GR}, -E_{D1}F_{GG} \\
\end{bmatrix}
\]

where \( E_{D1} = E_1 \times \cdots \times E_1 \), \( E_{D2} = E_1 \times \cdots \times E_1 \).

The matrices \( D_{D1} \) and \( E_{D2} \) are of the same dimension.

The matrix \( Q_{+} \) is a fundamental cut-set matrix of \( G \) for \( T_{pmax}(C) \).

The matrix \( D_{+} \) is a fundamental cut-set matrix of \( G \) for \( T_{pmax}(E) \).

\( \Phi_{pmax}(C) \) (denoted by \( G_{+} \)) for \( T_{pmax}(E) \). The matrix \( D_{+} \) is diagonal.

The matrix \( Q_{+} \) is a fundamental cut-set matrix of \( G \) for \( T_{pmax}(C) \).

The matrix \( D_{+} \) is a fundamental cut-set matrix of \( G \) for \( T_{pmax}(E) \).

\text{Since} \( g_0^{-1}(j,k) \) is written as

\[
g_0^{-1}(j,k) = \text{Cof}.G_0(k,j)/|G_0|, \quad (4.4.11)
\]

if \( g_0^{-1}(j,k) \neq 0 \), \( \text{Cof}.G_0(k,j) \neq 0 \). The \( \text{Cof}.G_0(k,j) \) is the determinant of the matrix obtained by omitting the \( k \)-th row from \( Q_{+} \) and the \( j \)-th colmn from \( Q_{+} \) in (4.4.10), which corresponds to contracting \( g_k \) in \( G_+ \) and \( g_j^+ \) in \( G_+^+ \). Therefore the terms in the expansion of \( \text{Cof}.G_0(k,j) \) correspond to common trees of \( G_+[g_k] \) and \( G_+^+[g_j^+] \). Moreover, there exists at least one independent common tree in \( G_+[g_k] \) and \( G_+^+[g_j^+] \), which is denoted by \( T_o \).

Since \( g_k \) is in the loop\( (S_i) \), a set \( \{S_i, T_o\} \) is a tree in \( G_+ \), where \( G_+ = \Phi_0 \Phi_{pmax}(C) \Phi_{pmax}(C) - S_i - S_i \). Since \( g_j \) is in the loop\( (S_i) \), (that is, \( g_j^+ \) in the loop\( (S_i) \) of \( G_+^+ \), a set \( \{S_i, T_o\} \) is a tree in \( G_+^+ \).

There are two cases, one of which is that \(|T_o - T_o \land T_o^+| \) is odd, and the other of which is that \(|T_o - T_o \land T_o^+| \) is even.

i) The case where \(|T_o - T_o \land T_o^+| \) is odd.

Suppose that the capacitor-branches \( S_i \) and \( S_i \) are a pair of
Correspondence of capacitors with a pair of gyrator-branches

Table 4-4.1
Correspondence of capacitors with a pair of gyrator-branches

Provisional gyrator-branches $g_m$ and $g_m^+$ in $G_1$ and $G_1^+$, respectively. (See Table 4-4.1.) Then a set $\{g_m, T_c\}$ is an independent common tree in $G_1$ and $G_1^+$. From Corollary of Lemma 4-3.1, in $G_1$ there exists a proper tree (denoted by $T_{PL}$) which contains $g_m$ and $g_m^+$, that is, $S_i$ and $S_l$. Consequently a set $\{T_{pmax}(C), T_{PL}\}$ is a proper tree of $G$, which contains more capacitor-branches than $T_{pmax}$. This contradicts that $T_{pmax}$ is a maximum proper tree.

ii) The case where $|T_c - T_c \cap T_c^+|$ is even.

Consider the graph which is union of $G_1$ and the graph which consists of parallel provisional gyrator-branches $g_m$ and $g_n$.

Table 4-4.2 Graph $G_2$
Suppose that the capacitor-branch $S_i$ is $g_m^+$ and $S_l$ is $g_n^+$ in the graph, which is denoted by $G_2$. Then $G_2^+$ is union of $G_1^+$, where $S_i$ is supposed to be $g_m$ and $S_l$ is supposed to be $g_n$, and the graph which consists of parallel provisional gyrator-branches $g_m^+$ and $g_n^+$. A set \{\(g_m^+, g_n^+, \mathcal{T}_c\}\} is an independent common tree of $G_2$ and $G_2^+$. (See Table 4-4.2.) From Corollary of Lemma 4-3.1, in $G_2$ there exists a proper tree which contains $g_m^+$. Consequently there exists a proper tree (denoted by $T_{p2}$) including $S_i$ in $G_1$. Then a set \(\{T_{p_{\text{max}}}(C), T_{p2}\}\) is a proper tree in $G$, which contains more capacitor-branches than $T_{p_{\text{max}}}$. This contradicts that $T_{p_{\text{max}}}$ is a maximum proper tree.

From the above discussion, we obtain
\[ F_{CL} G^{-1}_0 F_{CL} = 0. \] \hspace{1cm} (4-4.12)
Q.E.D.

We obtain

Lemma 4-4.2

The following equality holds.
\[ A_{SC}' = F_{CC} - F_{CL} R^{-1} F_{CL} G^{-1}_0 F_{CL} \]

for a maximum proper tree $T_{p_{\text{max}}}$,

where $A_{SC} = F_{CC} - F_{CL} G^{-1}_0 F_{CL} R^{-1} F_{EC}$.

Proof: Let $A_{CE}$ denote $F_{CL} G^{-1}_0 F_{EC}$. If $A_{CE}$ is a zero matrix, $A_{SC} = F_{CC}$, then $A_{SC}' = F_{CC}'$.

Assume that $A_{CE}$ is not a zero matrix. Then there exist $i$ and $j$ such that $a_{CE}(i,j) \neq 0$. If $a_{CE}(i,j) \neq 0$, there exist $k$ and $l$ such that
\[ f_{CL}(i,k) g^{-1}_0(k,l) f_{CL}(j,l) \neq 0. \] \hspace{1cm} (4-4.13)

If the branch corresponding to $f_{CL}(j)$ is a resistor-branch, it
can be proved that $T_{\text{pmax}}$ is not a maximum proper tree in the similar manner in the proof of Lemma 4-4.1. Then this branch is a gyrator-branch. The branch corresponding to $f_{\Sigma \Sigma}^{i}(k)$ is evidently a gyrator-branch.

Assume that $A_{\Sigma \Sigma}^{-1}F_{\Sigma \Sigma}$ is not a zero matrix, then there exist $i, j, k, l, m$ and $n$ such that

$$f_{\Sigma \Sigma}^{i}(i, k)g_{0}^{-1}f_{\Sigma \Sigma}^{j}(j, l)r_{\Sigma \Sigma}^{-1}(j, m)f_{\Sigma \Sigma}^{(n, m)} \neq 0. \quad (4-4.14)$$

The element $r_{\Sigma \Sigma}^{-1}(m)$ corresponds to a gyrator-branch since $r_{\Sigma \Sigma}^{-1}(j)$ corresponds to a gyrator-branch. Consequently the branches corresponding to nonzero-elements of $F_{\Sigma \Sigma}^{-1}F_{\Sigma \Sigma}^{i}F_{\Sigma \Sigma}^{j}F_{\Sigma \Sigma}$ are a capacitor-branch and gyrator-branches.

Let $\{A\}_{g}$ denote the partial matrix concering only gyrator-branches of $A$. Then we obtain

$$\{G_{0}^{-1}F_{\Sigma \Sigma}^{i}R_{\Sigma \Sigma}^{-1}\}_{g}^{'}=\{R_{\Sigma \Sigma}^{-1}F_{\Sigma \Sigma}^{i}G_{0}^{-1}\}_{g}^{'} \quad (4-4.15)$$

Since $G_{0}$ is given as

$$G_{0}=G_{\Sigma \Sigma}^{2}+F_{\Sigma \Sigma}^{i}R_{\Sigma \Sigma}^{-1}F_{\Sigma \Sigma}, \quad (4-4.16)$$

by multiplying Eq. (4-4.16) by $F_{\Sigma \Sigma}^{-1}G_{0}^{i}$ from the left, we obtain

$$F_{\Sigma \Sigma}^{2}G_{0}^{-1}G_{0}^{i}=R_{0}^{-1}R_{\Sigma \Sigma}F_{\Sigma \Sigma}^{i}. \quad (4-4.17)$$

Then we obtain

$$\{G_{0}^{-1}F_{\Sigma \Sigma}^{i}R_{\Sigma \Sigma}^{-1}\}_{g}^{'}=\{R_{0}^{-1}F_{\Sigma \Sigma}^{i}G_{0}^{-1}\}_{g}^{'} \quad (4-4.18)$$

Consequently

$$\{F_{\Sigma \Sigma}^{-1}F_{\Sigma \Sigma}^{i}R_{\Sigma \Sigma}^{-1}F_{\Sigma \Sigma}\}_{g}^{'}=F_{\Sigma \Sigma}^{-1}R_{0}^{-1}F_{\Sigma \Sigma}^{-1}F_{\Sigma \Sigma}. \quad (4-4.19)$$

Then we obtain

$$A_{\Sigma \Sigma}^{i}=F_{\Sigma \Sigma}^{-1}F_{\Sigma \Sigma}^{i}R_{0}^{-1}F_{\Sigma \Sigma}^{-1}F_{\Sigma \Sigma}. \quad (4-4.20)$$

Q.E.D.

From Lemma 4-4.1, $v_{10}$ can be eliminated from Eq. (4-4.7).

From Lemma 4-4.2, we obtain
\[ C_{\nu C_2} = -g_{\nu C_2} + P u, \quad (4-4.21) \]

where

\[ C = C^2 + A^T_{SC} C^T_{SC} \]

\[ \Phi = F_{SC} R_{0}^{-1} F_{SC} \]

\[ F = [A_{SC}^T C^T, 1, F_{SC} R_{0}^{-1}, -A_{SC}^T C^T_{SC} G_{0}^{-1} F_{SC} R_{0}^{-1}, -F_{SC} R_{0}^{-1} F_{SC} G_{0}^{-1}] \]

\[ u = \begin{bmatrix} \dot{e}_{C1} \\ j_{C2} \\ e_{\Sigma 1} \\ \dot{e}_{\Sigma 1} \\ j_{\Sigma 2} \\ \dot{j}_{\Sigma 2} \end{bmatrix} \]

If there do not exist special relations among the network-element-values, the matrix \( C \) is nonsingular and a state equation is obtained as

\[ \dot{v}_{C_2} = Av_{C_2} + Bu, \quad (4-4.22) \]

where \( A = C^{-1} \), \( B = C^{-1} P \).

The state equation may have no higher than the first order-derivatives of input functions as similarly as that of an RLCT network may have. Therefore it is concluded that in a passive network, its state equation has no higher than the first order-derivatives of input functions unless there exist special relations among the network-element-values.

Property 4-4.1

The matrix \( C \) is positive definite unless there are no special relations among the network-element-values, however, \( R_{0} \) is not always positive definite.
Proof: The matrix $C$ is given by

$$C = C_2 + A_{SC}^t C_1 A_{SC}$$

$$= [1, A_{SC}^t] \begin{bmatrix} C_2 & 0 \\ 0 & C_1 \end{bmatrix} A_{SC}$$

then it is positive definite.

The matrix $R_0$ is given by

$$R_0 = R_L + F_{\Sigma L} R_{\Sigma E} F_{\Sigma E}^t$$

$$= [1, F_{\Sigma L}] \begin{bmatrix} R_L & 0 \\ 0 & R_{\Sigma} \end{bmatrix} F_{\Sigma E}$$

where the second matrix is not always diagonal. It is sum of matrices which are symmetric positive definite and skew-symmetric.

Q.E.D.

Property 4-4.2

If the state equation has derivatives of input functions, there exists at least one set (denoted by $S$) of branches containing

i) capacitor- and voltage source-branches only, or

ii) capacitor- and current source-branches only. The set $S$ is

$$S = B_1 \bigoplus Q_2^+ \bigoplus B_3 \bigoplus \cdots \bigoplus Q_{2n}^+ \bigoplus B_{2n+1}$$

for i)

$$S = B_1 \bigoplus Q_2^+ \bigoplus \cdots \bigoplus Q_{2n}^+$$

for ii)

where $B_{2i-1}$ "$Q_{2i}$" is a loop "cut-set" in $G^*$. 

Proof: See Appendix V.

Q.E.D.

Property 4-4.3

If the matrix $g$ in Eq. (4-4.21) is singular without special relations among the network-element-values, there does not
exist a tree in \( G \) (all capacitor-branches) which is a proper tree of \( G \).

**Proof:** See Appendix V.

Q.E.D.

4-5 **EXAMPLE**

Fig. 4-5.1 shows an example for the results in this chapter.

Choose a maximum proper tree \( T_{p_{\text{max}}} \) in the network as

\[
T_{p_{\text{max}}} = \{ C_3, C_4, C_5, R_2, R_3, g_2, g_2^+, g_3, g_3^+ \}.
\]

The matrices \( G_0^{-1} \) and \( R_0^{-1} \) for \( T_{p_{\text{max}}} \) are obtained as

\[
G_0^{-1} = \begin{bmatrix}
R_2 & 0 & 0 & 0 & 0 & 0 \\
0 & R_3 & 0 & a_1 & 0 & -a_2 \\
0 & 0 & 0 & r_2 & 0 & 0 \\
0 & a_1 & -r_2 & a_3 & -a_4 & -a_5 \\
0 & 0 & 0 & a_4 & 0 & a_6 \\
0 & -a_2 & 0 & -a_5 & -a_6 & a_7
\end{bmatrix}
\]

![Fig. 4-5.1 Example](image-url)
\[
R_0^{-1} = \frac{1}{R_1^2 r_2^2 (r_3 + r_4)^2}
\begin{vmatrix}
1 & b_1, -b_2, 0, 0, 0 \\
-b_2, b_3, -b_4, b_5, -b_6 \\
0, b_4, 0, 0, 0, \\
0, b_5, 0, b_7, -b_8 \\
0, b_6, 0, -b_9, 0
\end{vmatrix}
\]

where

\begin{align*}
a_1 &= \frac{R_3}{r_2 (r_3 + r_4)} \\
a_2 &= \frac{R_3}{(r_3 + r_4)^2} \\
a_3 &= \frac{R_3}{r_2 (r_3 + r_4)^2 + r_3 r_4} \\
a_4 &= \frac{R_3}{r_2 (r_3 + r_4)^2} \\
a_5 &= \frac{R_3}{r_2 (r_3 + r_4)^2} \\
a_6 &= \frac{1}{(r_3 + r_4)^2} \\
b_1 &= \frac{r_1^2 (r_3 + r_4)}{r_3 + r_4} \\
b_2 &= \frac{r_1 r_2}{r_3 + r_4} \\
b_3 &= \frac{r_1 R_2 (r_3 + r_4)^2 + r_3 r_4}{r_3 + r_4} \\
b_4 &= \frac{R_1}{r_2 (r_3 + r_4)} \\
b_5 &= \frac{R_1}{r_2 (r_3 + r_4)} \\
b_6 &= \frac{R_1}{r_2 (r_3 + r_4)} \\
b_7 &= \frac{R_1}{r_2 (r_3 + r_4)} \\
b_8 &= \frac{R_1}{r_2 (r_3 + r_4)}
\end{align*}

Since \( F_{\Sigma E} \) is given as

\[
F_{\Sigma E} = \begin{vmatrix}
0, 0, 0, 0, 0 \\
0, 0, 0, 0, 1
\end{vmatrix}
\]

it is ascertained that Lemma 4-4.1, that is, \( F_{\Sigma E} G_0^{-1} F_{\Sigma E} \equiv 0 \) is true.

Since \( F_{\Sigma C} \) is given as

\[
F_{\Sigma C} = \begin{vmatrix}
0, 0, 1, 0, 0 \mid 1, 1, 0 \\
0, 0, 0, 0, 0 \mid -1, -1, 0 \\
0, 0, 0, 0, 0
\end{vmatrix}
\]

the matrix \( g \) is given as

69
\[
g = \begin{vmatrix}
  b_3 - 2b_5 + b_7 & -b_2 + b_3 - 2b_5 + b_7 & -b_2 \\
  -b_2 + b_3 - 2b_5 + b_7 & b_1 - 2b_2 + b_3 - 2b_5 + b_7 & b_1 - b_2 \\
  -b_2 & b_1 - b_2 & b_1
\end{vmatrix},
\]

it is singular. This ascertains that Property 4-4.3 holds.

The network has a unique solution if and only if \( r_3 + r_4 \neq 0 \).

The matrix \( C \) is given as
\[
C = \begin{vmatrix}
  c_3 + d_1 c_1 + d_2 c_2, & d_1 d_2 c_1 + d_2^2 c_2, & d_3 c_2 \\
  d_1 d_2 c_1 + d_2^2 c_2, & c_4 + d_2^2 c_1 + d_3^2 c_2, & d_3 c_2 \\
  d_3 c_2, & d_3 c_2, & c_5 + c_2
\end{vmatrix},
\]

where
\[
d_1 = r_1^{-1} r_2, \quad d_2 = d_1 - 1, \quad d_3 = d_2 / (r_3^{-1} r_4 + 1).
\]

The matrices \( C \) and \( g \) lead to the matrix \( A \) of the state equation.

Fig.4-5.2 shows an example of networks which have no proper tree. The voltage across "the current through" \( j \)-gyrator-port is denoted by \( v_j \), \( i_j \). The voltage- and current-relations of the gyrators are
\[
\begin{align*}
  v_1 &= 0, r_1 \begin{bmatrix} i_1 \end{bmatrix}, &
  v_2 &= 0, r_2 \begin{bmatrix} i_2 \end{bmatrix}, &
  v_3 &= 0, r_3 \begin{bmatrix} i_3 \end{bmatrix}, \\
  v_1' &= -r_1, 0 \begin{bmatrix} i_1' \end{bmatrix}, &
  v_2' &= -r_2, 0 \begin{bmatrix} i_2' \end{bmatrix}, &
  v_3' &= -r_3, 0 \begin{bmatrix} i_3' \end{bmatrix}.
\end{align*}
\]

Fig.4-5.2 Example of network without proper tree.
Then the following equalities hold.

\[ v_3 = R_3 i_3', \quad i_3' = i_1 + i_1', \quad v_3 = v_1 - v_1' = R_1 (i_1 + i_1') . \]

Then \( R_3 = R_1 \). Similarly \( R_3 = -R_2 \) is obtained.

Consequently even if \( R_3 = R_1 = -R_2 \), or the gyrators 2 and 3 may be removed, the current are indeterminate.

It can be said that the network 1) has no physical meaning or 2) does not satisfy the Kirchhoff's laws.

4-6 CONCLUDING REMARKS

The conjectures [6] given by E.S.Kuh and et al. for the solvability and the order of complexity of RCG networks are proved in Section 4-3.1. A necessary and sufficient condition for the solvability is obtained in Section 4-3.2. It depend on the network-topology and resistance values of the gyrators. Moreover a sufficient condition for the solvability depending on the network-topology is obtained in Section 4-3.2. An explicit form of state equations, whose state variables are voltages across the twig-capacitors of a maximum proper tree, is obtained in Section 4-4. It is of the same form as state equations of networks satisfying special assumption given by J.Tow [15].

The concept 'independent common tree' is given in order to obtain the above results. It is obtained from an algebraic point of view. Let us show the concept from a graphical point of view. Consider a common tree \( T_c \) of \( G \) and \( G^+ \). Let \( K_2 \) be \( T_c - T_c \wedge T_c^+ \).

From Property 4-3.2, if \( |K_2| \) is odd, \( T_c \) is not an independent common tree. Assume that \( |K_2| \) is even and that \( K_2 \Delta \{a_1', \ldots, a_{2n}'\} \), where \( a_i' \) is a gyrator-branch. Let \( b_i^+(k) \) denote \( a_i \) or \( a_i^+ \). Then
consider $2^{2n}$ sets as $S_k = \{b_1^{(k)}, \ldots, b_{2n}^{(k)}\}$, where $k=1,\ldots,2^{2n}$.

Let $\mathcal{S}$ be $\{S_1, \ldots, S_{2^{2n}}\}$. Let $S_1$ denote a subset of $\mathcal{S}$ whose elements are common trees of $G_\mathcal{S}$ and $G_\mathcal{S}^+$, where $G_\mathcal{S} = G(T_c - \mathcal{K}_2; T_c - \mathcal{K}_2^+)$. A set (one element of $S_1$) $\{T_c, T_c^+\}$ is a common tree of the same class of $T_c$. The number of elements in $S_1$ which contain even"odd" number of mark $+$ is denoted by $c"d"$. If $c \neq d$, the common tree $T_c$ is an independent common tree.

The above results are based on the concept 'proper tree'.

A physical meaning of the concept is as follows. Consider a network equation one of whose unknown variables is voltage across"current through" one port of a gyrator. It is necessary that current through"voltage across" the other port of the gyrator be determined from the network-topology. Considering this from a graphical point of view, it is necessary that the branch corresponding to one port of the gyrator is a link"twig" and that to the other branch is also a link"twig".
CHAPTER 5 ACTIVE NETWORK (4)-(8)

5-1 INTRODUCTION

In the preceding chapters, the solvability, the order of complexity and explicit forms of state equations in passive networks containing passive elements, that is, resistors, inductors, capacitors, ideal-transformers and gyrators, are studied. The networks considered in this chapter contain both passive elements and active elements such as transistors and electric tubes. It is well-known that in the linear operations, a network containing active elements is equivalent to a network containing dependent voltage-sources and dependent current-sources, the voltages and the currents of which depend on one or more of the voltages across or currents through the network-elements.

Several papers [18]-[21] discussing on the problems of the solvability and the order of complexity of active networks has been published. J.Tow [18] has introduced two graphs $G_v$ and $G_i$ obtained from the original network; the graph $G_v$ specifies the voltage relations of the network, and the graph $G_i$ specifies the current relations of the network. He has derived a condition for the solvability and the order of complexity by means of the above two graphs $G_v$ and $G_i$. E.J.Purslow [19][20] has approached the problems from an algebraic point of view. T.Ozawa [21] has considered, by means of the two graph-method, the problems for networks whose topology is of wider varieties than that considered by J.Tow. From a topological point of view, it can be said that the two graph-method has given the solutions of the problems of solvability and the order of complexity.
Several papers\textsuperscript{[22]-[26]} have discussed on formulating state equations for active networks. In the network considered in the previous chapters, the network-elements corresponding to state variables can be determined from their network-topology only. Since the state equations are obtained by an algebraic method, in the above formulations of state equations of active networks, the network-elements corresponding to state variables and the degree of the state equations obtained are not determined until the state equations are obtained.

There are few studies\textsuperscript{[20]} on network-equations without unique solutions.

Here a formulation of state equations for networks with unique solutions and canonical forms of network-equations without unique solutions are considered from the network-topological point of view.

In Section 5-2, the dependent sources and network topology in active networks to be considered are described. In Section 5-3, the results with respect to the problems of the solvability and the order of complexity are described. In Section 5-4, the state equations, the formulation of which depends strongly on the topological condition for the network-solvability and on the order of complexity are studied. In Section 5-5, a canonical form of network-equations without unique solutions are considered by means of matrix pencil.
5-2.1 ACTIVE NETWORK ELEMENT

There are four types of dependent sources, the voltage- and current-relations of which are given, respectively, by

\[
\begin{align*}
|v_o| &= u, 0 \quad |v_i| \\
|i_i| &= 0, 0 \quad |i_o| \\
\end{align*}
\]

(5-2.1)

\[
\begin{align*}
|v_o| &= 0, r \quad |i_o| \\
|i_i| &= 0, 0 \quad |v_i| \\
\end{align*}
\]

(5-2.2)

\[
\begin{align*}
|i_o| &= 0, h \quad |v_o| \\
|i_i| &= 0, 0 \quad |v_i| \\
\end{align*}
\]

(5-2.3)

\[
\begin{align*}
|i_o| &= 0, g \quad |v_o| \\
|i_i| &= 0, 0 \quad |v_i| \\
\end{align*}
\]

(5-2.4)

where the subscripts \( o \) and \( i \) signify out-put and in-put, respectively.

The network-elements specified by Eq. (5-2.1)-(5-2.4) are called voltage-controlled voltage source, current-controlled voltage source, current-controlled current source and voltage-controlled current source, respectively.

A current-controlled voltage source can be replaced by the cascade connection of a current-controlled voltage source and a voltage-controlled current source. Active networks considered here may be assumed to contain only current-controlled voltage sources and voltage-controlled current source but no other dependent sources. Furthermore, network-elements such as ideal transformers, gyrators and negative-impedance convertors which appear in general active networks can be replaced by combinations of the dependent sources. Inductors can be replaced by capacitors and dependent sources.
Resistors can be replaced by the dependent sources. Therefore,

Remark

The network considered in this chapter contains only capacitors, independent sources, current-controlled voltage sources and voltage-controlled current sources. The number of these elements are $n_C$, $n_\beta$ and $n_\delta$, respectively, and let $n_D$ and $n_e$ be $n_\beta+n_\delta$ and $n_C+n_D$, respectively.

A current-controlled voltage source and a voltage-controlled current source are symbolically shown in Fig.5-2.1(a) and (c). Their expressions used here are shown in Fig.5-2.1(b) and (d), where $\alpha$, $\beta$, $\gamma$ and $\delta$ express a current sensor, a controlled voltage sources, a voltage sensor and a controlled current source, respectively. From this, a current-controlled voltage source"a voltage-controlled current source" may be considered as a voltage source controlled by current through a current sensor"a current source controlled by voltage across a voltage sensor".

The voltage and current relation of controlled sources and sensors is given by

(a) a current-controlled voltage source and (b) its expression
(c) a voltage-controlled current source and (d) its expression
\[ H_{VD} = H_{ID}^i D^j \]  

where

\[
\begin{align*}
    v_D &= \begin{bmatrix} v_\beta^j \\ v_\gamma^j \end{bmatrix}, \\
    i_D &= \begin{bmatrix} i_\alpha^j \\ i_\delta^j \end{bmatrix}, \\
    H_{VD} &= \begin{bmatrix} 1, 0 \\ 0, G_\gamma \end{bmatrix}, \\
    H_{ID} &= \begin{bmatrix} R_\alpha, 0 \\ 0, 1 \end{bmatrix}.
\end{align*}
\]

The vectors \( v_\beta \) and \( v_\gamma \) consist of voltages across controlled voltage sources and voltage sensors, respectively. The vectors \( i_\alpha \) and \( i_\delta \) consist of currents through current sensors and controlled current sources, respectively. The matrices \( R_\alpha \) and \( G_\gamma \) are diagonal ones of the dimension \( n_\beta \) and \( n_\delta \), respectively.

5-2.2 ACTIVE NETWORKS AND GRAPHS

5-2.2.1 VOLTAGE AND CURRENT GRAPHS

Let graph \( G^* \) correspond to an active network \( N^* \) which contains capacitors, current sensors, controlled voltage sources, voltage sensors, controlled current sources and independent sources. Graph \( G \) is obtained from \( G^* \) by contracting independent voltage sources and deleting independent current sources.

Now, two graphs are introduced, which are the voltage and the current graphs as denoted by \( G_V \) and \( G_I \), respectively. The voltage graph \( G_V \) is obtained from \( G \) by contracting all the current sensors \( \alpha \) and deleting all the current sources \( \delta \). The current graph \( G_I \) is obtained from \( G \) by deleting all the voltage sensors \( \gamma \) and contracting all the voltage sources \( \beta \). The graphs \( G_V \) and \( G_I \).
represent the voltage and current restrictions of the network.

Every branch of $G_V$ and $G_I$ has one to one correspondence to a branch of each other, that is, a capacitor-branch $C$ in $G_V$ to the capacitor-branch $C$ in $G_I$, a voltage source-branch $\beta$ in $G_V$ to its current sensor-branch $\alpha$ in $G_I$, and a voltage sensor-branch $\gamma$ in $G_V$ to its current source-branch $\delta$ in $G_I$. Every branch in $G_V$ and its corresponding branch in $G_I$ are assumed to have the same label.

Let $B_V$ and $Q_I$ denote a basic loop matrix of $G_V$ and a basic cut-set matrix of $G_I$, respectively. Then the Kirchhoff's voltage and current laws state

$$B_V v = e \quad (5-2.6)$$
$$Q_I i = j \quad (5-2.7)$$

where

$$v = \begin{bmatrix} v_C \\ v_D \end{bmatrix}, \quad i = \begin{bmatrix} i_C \\ i_D \end{bmatrix}.$$

Partition $B_V$ and $Q_I$ as

$$B_V = [B_{VC}, B_{VD}], \quad (5-2.8)$$
$$Q_I = [Q_{IC}, Q_{ID}] \quad (5-2.9)$$

Then Eq. (5-2.6) and (5-2.7) are rewritten, respectively, as

$$[B_{VC}, B_{VD}] v_C = e, \quad [Q_{IC}, Q_{ID}] i_C = j \quad (5-2.10)$$

Definition 5-2.1

Let $T_V$ and $T_I$ denote trees of $G_V$ and $G_I$, respectively. A tree-pair of $T_V$ and $T_I$ such that $|T_V \cap T_I|$ is maximum is called a maximum tree-pair of $G_V$ and $G_I$, and is denoted by $\{T_V, T_I\}$.\(^\oplus\)

\(^\oplus\)It is called MCT in [10].
Definition 5-2.2

If $T_V = T_I$, $T_C$ is called a common tree of $G_V$ and $G_I$, where $T_V = T_I \Delta T_C$.

Definition 5-2.3

A maximum tree-pair $\{T_V : T_I\}$ such that $|T_V(C) \cap T_I(C)|$ is maximum "minimum" is called a normal "primitive" maximum tree-pair, which is denoted by $\{T^N_V : T^N_I\}$.

Definition 5-2.4

A common tree $T_C$ such that $|T_C(C)|$ is maximum is called a normal common tree, which is denoted by $T^N_C$.

5-2.2.2 INDUCED DIGRAPH

In Section 5-2.2.1, two graph $G_V$ and $G_I$ are introduced. Here let us introduce a directed graph or digraph induced from $G_V$ and $G_I$, which specifies voltage and current relation.

Let $t_i \{i = 1, 2, \ldots\}$ and $l_j \{j = 1, 2, \ldots\}$ denote twigs and links, respectively, for a common tree $T_C$.

Definition 5-2.5

A digraph obtained by the following procedures is called an induced digraph of $G_V$ and $G_I$ for a common tree $T_C$, which is denoted by $G^\#$.

1) Every branch of $G_V$ and $G_I$ corresponds to a node of $G^\#$.

2) There exists a directed branch in $G^\#$ from a node corresponding to a link $l_i$ (denoted by $V(l_i)$) to a node corresponding to a twig $t_j$ (denoted by $V(t_j)$), where $t_j$ is in the loop $l_i$ in $G_V$. 

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3) There exists a directed branch in $G^\#$ from a node $V(t_i)$ to a node $V(l_j)$, where $l_j$ is in the cut-set($t_i$) in $G_I$.

An example for $G^\#$ is shown in Fig.5-2.2. For the network shown in Fig.5-2.2(a), its voltage and its current graphs are shown in Fig.5-2.2(b) and (c), respectively. Its induced graph for a common tree $T_o = \{1, C_2, C_4\}$ is shown in Fig.5-2.2(d).

![Diagram](image)

From Definition 5-2.5, some properties for $G^\#$ are obtained.

Property 5-2.1

The induced digraph of $G_V$[some twigs $t_i{i=1,\cdots}$;some links $l_j{\{j=1,\cdots\}}$] and $G_I$[the twigs;the links] is obtained by deleting nodes $V(t_i)$ and $V(l_j)$ from $G^\#$.

@ 'Deleting nodes' means that the nodes and the directed branch touching the nodes are removed.
The directed branch from $V_i$ to $V_j$ in $G^\#$ is denoted by $d(V_i, V_j)$, where if $V_i$ corresponds to a twig "link", $V_j$ corresponds to a link "twig".

The directed loop in $G^\#$ containing $d(V_1, V_2)$, $d(V_2, V_3)$, ..., $d(V_{n-1}, V_n)$ and $d(V_n, V_1)$ is denoted by $diloop\{V_1, V_2, ..., V_n, V_1\}$. There are always diloops if there are neither self loop nor bridge in $G_V$ and $G_I$. If the node $V_i$ corresponds to a twig "link", the node $V_{i+1}$ corresponds to a link "twig" in the $diloop\{V_1, V_2, ..., V_n, V_1\}$. Therefore the number of the branches "nodes" in the diloop is even.

The tree-transformation in $G_V$ and $G_I$ may be considered such as $T_c-t_1+l_1$ is a tree in $G_V$, $T_c-t_1+l_2$ is a tree in $G_I$, $T_c-(t_1+t_2)+l_1+l_2$ is a tree in $G_V$, $T_c-(t_1+t_2)+(l_2+l_3)$ is a tree in $G_I$, and so on.

If another common tree $T_{c1}=T_c-(t_1+...+t_n)+(l_1+...+l_n)$ can be obtained by transforming links $l_i$ "twigs $t_i$" $\{i=1, ..., n\}$ to twigs "links" in $G_V$ and $l_{i+1}$ "twigs $t_i$" (where $n+1 \Delta l$) to twigs "links" in $G_I$, there exists a directed loop containing nodes $V(l_i)$ and $V(t_i)$ in $G^\#$, because $t_i$ is in the loop($l_i$) in $G_V$ and $l_{i+1}$ is in the out-set($t_i$) in $G_I$. This proves

Property 5-2.2

If another common tree $T_{c1}$ represented by $T_{c1}=T_c-(t_1+...+t_n)+(l_1+...+l_n)$ can be obtained from $T_c$, then there exists a directed loop represented by $diloop\{V(l_1), V(t_1), ..., V(l_n), V(t_n), V(l_1')\}$ in $G^\#$.

The converse of Property 5-2.2 is not always true. Consider
this in an example shown in Fig. 5-2.3. A common tree is assumed
to be $T_c = \{1, 2\}$ in the graphs $G_V$ and $G_I$ shown in Fig. 5-2.3(a).
Then the induced digraph $G^\#$ for $T_c$ is obtained as shown in Fig. 5-
2.3(b). There exists a $di$-loop $\{V(3), V(1), V(3)\}$ and set $\{3, 2\}$ is
a common tree. However, there exists a $di$-loop $\{V(4), V(1), V(3),
V(2), V(4)\}$, but set $\{3, 4\}$ is not a common tree.

Property 5-2.3

If another common tree represented by $T_{c_1} = T_c - t + l$ can be
obtained from $T_c$, there exists a $di$-loop $\{V(t), V(l), V(t)\}$ in $G^\#$ for
$T_{c_1}$, and the converse is true.

Proof: Consider the subgraphs $G_V[T_c - t; T_c - l]$ and $G_I[T_c - t; T_c - l]$, which are denoted by $\hat{G}_V$ and $\hat{G}_I$, respectively. The induced sub-

(a) $G^\#$

(b) $G_I$

Fig. 5-2.4 Example for the proof of Property 5-2.3

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digraph (denoted by \( \tilde{G}^d \)) of \( G_V \) and \( G_I \) for a common tree \( \{t\} \) is shown in Fig.5-2.4(a), then \( G_V \) and \( G_I \) are the graphs shown in Fig.5-2.4(b). Then the converse is true.

Q.E.D.

Consider the case where there exists a \textit{diloop} \( \{V(l_1), V(t_1), \ldots, V(l_n), V(t_n), V(l_1)\} \) in \( G^d \) for \( T_c \), but \( T_c-t_1+t_2+\cdots+t_n+l_1+l_2+\cdots+l_n \) is not a common tree. Let two graphs \( \tilde{G}_V \) and \( \tilde{G}_I \) be \( G_V[T_c-t_1+\cdots+t_n]; \tilde{G}_I[T_c-t_1+\cdots+t_n] \) and \( G_I[T_c-t_1+\cdots+t_n]; \tilde{G}_I[T_c-t_1+\cdots+t_n] \), respectively. The induced directed graph of \( \tilde{G}_V \) and \( \tilde{G}_I \) for a common tree \( \{t_1, \ldots, t_n\} \) is denoted by \( \tilde{G}^d \). Suppose that \( T_c-t_1+\cdots+t_k+l_1+l_2+\cdots+l_{k+1} \) is not a tree in \( \tilde{G}_V \) or \( T_c-t_1+\cdots+t_k+l_2+\cdots+l_{k+1} \) is not a common tree in \( \tilde{G}_I \) for a certain minimum integer \( k \). Then there exists at least one loop consisting only of \( l_k \) and a subset of \( \{l_1, \ldots, l_{k-1}\} \) either in \( \tilde{G}_V \), or of \( l_{k+1} \) and a subset of \( \{l_2, l_3, \ldots, l_k\} \) in \( \tilde{G}_I \). Assume that there exists such a loop in \( \tilde{G}_V \). Let \( \{l_{i_1}, \ldots, l_{i_m}\} \) be the loop \( \{k \geq m \geq 2\} \). Then there exist corresponding twigs \( t_{i_j} \{j=1, \ldots, m\} \), each fundamental cut-set for which contains more than one links of \( \{l_{i_1}, \ldots, l_{i_m}\} \) in \( \tilde{G}_V \). Conversely each link \( l_{i_j} \{j=1, \ldots, m\} \) is contained in more than one cut-sets of cut-set \( (t_{i_j}) \) in \( G_V \) (that is, each loop \( l_{i_j} \) contains more than one twigs \( t_{i_j} \)). Then in \( \tilde{G}^d \), there exist diloops which consist only some nodes of \( \{V(l_1), V(t_1), \ldots, V(l_n), V(t_n)\} \). If \( m=2 \), there may exist \textit{diloop} \( \{V(l_1), V(t_1), V(l_1), V(l_2), V(t_1), V(l_2), V(l_3), \ldots, V(t_1), V(l_2)\} \) in \( G^d \). The number of such diloops is more than \( m-1 \). The nodes in such diloops cover the nodes in the \textit{diloop} \( \{V(l_1), V(t_1), \ldots, V(l_n), V(t_n), V(l_1)\} \). If there exists a loop in \( \tilde{G}_I \), the similar argument holds.
Then we obtain

Property 5-2.4

If there exists a $\text{diloop}\{V(l_1), V(t_1), \cdots, V(l_n), V(t_n), V(l_1)\}$ (denoted by $D_L$) in $G'_V$ of $G$ for a common tree $T_c$, but if the set, $\{T_c-(t_1+\cdots+t_n)+(l_1+\cdots+l_n)\}$, is not a common tree of $G_V$ and $G_I$, then there exist diloops which consists only of some nodes in $D_L$, furthermore the nodes in all such diloops cover the nodes in $D_L$.

From Property 5-2.2-4, the next property holds.

Property 5-2.5

If, in $G'_V$ for $T_c$, there exists a $\text{diloop}\{V(l_1), V(t_1), \cdots, V(l_n), V(t_n), V(l_1)\}$ which contains no sub-diloop, then $T_c-(t_1+\cdots+t_n)+(l_1+\cdots+l_n)$ is a common tree.

5-3 SOLVABILITY AND THE ORDER OF COMPLEXITY

Consider the solvability and the order of complexity for the networks satisfying Assumption A in Chapter 2. From Eq. (5-2.5) and (5-2.10), a network equation of an active network is given by

\[
\theta(p) = \begin{vmatrix}
V & \Delta & C_p, 0, -1, 0, 0
\end{vmatrix}
\begin{vmatrix}
V_C = 0,
V_D = 0,
V_{V_C, V_D}, 0, 0
i_C = e,
0, 0, Q_{IC}, Q_{ID}
\end{vmatrix}
\begin{vmatrix}
i_D = j
\end{vmatrix}
\]  

(5-3.1)
If and only if the coefficient matrix $\theta(p)$ is nonsingular, the network has a unique solution.

We have

Theorem 5-3.1[18][21]

If a network has a unique solution, there exists a common tree in its voltage and current graphs.

The converse of the theorem is true unless there are no special relations among the network-element-values. It can be said that Theorem 5-3.1 is a necessary and sufficient condition for the topological solvability.

The concept 'common tree' is given from the relations between network-topology and nonsingular $\theta(p)$. In other words, since the current through a twig is given by the currents through some of the links in $G_f$, and the voltage across a link is given by the voltages across some of the twigs in $G_v$, the voltage across" the current through" a network-element" can be given by the current through"the voltage across" the network-element if there exists a common tree in $G_v$ and $G_f$.

For the order of complexity, we have

Theorem 5-3.2[18][21]

The order of complexity of an active network satisfying the remark in Section 5-3.2 is not greater than $|T^N_v(C)|$. 

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5-4 STATE EQUATION

Several papers\cite{20,22,24,25} discussing on the formulations of the state equations for linear active networks have been published. The outlines of the formulations are as follows;

Step 1. Obtain a set of network-equations by applying Kirchhoff's voltage and current laws and from the voltage-current relations of the network-element.

Step 2. By eliminating voltage- and current-variables of the non-reactive network-elements, obtain a differential equation in the form of
\[ A\dot{x} = Bx + Cu, \quad (5-4.1) \]
where \( x \) is a vector whose elements are the voltages across the capacitors and the current through the inductors.

Step 3. Set \( i=0, \) and \( A_0 = A, \ B_0 = B, \ C_0 = C, \ x_0 = x, \ (f) = u, \) then Eq. (5-4.1) becomes
\[ A_i \dot{x}_i = B_i x_i + \sum_{k=0}^{\infty} C_{i,k} (U). \quad (5-4.2) \]

Step 4. If \( A_i \) is nonsingular, go to Step 5. If \( A_i \) is singular, transform Eq. (5-4.2) into the following form;
\[
\begin{vmatrix}
A_i,1 & A_i,2 \\
0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
\dot{x}_i,1 \\
\dot{x}_i,2 \\
\end{vmatrix}
=
\begin{vmatrix}
B_i,1 & B_i,2 \\
B_{i+1,1} & B_{i+1,2} \\
\end{vmatrix}
\begin{vmatrix}
x_i,1 \\
x_i,2 \\
\end{vmatrix}
+ \sum_{k=0}^{\infty} C_{i+k} (U). \quad (5-4.3)
\]
Then by eliminating \( x_{i+2} \) from Eq. (5-4.3), obtain
\[ A_{i+1} \dot{x}_{i+1} = B_{i+1} x_{i+1} + \sum_{k=0}^{\infty} C_{i+k} (U). \]
Set \( i = i+1, \) \( x_{i+1} = x_{i+1} \), and go to Step 4.

Step 5. Obtain a state equation as
\[ \dot{x}_i = A_i x_i + B_i x_i + \sum_{k=0}^{\infty} C_{i,k} (U). \quad (5-4.4) \]
For the network considered in the preceding chapters, the state equations can be formulated by use of the trees given the order of complexity if the topological solvability conditions are satisfied. The state variables and the order of the formulated state equations are determined from the trees. However, the state variables and the order of the state equations derived by the above formulation for active networks cannot be determined until the state equations are obtained. One reason for this is that the formulation depends on only algebraic operations.

In this section, a formulation of state equations for the networks, which satisfy the following Assumption 5-4.1, is given. This formulation is based on the network-topological condition. The state variables can be determined before the state equations are obtained.

Assumption 5-4.1

The network-solvability, the order of complexity and the rank of matrices which appear in the process of deriving the state equations depend on the network-topology only.

The networks considered here contain only capacitors, current-controlled voltage sources, voltage-controlled current sources and independent sources as mentioned in Section 5-2.

By using the fundamental loop matrix of $G_V$ and the fundamental cut-set matrix of $G_I$ for a normal common tree $T_c^N$, the Kirchhoff's voltage and current laws lead to
where $B_{..}$ and $Q_{..}$ are the submatrices of the characteristic parts of the fundamental loop matrix of $G_V$ and the fundamental cut-set matrix of $G_I$, respectively. The subscript "$1_2"$ denotes link-"twig-"dependent sources. The subscript "$S"C" denotes link-"twig-"capacitors.

The voltage and current relations of dependent sources are given by

\[
\begin{align*}
\mathbf{v}_{B_1} &= \begin{bmatrix} R_{\alpha_1} & 0 \\ 0 & R_{\alpha_2} \end{bmatrix} \mathbf{i}_{\alpha_1}, \\
\mathbf{v}_{B_2} &= \begin{bmatrix} 0 & 0 \end{bmatrix} \mathbf{i}_{\alpha_2}, \\
\mathbf{i}_{\delta_1} &= \begin{bmatrix} G_{Y_1} \\ 0 \end{bmatrix} \mathbf{v}_{Y_1}, \\
\mathbf{i}_{\delta_2} &= \begin{bmatrix} 0 \\ G_{Y_2} \end{bmatrix} \mathbf{v}_{Y_2},
\end{align*}
\]

where the submatrices $R_{..}$ and $G_{..}$ are diagonal.

Eliminating the voltages across and the currents through the dependent sources from Eq.(5-4.5)-(5-4.8), we obtain

\[
\begin{bmatrix} 1, -F \\ 0, B \end{bmatrix} \mathbf{i}_C = \begin{bmatrix} 0, 0 \\ -A, 1 \end{bmatrix} \mathbf{v}_S + \begin{bmatrix} \mathbf{e}_C \\ \mathbf{j}_C \end{bmatrix},
\]

where

\[
\begin{align*}
A &= B_{SC} + B_{S2} R_{12}^{-1} Q_{21} R_{11}^{-1} B_{1C}, \\
B &= B_{S2} R_{12}^{-1} Q_{2S}, \\
D &= Q_{C1} R_{11}^{-1} B_{1C}, \\
F &= Q_{CS} + Q_{C1} R_{12}^{-1} B_{12} G_{22}^{-1} Q_{2S}.
\end{align*}
\]

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\[ \begin{align*}
\dot{e}_C &= Q_{C1} R^{-1} e_1 \\
j_C &= Q_{C1} R^{-1} B_{12} G_2^{-1} j_2 + j_C \\
\dot{e}_S &= B_{S2} g^{-1} Q_{21} R_1^{-1} e_1 + e_S \\
j_S &= B_{S2} g^{-1} j_2 \\
g &= G_2 - Q_{21} R_1^{-1} B_{12}, \quad R = R_1 - B_{12} G_2^{-1} Q_{12} \\
G_2 &= \begin{bmatrix} R_{\alpha_2}, 0 \\ 0, G_{Y_2} \end{bmatrix}, \quad R_1 = \begin{bmatrix} R_{\alpha_1}, 0 \\ 0, G_{\delta_1}^{-1} \end{bmatrix} \\
B_{S2} &= \left[ B_{\beta_2}, B_{\gamma_2} \right] (S), \quad B_{1C} = \left[ B_{\beta_1}, B_{Y_1} \right] (C) \\
B_{12} &= \left[ B_{\beta_1 \beta_2}, B_{\beta_1 Y_2} \right] (\beta_1), \quad Q_{21} = \left[ Q_{\alpha_2 \alpha_1}, Q_{\alpha_2 \delta_1} \right] (\alpha_2) \\
 & \quad \quad \quad \quad B_{\gamma_1 \gamma_2} (\gamma_1), \quad Q_{\delta_2 \alpha_1}, Q_{\delta_2 \delta_1} (\delta_2) \\
Q_{C1} &= \left[ Q_{\alpha_1}, Q_{C\delta_1} \right] (C), \quad Q_{2S} = \left[ Q_{\alpha_2 S}, Q_{\delta_2 S} \right] (S) \\
\dot{e}_1 &= \begin{bmatrix} e_{\beta_1} \\ e_{\gamma_1} \end{bmatrix}, \quad \dot{j}_2 = \begin{bmatrix} j_{\alpha_2} \\ j_{\delta_2} \end{bmatrix}.
\end{align*} \]

The voltage-current relation of capacitors is given by
\[ \begin{bmatrix} C_S, 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v_S \\ v_C \end{bmatrix} = \begin{bmatrix} i_S \\ i_C \end{bmatrix}. \]

The network-equation is given by Eq. (5-4.9) and (5-4.11).

If the matrix \( B \) in Eq. (5-4.9) is a zero matrix, it is easy to derive a state equation with the state variables \( v_C \). For passive networks considered in the preceding chapters, the matrix \( B \) is always a zero matrix, however, for active networks, it may not be a zero one.
Now let us examine the relations between the matrix $B$ and the network-topology. The matrix $B$ given in Eq. (5-4.10) is rewritten as

$$B = B_{S2} g^{-1} Q_{2S},$$  \hspace{1cm} (5-4.12)$$

and the matrix $g$

$$g = \begin{pmatrix} (B_2)^+ + (\gamma_2) \\ 1 & Q_{21} \end{pmatrix} [C_2, 0 \mid 1 \mid 0, R_1^{-1} \mid -B_{12}]$$

$\triangleleft Q_{ID} G^{Q}_{VD}$. \hspace{1cm} (5-4.13)

The first matrix of the right hand side of Eq. (5-4.13) is the fundamental cut-set matrix with respect to $T^N_C(a, \delta)$ for $G_I[T^N_C(C); T^N_C(C)]$ (denoted by $G_{ID}$). The last matrix of right hand side of Eq. (5-4.13) is the fundamental cut-set matrix with respect to $T^N_C(\beta, \gamma)$ for $G_V[T^N_C(C); T^N_C(C)]$ (denoted by $G_{VD}$). The second one is diagonal matrix. Then, if the $(i,j)$ element of $g^{-1}$ is nonzero, there exists a common tree in $G_{VD}[\text{twig}(q_{VD}(i))]$ and $G_{ID}[\text{twig}(q_{ID}(j))]$ since $g^{-1}(i,j) = \text{Cof}. g(i,j)/|g|$. Then if a $(k,k)$ element of $B$ is nonzero, there exist $\text{twig}(q_{VD}(i))$ in the loop($\text{link}(b_{2S}(k))$) in $G_V$, and $\text{link}(q_{2S}(k))$ in the cut-set($\text{twig}(q_{VD}(j))$) in $G_I$.

Therefore there exists a common tree in $G_V[T^N_C(C) + \text{link}(b_{2S}(k))]; T^N_C(C) - \text{link}(b_{2S}(k))$ and $G_I[T^N_C(C) + \text{link}(q_{2S}(k)); T^N_C(C) - \text{link}(q_{2S}(k))]$.

Then the capacitor corresponding to the $\text{link}(b_{2S}(k))$ in $G_V$ is identical with that corresponding to the $\text{link}(q_{2S}(k))$. Thus if every diagonal element of $B$ were nonzero, $T^N_C$ could not be a normal common tree, and we have
Lemma 5-4.1

Every diagonal element of $B$ in Eq. (5-4.9) is zero.

In order to examine properties of $B$, let us introduce a matrix $B^b = \{b^b(i,j)\}$ defined as follows:

$$b^b(i,j) = 1 \text{ if } b(i,j) \neq 0, \text{ and } b^b(i,j) = 0 \text{ if } b(i,j) = 0. \quad (5-4.14)$$

The matrix $B^b$ is square and its elements are 0 or 1. Then a digraph can be considered, whose adjacency matrix is $B^b$. It is denoted by $G^b$ and called the Boolean digraph for $B$. The nodes in $G^b$ correspond to the link-capacitors. The digraph $G^b$ is a subgraph of $G^#$ and obtained from $G^#$ by deleting twig-capacitor-nodes and contracting dependent source-nodes.

Lemma 5-4.2

The directed graph $G^b$ has no directed loop.

Proof: Assume that $G^b$ had a directed loop the length of which is $l^d$.

From Lemma 5-4.1, there is no loop of length 1.

Assume that $l^d = n$. Let $S_1, S_2, \ldots$ and $S_n$ denote the link-capacitors corresponding to the nodes of the directed loop in $G^b$. Then corresponding to nonzero $(i_k, j_k)$ elements of $g^{-1}$, there exist twig-dependent source-branches $b^b_{i_k}$ in the $\text{loop}(S_k)$ of $G_Y$ and twig-dependent source-branches $b^b_{j_k}$ in the $\text{loop}(S_k)$ of $G_I$.

With loos of generality, it can be assumed that twig-dependent source-branches can be chosen such that $b^b_{i_k}$ is not identical with $b^b_{i_l}$, or $b^b_{j_k}$ is not identiacl with $b^b_{j_l}$ \{k, l=1, \ldots, n. k\neq l\} since if

@ Node-contraction is the same as node-elimination in flow graphs.
such branches cannot be chosen, there exists a directed loop of the length less than $n$. (See Fig. 5-4.1)

From $g^{-1}(i_k, j_k) \neq 0$, there exists at least one common tree in $G_{VD}[b_1, \ldots, b_n]$ and $G_{ID}[b_1, \ldots, b_n]$. Then there exists at least one common tree in $G_V[T^N_\alpha(C) + S_1 + \cdots + S_n; T^N_\alpha(C) - (S_1 + \cdots + S_n)]$ and $G_I[T^N_\alpha(C) + S_1 + \cdots + S_n; T^N_\alpha(C) - (S_1 + \cdots + S_n)]$ from Property 5-2.5. However, this contradicts that $T^N_\alpha$ is a normal common tree. Therefore the directed graph $G^b$ has no directed loop.

Q.E.D.

From Lemma 5-4.2, we have

Lemma 5-4.3

The matrix $B^b"B"$ is a nilpotent matrix.

Proof: Since $G^b$ has no directed loop, there exists a finite number $m$ such that

$$(B^b)^m = 0. \quad (5-4.15)$$

Consequently the matrix $B$ is a nilpotent matrix.

Q.E.D.

From Lemma 5-4.3, the matrix $B$ can be transposed into a strictly triangular matrix $P_1BP_1^t$, where $P_1$ is a permutation-matrix.
Assume that Eq.(5-4.9) and (5-4.11) can be rewritten as

\[
\begin{bmatrix}
1, -F_1, -F_2 \\
0, B_1, B_2 \\
0, 0, 0
\end{bmatrix}
\begin{bmatrix}
\dot{v}_C \\
\dot{v}_{S1} \\
\dot{v}_{S2}
\end{bmatrix}
= \begin{bmatrix}
D, 0, 0 \\
-A_1, 1, 0 \\
-A_2, 0, 1
\end{bmatrix}
\begin{bmatrix}
v_C \\
v_{S1} \\
v_{S2}
\end{bmatrix}
+ \begin{bmatrix}
\dot{e}_C \\
\dot{e}_{S1} \\
\dot{e}_{S2}
\end{bmatrix}
+ \begin{bmatrix}
\dot{j}_C \\
\dot{j}_{S1} \\
\dot{j}_{S2}
\end{bmatrix},
\tag{5-4.16}
\]

and

\[
\begin{bmatrix}
C, 0, 0 \\
0, C_{S1}, 0 \\
0, 0, C_{S2}
\end{bmatrix}
\begin{bmatrix}
\frac{d}{dt}v_C \\
v_{S1} \\
v_{S2}
\end{bmatrix}
= \begin{bmatrix}
\dot{v}_C \\
\dot{v}_{S1} \\
\dot{v}_{S2}
\end{bmatrix},
\tag{5-4.17}
\]

where

\[
[F_1, F_2] = FP_1 \times P_1, \quad [B_1, B_2] = P_1 BP_1, \quad [A_1] = P_1 A, \quad [v_{S1}] = P_1 v_S
\]

The matrix \( B_j \) is strictly triangular.

The link-capacitors \( S_2 \) are such that there exist no paths from the nodes \( S_2 \) to other link-capacitor-nodes in \( G^b \), that is, no paths not containing twig-capacitor-nodes in \( G^\# \).

Eliminating \( v_{S2} \) and \( i_{S2} \) from Eq.(5-4.16), we obtain

\[
\begin{bmatrix}
1-F_2C_{A_2C}^{-1} \\
-F_1 \\
-B_2C_{S2}A_2C_{C}^{-1}
\end{bmatrix}
\begin{bmatrix}
\dot{i}_C \\
\dot{i}_{S1} \\
\dot{i}_{S2}
\end{bmatrix}
= \begin{bmatrix}
D, 0, 0 \\
-A_1, 1, 0 \\
-A_2, 0, 1
\end{bmatrix}
\begin{bmatrix}
v_C \\
v_{S1} \\
v_{S2}
\end{bmatrix}
+ \begin{bmatrix}
\dot{e}_C \\
\dot{e}_{S1} \\
\dot{e}_{S2}
\end{bmatrix}
+ \begin{bmatrix}
\dot{j}_C \\
\dot{j}_{S1} \\
\dot{j}_{S2}
\end{bmatrix}
+ \begin{bmatrix}
\dot{F}_2 \\
\dot{F}_2 \\
\dot{F}_2
\end{bmatrix}
\begin{bmatrix}
C_{S2} \dot{e}_{S2} \\
C_{S2} \dot{e}_{S2} \\
C_{S2} \dot{e}_{S2}
\end{bmatrix}.
\tag{5-4.18}
\]

Multiplying both sides of Eq.(5-4.18) on the left by the non-singular matrix,

\[
\begin{bmatrix}
1 \\
0 \\
-B_2C_{S2}A_2(C_{C}^{-1}F_2C_{S2}A_2^{-1})^{-1}
\end{bmatrix},
\tag{5-4.19}
\]

we obtain
\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
1 & -F_2 & 0 & | & 1 & -D & | & 0 & | & v_c & + & \zeta_c & + & 3_c & + & -F_2 & C_{S2} (\zeta_{S2} + 3_{S2}) \\
0 & 1 & \tilde{B}_1 & 1 & \tilde{i} & S_1 & 1 & \tilde{i} & S_1 & 1 & \tilde{i} & S_1 & 1 & \tilde{B}_2 \\
\end{array}
\]

(5-4.20)

where
\[
\begin{align*}
\tilde{B}_1 &= B_2 C_{S2} A_2 (C - F_2 C_{S2} A_2)^{-1} F_1 \\
1 &= A_1 + B_2 C_{S2} A_2 (C - F_2 C_{S2} A_2)^{-1} D \\
\tilde{\zeta}_{S1} &= -B_2 C_{S2} A_2 (C - F_2 C_{S2} A_2)^{-1} \zeta_c + \tilde{\zeta}_{S1} \\
\tilde{3}_{S1} &= -B_2 C_{S2} A_2 (C - F_2 C_{S2} A_2)^{-1} 3_c + \tilde{3}_{S1} \\
\tilde{B}_2 &= B_2 C_{S2} A_2 (C - F_2 C_{S2} A_2)^{-1} F_2 + B_2.
\end{align*}
\]

Let us examine properties of the matrix \( \tilde{B} \) defined as:
\[
\tilde{B} = B_1 + \tilde{B}_1. 
\]

(5-4.21)

Lemma 5-4.4

If there exist nonzero elements in \( \tilde{B} \), then the induced digraph \( G' \) has subgraphs such as shown in Fig.5-4.2, where \( S \) denote link-termcapacitors.

Proof: As described in the proofs in Lemma 5-4.1 - 5-4.3, the graph shown in Fig.5-4.2(a) is obtained for nonzero elements of \( B_1 \).

Assume that there exists a nonzero \((m,l)\) element of the matrix \( B_1 \). Since \( B_1 \) is written as
\[
\tilde{B}_1 = B_2 C_{S2} A_2 C^{-1} F_1, \text{ (where } \tilde{C} = C - F_2 C_{S2} A_2 \)
\]
(5-4.22)

there exist \( k, i \) and \( j \) such that
\[
b_2(m,k) C_{S2}(k,k) a_2(k,i) C^{-1}(i,j) f_1(j,l) \neq 0. 
\]
(5-4.23)

Let \( S^m_1 \) and \( S^k_2 \) denote the link-capacitors corresponding to \( b_2(m) \) and \( b_2(k) \), then the induced digraph \( G' \) should include a sub-
Fig. 5-4.3 Subgraph corresponding to nonzero element of $B_2$.

Fig. 5-4.4 Subgraphs corresponding to $a_2(k,j)\neq 0$ and $f'_1(j,l)\neq 0$.

Fig. 5-4.2 Subgraphs in $G^\#$ for Lemma 5-4.4.

The graph shown in Fig. 5-4.3 because $b_2(m,k)\neq 0$.

Let $C_i$ and $C_j$ denote the twig-capacitors corresponding to $a_2(i)$ and $f'_1(j)$, respectively, and $S^L_1$, the link-capacitor to $f'_1(l)$. Then $G^\#$ should include a subgraph shown in Fig. 5-4.4 because $a_2(k,i)\neq 0$ and $f'_1(j,l)\neq 0$.

Consider $\tilde{C}^{-1}_C(i,j)$ defined in Eq. (5-4.22). The matrix $C_C$ can be rewritten as

$$
\tilde{C}_C = \begin{pmatrix}
1 & F_2 \\
0 & C_{S^2} & -A_2
\end{pmatrix}
$$

Similarly to that with respect to the relation between $g^{-1}(i,j)\neq 0$ in Eq. (5-4.13) and the network-topology, the following argument holds. Since $\tilde{C}^{-1}_C(i,j)\neq 0$, there exists a link-capacitor $S^O_2$ and a common tree in $G_V[C_i;T^N_2(C)-S^O_2]$ and $G_I[C_i;T^N_2(C)-S^O_2]$. The common tree contains $T^N_2(C)-C_i-C_j$ and $S^O_2$. Therefore if
there exists a subgraph in \( G^\# \) such as shown in Fig. 5-4.5.

From Fig. 5-4.3 - 5-4.5, we see that \( G^\# \) includes graphs in Fig. 5-4.2 if there exists a nonzero element of \( J^B \).

Q.E.D.

Lemma 5-4.5

The graph \( \_1G^b \) has no directed loop and the \( \_1B \) is a nilpotent matrix, where \( \_1G^b \) is the Boolean digraph for \( \_1B \).

Proof: If there is no sequence \( i_0, i_1, \cdots \) and \( i_n \{ n=1, \cdots \} \) such that all \( (i_0, i_1), (i_1, i_2), \cdots, (i_{n-1}, i_n) \) and \( (i_n, i_0) \) elements of \( \_1B \) are nonzero, \( \_1G^b \) has no directed loop, and then the matrix \( \_1B \) is a nilpotent matrix.

Assume that all \( (i_0, i_1), (i_1, i_2), \cdots, (i_{n-1}, i_n) \) and \( (i_n, i_0) \) elements of \( \_1B \) are nonzero. Then, from Lemma 5-4.4, the induced digraph \( G^\# \) includes a directed loop \( D_L \) such as is shown in Fig. 5-4.6, for \( n=1 \), as an example. No capacitor-node in the directed loop is not identical with another.

In order to examine the directed loop \( D_L \), let us consider the graphs \( G_Y \) and \( G_I \) obtained from \( G_Y \) and \( G_I \) by contracting all twigs and deleting all links except the capacitor- and the dependent source-
branches corresponding to the nodes in $D_L$. There exists a subset of $T_c^N$ (denoted by $T_c^N'$) such that $T_c^N'$ is a common tree of $G^I_V$ and $G^I_I$.

If the common cotree $T_c^N$ for $T_c^N'$ were a common tree, $T_c^N$ would not be a normal common tree since $T_c^N'$ contains more capacitor-branches than $T_c^N$. (See Fig.5-4.7). Then all $(i_0, i_1), (i_1, i_2), \ldots, (i_n, i_0)$ elements of $\widehat{\beta}$ are not nonzero.

If $T_c^N'$ is not a common tree, there exist some directed subloops in $(G^#)'$, where $(G^#)'$ is the induced graph corresponding to $G^I_V$ and $G^I_I$, from Property 5-2.4 in Section 5-2. Moreover there exists a directed subloop in $(G^#)'$ such that the number of the link-capacitor-nodes in the subloop is greater than that of the twig-capacitor-nodes. This is because if there exists a directed subloop where the number of the twig-capacitor-nodes is greater than (or equal to) that of the link-capacitor-nodes, for example, such as the directed subloop $\{V_j, \ldots, C, \ldots, S, \ldots, C, \ldots, V_j, V_j\}$ shown in Fig.5-4.7, where $V$ is a capacitor- or dependent source-node, then from Property 5-2.4, there exists at least one directed loop $D^I_L$ such that the number of the link-capacitor-nodes is greater than that of the twig-capacitor-nodes. (See Fig.5-4.7). Consider the graphs $G^V_V$ and $G^I_I$ obtained from $G^I_V$ and $G^I_I$ by contracting all twigs and deleting all links except the capacitor- and dependent source-branches corresponding to the nodes in the directed subloop $D^I_L$. Let $T_c^{N''}$
denote a subset of $T^N_c$, where $T^N_c$ is a common tree of $G^N_V$ and $G^N_I$. If $T^N_c$ were a common tree of $G^N_V$ and $G^N_I$, $T^N_c$ would not be a normal common tree. If $T^N_c$ is not a common tree, there exists a directed subloop $D^N_L$ in $(G^#)^N$, where the number of the link-capacitor-nodes is greater than that of the twig-capacitor-nodes.

By repeating the above discussion, we can conclude that if all $(i_0, \hat{\tau}_1), (i_1, \hat{\tau}_2), \ldots, (i_{n-1}, \hat{\tau}_n)$ and $(i_n, \hat{\tau}_0)$ elements of $\hat{\tau}_I$ are nonzero, there exists a directed loop $D^{(k)}_L$ and $(G^#)^{(k)}$, such that the number of the link-capacitor-nodes is greater than that of the twig-capacitor-nodes. From Property 5-2.5, it can be proved that $T^N_c(k)$ is a common tree of $G^{(k)}_V$ and $G^{(k)}_I$ obtained from $G_V$ and $G_I$ by contracting all twigs and deleting all links except the branches corresponding to the nodes in the directed loop $D^{(k)}_L$. Then, the common tree $T^N_c(b) + T^N_c(k)$, where $b$ is the set of twigs except those in $D^{(k)}_L$, would contain more capacitor-branches than $T^N_c$. This contradicts that $T^N_c$ is a normal common tree. Therefore there is no sequence $i_0, i_1, \ldots$ and $i_n$ such that all $(i_0, i_1), \ldots, (i_{n-1}, i_n)$ and $(i_n, i_0)$ elements of $\hat{\tau}_I$ are nonzero, and $G^b_I$ has no directed loop. Therefore the matrix $\hat{\tau}_I$ is a nilpotent matrix.

Q.E.D.

From Lemma 5-4.5, by a permutation matrix $P_2$, we obtain a strictly triangular matrix,

$$P_2 \hat{\tau}_2 P_2^t = \begin{bmatrix} \hat{\tau}_1 & \hat{\tau}_2 \\ 0 & 0 \end{bmatrix}.$$  \hspace{1cm} (5-4.25)

By use of the matrix $P_2$, Eq. (5-4.20) becomes
where

\[ P_{2i} = \begin{bmatrix} 2^i S_1 \\ 2^i S_2 \end{bmatrix}, \quad P_{2v} = \begin{bmatrix} 2^v S_1 \\ 2^v S_2 \end{bmatrix}, \quad P_{2a} = \begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix} \]

\[ P_{2l} = \begin{bmatrix} 2^l S_1 \\ 2^l S_2 \end{bmatrix}, \quad P_{2j} = \begin{bmatrix} \tilde{J}_1 \\ \tilde{J}_2 \end{bmatrix} \]

\[ P_{2b} = \begin{bmatrix} \tilde{B}_2 \\ \tilde{B}_3 \end{bmatrix} \]

- \[ F^\dagger_{12} = [-1 F_1, -1 F_2] \]

The vectors \( 2^i S_2 \) and \( 2^v S_2 \) can be eliminated from Eq. (5-4.26). Then the following Eq. (5-4.27), which is of similar form to Eq. (5-4.20), may be obtained.

\[ P_{2i} = \begin{bmatrix} 2^i S_1 \\ 2^i S_2 \end{bmatrix}, \quad P_{2v} = \begin{bmatrix} 2^v S_1 \\ 2^v S_2 \end{bmatrix}, \quad P_{2a} = \begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix} \]

\[ P_{2l} = \begin{bmatrix} 2^l S_1 \\ 2^l S_2 \end{bmatrix}, \quad P_{2j} = \begin{bmatrix} \tilde{J}_1 \\ \tilde{J}_2 \end{bmatrix} \]

\[ P_{2b} = \begin{bmatrix} \tilde{B}_2 \\ \tilde{B}_3 \end{bmatrix} \]

\[ e_{2^i} = \begin{bmatrix} e_{2^i 1} \\ e_{2^i 2} \end{bmatrix} \]

\[ e_{2^v} = \begin{bmatrix} e_{2^v 1} \\ e_{2^v 2} \end{bmatrix} \]

\[ e_{2^l} = \begin{bmatrix} e_{2^l 1} \\ e_{2^l 2} \end{bmatrix} \]

\[ e_{2^j} = \begin{bmatrix} e_{2^j 1} \\ e_{2^j 2} \end{bmatrix} \]

\[ e_{2^b} = \begin{bmatrix} e_{2^b 1} \\ e_{2^b 2} \end{bmatrix} \]

\[ e_{2^c} = \begin{bmatrix} e_{2^c 1} \\ e_{2^c 2} \end{bmatrix} \]

\[ e_{2^d} = \begin{bmatrix} e_{2^d 1} \\ e_{2^d 2} \end{bmatrix} \]

\[ e_{2^e} = \begin{bmatrix} e_{2^e 1} \\ e_{2^e 2} \end{bmatrix} \]

\[ e_{2^f} = \begin{bmatrix} e_{2^f 1} \\ e_{2^f 2} \end{bmatrix} \]

\[ e_{2^g} = \begin{bmatrix} e_{2^g 1} \\ e_{2^g 2} \end{bmatrix} \]

\[ e_{2^h} = \begin{bmatrix} e_{2^h 1} \\ e_{2^h 2} \end{bmatrix} \]

\[ e_{2^i} = \begin{bmatrix} e_{2^i 1} \\ e_{2^i 2} \end{bmatrix} \]

\[ e_{2^j} = \begin{bmatrix} e_{2^j 1} \\ e_{2^j 2} \end{bmatrix} \]

\[ e_{2^k} = \begin{bmatrix} e_{2^k 1} \\ e_{2^k 2} \end{bmatrix} \]

\[ e_{2^l} = \begin{bmatrix} e_{2^l 1} \\ e_{2^l 2} \end{bmatrix} \]

\[ e_{2^m} = \begin{bmatrix} e_{2^m 1} \\ e_{2^m 2} \end{bmatrix} \]

\[ e_{2^n} = \begin{bmatrix} e_{2^n 1} \\ e_{2^n 2} \end{bmatrix} \]

\[ e_{2^o} = \begin{bmatrix} e_{2^o 1} \\ e_{2^o 2} \end{bmatrix} \]

\[ e_{2^p} = \begin{bmatrix} e_{2^p 1} \\ e_{2^p 2} \end{bmatrix} \]

\[ e_{2^q} = \begin{bmatrix} e_{2^q 1} \\ e_{2^q 2} \end{bmatrix} \]

\[ e_{2^r} = \begin{bmatrix} e_{2^r 1} \\ e_{2^r 2} \end{bmatrix} \]

\[ e_{2^s} = \begin{bmatrix} e_{2^s 1} \\ e_{2^s 2} \end{bmatrix} \]

\[ e_{2^t} = \begin{bmatrix} e_{2^t 1} \\ e_{2^t 2} \end{bmatrix} \]

\[ e_{2^u} = \begin{bmatrix} e_{2^u 1} \\ e_{2^u 2} \end{bmatrix} \]

\[ e_{2^v} = \begin{bmatrix} e_{2^v 1} \\ e_{2^v 2} \end{bmatrix} \]

\[ e_{2^w} = \begin{bmatrix} e_{2^w 1} \\ e_{2^w 2} \end{bmatrix} \]

\[ e_{2^x} = \begin{bmatrix} e_{2^x 1} \\ e_{2^x 2} \end{bmatrix} \]

\[ e_{2^y} = \begin{bmatrix} e_{2^y 1} \\ e_{2^y 2} \end{bmatrix} \]

\[ e_{2^z} = \begin{bmatrix} e_{2^z 1} \\ e_{2^z 2} \end{bmatrix} \]

A subgraph of \( G \) which corresponds to the nonzero element of \( Q \) has the same property as that of \( Q \). Then it can be proved in the same manner as the proof in Lemma 5-4.5 that \( Q \) has no directed loop and \( Q \) is a nilpotent matrix. Therefore there exists a permutation matrix \( P_3 \) such that \( Q P_3 \) is strictly triangular. The vector \( 2^i S_1 \) and \( 2^j S_1 \) are partitioned into
Then the vectors $3^\nu S_2$ and $3^i S_2$ can be eliminated from Eq. (5-4.27), and an equation of similar form to Eq. (5-4.27) can be obtained.

Repeating the above procedure, we obtain

\[
\begin{bmatrix}
\text{nE, nF} & \text{nC} = & D, 0 & \text{vC} + \sum \left\{ \frac{(\Omega)}{n^\nu C}, \frac{(\Omega)}{n^\nu S} \right\}, \\
0, \tilde{\text{B}} & n^i S & -A, 1 & n^\nu S
\end{bmatrix}
\]

where $0^E=1, 0^F=F, \tilde{\text{B}}=\text{B}, 0^A=A$.

Now let us consider the properties of the induced directed graph when there are nonzero elements in the matrices $n^A, \tilde{n^B}, n^F$ and $D$.

Property 5-4.1

If there are nonzero elements in $A, B, F$ and $D$ in Eq. (5-4.9), there exist directed paths in $G^\#$, which are denoted by $P_A, P_B, P_F$ and $P_D$, respectively, as shown in Fig.5-4.8(a).

Proof: The matrix $A$ is written by

\[
A = B_{SC} + B_{S2} \ g^{-1} \ Q_{21} \ R_{11}^{-1} \ B_{1C}.
\]

Then the nonzero elements of $A$ correspond to directed paths, as shown in Fig.5-4.8(a), whose initial nodes are link-capacitor-nodes and terminal nodes are twig-capacitor-nodes. Similarly nonzero elements of $B, F$ and $D$ correspond to directed paths as shown in Fig.5-4.8(a). The initial"terminal" nodes of $P_A, P_F$ and $P_D$ are link-"link-", twig-"link-" and twig-"twig-" capacitor nodes, respectively.

Q.E.D.
Property 5-4.2

If there exists a nonzero element in $a_{ij}^A$ and $a_{ij}^B$ for $n=1, 2, \cdots$, there exists a directed path as shown in Fig. 5-4.8(b) for $n=1$ and Fig. 5-4.8(c) for $n \geq 2$. The paths are denoted by $p_A^i$ and $p_B^i$, respectively.

Proof: Examine the case of $n=1$. From Eq. (5-4.20), we have

$$A = A_1 + B_2 C S_2 A_2 (C - F_2 C S_2 A_2)^{-1} D$$

$$B = B_1 + B_2 C S_2 A_2 (C - F_2 C S_2 A_2)^{-1} F_2$$

where the matrices $B_1$ and $B_2$ are submatrices of $B$. The matrix $F$ is a submatrix of $F$. The matrix $D$ is invariant during the elimination-procedures.

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A nonzero element of \( (C_g^e F_s^e S_s^e A_s^e)^{-1} \) (cf. Lemma 5-4.4) corresponds to a path, which has twig-capacitor-nodes as its initial and terminal nodes and contains a link-capacitor-node. The path is represented by \( P_F + P_A \). Then the path corresponding to a nonzero element of \( I_A \) is represented by \( P_A \) or \( P_B + P_A + P_F + P_A + P_D \), as is shown in Fig.5-4.8(b). Similarly the path \( P_B \) is represented by \( P_B \) or \( P_B + P_A + P_F + P_A + P_F \), which is shown in Fig.5-4.8(b).

By induction, the path \( P_A \) is represented by \( n-1 \) \( P_A \) or
\[ n-1 P_B + n-1 P_A + P_F + P_A + P_D \] Similarly, the path \( P_B \) is represented by \( n-1 P_B \) or
\[ n-1 P_B + n-1 P_A + P_F + n-1 P_A + P_F \].

Q.E.D.

Let \( N_S(.P.)"N_C(.P.)" \) denote the number of link-"twig-"
capacitor-nodes in \( .P. \). The number \( N(.P.) \) is defined by
\[ N(.P.) \triangleq N_S(.P.) - N_C(.P.) \]

Property 5-4.3

The following equalities hold:
\[ n P_A = 0, \quad n P_B = 2, \quad N(F) = 0, \quad N(P) = 2. \]

Proof: From Property 5-4.1, the following equalities hold:
\[ N(F) = 0, \quad N(F) = 2, \quad N(F) = 0, \quad N(F) = 2. \]

From Property 5-4.2, we obtain
\[ n P_A = 0, \quad n P_B = 2. \]

Q.E.D.

From the above properties, we obtain
Lemma 5-4.6

The matrix $\hat{\mathbf{B}}_n$ is a nilpotent matrix.

Proof: If $\hat{\mathbf{B}}_n$ were not a nilpotent matrix, there would exist a directed loop in $G^\#$, which consists of some directed paths $\mathbf{P}_B$. There would exist more link-capacitor-nodes than twig-capacitor-nodes in the directed loop. Then in the similar manner to the proof of Lemma 5-4.5, it can be proved that the matrix $\hat{\mathbf{B}}_n$ is a nilpotent matrix.

Q.E.D.

From Lemma 5-4.6, repeating the elimination-procedure, we can obtain an equation of similar form to Eq. (5-4.29), where $\tilde{m}_0=0$. Therefore a state equation can be obtained whose state variables are the voltages across the twig-capacitors, that is,

$$\tilde{\mathbf{V}}_C = \mathbf{A}_v \mathbf{C}_0 \mathbf{u} + \cdots + \mathbf{B}_{m+1} \tilde{\mathbf{u}}^{(m+1)}.$$  \hspace{1cm} (5-4.30)

Note that the matrices $\mathbf{A}$ and $\mathbf{B}_i$ in this equation are different from those in Eq. (5-4.27).

Then we have

Theorem 5-4.1

An active network on the Assumption 5-4.1 which contains capacitors, voltage-controlled current-sources, current-controlled voltage-sources and independent sources has a state equation whose state variables are the voltages across the twig-capacitors in a normal common tree.

The state equation for a passive network may have first order-derivatives of inputs, $\dot{\mathbf{u}}$, but cannot have higher order-derivatives
than the first unless there are special relations among the
network-element-values. However, the state equation of an active
network may have the higher order-derivatives of inputs, \( \{ \mathbf{u} \} \),
even if there is no special relation among the network-element-
values.

Let \( \mathcal{N}_E \) represent Eq. (5-4.9) and (5-4.11). The equations
obtained from \( \mathcal{N}_E \) by the elimination of voltages across and
currents through the link-capacitors not corresponding to the
initial nodes of \( \mathcal{I} \), are represented by \( \mathcal{N}_E^1 \) (cf. Eq. (5-4.26)).
The equation \( \mathcal{N}_E^n \) represents Eq. (5-4.29) plus the voltage- and
current-relations of capacitors.

If there exists a link-capacitor \( S_k \) \( \{ k \geq 1 \} \) whose voltage and
current are contained in \( \mathcal{N}_k \) but not in \( \mathcal{N}_{k-1} \), then there exists
a link-capacitor \( S_{k-1} \) such that 1) there exists a path \( \mathcal{P}_{k-1} \) from
\( S_k \) to \( S_{k-1} \) in \( G^\# \), 2) its voltage and current are contained in
\( \mathcal{N}_{k-1} \) but not in \( \mathcal{N}_k \), and 3) \( S_{k-1} \) does not correspond to the
initial nodes of \( \mathcal{I} \). Similarly for \( S_{k-1} \), there exists a link-
capacitor \( S_{k-2} \) such that 1) there exists a path \( \mathcal{P}_{k-2} \) from \( S_{k-1} \)
to \( S_{k-2} \) in \( G^\# \), 2) its voltage and current are contained in \( \mathcal{N}_{k-2} \)
but not in \( \mathcal{N}_{k-1} \), and 3) \( S_{k-2} \) does not correspond to the initial
nodes of \( \mathcal{I} \). And so on. Consequently for \( S_k \), there exists a
link-capacitor \( S_0 \) such that 1) there exists a path \( P \) (represented
by \( P = \mathcal{P}_{k-1} \cdots \mathcal{P}_0 \)) from \( S_k \) to \( S_0 \) in \( G^\# \), 2) its voltage and current
are contained in \( \mathcal{N}_0 \) but not in \( \mathcal{N}_1 \), and 3) \( S_0 \) does not correspond
to the initial nodes of every \( \mathcal{P}_B \) \( \{ l = 1, 2, \cdots \} \). For the path \( P \),
\( N(P) = k+1 \).
Definition 5-4.1

A directed path in \( G^\# \) whose initial and terminal nodes link-capacitor-nodes is called an \( l \)-path and denoted by \( L_d \). A maximum \( l \)-path is an \( l \)-path such that \( N(L_d) \) is maximum. It is denoted by \( L_{md} \). The number \( l_m \) is defined by \( l_m = N(L_{md}) \).

If a state equation is obtained from \( NE_0 \) by \( l_m \)-repetitions of the eliminating procedure of the voltages across and the currents through link-capacitors, then there exists an \( l \)-path \( L_d \) with \( N(L_d) = l_m \), and the state equation may have \( l_m \)-th order-derivatives of the inputs. Then we obtain

Theorem 5-4.2

In deriving a state equation from the network equations (Eq. (5-4.9) and (5-4.11)), we need not repeat more than \( l_m \) times of the eliminating procedure. The order of derivatives of inputs is not more than \( l_m \).

For example, consider the derivation of a state equation for the network whose induced digraph has the subgraph shown in Fig.5-4.9. In this digraph, the directed paths \( P_A, P_B, P_F \) and \( P_D \) are

\[
P_A: S_3-C_1, \quad S_4-C_2
\]

Fig.5-4.9 Example of subgraph of \( G^\# \).
Then the equation $NE_1$ does not contain the voltages across and the currents through the link-capacitors $S_3$, $S_4$ and $S_7$, which do not correspond to the initial nodes of $P_B$. The matrix $\bar{I}$ may be

$$
\begin{array}{cccc}
S_1 & S_2 & S_5 & S_6 \\
\hline
S_1 & \ast_1 \\
S_2 & & \ast_2 \\
S_5 & & & \ast_3 \\
S_6 & & & \\
\end{array}
$$

(where the marks $\ast_i$ mean nonzero. \{i=1,2,3\})

where $P_B$ are $S_1-S_2$, $S_2-S_3-C_1-S_4-C_2-S_5$ and $S_5-S_6$ for $\ast_1$, $\ast_2$ and $\ast_3$, respectively. Then the equation $NE_2$ does not contain the voltage across and the current through the link-capacitor $S_6$.

The link-capacitors $S_2$ and $S_5$ correspond to the initial nodes of $P_B$.

By induction, the equation $NE_3$ does not contain the voltage across and the current through the link-capacitor $S_5$.

Consequently the equation $NE_4$ does not contain the voltages and the currents through all link-capacitors. Then the state equation may have at most sixth order-derivatives of inputs. The number $l_m$ is five in the subgraph shown in Fig.5-4.9.

Consider the matrices appearing in the formulation of the state equation. Nonsingular $R$ and $g$ guarantee the existence of a hybrid matrix, whose rows correspond to the currents through the twig-capacitors and the voltages across the link-capacitors,
and whose columns correspond to the voltages across the twig-capacitors and the currents through the link-capacitors.
The matrices $R$ and $g$ correspond to the loop resistance matrix and the cut-set conductance matrix, respectively, in RLC networks. The matrix $B(AB_{S_2}^{-1}Q_{S_2})$ may be regarded as a resistance matrix of a non-reactive network, whose ports are connected to the link-capacitors. Nonsingular $\tilde{C}_C(A^T C_C - F_{S_2} C_{S_2} A_{S_2})$ guarantees the existence of a state equation for the network when the link-capacitors except $S_2$ are deleted from the network.

5-5 A CANONICAL FORM OF NETWORK-EQUATIONS WITH NO UNIQUE SOLUTION

A state equation of a network with a unique solution is obtained in Section 5-4. It is a canonical form of the network-equation with a unique solution. In this section, is studied a canonical form of a network-equation with no unique solution. (that is, there exists no common tree.) The matrix pencil \[ A + \lambda B \] is available for the study. (See Appendix I and II.) A matrix pencil is represented by

\[ A + \lambda B, \]  

(5-5.1)

where $A$ and $B$ are $m \times n$ constant matrices and $\lambda$ is a parameter. In Appendix I, it is shown that pencil $A + \lambda B$ is strictly equivalent to a canonical quasi-diagonal form, $A + \lambda B$. In Appendix II, it is shown that in accordance with the canonical form, a system of $m$ linear differential equations of the first order in $n$ unknown functions with constant coefficients splits into the five cases of subsystems. The subsystems belonging to Case 1) are not

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consistent with no special relation among the independent source-values, those to Case 2) are always consistent but indeterminate, those to Case 3) are not consistent with no special relation of the derivative forms among the independent source-values, those to Case 4) may have the derivative parts of the source-values, and those to Case 5) posses the unique solution.

The coefficient matrix \( \Theta(p) \) in Eq. (5-3.1) is written as

\[
\Theta(p) = \Theta_1 + p \Theta_2,
\]

where

\[
\Theta_1 = \begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & H_{VP} & 0 & -H_{ID} \\
B_{VC} & B_{VP} & 0 & 0 \\
0 & 0 & Q_{IC} & Q_{ID}
\end{bmatrix}
\]

\[
\Theta_2 = \text{diag.}\{C,0,0,0\}.
\]

Then \( \Theta(p) \) is a pencil.

Let us classify branches in accordance with a normal maximum tree pair, as shown in Table 5-5.1.

The branches belonging to Type 1 are links in \( G_V \) and \( G_I \), those to Type 2 are twigs in \( G_V \) and \( G_I \), those to Type 3 are links in \( G_V \) but twigs in \( G_I \), and those to Type 4 are twigs in \( G_V \) but links in \( G_I \).

Since the network considered here has no common tree, there exists at least one branch belong-
ing to Type 3 or 4. Then the canonical quasi-diagonal form
of the pencil (5-5.2) should contain quasi-diagonal pencil 0,
L or $L'$. 

Let us assume that there exist branches of Type 3, and examine
the column rank of the matrix $M_\epsilon$. The matrix $M_\epsilon \{\epsilon=0\}$ is
written as

$M = \begin{bmatrix}
(n_C) & (n_D) & (n_C) & (n_D) \\
0 & 0 & -1 & 0 & (n_C) \\
0 & H_{VD} & 0 & -H_{ID} & (n_D) \\
B_{VC}, B_{VD}, & 0 & 0 & (\mu(G_V)) \\
0 & 0 & Q_{IC}, & Q_{ID} & (\nu(G_I)) \\
C & 0 & 0 & 0 & (n_C) \\
0 & 0 & 0 & 0 & (n_D) \\
0 & 0 & 0 & 0 & (\mu(G_V)) \\
0 & 0 & 0 & 0 & (\nu(G_I)) \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$ \hspace{1cm} (5-5.3)

\[ \begin{align*}
\text{i) Assume that there is no dependent source-branch belonging} \\
\text{to Type 4. Let us choose pivot-terms of a minor determinant of} \\
\text{M}_0 \text{ as follows:} \\
\text{1) In } [B_{VC}, B_{VD}], \text{ elements corresponding to the branches} \\
\text{belonging to Type 1 and 3. } (n_1+n_3 \text{ elements}) \\
\text{2) In } [Q_{IC}, Q_{ID}], \text{ elements to Type 2 and 3. } (n_2+n_3 \text{ elements}) \\
\text{3) In } H_{VD}, \text{ elements to Type 2. } (n_{2D} \text{ elements}) \\
\text{4) In } -H_{ID}, \text{ elements to Type 1. } (n_{1D} \text{ elements}) \\
\text{5) In } -1, \text{ elements to Type 1. } (n_{1C} \text{ elements}) \\
\text{6) In C, elements to Type 2 and 4. } (n_{8C}+n_{4C} \text{ elements}) \\
\end{align*} \\
\text{The independent } 2n_\epsilon \text{ pivots can be chosen in the above procedure.} \\
\text{Then the rank of } M_0 \text{ is } 2n_\epsilon. \text{ If the pivot-terms of a minor}
\]
determinant of $M_{\varepsilon}$ \{\varepsilon=1,2,\cdots\} are chosen as above, the rank of $M_{\varepsilon}$ is $2(\varepsilon+1)n_{\varepsilon}$. The canonical quasi-diagonal form of the coefficient matrix $\theta(p)$ contains no submatrix $L_{\varepsilon}$, but may contain submatrices $L'_{\eta}$.

ii) Consider the case where there are branches belonging to Type 4

Definition 5-5.1

A Type 2 capacitor-branch in loop(b) in $G_I$ (where b is a dependent source-branch belonging to Type 4) is called an e-capacitor-branch.

Let us choose pivot-terms of a minor determinant of $M_{\theta}$ as follows:

1) In $[B_{VC},B_{VD}]$, elements corresponding to the branches belonging to Type 1 and 3.

2) In $C$, elements to Type 2 and 4.

3) In $H'_{VD}$, elements to Type 2 and 4.

4) In $H'_{ID}$, elements to Type 1.

5) In $Q'_{ID}$, elements to Type 2 and 3.

6) In $Q'_{ID}$, elements corresponding to Type 4 dependent source-branches if the fundamental loops determined by those dependent source-branches contain e-capacitor-branches.

7) In $Q'_{IC}$, elements to Type 2 and 3 except e-capacitor-branches.

8) In $I$, elements corresponding to the capacitor-branches not chosen in 7).

If there is not an e-capacitor-branches for every Type 4
dependent source-branch, then the rank of $M_0$ is less than $2n_e$.

From i) and ii), we have

Lemma 5-5.1

If there is no branch of Type 4, the rank of $M_\varepsilon \{\varepsilon=0,1,\ldots\}$ is $2(\varepsilon+1)n_e$.

If there exists an e-capacitor-branch for every dependent source-branch, the rank of $M_0$ is $2n_e$. Otherwise it is less than $2n_e$.

In Fig.5-5.1, the shaded portions are independent pivots, and

![Diagram](image-url)

Fig.5-5.1 $M_0$ and independent pivots.
$C_1$ are e-capacitor-branches for $b_1$.

The matrix $M_0$, for example, for the network shown in Fig.5-5.2, is written as

$$M_0 = \begin{bmatrix}
0, 0, 0, -1, 0, 0 \\
0, g_2, 0, 0, -1, 0 \\
0, 0, g_3, 0, 0, -1 \\
1, 1, 1, 0, 0, 0 \\
0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0
\end{bmatrix} \quad (5-5.4)$$

The rank of $M_0$ is $2n_e=6$. The capacitor $C_1$ is an e-capacitor for the Type 4 branch 2 in the normal maximum tree-pair $\{T^N_V, T^N_I\}$, where $T^N_V=\{C_1, 2\}$ and $T^N_I=\{C_1\}$.

![Diagram](image.png)

Fig.5-5.2 Example

(a) Original network, (b) its voltage graph and (c) its current graph.

The rank of $M_0$ for the network shown in Fig.5-5.3 is 13, not $2n_e=14$. There is no e-capacitor-branch for the Type 4 branch 6.
in the normal maximum tree-pair \( T^N_V : T^N_I \) where \( T^N_V = \{C_1, C_2, 5, 6\} \) and \( T^N_I = \{C_1, C_2, 5\} \).

Consider the case where the rank of \( M_{\epsilon-1} \) is \( 2\epsilon n_{\epsilon} \) and that of \( M_\epsilon \) is less than \( 2(\epsilon+1)n_{\epsilon} \). Let \( M_\epsilon \) be

\[
M_\epsilon = \begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_2 \\
\theta_2 \\
\theta_1 \\
\theta_2
\end{bmatrix}
\]

where \( B_i = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \quad i = 0, 1, \ldots, \epsilon-1, \epsilon. \)

\[ M_0 = B_0 \]

Fig. 5.5.3 Example
(a) Original network, (b) its voltage graph and (c) its current graph
The pivots in $B_0$ are chosen as mentioned above. Consider the pivots in $B_I$. The elements of $C$ in $B_I$ corresponding to the e-capacitor-branches $C_{e0}$ cannot be chosen as pivots because the elements of $-I$ in $B_0$ corresponding to those are pivots, that is, the e-capacitors are links for the tree which determines pivots in $B_I$. Then the graphs to be considered in choosing the pivots in $B_I$ are $G_V(C_L0)$ (denoted by $G_{VI}$) and $G_I(C_L0)$ (denoted by $G_{II}$), where $C_L0 = C_{e0} +$ link capacitor-branches. In the similar manner of choosing the pivots of $B_0$ for a normal maximum tree pair of $G_V$ and $G_I$, the pivots of $B_I$ can be chosen for a normal maximum tree pair of $G_{VI}$ and $G_{II}$ as follows:

1) In $[B_{VC}, B_{VD}]$, the elements corresponding to the branches belonging to Type 1 and 3, and the elements corresponding to $C_L0$.  
2) In $C$, elements to Type 2 and 4.  
3) In $H_{VD}$, elements to Type 2 and 4.  
4) In $-H_{ID}$, elements to Type 1.  
5) In $Q_{ID}$, elements to Type 2 and 3.  
6) In $Q_{ID}$, elements to Type 4-dependent source-branches if the fundamental loops determined by those dependent source-branches contain e-capacitor-branches.  
7) In $Q_{IC}$, elements to Type 2 and 3 except e-capacitor-branches.  
8) In $l$, elements to the capacitor-branches not chosen in 7).

If there is no e-capacitor-branches, the rank of $M_I$ is less than $4n_e$, and in this case $\epsilon = 1$. Otherwise there exist an e-capacitor-branch for every Type 4 dependent source-branch in $G_{VI}$ and $G_{II}$. Then the rank of $M_I$ is $4n_e$.

Consider pivots of $M_\epsilon \{\epsilon = 2, \cdots\}$. Since the rank of $M_\epsilon$ is less
than $2(\epsilon+1)n_e$, there is no e-capacitor-branch in $G_{V\epsilon}$ and $G_{I\epsilon}$.

A normal maximum tree pair in $G_{V\ell}$ and $G_{I\ell}$ is considered when the pivots of $B_\ell$ is chosen. It is a maximum tree pair in $G_V$ and $G_I$. Examine the maximum tree pair. For simplicity, let us assume that there exists only one branch of Type 4, $b_1$, in $G_V$ and $G_I$. The maximum tree pair for the pivots in $B_0$ is a normal maximum tree pair $\{T'_V:T'_I\}$. The maximum tree pair for the pivots in $B_1$ is a normal maximum tree pair of $G_{V1}\Delta G_{V\ell}\{C_S,C_1\}$ and $G_{I1}\Delta G_{I\ell}\{C_S,C_1\}$, where $C_S$ is a set of link-capacitor-branches belonging to Type 1 and $C_1$ is an e-capacitor. The maximum tree pair for the pivots in $B_2$ is a normal maximum tree pair of $G_{V2}\Delta G_{V\ell}\{C_S,C_1,C_2\}$ and $G_{I2}\Delta G_{I\ell}\{C_S,C_1,C_2\}$, where $C_2$ is an e-capacitor in $G_{V1}$ and $G_{I1}$. By continuing the above, the maximum tree pair for the pivots in $B_{\epsilon-1}$ is a normal maximum tree pair of $G_V\{C_S,C_1,\ldots,C_{\epsilon-1}\}$ and $G_I\{C_S,C_1,\ldots,C_{\epsilon-1}\}$, where $C_i$ is an e-capacitor in $G_V\{C_S,C_1,\ldots,C_{\ell-1}\}$ and $G_I\{C_S,C_1,\ldots,C_{\ell-1}\}$. Since the rank of $M_\epsilon$ is less than $2(\epsilon+1)n_e$, there exists no capacitor-branch $C_\epsilon$ such that a normal maximum tree pair of $G_V\{C_S,C_1,\ldots,C_{\epsilon-1},C_\ell\}$ and $G_I\{C_S,C_1,\ldots,C_{\epsilon-1},C_\ell\}$ is a maximum tree pair of $G_V$ and $G_I$. Consequently in determining the pivots of $M_\epsilon$, a normal maximum tree pair of $G_V$ and $G_I$ is chosen to determine the pivots of $B_\theta$, and a primitive maximum tree pair in Definition 5-2.3, to determine those of $B_{\epsilon-1}$.

Meanwhile two graphs are uniquely partitioned into the pair subgraphs for an arbitrary maximum tree pair [21]. The partitioned subgraphs have one to one correspondence to the branches belonging to Type 3 or 4. Then we obtain
Lemma 5-5.2

A minimal index of the column dependence $e_k$ of the pencil (5-5.2) is given by

$$e_k = |T^N_{V}(C)| - |T^P_{I}(C)|$$

for the partitioned graph of $G_V$ and $G_I$ which has only one branch belonging to Type 4.

Consider the row dependence of the pencil (5-5.2). For this, the column dependence of the pencil $\theta'_I + p\theta'_2$. The matrix $M_0$ for $\theta'_I + p\theta'_2$ is written as

$$M_0 = \begin{vmatrix} 0, 0, B'_{VC}, 0 \\ 0, H'_{VD}, B'_{VD}, 0 \\ -1, 0, 0, Q'_I \\ 0, H'_{ID}, 0, Q'_ID \\ C, 0, 0, 0 \\ 0, 0, 0, 0 \end{vmatrix}$$

Let us choose pivots of a minor determinant of $M_0$ as follows:

1) In $\begin{vmatrix} Q'_I \\ Q'_ID \end{vmatrix}$, elements corresponding to the branches belonging to Type 2 and 3.

2) In $H'_{ID}$, elements to Type 1 and 4.

3) In $H'_{VD}$, elements to Type 2 and 3.

4) In $1$, elements to Type 1 and 4.

5) In $C$, elements to Type 2 and 3.

6) In $\begin{vmatrix} B'_{VC} \\ B'_{IC} \end{vmatrix}$, elements to Type 1 and the capacitor-branches which is in the loop $(b)$ in $G_V$ (where $b$ is a branch belonging to Type 3).
Between the above pivot-terms and those for the case of \( \theta_1 + p \theta_2 \), there are some correspondences shown in Table 5-5.2.

Then we obtain

Lemma 5-5.3

A minimal index \( \eta_k \) of the row dependence of the pencil (5-5.2) is given by

\[
\eta_k = |T^N_I(C)| - |T^P_V(C)|
\]

for the partitioned graph of \( G_V \) and \( G_I \) which has only one branch belonging to Type 3.

Examine a canonical form of the network-equation for the network shown in Fig.5-5.4, where the voltage and the current graphs are shown in Fig.5-5.4 (a) and (b), respectively. The graphs \( G_{V1}, G_{I1}, G_{V2} \) and \( G_{I2} \) are derived from \( G_V \) and \( G_I \) for the branches belonging to Type 4. The graphs \( G_{V3} \) and \( G_{I3} \) are partitioned for the branches belonging to Type 3. The graphs \( G_{V4} \) and \( G_{I4} \) has common trees.

The network-equation for the network shown in Fig.5-5.4 is strictly equivalent to

\[
\begin{array}{c|cccccc|c}
0, p, 1, 0, 0, 0 & \theta_1 + p \theta_2 \\
0, 0, p, 1, 0, 0 & \theta_1 + p \theta_2 \\
0, 0, 0, 0, p, 0 & 0 \\
0, 0, 0, 0, 1, p & 0 \\
0, 0, 0, 0, 1, 1 & 0 \\
0 & |A_0^+ p B_0| & 117
\end{array}
\]
where $A_0 + pB_0$ is a regular pencil.

The method of obtaining a canonical form of a network-equation is given in this section. To obtain a canonical form of the regular pencil $A_0 + pB_0$ is to obtain a state equation. It is discussed in Section 5-4.

5-6 CONCLUDING REMARKS

In Section 5-2, the induced digraph induced from the voltage and current graphs is defined, and the properties of the digraph are considered. They are useful in formulating the state equation.

In Section 5-3, the solvability and the order of complexity are considered.

In Section 5-4, a state equation
whose state variables are the voltage across the twig-capacitor for a normal common tree is formulated. It may include higher order-derivatives of the inputs than the first order. The relation between the higher order-derivatives and the network-topology is studied.

In Section 5-5, a canonical form of a network-equation with no unique solution is studied by means of 'matrix-pencil'. It gives the properties of the network. It may be useful for the piecewise linear analysis of a nonlinear network such as the network-topology of some piecewise linearized networks is different from that of others.

The considerations in this chapter deal with the network-topology only. However the problems on the solvability and the formulation of a state equation depend not only on the network-topology but also the network-element-values. The problems considering the both are not studied. They may have no fruitful result.
CHAPTER 6  LINEAR NETWORKS CONTAINING PERIODICALLY OPERATED SWITCHES (9)-(15)

6-1  INTRODUCTION

Periodically interrupted electric networks (networks containing periodically operated switches) play an important role in many circuit applications [28][33].

Methods for analyzing such networks have been provided by many authors [28]-[32]. S. Hayashi [28] analyzed such circuits by means of matrix-operational calculus and introduced two kinds of initial values. W.R. Bennett [29] gave steady state solutions with sinusoidal inputs. Y. Sun and I.T. Frish [30] analyzed circuits containing switches, resistors and capacitors, and succeeded in obtaining circuits with larger time constants than those containing no switches. A. Fetweis [31] gave an analysis by means of z-transformation, and M. L. Liou [32], by means of state variables.

The above mentioned methods are applicable to the networks with certain restrictions on the inputs [28] or on the network-structures [29][30]. But the formal relations between two kinds of initial values and the stability problem (that is, the problem whether the circuits have steady state solutions or not) are not studied. The change of network-topology due to switch-operations is not discussed.

In Section 6-2, changes of network-topology due to switch-operations are studied.

In Section 6-3, analysis of networks containing periodically operated switches is studied by state variable-approach.

In Section 6-4, the initial values of the second kind [28]
are derived formally from those of the first kind, in case of RLC, RLCT and RCG networks. In active networks, the relation between the initial values of the first and second kinds is discussed.

In Section 6-5, the mathematical meaning of the two kinds of initial values is discussed.

In Section 6-6, the problem on the stability is studied.

6-2 NETWORK-TOPOLOGY AND RESTRICTIONS

6-2.1

The Fig. 6-2.1 shows a general network containing periodically operated switches, where the switches represented by $S$ are closed and opened with a period $T$. By switch-operations, $n$ circuit-modes are produced periodically. Each circuit-mode is produced again after the period $T$, and its interval is called a stage. (See Fig. 6-2.2)

The graph corresponding to the network containing periodically operated switches is denoted by $G^*$. Note that the graph $G^*$ contains branches corresponding to switches (switch-branches).

![Network N](image)

Fig. 6-2.1

Network containing periodically operated switches.

(a) Circuit-mode; switch is closed.

(b) Circuit-mode; switch is opened.

Fig. 6-2.2
The graph obtained from $G^*$ by contracting voltage source-branches and deleting current source-branches is denoted by $G$.

In general, the graph $G$ may be separable, i.e. there may be some network-elements whose currents and voltages do not depend on any switch-operation. In our analysis, only a connected component of $G$ containing some switch-branches may be considered. For example, the graph $G$ corresponding to the network shown in Fig.6-2.3(a) is shown in Fig.6-2.3(b). Then only a network shown in Fig.6-2.3(c) is considered.

The symbol $G(S_1, \ldots, S_k)$ denotes the graph corresponding to the network, where $S_i = 1$ if $S_i$ is closed, $0$ if $S_i$ is opened, and $k$ is the number of switches. The symbol $G_i$ denotes the graph corresponding to the $i$-th circuit mode.

In the state variable-approach, a normal tree, a maximum proper tree or a normal common tree is chosen for every circuit mode if the circuit mode is of an RLC, RLCT "RCG" or active network, respectively.

\[\begin{array}{c}
\begin{array}{c}
\text{(a) Original network.}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{(b) Its corresponding graph.}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{(c) Network to be considered.}
\end{array}
\end{array}
\end{array}
\end{array} \]
Assume that every circuit mode is of an RLC network. Let us choose a normal tree (denoted by $T_b$) in $G(1, \cdots, 1)$. A normal tree $T_o$ in $G(0, \cdots, 0)$ can be obtained such that it includes $T_b$. Then a normal tree (denoted by $T_i$) in $G_i$ containing all twigs in $T_b$ and no links for $T_o$ can be chosen. Such a tree is called a basic tree for $T_b$ and $T_o$. Then the voltages across the twig-capacitors in $T_b$ and the currents through the link-inductors for $T_o$ can always be state variables in every circuit mode. However, the state variables in some circuit mode are not only these voltages and currents since generally the order of complexity of one circuit mode may be different from that of another.

Definition 6-2.1

Let $T$ be a basic tree of $G(1, \cdots, 1, \hat{0},1, \cdots, 1)$ for $T_b$ and $T_o$. If the number of the twig-capacitors in $T$ is not equal to that in $T_b$, the switch $S_i$ is called a $C$-switch, if the number of the link-inductors in $T$ is not equal to that in $T_o$, the switch $S_i$ is called an $L$-switch, and otherwise, the switch $S_i$ is called an $R$-switch.

From Definition 6-2.1, if all switches are $R$-switches, the order of complexity and a set of state variables do not vary in switch-operations.

Consider how the network-topology changes by switch-operation if the network contains only one $R$-switch $S_i$. Then the characteristic part of the fundamental loop matrix of $G$ is given by
\[(S_1)(C) \quad (G) \quad (T)\]

\[
\begin{bmatrix}
S & 0 & F_{SC}, 0, 0 \\
R & f_{Rx} & F_{RC}^T, F_{RG}, 0 \\
y & f_{yx} & F_{yC}, f_{yG}, 0 \\
L & f_{Lx} & F_{LC}, F_{LG}, F_{LT}
\end{bmatrix}
\]

(6-2.1)

where the switch-branch \(S_1\) is a twig. The subscripts \(x\) and \(y\) denote switch \(S_1\) and the resistor which becomes a twig when \(S_1\) becomes a link. The matrix obtained from (6-2.1) by deleting the column for \(S_1\) is the characteristic part of the loop matrix of \(G(1)\).

Then the characteristic part of the fundamental loop matrix, when \(S_1\) is chosen as a link, is

\[
\begin{bmatrix}
(C) & (G) & (y) & (T)
\end{bmatrix}
\]

\[
\begin{bmatrix}
(S) & F_{SC} & 0 & 0 & 0 \\
(R) & F_{RC} & f_{Rx} f_{yC} & f_{RG} & f_{yx} f_{yG} & f_{yx} f_{yG} & 0 \\
(L) & f_{LC} & f_{Lx} f_{yC} & f_{LG} & f_{yx} f_{yG} & f_{yx} f_{yG} & F_{LT} \\
(S_1) & -f_{yx} f_{yC} & -f_{yx} f_{yG} & f_{yx} & 0
\end{bmatrix}
\]

(6-2.2)

by Lemma A-3.2. (See Appendix A-3.2.) The matrix obtained from (6-2.2) by deleting the row for \(S_1\) is the characteristic part of the loop matrix of \(G(0)\).

In the network containing more than one \(R\)-switch, the variations in network-topology caused by switch-operations are complicated unless the network satisfies the following assumption.

Assumption 6-2.1

There is no intersection of \(S_{L1}\) and \(S_{L2}\), where \(S_{L1} \cap S_{L2}\) is a subset of links \(T_b\) each of which can become a twig of a tree for
Consider a network containing $n$ $R$-switches. The characteristic part of the fundamental loop matrix for $T_s$ is written as

$$G(1,\ldots,1,\hat{i},1,\hat{j},1,\ldots,1)"G(1,\ldots,1,\hat{i},1,\ldots,1)\{i, j=1, \quad i \neq j\}.$$ 

From the matrix (6-2.3), the characteristic part of the fundamental loop matrix when $S_i$ and $S_j$ are links is written as the following matrix (6-2.4), where the elements for switches are eliminated, and the superscript $i$ of $F_{RC}^i$ denotes a characteristic part corresponding to the resistor-branches for which the fundamental loops contain the switch $S_i$. 

$$
\begin{bmatrix}
F_{SC}, & 0, & 0 \\
F_{RC}, F_{RG}, & 0 \\
F_{RC}, F^1_{RG}, & 0 \\
F_{RC}, F^1_{RG}, & 0 \\
\vdots & \vdots \\
F^n {RC}, F^n_{RG}, & 0 \\
F^n {RC}, F^n_{RG}, & 0 \\
F_{LC}, F_{LG}, F_{LT} \\
F^1_{LC}, F^1_{LG}, F^1_{LT} \\
\vdots & \vdots \\
F^n_{LC}, F^n_{LG}, F^n_{LT}
\end{bmatrix}
$$

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\[
\begin{array}{c|c|c|c|c|c}
F_{SC} & 0 & 0 & 0 & 0 & 0 \\
F_{RC} & F_{RG} & & & & \\
F_{RC} & 1 & F_{RG} & & & \\
\vdots & & & \ddots & & \vdots \\
F_{RC} & 1 & F_{RG} & \cdots & F_{RG} & \cdots \\
F_{RC} & 1 & F_{RG} & \cdots & F_{RG} & \cdots \\
F_{JC} & F_{LG} & & & F_{LG} & \\
F_{JC} & 1 & F_{LG} & & F_{LG} & \\
\vdots & & \ddots & & \vdots & \vdots \\
F_{JC} & 1 & F_{LG} & \cdots & F_{LG} & \cdots \\
F_{JC} & 1 & F_{LG} & \cdots & F_{LG} & \cdots \\
\end{array}
\]
Consider a network satisfying Assumption 6-2.1. Then its graph satisfies:

Condition 6-2.1
There exists a tree such that several specified branches are twigs and there is no fundamental loop containing two or more of the specified branches.

The dual condition of Condition 6-2.1 is

Condition 6-2.2
There exists a tree such that several specified branches are links and there is no fundamental cut-set containing two or more of the specified branches.

Assume that the number of specified branches is two. We have some definitions.

Definition 6-2.2
The branches specified to be twigs are called r-branches (reserved branches). The nodes which touch r-branches are called r-nodes.

Definition 6-2.3
A tree (containing r-branches) is called a p-tree (partitioning tree) if every fundamental loop with respect to the tree contains less than two of r-branches.
Definition 6-2.4

A path from a node $i$ to $j$ which does not contain r-branches and r-nodes except the r-nodes $i$ and $j$ is denoted by $p(i,j)$.

A node is called a p-node if it is not an r-node and if there exist paths from it to more than three r-nodes containing no r-branch.

A path $p(i,j)$ containing no p-node is denoted by $d-p(i,j)$ (direct path).

From the definitions, we obtain

Lemma 6-2.1

If a node touching a branch is a p-node, the other node touching the branch is an r-node or a p-node.

Proof: If the node is not an r-node, there exists a path from the node to the p-node. Then the node is a p-node.

Q.E.D.

Consider the case where an r-node touching an r-branch $1$ is identical to an r-node touching r-branch $2$. Without loss of generality, the other r-nodes can be assumed to be labeled with $1'$ and $2'$. Then we obtain

Theorem 6-2.1

In the case where the r-node $1$ is identical to the r-node $2$, there exists a p-tree if and only if there exist no $d-p(1',2')$.

Proof: Either there is a $p(1',2')$ or not.

i) If there is no $p(1',2')$, then the graph is separable and
the r-node 1"2" is a cut-node. Then any tree containing the r-branches 1 and 2 is a p-tree.

ii) If there exists a p(1',2'), there exists at least one p-node in the p(1',2'). Then a tree containing the r-branches 1 and 2, such that every p-node is connected to the r-node 1 with a path in it even if the r-branches are deleted, can be chosen.

If a node touching a link for the tree is a p-node, the other node touching the link is a p-node or an r-node from Lemma 6-2.1. Then the two nodes are connected in the tree if either of the two r-branches is deleted. Then the fundamental loop for the link does not contain both of the r-branches. If either node touching a link in p(1',2') were not a p-node, (when the fundamental loop for the link contains both of the r-branches,) there would exist a d-p(1',2'). This contradicts the assumption, and therefore the tree is a p-tree.

If there exists a d-p(1',2'), the fundamental loop for a link in the d-p(1',2') with respect to any tree contains both of the r-branches. Then there does not exist a p-tree.

Q.E.D.

Consider the case where all r-nodes are different. The following theorem covers the case since the r-nodes can be arbitrarily labeled.

Theorem 6-2.2

Suppose that all r-nodes are different. i) If there exist all d-p(1,2), d-p(1',2'), d-p(1',2) and d-p(1,2'), or ii) if there exists d-p(1,2) and d-p(1',2') and neither of p(1',2) and p(1,2'),
then there is no p-tree. The converse is also true.

**Proof:** In the case where there exist all $d-p(1,2)$, $d-p(1',2')$, $d-p(1',2)$ and $d-p(1,2')$, even if the r-branches are deleted in any tree $T$ containing the r-branches, there exists a path from the r-node 1 to 2, from 1' to 2', from 1' to 2 or from 1 to 2' in the subtree $T\{-r\text{-branches}\}$. Assume that there exists a path from 1 to 2. A fundamental loop for a link in the $d-p(1',2')$ contains both of the r-branches. Therefore there exists no p-tree. Similarly there exists a path from 1' to 2', from 1' to 2 or from 1 to 2' in $T\{-r\text{-branches}\}$.

In the case where there exist $d-p(1,2)$ and $d-p(1',2')$ and there exists neither $p(1',2)$ nor $p(1,2')$, there exists either a path from the r-node 1 to 2 or from 1' to 2' in a subtree where the r-branches are deleted in any tree containing the r-branches. Assume that there exists a path from 1 to 2. Then a fundamental loop for a link in the $d-p(1',2')$ contains both of the r-branches. Therefore there exists no p-tree.

The cases which are not covered by the case where there exist all $d-p(1,2)$, $d-p(1',2')$, $d-p(1',2)$ and $d-p(1,2')$, and the case where there exist $d-p(1,2)$ and $d-p(1',2')$ and there does not exist either $p(1',2)$ or $p(1,2')$ are as follows:

Case 1) None of $d-p(1,2)$, $d-p(1',2')$, $d-p(1',2)$ and $d-p(1,2')$ exists.

Case 2) There exists only $d-p(1,2)$ and there is no other d-path.

Case 3) There exist $d-p(1,2)$ and $d-p(1,2')$, and there does not exist another d-path.

Case 4) There exist $d-p(1,2)$, $d-p(1,2')$ and $d-p(1',2)$, and there is no $d-p(1',2')$.  

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Case 5) There exist $d-p(1,2)$ and $d-p(1',2')$, and there exists either $p(1',2)$ or $p(1,2')$.

We prove that there exists a p-tree in the five cases.

Case 1) There exists either of the paths $p(i,j) \{i=1,1'; j=2,2'\}$ since the graph considered is connected. Then there exists at least a p-node, and a node in $p(i,j)$ is either an r-node or a p-node. There exists a tree containing both of the r-branches, such that there is a path from a p-node to another in the tree even if either of the r-branches is deleted. Then a fundamental loop for a link touching a p-node does not contain both of the r-branches. If neither of nodes touching a link were a p-node and if the fundamental loop for the link contained both of the r-branches, there would exist a $d-p(i,j) \{i=1 \text{ or } 1'; j=2 \text{ or } 2'\}$. This contradicts the assumption of Case 1). There exists a p-tree.

Case 2) Since there exists a path from a p-node to r-node 1 or 2 which does not contain r-nodes except the terminal nodes, there exists a tree containing both of the r-branches and all branches of a $d-p(1,2)$, such that there exists a path from a p-node to another p-node in the tree even if either of the r-branches is deleted. Then a fundamental loop for a link touching a p-node does not contain both of the r-branches. Assume that neither node touching a link is a p-node. The link is either contained in a $p(1,2)$ or not. In the former, the fundamental loop for the link could not contain either of the r-branches since all branches in a $d-p(1,2)$ are twigs. In the latter, the fundamental loop for the link would contain both of the r-branches. Then there would exist a $d-p(1',2')$, $d-p(1,2')$ or $d-p(1',2)$. This contradicts the
assumption of Case 2). Thus there exists a p-tree.

Case 3) There exists a tree containing both of the $r$-branches and all branches in a $d-p(1,2')$, and a path from a p-node to another p-node exists in the tree even if either of the $r$-branches is deleted. Then a fundamental loop for a link touching a p-node does not contain both of the $r$-branches. If the link is contained in a $d-p(1,2)$ or another $d-p(1,2')$, the fundamental loop for the link does not contain both of the $r$-branches since all branches of a $d-p(1,2')$ are twigs. If the link is not contained in a $d-p(1,2)$ or a $d-p(1,2')$ and if the fundamental loop for the link contains both of the $r$-branches, then there exists a $d-p(1,2)$.

This contradicts the assumption of Case 3). There exists a p-node.

Case 4) There exists a tree containing both of the $r$-branches and all branches in a $d-p(1,2)$, and a path from a p-node to another p-node exists in the tree even if either of the $r$-branches is deleted. Similar arguments to those in Case 1)-3) lead to that the tree is a p-tree.

Case 5) There exists a tree containing both of the $r$-branches and all branches of a $p(1',2)$ or a $p(1,2')$, and a path from a p-node to another p-node exists in the tree even if either of the $r$-branches is deleted. Similar arguments to those in Case 1)-4) lead to that the tree is a p-tree.

Q.E.D.

The problem whether there exists a p-tree or not is solved for a graph where the number of $r$-branches is two. The problem where the number of $r$-branches is more than two is considered. This
problem is not solved. But a heuristic algorithm for a p-tree is obtained in the next paragraph. The proof of the algorithm is not given.

In order to consider condition 6-2.2, we have some definitions.

Definition 6-2.4

The branches specified to be links are called s-branches and the nodes which touch s-branches are called s-nodes.

Definition 6-2.5

A tree (containing no s-branches) is called a d-tree if every fundamental cut-set with respect to it contain less than two s-branches.

For a graph with two s-branches $S_1$ and $S_2$, we have

Theorem 6-2.3

A tree which contains no s-branch is denoted by $T$. Assume that there exists a twig $t_0$ such that the fundamental cut-set for $t_0$ with respect to $T$ (cut-set($t_0$)) contains two s-branches.

(If the tree $T$ does not satisfy the assumption, it is a d-tree.)

If the link $l_1$ and the twig $t_1$ exists such that

i) The link $l_1$ is in cut-set($t_0$),

ii) The twig $t_1$ is in loop($l_1$), and cut-set($t_1$) contains only one s-branch (e.g. $S_1$),

iii) There is no twig in loop($l_1$) such that the fundamental cut-set for the twig contains $S_2$ but does not contain $S_1$,
then the number of the twigs in the tree $T-t_1^1+l_1^1$ such that the fundamental cut set for each of the twigs with respect to $T-t_1^1+l_1^1$ contains two s-branches is less than that in $T$.

**Proof:** Let the fundamental cut-set for a twig $t$ with respect to $T-t_1^1+l_1^1$ be $C(t)^1=C_1(t)^1$. Then if a twig $t$ is in loop($l_1^1$) with respect to $T$ but is not $t_1^1$, (a) $C_1(t)^1=C(t)^1+C(t_1^1)$. Otherwise (b) $C_1(t)^1=C(t)^1$, and (c) $C_1(t_1^1)^1=C(t_1^1)$. From iii) $C_1(t)^1$ in (a) does not contain two r-branches. Then the theorem is proved.

Q.E.D.

**Corollary**

If the branches as the link $l_1^1$ and the twig $t_1^1$ do not exist, and if $T$ is not a d-tree, then there is no d-tree.

6-2.3 ALGORITHM FOR p-TREES AND d-TREES

i) The number of r-branches is two.

By Theorem 6-2.1 and 6-2.2, it can be determined whether there exists a p-tree or not. If there exists a p-tree, it can be obtained by choosing a tree as described in the proofs of the theorems.

ii) The number of s-branches is two.

By Theorem 6-2.3, it can be determined whether there exists a d-tree or not. If there exists a d-tree, a d-tree can be obtained by tree-transformations mentioned in Theorem 6-2.3.

iii) The number of r-branches is more than two.

In this case, we have no necessary and sufficient condition for the existence of a p-tree. However, the following theorem gives a necessary condition.
Theorem 6-2.4

Let \( n_n \) and \( n_b \) denote the numbers of nodes and branches in a graph, respectively. Let \( M_{ct} \) be minimum number of branches in a cut-set among the cut-sets containing only one \( r \)-branch \( r_i \). Then if there exists a p-tree, the following inequality holds:

\[
\sum_{r_i \in R} M_{ct} > n_n - (|R| + 1),
\]

where \( R \) is the set of \( r \)-branches.

Proof: Since there exists a p-tree (i.e., there is no intersection among the cut-sets for \( r \)-branches), let the number of the branches of the fundamental cut-set of the \( r \)-branches \( r_i \) for the p-tree be \( F_{ci} \), then

\[
\sum_{r_i \in R} M_{ct} > n_n - \sum_{r_i \in R} F_{ci}.
\]

Then the right hand of Eq. (6-2.6) is not less than the rank of \( G[r\text{-branches}] \). Consequently

\[
\sum_{r_i \in R} F_{ci} > n_n - (|R| + 1).
\]

The theorem is proved.

We have a hueristic algorithm of obtaining a p-tree. It is shown in Appendix VI.

6-2.4 CONCLUDING REMARK

The similar problems concerning p-tree or d-tree are

i) the problem of choosing a tree such that there are dummy branches in the different subgraphs for the fundamental loops[37].

ii) the dual problem of i).
iii) the problem of maximizing the sparsity of the admittance matrix $\gamma^{[38]}$.

6-3 ANALYSIS OF NETWORKS CONTAINING PERIODICALLY OPERATED SWITCHES

Method of analyzing such networks have already been provided. Matrix operational calculus$^{[28]}$, $z$-transformation$^{[31]}$ and state variable-approaches$^{[32]}$ are examples of such methods. The state variable-approaches fit the computer-aided analysis and have some informations of the network topology. Here a state variable-approach to the analysis of networks containing periodically operated switches is studied. Such networks are assumed to satisfy the following Restriction A.

Restriction A

Every circuit mode satisfies Assumption A in Chapter 2, and does not contain impulsive voltage- or current-sources.

For an RLC network, as mentioned in Section 6-2, the graph corresponding to every circuit mode has a normal tree, which contains all twigs of a normal tree $T_0$ in $G(1,\cdots,1)$ and no links of a normal tree $T_0$ in $G(0,\cdots,0)$. Therefore a set of state variables for some circuit mode can be chosen so that the set covers a set of state variables for the circuit mode whose order of complexity is minimum among the all circuit modes. This state variables can be covered by state variables of the circuit mode whose order of complexity is maximum among all the circuit modes.

Let us consider a network with two circuit modes. The period
of the modes is denoted by \( T \). Assume that a set of state variables for the first circuit mode covers that for the second.

The state equations for the first and the second circuit modes for the \( n \)-th stage may be obtained, respectively, as

\[
\dot{x}_{1,n}(t) = A_1 x_{1,n}(t) + B_1 u(t + nT), \quad \text{at } nT < \tau < nT + t_1, \quad (6-3.1)
\]

where \( t = \tau - nT \), and

\[
\dot{x}_{2,n}(t) = A_2 x_{2,n}(t) + B_2 u(t + nT + t_1), \quad \text{at } nT + t_1 < \tau < (n+1)T, \quad (6-3.2)
\]

where \( t = \tau - (nT + t_1) \). The vectors \( x_{i,n} (t) \) \( \{i=1,2\} \) and \( u(t) \) are the state and the input vector, respectively. The matrices \( A_i \) and \( B_i \) are constant.

Assume that \( B_i \) contains no derivative term. (It will be shown in the following Section 6-4 that the assumption is reasonable.) Then the solution of Eq.(6-3.1) and (6-3.2) are obtained, respectively as

\[
x_{1,n}(t) = e^{tA_1} x_{1,n}(+0) + e^{tA_1} \int_{nT}^{nT+t} e^{-(\tau - nT)A_1 B_1} u(\tau) d\tau, \quad (6-3.3)
\]

\[
x_{2,n}(t) = e^{tA_2} x_{2,n}(+0) + e^{tA_2} \int_{nT+t_1}^{(n+1)T} e^{-(\tau - (nT + t_1))A_2 B_2} u(\tau) d\tau, \quad (6-3.4)
\]

where \( x_{i,n}(+0) \) \( \{i=1,2\} \) are the initial value vectors of the second kind.

Consider the input functions,

\[
u(\tau) = u e^{j\omega t + \phi}, \quad (6-3.5)
\]

where \( u \) is a constant vector. Then Eq.(6-3.3) and Eq.(6-3.4) become, respectively,

\[
x_{1,n}(t) = e^{tA_1} x_{1,n}(+0) + (j\omega A_1)^{-1} (e^{j\omega t} - e^{tA_1}) B_1 u e^{j\omega nT + \phi}, \quad (6-3.7)
\]

\[
x_{2,n}(t) = e^{tA_2} x_{2,n}(+0) + (j\omega A_2)^{-1} (e^{j\omega t} - e^{tA_2}) B_2 u e^{j(\omega nT + t_1 + \phi)}, \quad (6-3.8)
\]
i) Since the set of state variables for the first circuit mode covers that of the second, some voltages across the capacitors and/or some currents through the inductors may be state variables but not those for the second circuit mode. However, such voltages or currents for the second circuit mode can be expressed in terms of the state variables. A network-element corresponding to the \( j \)-th element of \( x_{1,n}(t) \) can be identical to that of \( x_{2,n}(t) \).

ii) The initial value vectors of the second kind, \( x_{1,n}(+0) \) and \( x_{2,n}(+0) \), can be derived from \( x_{2,n}(t_2) \) and \( x_{1,n-1}(t_1) \), respectively. (This shall be mentioned in Section 6-4.)

From the discussion i) and ii), Eq. (6-3.7) and (6-3.8) can be written, respectively, as

\[
x_{1,n}(t) = H_1(t)x_{2,n-1}(t_2) + I_1(t)e^{j\omega nt}, \quad (6-3.9)
\]

\[
x_{2,n}(t) = H_2(t)x_{1,n}(t_1) + I_2(t)e^{j\omega nt + t_1}, \quad (6-3.10)
\]

where \( T = t_1 + t_2 \).

The above equations lead to

\[
x_{1,n}(t_1) = D_1x_{1,n-1}(t_1) + S_1e^{j\omega nT}
\]

\[
x_{2,n}(t_2) = D_2x_{2,n-1}(t_2) + S_2e^{j\omega nT + t_1},
\]

where \( D_1 = H_1(t_1)H_2(t_2) \), \( S_1 = H_1(t_1)I_2(t_2)e^{j\omega t_2 + I_1(t_1)} \),

\( D_2 = H_2(t_2)H_1(t_1) \), \( S_2 = H_2(t_2)I_1(t_1)e^{-j\omega t_1 + I_2(t_2)} \).

Consequently, \( x_{1,n}(t_1) \) and \( x_{2,n}(t_2) \) are expressed by \( x_{1,1}(t_1) \) and \( x_{2,1}(t_2) \), respectively, as

\[
x_{1,n}(t_1) = D_1^{-1}x_{1,1}(t_1) + (1 - D_1^{-1})e^{-j\omega T} - 1 e^{j\omega nT} - D_1^{-1}S_1, \quad (6-3.11)
\]

\[
x_{2,n}(t_2) = D_2^{-1}x_{2,1}(t_2) + (1 - D_2^{-1})e^{-j\omega T} - 1 e^{j\omega nT} - D_2^{-1}S_2 e^{j\omega T}.
\]

The vectors \( x_{i,n}(t_1) \) \( \{i = 1, 2\} \) can be partitioned into three
vectors as

\[ x_{i,n}(t_i) = x_{i,n}(t_i) + \{x_{i,n}(t_i)\}_o + \{x_{i,n}(t_i)\}_s \]  \tag{6-3.13}

where

\[
\{x_{i,n}(t_i)\}_t = D_i^{n-1} x_{i,1}(t_i)
\]

\[
\{x_{1,n}(t_i)\}_o = -(1-D_1 e^{-j\omega T}) D_1^{n-1} S_1
\]

\[
\{x_{2,n}(t_i)\}_o = -(1-D_2 e^{-j\omega T}) D_2^{n-1} S_2 e^{j\omega T}
\]

\[
\{x_{1,n}(t_i)\}_s = (1-D_1 e^{-j\omega T})^{-1} S_1 e^{j\omega T}
\]

\[
\{x_{2,n}(t_i)\}_s = (1-D_2 e^{-j\omega T})^{-1} S_2 e^{j\omega T + t_1}
\]  \tag{6-3.14}

The terms \(x_{i,n}(t_i)\)_t, \(x_{i,n}(t_i)\)_o and \(x_{i,n}(t_i)\)_s are called the transient state terms for the initial values, the transient state terms for the input voltages and currents, and the steady state terms, respectively.

If \(D_1^{n} \rightarrow 0\) and \(D_2^{n} \rightarrow 0\) as \(n \rightarrow \infty\), then \(x_{i,n}(t_i)\) contain only steady state terms at \(n = \infty\).

Let \(\{x_{1,n}(t)\}_s\) and \(\{x_{2,n}(t)\}_s\) denote steady state solutions of the network, then \(\{x_{1,n}(t)\}_s\) and \(\{x_{2,n}(t)\}_s\) are given, respectively, as

\[
\{x_{1,n}(t)\}_s = \{H_1(t)(1-D_2 e^{-j\omega T})^{-1} S_2 e^{j\omega T_1 + f_1(t)}\}_s e^{j\omega T}
\]

\[
\Delta f_1(t) e^{j\omega T} \]  \tag{6-3.15}

\[
\{x_{2,n}(t)\}_s = \{H_2(t)(1-D_1 e^{-j\omega T})^{-1} S_1 e^{-j\omega T} + f_2(t)\}_s e^{j\omega T_1 + t_1}
\]

\[
\Delta f_2(t) e^{j\omega T} \]  \tag{6-3.16}

The time origin in the function \(x_{i,n}(t)\) is at the start of the \(i\)-th circuit mode for the \(n\)-th stage. Here, the time origin is moved at \(\tau = 0\). Let \(x(t)\) denote the steady state solution of the network as
From Eq. (6-3.16), we obtain

\[ x(\tau) = \begin{cases} x_1(\tau-nT) & \text{if } nT \leq \tau < nT + T_1 \\ x_2(\tau-nT-t_1) & \text{if } nT + T_1 \leq \tau < (n+1)T. \end{cases} \quad (6-3.17) \]

From (6-3.15) and (6-3.16), the vector \( x(\tau) \) does have a quasi periodicity as

\[ x(\tau + T) = x(\tau) e^{j\omega T}. \quad (6-3.19) \]

Then the vector \( f(\tau) \) written as

\[ f(\tau) = x(\tau) e^{-j\omega \tau} \quad (6-3.20) \]

is periodic, since

\[ f(\tau + T) = x(\tau + T) e^{-j\omega (\tau + T)} = x(\tau) e^{-j\omega T} = x(\tau). \quad (6-3.21) \]

Then we obtain

\[ x(\tau) = f(\tau) e^{j\omega \tau} = e^{j\omega \tau} \sum_{m=-\infty}^{\infty} c_m e^{j2\pi \frac{m}{T}}, \quad (6-3.22) \]

where

\[ c_m = \frac{1}{T} \int_{nT}^{(n+1)T} f(\tau) e^{-j2\pi \frac{m}{T} \tau} d\tau = \frac{1}{T} \left\{ \int_{0}^{T} f_1(\tau) e^{-j2\pi \frac{m}{T} \tau} d\tau + \int_{t_1}^{T} f_2(\tau) e^{-j2\pi \frac{m}{T} (\tau-t_1)} d\tau \right\}. \]

If there exist integers \( p \) and \( q \) such that \( pt = q \frac{2\pi}{\omega} \), the steady state solution is periodic with period \( pt \). In general, however, the steady state solution is not periodic, but quasi periodic.

If \( u(\tau) \) is a constant function, the steady state solution is periodic with period \( T \). For example consider the network shown
in Fig.6-3.1, where in the first circuit mode the switch S is open, and in the second circuit mode S is closed. The period is \( T \). If \( e \) is constant and \( e=E \), the network has a periodic solution with period \( T \). If \( e=Ee^{j\omega t+\theta} \), and if \( mt=2\pi n/\omega \), then the network has a periodic solution with period \( mT \), otherwise it has no periodic solution but a quasi periodic solution.

The following method of analyzing such networks is simpler than the above. It is, however, unsuitable to study the problem on stability. Let \( x \) and \( u \) denote the state and the input vectors, respectively. In most cases, \( u \) is a solution of a linear differential equation. For example if \( u=a e^{j\omega t} \), \( u \) is a solution of \( \dot{u}=j\omega u \). Then we have a differential equation as

\[
\begin{bmatrix}
\dot{x} \\
\dot{u}
\end{bmatrix} = \begin{bmatrix} A, B \end{bmatrix} \begin{bmatrix} x \\
0, j\omega \end{bmatrix} u,
\]

\[
\dot{y} = \widetilde{A}y,
\]

where \( y = \begin{bmatrix} x \\
u \end{bmatrix} \), \( \widetilde{A} = \begin{bmatrix} A, B \\
0, j\omega \end{bmatrix} \).

The Eq. (6-3.23) can be considered a state equation without input functions. The solution of Eq. (6-3.23) is
\[ y(t) = e^{tA} y(0), \] (6-3.24)

where
\[
e^{tA} = \begin{cases} 
  e^{tA}, & e^{tA} - e^{j\omega_1 t} \begin{pmatrix} A - j\omega_1 \end{pmatrix}^{-1} \begin{pmatrix} e^{tA} - e^{j\omega_1 t} \end{pmatrix} B, \\
  0, & e^{j\omega_1 t} 
\end{cases} \] (6-3.25)

Consider a network containing periodically operated switches with two circuit modes, then we obtain

\[ y_1, n(t) = H_1(t) y_2, n-1(t_2) \] (6-3.26)
\[ y_2, n(t) = H_2(t) y_1, n(t_1), \] (6-3.27)

which correspond to Eq.(6-3.9) and (6-3.10), respectively. Then we obtain

\[ y_1, n(t_1) = D_1^{n-1} y_2, 1(t_1) \]
\[ y_2, n(t_2) = D_2^{n-1} y_2, 1(t_2), \]

which correspond to Eq.(6-3.11) and (6-3.12), respectively. Let us define \( D_1 \) and \( D_2 \) as

\[ D_1 e^{j\omega_n T} \Delta \lim_{n \to \infty} D_1^n, \quad D_2 e^{j\omega_n T} \Delta \lim_{n \to \infty} D_2^n. \]

then we obtain the steady state solution as

\[ \{ y_1, n(t) \}_s = H_1(t) D_2 y_2, 1(t_2) e^{j\omega_n T} \] (6-3.28)
\[ \{ y_2, n(t) \}_s = H_2(t) D_1 y_1, 1(t_1) e^{j\omega_n T}, \] (6-3.29)

which correspond to Eq.(6-3.15) and (6-3.16), respectively.

This method is essentially the same as the previous method. This method, however, makes it evident that networks have periodic or quasi periodic solutions. The variables \( y_{i,n} \) may be state variables of the \( i \)-th circuit mode, or may be voltages across all the capacitors and currents through all the inductors.
6-4 EXPLICIT DEDUCTION OF THE INITIAL VALUES OF THE SECOND KIND FROM THOSE OF THE FIRST

6-4.1 THE DERIVATIVES IN THE MATRIX B

In Section 6-3, the analysis of the circuits is studied on the assumption that there exists no derivative term in $B_i$ of Eq. (6-3.3) and (6-3.4). In general, however, there exist derivatives in $B_i$ as mentioned in Chapter 2, 3, 4 and 5, then at $t=0$, there exist impulsive terms $\left(\frac{du}{dt}\right)_{t=0}$. Here it is shown that the matrix $B$ without impulsive terms can be obtained by choosing initial values properly as follows.

In RLC networks there exists derivatives in $B_i$ when loops containing only capacitor- and voltage-source-branches (capacitor- and voltage-source-only loops) are produced by closing $C$-switches, or when cut-sets containing only inductor- and current-source-branches (inductor- and current-source-only cut-sets) are produced by $L$-switches.

Consider the case where capacitor- and voltage-source-only loops are produced at $t=0$ by closing some $C$-switches. At $t=+0$, the Kirchhoff's voltage law in the loops states

$$\left[1,F_{SC}\right] \begin{bmatrix} v_{S}^{t=0} \\ v_{C}^{t=0} \end{bmatrix} = e_S$$  \hspace{1cm} (6-4.1)

where the matrix $F_{SC}$ is the same as is used in Chapter 2.

Let $q$ denote a vector of electric charges associated with the elements as noted by the subscripts. Then we obtain

$$\left[-F_{SC}^t,1\right] \begin{bmatrix} q_{S}^{t=0} \\ q_{C}^{t=0} \end{bmatrix} = 0.$$  \hspace{1cm} (6-4.2)

@ The case, where the right hand term is nonzero, is considered in Section 6-7.
Since
\[
\begin{vmatrix}
  C_1, & 0 \\
  0, & C_2 \\
\end{vmatrix}
\begin{vmatrix}
  v_s^+ \\
  v_c^+ \\
\end{vmatrix} =
\begin{vmatrix}
  q_s^+ \\
  q_c^+ \\
\end{vmatrix},
\]  
(6-4.3)

the initial voltages of the second kind across the twig-capacitors are explicitly given as
\[
v_c^+ = C^{-1}_{SC} I^t_{C1eS}.
\]  
(6-4.4)

When inductor- and current-source-only cut-sets are produced by opening some \( L \)-switches, the initial currents of the second kind through the link-inductors are obtained as
\[
i_l^+ = -L^{-1}_{L1} I^t_{L2j}.
\]  
(6-4.5)

By choosing Eq. (6-4.4) and (6-4.5) as the initial values, the state equations without impulsive term in \( B \), can be obtained (i.e. the terms \( B \frac{dv}{dt} \mid t>0 \) can be replaced by the initial values and \( B \frac{dv}{dt} \mid t>0 \)

In RLCT networks, the vectors \( v_c^+ \) and \( i_l^+ \) are given as
\[
v_c^+ = C^{-1}_{SC} I^t_{C1eS},
\]  
(6-4.6)
\[
i_l^+ = -L^{-1}_{L1} I^t_{L2j}.
\]  
(6-4.7)

In RCG networks, the vector \( v_c^+ \) is given as
\[
v_c^+ = -A^t_{SC} I^t_{C1eC1} - C^{-1}_{SC} I^t_{C1eC1} F_{CL} G^{-1}_{CL} I^t_{C1eC1} F_{CL} R^{-1}_{CL} I^t_{C1eC1} - C^{-1}_{SC} I^t_{C1eC1} F_{CL} G^{-1}_{CL} I^t_{C1eC1} F_{CL} R^{-1}_{CL} I^t_{C1eC1}.
\]  
(6-4.8)

In active networks, the initial values cannot be obtained explicitly as those in passive networks. If the matrix \( B \) in Eq. (5-4.9) is zero, the equalities corresponding to Eq. (6-4.1) and (6-4.2) can be obtained as
\[
[-A, I] \begin{vmatrix}
  v_s^+ \\
  v_c^+ \\
\end{vmatrix} + \begin{vmatrix}
  \widetilde{e}_S \\
  \widetilde{j}_S \\
\end{vmatrix} = 0
\]  
(6-4.9)

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\[
\begin{align*}
[1,-F]\begin{pmatrix} q_{C}^{+0} \\ q_{S}^{+0} \end{pmatrix} &= 0, \\
\begin{pmatrix} q_{C}^{+0} \\ q_{S}^{+0} \end{pmatrix} &= 0,
\end{align*}
\]  \hspace{1cm} (6-4.10)

where the matrices A and F are the same as are used in Chapter 5.

Then we obtain

\[
v_{C}^{+0} = -(C_{C} - FC_{S}A)^{-1}F(e_{S} + j_{S}).
\]  \hspace{1cm} (6-4.11)

### 6-4.2 THE INITIAL VALUES OF THE FIRST AND THE SECOND KINDS

In Section 6-3, the networks are analyzed on the assumption that the initial values at a circuit mode can be obtained from the final values at the previous circuit mode, that is, the initial values of the second kind can be obtained those of the first. The methods for obtaining the initial values of the second kind from those of the first were provided by S. Hayashi\(^{[28]}\) and others\(^{[32]}\). However, they are not formulated. Here we derive a formula for RLC, RLCT and RCG networks.

**RLC networks**

When capacitor-only loops are produced by closing C-switches, the initial voltages of the second kind across the capacitors in the loops may be not equal to those of the first. Consider the network where the capacitor-only loops are produced at \(t=0\) by closing C-switches. Kirchhoff's voltage law states

\[
\begin{align*}
[1,F_{SC}]\begin{pmatrix} v_{C}^{+0} \\ v_{S}^{+0} \end{pmatrix} &= 0, \\
\begin{pmatrix} v_{C}^{+0} \\ v_{S}^{+0} \end{pmatrix} &= 0,
\end{align*}
\]  \hspace{1cm} (6-4.11)

where the matrix \(F_{SC}\) is the same as is used in Chapter 2.

Let \(q_{C}^{+0}\) and \(q_{S}^{+0}\) denote the vectors of the initial electric charges of the first and the second kinds, respectively. From the law of conservation of electric charges, we obtain
The relations between the vectors of electric charges and voltages across capacitors are given as

\[ C_1 q^+ = q_S^+ , \quad C_2 q^+ = q_C^+ \]
\[ C_1 q^- = q_S^- , \quad C_2 q^- = q_C^- \]

From Eq. (6-4.12), (6-4.13) and (6-4.14), the explicit relation between the initial voltages across the capacitors of the first and the second kinds are obtained as

\[ v_C^+ = -C^{-1} F_{SC} q_S^- + C^{-1} C_2 v_C^- \]

When inductor-only cut-sets are produced by opening \( L \)-switches, the initial currents of the second kind through inductors may not equal to those of the first. The initial currents of the second kind through inductors are derived from those of the first as

\[ i_L^+ = L^{-1} L_1 i_1^- + L^{-1} i_L L_2 i_2^- \]

RLCT networks

An RLCT network can be treated as an RLC network, as mentioned in Chapter 3, by regarding every matrix \( \hat{F} \) as the corresponding matrix \( F \). Therefore the initial values of the second kind are obtained from those of the first as

\[ v_C^+ = -C^{-1} \hat{F}_{SC} C_1 v_S^- + C^{-1} C_2 v_C^- \]
\[ i_L^+ = L^{-1} L_1 i_1^- + L^{-1} \hat{F}_{L} L_2 i_2^- \]

RCG network

An RCG network can be treated as an RLC network by regarding the matrix \( A_{SC} \) as \( F_{SC} \), therefore the initial voltages of the second kind across the capacitors are

\[ v_C^+ = -C^{-1} A_{SC} C_1 v_S^- + C^{-1} C_2 v_C^- \]
where the matrices are those used in Chapter 4.

**Active networks**

The initial values of the second kind cannot explicitly be obtained from those of the first unless the matrix \( B \) in Eq.(5-4.9) is zero. Consider the case of \( B=0 \). From Eq.(5-4.9), we obtain

\[
\begin{bmatrix} -A, 1 \end{bmatrix} \begin{bmatrix} v^+ \cr v^- \end{bmatrix} = 0 \tag{6-4.20}
\]

\[
\begin{bmatrix} 1, -F \end{bmatrix} \begin{bmatrix} q^+ \cr q^- \end{bmatrix} = \begin{bmatrix} 1, -F \end{bmatrix} \begin{bmatrix} q^- \cr q^- \end{bmatrix}, \tag{6-4.21}
\]

which correspond to Eq.(6-4.12) and (6-4.13), respectively. Then we obtain

\[
v^+_c = c^{-1}(C_Cv^+_c - FC_Sv^-_s), \tag{6-4.22}
\]

where \( C = C_C - FC_SA \).

Consider the case of \( B \neq 0 \). In the process of deducing state equations from network equations, Eq.(5-4.9) and (5-4.11), we obtain the equalities,

\[
-A_2v^+_c + v^-_s = 0 \quad \text{(in Eq.(5-4.16))}
\]

\[
-A_1v^+_s + v^-_s = 0 \quad \text{(in Eq.(5-4.26))} \tag{6-4.23}
\]

if the voltage- and the current-sources are neglected.

Let Eq.(6-4.23) be represented by one matrix equality as

\[
\tilde{A}v^+_c + v^-_s = 0. \tag{6-4.24}
\]

Then the initial voltages of the second kind are obtained, by replacing \( \tilde{A} \) with \( A \) in Eq.(6-4.22), as

\[
v^+_c = c^{-1}(C_Cv^+_c - FC_Sv^-_s), \tag{6-4.25}
\]

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where \( C = C_c - F \tilde{C} \). Note that the rows of \( \tilde{A} \) must be rearranged such that the link-capacitors corresponding to the columns of \( F \) are identical with those to the rows of \( A \).

6-5 TWO KINDS OF INITIAL VALUES AND APPROXIMATE SOLUTIONS OF NETWORKS

6-5.1 INTRODUCTION

As mentioned in Section 6-4, the initial values of the second kind are not always equal to those of the first, that is, the voltage across a capacitor or the current through an inductor may jump instantaneously from an initial value of the first kind to that of the second. From a topological point of view, the above phenomenon arises when capacitor-only loops or capacitor- and independent voltage source-only loops"inductor-only cut-sets or inductor- and independent current source-only cut-sets" are produced by closing \( C \)-switches"open \( L \)-switches" in RLC networks. A condenser"coil" used in an actual network is equivalent to a capacitor"inductor" with a stray series-resistor of a small value and with a stray parallel-resitor of a large value. Then in its corresponding graph, there exists a normal tree containing all the capacitor-branches but no inductor-branch. In such a network containing periodically operated switches, the state variables for all the circuit modes are the same. The initial values in a circuit mode are always equal to the final values in the previous circuit mode, that is, the initial values of the second kind are always equal to those of the first. However in order to simplify
the analysis, the above mentioned stray resistors connected in series and in parallel with capacitors and inductors are often neglected. In such networks, the initial values of the second kind are not always equal to those of the first. It is known that the stored energy in reactive elements decreases instantaneously at switch-operations.

Let us consider the networks $N$ and $N_e$ defined as

$N$: a network with stray series-resistor and with stray parallel-resistors

$N_e$: a network obtained from $N$ by contracting the stray series-resistors and deleting the stray parallel-resistors.

Let $A$ and $A_e$ denote the $A$-matrices in the state equations for $N$ and $N_e$, respectively. The matrix $A$ has very large eigen values due to the stray series-resistors $r$ with capacitors and the stray parallel-resistors $g$ with inductors. The matrix $A_e$ has no such eigen values. In general if a matrix $A$ has eigen values which are much larger than the others, it is difficult to obtain the numerical solution of $e^{tA}$ by the usual computations. On the contrary, it is not difficult to obtain that of $e^{tA_e}$. Then it is easier to obtain the solution for $N_e$ than for $N$. It is shown in the following sections that the solution for $N_e$ is an approximation for $N$.

In Section 6-5.2, the propositions for the analytical proof of the above point are posed.

In Section 6-5.3 and 4, the propositions in Section 6-5.2 are proved.
6-5.2 THE MATRICES A AND $A_e$

Let us pose the following propositions.

Proposition 6-5.1

The voltages across capacitors and the currents through inductors in $N$ at time $t \approx 0$ are nearly equal to those initial values of the second kind in $N_e$.

Proposition 6-5.2

The voltages across capacitors and the currents through inductors in $N$ at time $t \gg 0$ are nearly equal to those in $N_e$.

It is evident that if the propositions are proved the solution for $N_e$ is an approximation for $N$.

Let us derive the matrices $A$ and $A_e$ as described below.

The fundamental loop matrix of graph $G$ corresponding to $N$ for a normal tree is written by

$$
\begin{pmatrix}
(r) & (R) & (L) & (T) & (S) & (C) & (G) & (g) \\
(r) & 1, 0, 0, 0 & 1 & F_{r2} & 0 & 0 \\
(R) & 0, 1, 0, 0 & 0 & F_{R2} & F_{RG} & 0 \\
(L) & 0, 0, 1, 0 & 0 & F_{I2} & F_{LG} & F_{lg} \\
(T) & 0, 0, 0, 1 & 0 & 0 & 0 & -1
\end{pmatrix}
$$

(6-5.1)

where the first rows correspond to stray link-resistor-branches $r$ in series with capacitor-branches $S$. The last columns correspond to stray twig-resistor-branches $g$ in parallel with inductor-branches $G$.

Consider the graph $G_e$ corresponding to $N_e$. The graph $G_e$ is obtained from $G$ by contracting the stray series resistor-branches
and deleting the stray parallel ones. The fundamental loop matrix of \( G_e \) may be written by

\[
\begin{bmatrix}
1,0,0,F_{SC}, & 0 , 0 \\
0,1,0,F_{RC}, & F_{RG}, 0 \\
0,0,1,F_{LC}, & F_{LG}, F_{LR}
\end{bmatrix}
\]

(6-5.2)

Since there exists a normal tree such that \( S \) are links and \( \Gamma \) twigs, the relation (6-5.3) for the submatrices (6-5.1) and (6-5.2) can be established.

\[
F_{r2} = F_{SC}, \quad F_{R2} = F_{RC}, \quad F_{12} = F_{LC}, \quad F_{1G} = F_{LG}, \quad F_{1g} = F_{LR}.
\]

(6-5.3)

The matrix \( A_e \) is given as

\[
A_e = \begin{bmatrix}
c^{-1}, & 0 \\
0 , L^{-1} & -H', -z
\end{bmatrix}
\]

(6-5.4)

The matrices on the right hand side of (6-5.4) are the same as are used in Chapter 2.

Let \( r"g" \) denote the resistance matrix of the resistor-branches \( r" \) the conductance matrix of the resistor-branches \( g" \). Then the matrix \( A \) may be written as

\[
A = \begin{bmatrix}
c^{-1}, & 0 , 0 , 0 \\
0 , c^{-1}\!, & 0 , 0 & -r^{-1} & -r^{-1}F_{r2}
\end{bmatrix}
\]

\[
0 , 0 , L^{-1}, 0 \\
0 , 0 , 0 , L^{-1}_2
\]

\[
F_{12}r^{-1}, -(y+g^{-1}_F F_{1g} F_{1g}'), H \\
0 , 0 , 0 , L^{-1}_2
\]

(6-5.5)
where $C_1$ and $C_2$; $L_1$ and $L_2$" are the capacitance matrices"the inductance matrices" for the link- and the twig-capacitors"inductors", respectively, in $N_e$. The left upper and the right lower parts of $A$ are of large values.

6-5.3 THE PROOF OF PROPOSITION 6-5.1

Let us give the proof of Proposition 6-5.1 in Section 6-5.2.

The state equation of $N$ is given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (6-5.6)$$

where $x(t)$ is the vector whose elements are the voltages across the capacitors and the currents through the inductors.

The solution of Eq.$(6-5.6)$ may be given by

$$x(t) = e^{tA}x(+0) + \int_0^t e^{(t-\tau)A}Bu(\tau)\,d\tau. \quad (6-5.7)$$

The matrix $e^{tA}$ is given by a form of an infinite series as

$$e^{tA} = \sum_{n=0}^{\infty} \frac{1}{n!}(tA)^n. \quad (6-5.8)$$

The matrix $e^{tA}$ at $t=0$ is nearly equal to $1+tA$. This matrix has a form of

$$\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}$$

where the shaded portions are the submatrices for the stray resistors $r$ and $g$.

The submatrices not corresponding to the stray resistors in (6-5.1) are approximated by zero matrices. Then we obtain

$$1_{C} - F_{r}^{!} 1_{r} \approx 0, \quad (6-5.10)$$
where \( i \) is the current vector associated with the elements as denoted by the subscripts.

From (6-5.1), we obtain
\[
i_S - i_r = 0. \quad (6-5.11)
\]

From Eq. (6-5.10) and (6-5.11), we obtain
\[
i_C - F_{r2}^i i_S = 0, \quad \text{(for } t=0). \quad (6-5.12)
\]

Integrating Eq. (6-5.12) from \( t=-0 \) to \( t=+0 \), we obtain the following equation for the electric charges;
\[
q_C^0 - F_{r2}^i q_S^0 = q_C^+ - F_{r2}^i q_S^+. \quad (6-5.13)
\]

From Eq. (6-5.3), we see that Eq. (6-5.13) is identical with Eq. (6-4.13). Then the voltages across the capacitors at \( t=0 \) in \( N \) are nearly equal to the initial voltages of the second kind across the capacitors in \( N_e \).

Similarly it can be proved that the currents through the inductors at \( t=0 \) in \( N \) are nearly equal to the initial currents of the second kind through the inductors in \( N_e \).

Consequently a proof is given to Proposition 6-5.1.

6-5.4 A PROOF OF PROPOSITION 6-5.2

Let us define the matrix \( P \) as
\[
P = \begin{pmatrix}
(1+F_{r2}^i F_r^f)^{-1}, & (1+F_{r2}^i F_r^f)^{-1} F_{r2}^1, & 0, & 0 \\
-(1+F_{r2}^i F_r^f)^{-1} F_{r2}^1, & (1+F_{r2}^i F_r^f)^{-1}, & 0, & 0 \\
0, & 0, & (1+F_{1g}^i F_{1g}^f)^{-1}, & (1+F_{1g}^i F_{1g}^f)^{-1} F_{1g}^1 \\
0, & 0, & -(1+F_{1g}^i F_{1g}^f)^{-1} F_{1g}^1, & (1+F_{1g}^i F_{1g}^f)^{-1}
\end{pmatrix} \quad (6-5.14)
\]

The inverse matrix of \( P \) is given as

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The matrix $A$, (6-5.5) is similar to the following matrix,

\[
PAP^{-1} = \begin{pmatrix}
-(1+F'_{r2}F''_{r2})^{-1}\{(C^{-1}_{r2}+F'_{r2}C^{-1}_{r2}F'_{r2})P'A_{r2}\}^{-1}((1+F_{r2}C^{-1}_{r2})+F'_{r2}C^{-1}_{r2}yF'_{r2}), \\
(1+F'_{r2}F''_{r2})^{-1}\{(F'_{r2}C^{-1}_{r2}-C^{-1}_{r2}F'_{r2})P'A_{r2}\}^{-1}((1+F_{r2}C^{-1}_{r2})-C^{-1}_{r2}yF'_{r2}) \\
-(1+F'_{lg}F'_{lg})^{-1}L^{-1}_{r2}HF'_{r2} \\
(1+F'_{lg}F'_{lg})^{-1}F'_{lg}L^{-1}_{r2}HF'_{r2}
\end{pmatrix}
\]

Let $\tilde{A}$ denote the matrix obtained by omitting the underlined terms in (6-5.6). The matrix $A$ is nearly equal to $PAP^{-1}$ since the values of $r^{-1}$ and $g^{-1}$ are much larger than those of the other terms in $PAP^{-1}$. The matrix $QAQ^{-1}$ in (6-5.17) is similar to $\tilde{A}$.

\[
QAQ^{-1} = \begin{pmatrix}
A_{11}, A_{12}, A_{13}, A_{14}, \\
A_{21}, A_{22}, A_{23}, A_{24}, \\
A_{31}, A_{32}, A_{33}, A_{34}, \\
A_{41}, A_{42}, A_{43}, A_{44}
\end{pmatrix}
\]
where \( A_{ij} \) are shown in Appendix VII and the matrix \( Q \), which is determined by the network-topology and network-element values, is

\[
Q = \begin{pmatrix}
1 & 0 \\
(1 + F_{12} F_{21})^{-1} (1 + F_{21}^{-1} C_{1} F_{12}^{-1} C_{2})^{-1} (1 + F_{12} F_{21})^{-1} & 1 \\
0 & 0 \\
0 & 0 \\
0, & 0 \\
1 & (1 + F_{1g} F_{g1})^{-1} (L_{1}^{-1} F_{1g} L_{1}^{-1})^{-1} (1 + F_{1g} F_{g1}) & 1
\end{pmatrix}
\]

The submatrices \( A_{ij} \), except \( A_{11} \) and \( A_{44} \), do not contain \( r^{-1} \) and \( g^{-1} \). Therefore the matrix \( QAQ^{-1} \) is nearly equal to a matrix such as

\[
QAQ^{-1} = \begin{pmatrix}
A_{11}, 0, 0, 0 \\
0, A_{22}, A_{23}, 0 \\
0, A_{32}, A_{33}, 0 \\
0, 0, 0, A_{44}
\end{pmatrix}
\]

Then the eigenvalues of \( A \) are nearly equal to those of \( A_{11}, A_{44} \) and \( |A_{22}, A_{23}| \). The eigenvalues of \( A_{11} \) and \( A_{44} \) are very large.

Now we show the following equality,

\[
\begin{pmatrix}
A_{22}, A_{23} \\
A_{32}, A_{33}
\end{pmatrix} = \begin{pmatrix}
C^{-1}, 0 \\
0, L^{-1}
\end{pmatrix} \begin{pmatrix}
y, H \\
-H, -z
\end{pmatrix}
\]

From Lemma A-3-3, we obtain

\[
(C_{1}^{-1} + F_{r2} C_{2}^{-1} F_{r2})^{-1} = C_{1}^{-1} C_{2} F_{r2} (C_{2} + F_{r2} C_{1} F_{r2})^{-1} F_{r2} C_{1}.
\]

Eq. (6-5.3) leads to
Similarly we obtain
\[(L_1^{-1}+F'_1 F_1^{-1})^{-1} = L_1^{-1} - L_1 F_1^{-1} L_1^{-1} F_1 L_1^{-1}.\]  \hspace{1cm} (6-5.23)

From Eq. (6-5.22) and (6-5.23), we obtain
\[ (1+F'_2 F_2^{-1})^{-1} \left\{ (F'_2 F_2^{-1} - F'_1 F_1^{-1}) (C_1^{-1}+F'_2 F_2^{-1})^{-1} F'_2 C_2^{-1} + C_2^{-1} \right\} \]
\[ = C_2^{-1}, \quad (6-5.24) \]
\[ (1+F'_1 F_1^{-1})^{-1} \left\{ (L_1^{-1} F_1^{-1} L_1^{-1}) (L_1^{-1}+F'_1 F_1^{-1})^{-1} F'_1 L_1^{-1} - L_1^{-1} \right\} \]
\[ = L_1^{-1}. \quad (6-5.25) \]

Therefore Eq. (6-5.20) holds. Then this completes the proof of Proposition 6-5.2.

6-5.4 EXAMPLE

Consider a network shown in Fig. 6-5.1. Opening the resistor g, and shorting the resistor r, we obtain a network \( N_e \). There is no capacitor-only loop or inductor-only cut-set in \( G \) corresponding to \( N \). There are a capacitor-only loop and an inductor-only cut-set in \( G_e \) corresponding to \( N_e \).

The eigenvalues of \( A_e \) and \( A \) for several values of the resistor \( r \) and \( g \) are shown in Table 6-5.1. Those eigenvalues are obtained by the method of Danilevsky.

The voltage across \( C_2 \) and the current through \( L_1 \) for time \( t \) are computed for the initial values shown in Table 6-5.2.

The results are shown in Fig. 6-5.2 - 6-5.5.

Table 6-5.1 shows that, in Case (1) and (2), the eigenvalues of \( A \) except those with large negative values are nearly equal to
Table 6-5.1  Eigen values of $A_0$ and $A$

<table>
<thead>
<tr>
<th>$N_e$</th>
<th>Case (1)</th>
<th>Case (2)</th>
<th>Case (3)</th>
<th>Case (4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.1216325</td>
<td>-0.1216239</td>
<td>-0.1215665</td>
<td>-0.1213041</td>
<td>-0.1209806</td>
</tr>
<tr>
<td>-4.692572</td>
<td>-4.692756</td>
<td>-4.694653</td>
<td>-4.707070</td>
<td>-4.733613</td>
</tr>
<tr>
<td>-0.03248342</td>
<td>-0.03248453</td>
<td>-0.03246332</td>
<td>-0.03238303</td>
<td>-0.03228288</td>
</tr>
<tr>
<td>-1.400931</td>
<td>-1.397356</td>
<td>-1.365844</td>
<td>-1.239227</td>
<td>-1.107467</td>
</tr>
<tr>
<td>-163.333</td>
<td>-163.3304</td>
<td>-36.64878</td>
<td>-18.28363</td>
<td></td>
</tr>
<tr>
<td>-3509.223</td>
<td>-359.2551</td>
<td>-79.3641</td>
<td>-44.5202</td>
<td></td>
</tr>
</tbody>
</table>

$r$ (ohm)   | $1 \times 10^{-3}$ | $1 \times 10^{-2}$ | $5 \times 10^{-2}$ | $1 \times 10^{-1}$  
$g$ (mho)    | $1 \times 10^{-3}$ | $1 \times 10^{-2}$ | $5 \times 10^{-2}$ | $1 \times 10^{-1}$  

Table 6-5.2  Initial values

<table>
<thead>
<tr>
<th></th>
<th>$V_{C_1}$</th>
<th>$V_{C_2}$</th>
<th>$V_{C_3}$</th>
<th>$I_{L_1}$</th>
<th>$I_{L_2}$</th>
<th>$I_{L_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t=-0$</td>
<td>-5.0 V</td>
<td>0.0 V</td>
<td>10.0 V</td>
<td>0.0 A</td>
<td>10.0 A</td>
<td>2.0 V</td>
</tr>
<tr>
<td>$t=+0$</td>
<td>35/11 V</td>
<td>-45/11 V</td>
<td>80/11 V</td>
<td>-12/7 A</td>
<td>22/7 A</td>
<td>-10/7 A</td>
</tr>
</tbody>
</table>

The values for $t=+0$ are computed from those for $t=-0$.
They are the initial values of the second kind in $N_e$. 

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those of \( A_e \). The voltages and the currents in \( N \) agree precisely with those in \( N_e \) except small \( t \), as shown in Fig.6-5.2 - 6-5.5. In case (3) and (4), however, the eigenvalues of \( A \) and those of \( A_e \) do not agree. The voltages and the currents in \( N \) are different from those in \( N_e \).

From the above discussions, it is concluded that if \( r \) and \( g \) are small, the voltages and the currents in the network \( N_e \) are nearly equal to those in \( N \) for all \( t \) except small \( t \). The voltages and the currents in \( N \) approach from the initial values of the first kind to those of the second kind in \( N_e \) in a very short
period of time. After that time, the network \( N_e \) gives an approximate solution to \( N \).

6-6 STABILITY

6-6.1 INTRODUCTION

In Section 6-3, on the assumption that \( D_{i}^n \to 0 \) as \( n \to \infty \), steady state solutions of linear networks containing periodically operated switches are obtained. Whether \( D_{i}^n \to 0 \) or not as \( n \to \infty \), is called the problem on stability. Only S. Hayashi [28] has discussed this problem. His approach of this problem, however, is the examination of the eigen values of \( D_i \). Here without examining eigenvalues, we get the conclusion that passive linear networks containing periodically operated switches always have steady state solutions.

6-6.2 PROPERTIES OF \( D_i \)

Property 6-6.1

Every eigen value of \( D_i \) is equal to that of \( D_j \) \( \{i \neq j\} \).

Proof: From Section 6-3, the matrices \( D_i \) and \( D_{i+1} \) are:

\[
D_i = H_i(t_i)H_{i+1}(t_{i+1}) \cdots \cdots H_{i-1}(t_{i-1})
\]

\[
D_{i+1} = H_{i+1}(t_{i+1}) \cdots \cdots H_i(t_i).
\]

Then the matrices \( D_i \) and \( D_{i+1} \) are written as

\[
D_i = H_i(t_i) \hat{H_i}
\]

\[
D_{i+1} = H_{i+1}(t_{i+1}) \cdots \cdots H_n(t_n)H_1(t_1) \cdots \cdots H_{i-1}(t_{i-1}).
\]

From Lemma A-3.4, the characteristic equation for \( D_i \) is the
same as that for $D_{i+1}$. Then the eigen values of $D_i$ are equal to those of $D_{i+1}$. Similarly the eigen values of $D_{i+1}$ are equal to those of $D_{i+2}$. By induction, the eigen values of $D_i$ are equal to those of $D_j$ ($j=1, \ldots, n. j \neq i$).

6-6.3 STABILITY OF A PASSIVE NETWORK SATISFYING ASSUMPTION 6-6.1

Now let us consider a passive network satisfying:

Assumption 6-6.1

A set of state variables for one circuit mode is identical with that of every other circuit mode.

It can be assumed that such a network has two circuit mode. The state equations of the first and the second circuit modes are, respectively,

$$\dot{x}_1 = A_1 x_1 + B_1 u_1, \quad (6-6.5)$$
$$\dot{x}_2 = A_2 x_2 + B_2 u_2. \quad (6-6.6)$$

From Assumption 6-6.1, the matrices $D_1$ and $D_2$ are written as

$$D_1 = e^{t_1 A_1} e^{t_2 A_2}, \quad D_2 = e^{t_2 A_2} e^{t_1 A_1}, \quad (6-6.7)$$

where $t_1$ and $t_2$ are periods of the first and the second circuit modes, respectively.

Since the network considered is passive, the real parts of all eigen values of the matrices $A_1$ and $A_2$ are nonpositive.

In order to examine whether $D^N_i \rightarrow 0$ as $n \rightarrow \infty$, consider matric differential equations defined by

$$\dot{x}_i = A_i x_i \quad \{i=1, 2\}. \quad (6-6.8)$$

The solutions of Eq. (6-6.8) are

$$\dot{x}_i(t) = e^{tA_i}x(0) \quad \{i=1, 2\}. \quad (6-6.9)$$
Let us define a function;

$$\Delta = x'(0)x(0) - x'(t)x(t).$$  \hspace{1cm} (6-6.10)

From Eq. (6-6.9), we get

$$\Delta = x'(0)[1 - (e^{tA_{i}})^{t}e^{tA_{i}}]x(0).$$  \hspace{1cm} (6-6.11)

For a linear passive network, the matrix $1 - (e^{tA_{i}})^{t}e^{tA_{i}}$ is positive semi-definite or positive definite.

Assume that the matrix $1 - (e^{tA_{i}})^{t}e^{tA_{i}}$ is positive definite.

From Lemma A-3.5, there exists a real nonsingular matrix $P_{i}$ such that

$$P_{i}P_{i}^{t} = 1 - (e^{tA_{i}})^{t}e^{tA_{i}}.$$  \hspace{1cm} (6-6.12)

Consider a function defined by

$$x(t+T) = e^{t_{1}A_{1}}e^{t_{2}A_{2}}x(t),$$  \hspace{1cm} (6-6.13)

where $T = t_{1} + t_{2}$,

and a function defined by

$$\tilde{\Delta} = x'(t+T)x(t+T) - x'(t)x(t).$$  \hspace{1cm} (6-6.14)

If $\Delta$ is negative, a system described by (6-6.13) is asymptotic stable [34].

Substituting Eq. (6-6.13) into Eq. (6-6.14), we obtain

$$\tilde{\Delta} = x'(t)((e^{t_{2}A_{2}})^{t}(e^{t_{1}A_{1}})^{t}e^{t_{1}A_{1}}e^{t_{2}A_{2}} - 1)x(t).$$  \hspace{1cm} (6-6.15)

From the matrix (6-6.12), Eq. (6-6.15) becomes

$$\tilde{\Delta} = -x'(t)[P_{2}P_{2}^{t} + Q^{t}Q]x(t),$$  \hspace{1cm} (6-6.16)

where $Q = P_{2}e^{t_{2}A_{2}}$.

Since the matrices $P_{2}P_{2}^{t}$ and $Q^{t}Q$ are positive definite, $\Delta$ is negative.

If the matrix $1 - (e^{tA_{i}})^{t}e^{tA_{i}}$ is positive semi-definite, then, from Property 2-3.4 in Chapter 2, the graph $G_{i}$ (corresponding to
the \( i \)-th circuit mode) contains capacitor-only cut-set and/or inductor-only loop, or the graph \( G_i \) may have connected-components containing only capacitor- and/or inductor-branches.

If both the matrices \( 1-(e^{tA_i})'e^{tA_i} \) are positive semi-definite, there exist capacitor-only cut-sets and/or inductor-only loops in both \( G_1 \) and \( G_2 \). Note that we have assumed that both graphs \( G_1 \) and \( G_2 \) cannot have a non-separable component containing capacitor- and/or inductor-branches only.

In case when there exists a capacitor-only cut-set or an inductor-only loop in both \( G_1 \) and \( G_2 \), \( G_1 \) and \( G_2 \) are converted to equivalent graphs containing no capacitor-only cut-set and no inductor-only loop by the star-delta transformation as shown in Fig.6-6.1. Then the matrices \( 1-(e^{tA_i})'e^{tA_i} \) are positive definite.

When at least one of the matrices \( 1-(e^{tA_i})'e^{tA_i} \) is positive semi-definite, it can be assumed without loss of generality that the matrix \( 1-(e^{tA_2})'e^{tA_2} \) is positive definite. Then there exists a real nonsingular matrix \( P_2 \) such that

\[
\begin{align*}
  y = P_2 x & \quad \text{for all } x
  \end{align*}
\]

Fig.6-6.1
Example of star-delta transformation
\[ P_2^2 = (e^{t_2 A_2})^t e^{t_2 A_2}. \]  

(6-6.17)

From Lemma A-3.6, we obtain a real matrix \( P_1 \) such that

\[ P_1^2 = (e^{t_1 A_1})^t e^{t_1 A_1}. \]  

(6-6.18)

In Eq. (6-6.16), the matrix \( Q \) may singular. Since \( P_2^2 \) is positive definite, however, \( \bar{\alpha} \) is negative.

Consequently we obtain

Theorem 6-6.1

The linear passive networks containing periodically operated switches which satisfy Assumption 6-6.1 possess steady state solutions.

6-6.4 STABILITY OF LINEAR PASSIVE NETWORKS CONTAINING PERIODICALLY OPERATED SWITCHES

In Section 6-6.3 is studied the stability of linear passive networks containing periodically operated switches which satisfy Assumption 6-6.1. Now let us consider the stability of the networks which do not satisfy Assumption 6-6.1. In such networks, the final values of a circuit mode may not be always equal to the initial values of the following circuit mode. If the total energy stored in all the reactive elements does not increase at the switch-operations, the network is stable since the network considered is passive. Let us considered the energy variations at the switch-operations.

As mentioned in Section 6-4, when \( C \)-switches are closed"L-switches are opened", the voltages across the capacitors"the currents through inductors" may vary.
Since the stability does not depend on the B-matrix of the state equations, networks where the voltage sources are contracted and the current sources are deleted in the original networks are considered here.

Let us consider the difference of the energy stored in capacitors when C-switches are closed. Let $P_0$ and $P_N$ denote the energy stored just before and just after closing C-switches, respectively. Then we obtain

$$P_0 = \frac{1}{2} \left( v_1^0, C_1 v_1, 0 + v_2^0, C_2 v_2, 0 \right)^T,$$

$$P_N = \frac{1}{2} \left( v_1^N, C_1 v_1, N + v_2^N, C_2 v_2, N \right)^T,$$

where the subscripts $1$ and $2$ denote link- and twig-capacitors, respectively.

A vector $v_0$ in case of an RLC network is defined as

$$v_0 = [1, F_{SC}]^T v_1,$$

where $[1, F_{SC}]$ is the matrix used in Chapter 2.

From (6-6.21), the difference of the energy stored in the capacitors is written as

$$P_0 - P_N = \frac{1}{2} v_0^T (C_1 - C_1 F_{SC} C_2^{-1} F_{SC}^T C_1) v_0.$$  (6-6.22)

From Lemma A-3.3, Eq. (6-6.22) becomes

$$P_0 - P_N = \frac{1}{2} v_0^T (C_1^{-1} + F_{SC} C_2^{-1} F_{SC}^T) v_0.$$  (6-6.23)

Since the matrix $C_1^{-1} + F_{SC} C_2^{-1} F_{SC}^T$ is positive definite, the total energy stored in the capacitors does not increase at closing C-switches.

Similarly it is proved that the total energy stored in the inductors does not increase at opening L-switches.

Consequently linear RLC networks containing periodically
operated switches have steady state solutions.

In case of linear RLCT networks containing periodically operated switches, it can be proved, by replacing $F_{SC}$ by $\tilde{F}_{SC}$ in Eq.(6-6.21)-(6-6.23), that they have steady state solutions.

In case of linear RCG networks containing periodically operated switches, it can be proved, by replacing $F_{SC}$ by $A_{SC}$ in Eq.(6-6.21)-(6-6.23), that they have steady state solutions.

Consequently we obtain

Theorem 6-6.2

Linear passive networks containing periodically operated switches have steady state solutions.

6-6.5 STABILITY OF ACTIVE NETWORKS CONTAINING PERIODICALLY OPERATED SWITCHES

In the preceding sections, is studied stability of linear networks containing periodically operated switches where all the circuit modes are stable. Here we considered linear networks containing periodically operated switches, where the $A$-matrices of state equations for some circuit modes have eigen values whose real parts are positive and those for the other modes have no such eigen values. It is examined whether such networks are stable or not.

For this consideration, it is convenient to derive the Jordan canonical forms of the matrix $D_i$ as

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\[ D_i = T^{-1} \begin{bmatrix} J_1, 0, \cdots, 0 & 1 \\ 0, J_2, \cdots, 0 \\ \cdots & \cdots \\ 0 & 0, \cdots, J_s \end{bmatrix} \]  \tag{6-6.24}

where \( J_k \) is a Jordan block as

\[ J_k = \begin{bmatrix} \lambda_k, 1 & 0 \\ 0, \lambda_k, 1 \\ \cdots \\ 0 & \cdots, 1 \\ & & \lambda_k \end{bmatrix} \]  \tag{6-6.25}

The dimension of the block \( J_k \) is assumed to be \( d_k \), and \( \lambda_k \) is an eigenvalue of \( D_i \). From Eq. \((6-6.24)\), \( D_i^n \) is written as

\[ D_i^n = T^{-1} \begin{bmatrix} J_1^n, 0 \\ J_2^n \\ \cdots \\ 0 & \cdots, J_s^n \end{bmatrix} \]  \tag{6-6.26}

where

\[ J_k^n = \begin{bmatrix} \lambda_k^n, n \lambda_k^{n-1}, \cdots, \lambda_k^n, 1 \\ 0 & \cdots, 1 \\ \cdots & \cdots & \cdots \\ \lambda_k \end{bmatrix} \]  \tag{6-6.27}

From Eq. \((6-6.24)-(6-6.27)\), it is concluded that

1) If the absolute value of \( \lambda_k \) is less than 1 for all \( k \), then \( J_k^n \to 0 \) as \( n \to \infty \), and \( D_i^n + 0 \) as \( n \to \infty \).

2) If there exists at least one eigenvalue such that \( |\lambda_k| > 1 \), \( J_k^n \) does not converge. Then the network is unstable.
3) If there exist some eigen values $\lambda_\ell$ such that $|\lambda_\ell|=1$, and if all other absolute values of $|\lambda_k|$ for $k \neq \ell$ are less than 1, we must consider two cases as follows:

a) If $d_\ell=1$ for all $\ell$, the matrix $D_\ell$ does not diverge.

b) If $d_\ell \neq 1$ for some $\ell$, the matrix $D_\ell$ diverges. Then the network is unstable.

The stability of such networks is a complex problem. Consider the stability of a simple example shown in Fig.6-6.2.

Let $A_1$ and $A_2$ denote the $A$-matrices of the state equations for the first and the second circuit modes, respectively. In the state equations, the state variables are the voltage across the capacitor and the current through the inductor. The matrices $A_1$ and $A_2$ are written, respectively, as

$$A = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix}. \quad (6-6.28)$$

**Fig.6-6.2 Example**

(a) Original network, where $v_{\alpha 1} = 2 R \ell_{\alpha 1}$

(b) The first circuit mode.

(c) The second circuit mode.
Fig. 6-6.3 Stable regions of the network shown in Fig. 6-6.2

<table>
<thead>
<tr>
<th></th>
<th>L (H)</th>
<th>C (F)</th>
<th>R (Ω)</th>
<th>τ</th>
<th>S</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0.01</td>
<td>0.05</td>
<td>2.0</td>
<td>0.01</td>
<td>100</td>
<td>0.0628</td>
</tr>
<tr>
<td>(b)</td>
<td>0.005</td>
<td>0.001</td>
<td>0.5</td>
<td>0.02</td>
<td>444.4</td>
<td>0.014</td>
</tr>
<tr>
<td>(c)</td>
<td>0.01</td>
<td>0.005</td>
<td>0.2</td>
<td>0.1</td>
<td>147</td>
<td>0.044</td>
</tr>
<tr>
<td>(d)</td>
<td>0.005</td>
<td>0.001</td>
<td>0.3</td>
<td>0.033</td>
<td>446.2</td>
<td>0.014</td>
</tr>
<tr>
<td>(e)</td>
<td>0.005</td>
<td>0.005</td>
<td>0.2</td>
<td>0.1</td>
<td>199.7</td>
<td>0.031</td>
</tr>
</tbody>
</table>
The network for the first circuit mode has no steady state solution, but that for the second one has a steady state solution.

The eigen values of $A_1$ are
\[ \lambda_1 = \frac{1}{\tau} \pm j\beta, \quad (6-6.29) \]
where \( \frac{1}{\tau} = R/2L = \sqrt{1/2C - 1/\tau^2} \).

The eigen values of $A_2$ are
\[ \lambda_2 = \frac{1}{\tau} \pm j\beta. \quad (6-6.30) \]

The graphs in Fig.6-6.3 show the stable regions in several values of the network-elements. It may be expected that the network will be stable for $t_1 < t_2$ since the networks for the two circuit modes are the same except the sign of the resistance. But from Fig.6-6.3, we see the network is not always stable for $t_1 < t_2$.

We investigate the stable regions in Fig.6-6.3 in detail.

The matrix $D_1$ is written as
\[
D_1 = e^{t_1 A_1} e^{t_2 A_2}
\]
\[ = e^{(t_1-t_2)/\tau} \begin{vmatrix} r_{11}, r_{12} \\ r_{21}, r_{22} \end{vmatrix} \]
\[ = a \begin{vmatrix} r_{11}, r_{12} \\ r_{21}, r_{22} \end{vmatrix}, \quad (6-6.31) \]

where
\[
a = e^{(t_1-t_2)/\tau}
\]
\[
r_{11} = \cos\beta_1 \cos\beta_2 \frac{1}{\beta_2} \sin\beta_1 \cos\beta_2 - \cos\beta_1 \sin\beta_2 \cdot (1 + \frac{2}{\beta_2^2}) \sin\beta_1 \times \sin\beta_2
\]
\[
r_{12} = \frac{1}{C} \cos\beta_1 \sin\beta_2 \cdot \frac{1}{C} \sin t_1 \sin t_2 + \frac{1}{C} \sin\beta_1 \cos\beta_2
\]

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The characteristic equation of \( D_1 \) is
\[
f(\lambda) = \lambda^2 - 2a(\cos \beta_1 \cos \beta_2 - (1+\frac{2}{\beta_1^2})\sin \beta_1 \sin \beta_2)\lambda + a^2 = 0, \quad (6.6.32)
\]
and its discriminant (denoted by \( D^D \)) is
\[
D^D/4 = a(\cos \beta_1 \cos \beta_2 - (1+\frac{2}{\beta_1^2})\sin \beta_1 \sin \beta_2)^2 - a^2. \quad (6.6.33)
\]
Let \( \lambda^D \) be an eigen value of \( D_1 \). If \( D^D < 0 \), the absolute value of \( \lambda^D \) can be written as
\[
|\lambda^D| = e^{(t_1-t_2)/\tau}, \quad (6.6.34)
\]
than the network is stable for \( t_1 < t_2 \).

Consider the case of \( D^D > 0 \). Since \( f(0) > 0 \), the conditions for
\[
|\lambda^D| < 1
\]
are
\[
i) \quad f(1) > 0 \quad \text{and} \quad 0 < a(\cos \beta_1 \cos \beta_2 - (1+\frac{2}{\beta_1^2})\sin \beta_1 \sin \beta_2) < 1
\]
or
\[
ii) \quad f(-1) > 0 \quad \text{and} \quad -1 < a(\cos \beta_1 \cos \beta_2 - (1+\frac{2}{\beta_1^2})\sin \beta_1 \sin \beta_2) < 0.
\]
Note that if the condition i) or ii) holds for \( t_1 > t_2 \), then \( D^D \)
must be negative.

The condition i) always hold in \( D^D > 0 \) (See Appendix VIII).

Next consider the condition ii). The shaded portions in Fig.6-6.4 show the domain of \( D^D > 0 \) and \( t_1 < t_2 \). For the region denoted by \( A \),
\[
\cos \beta_1 \cos \beta_2 - (1+\frac{2}{\beta_1^2})\sin \beta_1 \sin \beta_2 < -1. \quad (6.6.35)
\]
For the region denoted by \( B \),
\[
\cos \beta_1 \cos \beta_2 - (1+\frac{2}{\beta_1^2})\sin \beta_1 \sin \beta_2 > 1. \quad (6.6.36)
\]
The intersection-points of two curves corresponding to \( t_1 = t_2 \) and

\[
\cos \beta t_1 \cos \beta t_2 - (1 + \frac{2}{\beta^2 \tau^2}) \sin \beta t_1 \times \sin \beta t_2 = -1 \quad (6-6.37)
\]

are given by

\[
t_1 = t_2 = \frac{1}{\beta^2} \sin^{-1} \sqrt{\frac{2 \tau^2}{1 + \beta^2 \tau^2}}.
\]

(6-6.38)

The intersection-points of the two curves corresponding to \( t_1 = t_2 \) and \( f(-1) = 0 \) are given by (6-6.38). The intersection-points of the two curves corresponding to \( t_1 = t_2 \) and

\[
a \{ \cos \beta t_1 \cos \beta t_2 - (1 + \frac{2}{\beta^2 \tau^2}) \sin \beta t_1 \sin \beta t_2 \} = -1
\]

are given by (6-6.39). The intersection-points of the curves corresponding to Eq. (6-6.37), (6-6.39) and \( f(-1) = 0 \) are all on the line \( t_1 = t_2 \). The domain of \( t_1 < t_2 \) and

\[-1 < a \{ \cos \beta t_1 \cos \beta t_2 - (1 + \frac{2}{\beta^2 \tau^2}) \sin \beta t_1 \sin \beta t_2 \} < 0
\]

is in the region of \( t_1 < t_2 \) and \( f(-1) > 0 \). Therefore the stable regions are as shown in Fig. 6-6.3.

It is studied on the stability of a network, where the state equations of some circuit modes have steady state solutions, but those of others have none. It is concluded that the stability cannot be determined by the eigen values of the circuit modes only. It is neccessary to find the eigen values of the matrices \( D_t \). No simple way to investigate the stability is found.
6-7 CONCLUSION

Some considerations on networks containing periodically operated switches is studied from a topological point of view. In Section 6-2, it is shown that switch-operations vary the network topology. In networks satisfying some restrictions, it is easy to obtain fundamental loop and fundamental cut-set matrices without knowing the network-topology varied by switch-operations. A necessary and sufficient condition for networks with two switches to satisfy the restrictions, is obtained. A heuristic method for networks with more than two switches to satisfy the restrictions, is proposed.

In Section 6-3, the steady state solutions are obtained.

In Section 6-4, is given an explicit formula for obtaining initial values of the second kind from those of the first in RLC, RLCT and RCG networks. For active networks, the determining the initial values are very complex, but a systematic procedure for obtaining them is given. It is shown that state equations without derivatives of inputs can be obtained by choosing proper initial values. In Section 6-4.1, the initial values of the second kind are obtained on the condition that those of the fist are zero. If those of the first are nonzero, those of the second are represented by the sum of those obtained in Section 6-4.1 and 6-4.2.

In Section 6-5, it is shown that there are peculiar phenomena in the networks where initial values of the first kind are not equal to those of the second. It is shown that a complex network, whose solution is difficult to obtain, is approximated by a less complex network, in which the initial values of the first kind
are not equal to those of the second.

In Section 6-6, it is shown that linear passive networks containing periodically operated switches have always steady state solutions. The stability of active networks is also discussed.
Definition A-1.1

A matrix pencil $A+\lambda B$ is called regular if

1) $A$ and $B$ are square matrices of the same order $n$, and

2) The determinant $|A+\lambda B|$ does not vanish identically. In all other cases, the pencil is called singular.

Definition A-1.2

Two pencils of rectangular matrices $A+\lambda B$ and $A_1+\lambda B_1$ of the same dimension $m\times n$ connected by the equation,

$$P(A+\lambda B)Q=A_1+\lambda B_1,$$

where $P$ and $Q$ are constant square nonsingular matrices of order $m$ and $n$, respectively, are called strictly equivalent.

Theorem A-1.1

A singular pencil $A+\lambda B$ of dimension $m\times n$ is strictly equivalent to

$$\begin{bmatrix} L_\epsilon & 0 \\ 0 & A+\lambda B \end{bmatrix},$$

(A-1.1)

where

$$L_\epsilon = \begin{bmatrix} \epsilon+1 \\ \lambda, I \\ \vdots \\ \lambda, I \end{bmatrix}$$

if
\[ M_\varepsilon = \begin{bmatrix} A & B, & \varepsilon +1 \\ B, & A & 0 \\ & B & \varepsilon \\ 0 & & A \\ & B & \varepsilon +1 \end{bmatrix} \]

is of rank \( \rho_\varepsilon = (\varepsilon +1)n \), and

\[ M_\alpha = \begin{bmatrix} A & 0 & \alpha +1 \\ B & \alpha & \varepsilon +1 \\ 0 & A & \varepsilon +1 \end{bmatrix} \]

is of rank \( \rho_\alpha = (\alpha +1)n \) for \( 0 \leq \alpha \leq \varepsilon \).

**Theorem A-1.2**

A pencil is strictly equivalent to

\[ \{ h \Theta_{\varepsilon +1}, \cdots, L_{\varepsilon p} \eta_{\varepsilon +1}, \cdots, L_{\eta q} \eta_{\varepsilon +1}, A_\theta + \lambda B_\theta \}, \]

where

\[ L_\varepsilon = \begin{bmatrix} \lambda, & 1 & 0 \\ \lambda, & 1 & 0 \\ \lambda, & 1 & 0 \\ \lambda, & 1 & 0 \end{bmatrix}, \quad L_{\eta} = \begin{bmatrix} \lambda, & 0 \\ \lambda, & 0 \\ \lambda, & 0 \\ \lambda, & 0 \end{bmatrix} \]

\( \lambda_\theta + \lambda B_\theta \) is a regular pencil.

**Theorem A-1.3**

A regular pencil \( A + \lambda B \) can be reduced to a (strictly equivalent) canonical quasi-diagonal form

\[ \{ N(\mu_1), \cdots, N(\mu_n), J + \lambda I \}, \]

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where

\[ N(u_i) = \begin{pmatrix}
I & \lambda & 0 \\
I & \lambda & \ddots & u_i \\
\ddots & \ddots & \ddots \\
0 & \ddots & \lambda & 1
\end{pmatrix}
\begin{pmatrix}
u_i \\
\end{pmatrix}
\]

and J is a Jordan matrix.
APPENDIX II APPLICATION OF PENCIL

The results in Appendix I are applied to a system of \( m \) linear differential equations of the first order in \( n \) unknown functions with constant coefficients:

\[
Ax + B \frac{d}{dt} x = f(t), \quad (A-2.1)
\]

where \( A \) and \( B \) are \( m \times n \) matrices, and \( x \) and \( f(t) \) are vectors of dimension \( n \).

We introduce a new unknown vector \( z \) that is connected with the old \( x \) by a linear nonsingular transformation;

\[
x = Qz, \quad (A-2.2)
\]

By substituting \( Qz \) for \( x \) in Eq. (A-2.1) and multiplying (A-2.1) on the left by \( P \), we obtain

\[
\tilde{A}z + B \frac{d}{dt} z = \tilde{f}(t), \quad (A-2.3)
\]

where \( \tilde{A} = PAQ \) \( \tilde{B} = PBQ \) \( \tilde{f}(t) = Pf(t) \).

We choose the matrices \( P \) and \( Q \) such that the pencil \( A + \lambda B \) has a canonical quasi-diagonal form;

\[
\tilde{A} + \lambda \tilde{B} = \{ h_0, M_{g+1}^p, \ldots, M_{g+1}^p, M_{g+1}^q, \ldots, M_{g+1}^q, \ldots, M_{g+1}^q \}
\]

\[
(A-2.4)
\]

In accordance with the diagonal blocks in (A-2.4), the system of differential equations split into \( v = p - g + h + s + 2 \) separate systems of the form

\[
\begin{align*}
1 & 1 \\
0 & \tilde{f}
\end{align*}
\]

\[
(A-2.5)
\]

\[
L_{\epsilon} \frac{d}{dt} z = \tilde{f} 
\]

\[
(A-2.6)
\]

\[
L_{\eta}^{p-q+1+j} \frac{d}{dt} z = \tilde{f} 
\]

\[
(A-2.7)
\]
\[ N(u_k) \frac{d}{dt} p-g+q-h+1+k \frac{z}{f} = (J+q) \frac{d}{dt} v \cdot v \cdot \cdot \cdot = f \]  

(A-2.8)  

\[ (J+q) \frac{d}{dt} z = f \]  

(A-2.9)  

where  
\[ \begin{array}{c|c}
  z \\
  \hline
  \eta \\
  \hline
  \varepsilon \\
  \hline
  \omega \\
  \hline
  \gamma \\
\end{array} \begin{array}{c|c}
  \tilde{f} \\
  \hline
  f \\
\end{array} \]

1) The system (A-2.5) is not inconsistent if and only if  
\[ \frac{1}{\tilde{f}} = 0. \]  

(A-2.10)  

2) The system (A-2.6) is of the form  
\[ \begin{array}{c}
  \frac{d}{dt} z_1 + z_2 = \tilde{f}_1(t), \\
  \vdots \\
  \frac{d}{dt} z_\eta + z_{\eta+1} = \tilde{f}_\eta(t).
\end{array} \]  

Such a system is always consistent.  

3) The system (A-2.7) is of the form  
\[ \begin{array}{c}
  \frac{d}{dt} z_1 = \tilde{f}_1(t), \\
  \vdots \\
  \frac{d}{dt} z_\eta + z_{\eta+1} = \tilde{f}_\eta(t), \\
  z_\eta = \tilde{f}_{\eta+1}(t).
\end{array} \]  

(A-2.14)  

From all the equations (A-2.14) except the first, we determine  
\[ z_\eta = \tilde{f}_{\eta+1} \]  

\[ z_{\eta-1} = \tilde{f}_{\eta} \frac{d}{dt} \tilde{f}_{\eta+1} \]  

\[ \vdots \]  

\[ z_1 = \tilde{f}_2 \frac{d}{dt} \tilde{f}_3 + \cdots + (-1)^{\eta-1} \frac{d^{\eta-1}}{dt^{\eta-1}} \tilde{f}_{\eta+1}. \]  

(A-2.15)  

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By substituting this expression for $z_1$ into the first equation, we obtain the condition for consistency

$$f_1 - \frac{d}{dt} f_2 + \cdots + (-1)^n \frac{d^n}{dt^n} f_{n+1} = 0.$$  \hspace{1cm} (A-2.16)

4) The system (A-2.8) is of the form

$$N(u) \frac{d}{dt} z = \tilde{f}$$  \hspace{1cm} (A-2.17)

or more explicitly

$$\frac{d}{dt} z + z_1 = \tilde{f}_1, \cdots, \frac{d}{dt} u + u_{-1} = \tilde{f}_{u-1}, \quad z_u = \tilde{f}_u.$$  \hspace{1cm} (A-2.18)

Therefore we determine successively the unique solution

$$z_u = \tilde{f}_u$$

$$z_{u-1} = \tilde{f}_{u-1} - \frac{d}{dt} \tilde{f}_u$$  \hspace{1cm} (A-2.19)

$$z_1 = \tilde{f}_1 - \frac{d}{dt} \tilde{f}_2 + \cdots + (-1)^{u-1} \frac{d^{u-1}}{dt^{u-1}} \tilde{f}_u.$$  \hspace{1cm} 

5) The system (A-2.9) is of the form

$$Jz + \frac{d}{dt} z = \tilde{f}.$$  \hspace{1cm} (A-2.20)

Therefore we obtain the general solution

$$z = e^{-Jt} z_0 + \int_0^t e^{-J(t-\tau)} f(\tau) d\tau,$$  \hspace{1cm} (A-2.21)

where $z_0$ is a vector with arbitrary values.
Lemma A-3.1

Let A and B be m x n and n x m matrices, respectively, and the following equality holds.
\[ \det |1 + AB| = \det |1 + BA| \]  \hspace{1cm} (A-3.1.1)

**Proof:** Define a matrix D as
\[ \begin{pmatrix} 1, -B \\ A, 1 \end{pmatrix} \]

Then the matrix can be written as
\[ D = \begin{pmatrix} 1, 0 \\ A, 1 \end{pmatrix} \begin{pmatrix} 1, -B \\ 0, 1 \end{pmatrix} \begin{pmatrix} 1, 0 \\ 0, 1 \end{pmatrix} A, 1 \]

Then we obtain
\[ \det D = \det |1 + AB| = \det |1 + BA| \]  \hspace{1cm} (A-3.1.4)

Q.E.D.

Lemma A-3.2

Consider an equality as
\[ y = Ax, \]  \hspace{1cm} (A-3.2.1)

where A is a matrix and y and x are vectors.

Let us partition Eq. (A-3.2.1) as
\[ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} A_{11}, a_1, a_2 \\ a_2, a, a_3 \\ A_{21}, a_4, a_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x \end{pmatrix} \]  \hspace{1cm} (A-3.2.2)

Then the following equality holds.
\[ \begin{pmatrix} y_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_{11} - \frac{1}{a_1} a_2, \frac{1}{a_1} a_1, A_{12} - \frac{1}{a_1} a_3 \\ \frac{1}{a_2}, a, \frac{1}{a_3} \\ A_{21} - \frac{1}{a_4} a_2, \frac{1}{a_4} a_4, A_{22} - \frac{1}{a_4} a_3 \end{pmatrix} \begin{pmatrix} x_1 \\ y \end{pmatrix} \]  \hspace{1cm} (A-3.2.3)
Proof: From Eq. (A-3.2.2), we obtain
\[ x = \frac{1}{a}(-a_2 x_1 + y - a_3 x_2). \]  
(A-3.2.4)

Then we obtain
\[
\begin{align*}
| x_1 | & = 1, 0, 0 | x_1 |, \\
| x | & = -a_2/a_1, 1/a_1, -a_2/a_1 | y | \\
| x_2 | & = 0, 0, 1 | x_2 |
\end{align*}
\]  
(A-3.2.5)

and
\[
\begin{align*}
| y_1 | & = A_{11}-a_1 a_2/a_1, -a_1/a_1 A_{12}-a_2 a_4/a_1 | x_1 |, \\
| y | & = 0, 1, 0 | y | \\
| y_2 | & = A_{21}-a_2 a_4/a_1, -a_4/a_1 A_{22}-a_2 a_4/a_1 | x_2 |
\end{align*}
\]  
(A-3.2.6)

From Eq. (A-3.2.4) and (A-3.2.6), we have Eq. (A-3.2.3).

Q.E.D.

Lemma A-3.3

Let A and B be m×n and n×m matrices, respectively. Then the following equality holds.
\[ (1+AB)^{-1} = 1 - A(1+BA)^{-1} B \]  
(A-3.3.1)

Proof: From Eq. (A-3.1.3), if the matrix D is nonsingular, we obtain
\[ D^{-1} = | 1 - B(1+AB)^{-1} A, B(1+AB)^{-1} |, \]  
(A-3.3.2)

and
\[ D^{-1} = | (1+BA)^{-1}, B(1+BA)^{-1} |, \]  
(A-3.3.3)

From Eq. (A-3.3.2) and (A-3.3.3), we have Eq. (3.3.1).

Q.E.D.
Lemma A-3.4

Let $A$ and $B$ be $n \times n$ matrices. Then the eigenvalues of the matrix $AB$ are equal to those of $BA$.

Lemma A-3.5

A real symmetric matrix $A$ is positive definite if and only if there exists a real nonsingular matrix $P$ such that $A = PP'$. 

Lemma A-3.6

A real symmetric matrix $A$ written as $PP'$ is positive semidefinite and if $A$ is nonsingular, it is positive definite.
APPENDIX IV PROOFS OF PROPERTY 3-4.2 AND 3-4.3

Definition

The graphs obtained by the following procedure $I$"II" are called B-S-graphs"S-B-graphs" of $G$ and $M$.

Procedure I

i) Set $G_1=G$, $M_0=M$, $i=1$.

ii) If there exists a transformer-bridge in $G_i$, go to Step iii).

Otherwise go to Step vi).

iii) Set $M_i=M_{i-1}$[all the transformer-bridges in $G_i$].

iv) If there exists a transformer-self-loop in $M_i$, go to Step v).

Otherwise Set $i=i+1$, $G_i=G_{i-1}$ and go to Step vi).

v) Set $G_{i+1}=G_i$[all the transformer-self-loops in $M_i$], $i=i+1$

and go to Step ii).

vi) Stop.

The graphs $G_i$ and $M_{i-1}$ are B-S-graphs.

Procedure II is obtained from Procedure I by replacing bridges, self-loops, [ ] and { } with self-loops, bridges, { } and [ ], respectively.

Proof of property 3-4.2

It is assumed that $F_{SC}C \neq 0$. From Eq. (3-4.20), we obtain

$$
\tilde{F}_{SC}C e_s = (F_{SC} - F_i \tilde{M}F_i F_{SC}) C_i (e_s - F_{SC} \tilde{M}e_1)
$$

$$
= F_{SC}C e_s - F_i \tilde{M}F_i C e_s - F_{SC}C \tilde{M}e_1 + F_i \tilde{M}F_i C \tilde{M}e_1
$$

(A-4.1)

When the first term of Eq. (A-4.1) is nonzero, there exists at least one loop which consists of capacitor- and independent voltage source-branches only. Then the property holds.

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Consider the case where the second term of Eq. (A-4.1) is nonzero. Then the following procedure is examined.

i) In $G$, choose fundamental loops which contain the capacitor-branch corresponding to the nonzero element of the term.
   Select the link-transformer-branches in the loops.

ii) In $M$, choose twig-transformer-branches in loop (the link-transformer-branches).

iii) In $G$, choose link-capacitor- and link-transformer-branches in cut-set (the twig-transformer-branches).

iv) If link-capacitors are chosen and the corresponding elements of $e_S$ stop. Otherwise go to Step ii).

The above procedure converges since the second term of Eq. (A-4.1) is nonzero.

Since $F_{lG}^′M^′F_{St}^′ = 0$, in $G$ there exist neither twig-resistor- nor twig-inductor

Since $F_{St} MF_{lG} = 0$ and $F_{St} MF_{lT} = 0$, in $G$ there exist neither twig-resistor- nor twig-inductor-branches in loop (the link-transformer-branches considered in Step i) and ii)). Then the loops considered in Step i) and iii) contain capacitor- and transformer-branches only. Since the loops considered in Step ii) contain transformer-branches only, the ring sum of the loops considered in Step i), ii) and iii) contains capacitor- and twig-transformer-branches. However, the fundamental cut-sets cut-set (the twig-transformer-branches in the ring sum) in $G$ do not contain link-capacitor-branch. Then the twig-transformer-branches are bridges in the B-S-graphs. Therefore when the second term of Eq. (A-4.1) is nonzero, the property holds.
Similarly when the third and the fourth terms are nonzero, it can be proved that this property holds.

When $F_{L}L_{E}jF_{I} \neq 0$, it can be proved in the similar (dual) manner that this property holds.

Q.E.D.

Proof of Property 3-4.3

If the rank of $F_{RC}^{T}F_{LG}$ is less than the number of the twig-capacitors"the link-inductors" for the maximum proper tree, then the matrix-product $F_{RC}^{T}F_{RC}^{T}F_{LG}^{T}F_{LG}^{T}$ is singular. Then the matrix A is singular.

When the rank of $F_{RC}^{T}F_{LC}^{T}$ is less than the number of the twig-capacitors"the link-inductors" for the maximum proper tree, there exists at least one twig-capacitor-branch"link-inductor-branch" for a proper tree $T_{PL}$ such that $|T_{PL}(C)|+|T_{PL}(L)|$ is maximum. This corresponds to that there exists at least one twig-inductor-branch"link-capacitor-branch" in the maximum proper tree. Therefore we can prove this property in the same manner of the proof of Property 3-3.2.

Q.E.D.
APPENDIX V PROOFS OF PROPERTY 4-4.2 AND 4-4.3

Proof of Property 4-4.2

If the state equation has derivatives of input functions, from Eq.(4-4.21), we obtain

$$A'_{SC}C_1e_C1\neq 0,$$

$$A'_{SC}C_1F_{CL}G_0^{-1}R_1^{-1}e_{E1}\neq 0,$$ or

$$A'_{SC}C_1F_{CL}G_0^{-1}R_2^{-1}e_{E2}\neq 0.$$

Let $C$ denote a twig-capacitor corresponding to the nonzero term of the above.

1) Case of $A'_{SC}C_1e_C1\neq 0$

We have

$$A'_{SC} = F'_{CC} - F'_{SC} R_0^{-1} F_{CL} G_0^{-1} F'_{SC}.$$ 

When $F'_{CC} e_{C1} \neq 0$, there exists at least one loop which consists of capacitor- and independent voltage source-branches only.

Let us consider the case where $F'_{SC} R_0^{-1} F_{CL} G_0^{-1} F'_{SC} e_{C1} \neq 0$. Then there exists a link-gyrator-branch (denoted by $g_1$) in cut-set($C$).

There exists a twig-gyrator-branch (denoted by $g_2$) in loop($g_1^+$).

There exists a link-gyrator-branch (denoted by $g_3$) in cut-set($g_2^+$).

By repeating the above, for a certain integer $n$, there exists a twig-gyrator-branch $g_{2n}$ in loop($g_{2n-1}^+$) and there exists a link-capacitor-branch $S$ in cut-set($g_{2n}^+$). Let us call that $S$ is in CL($C$) or that $C$ is in LC($S$).

If there existed resistor-branches in cut-set($g_{2k}$) or CL($g_{2k}$) "loop($g_{2k-1}^+$) or LC($g_{2k-1}^+$)", there would exist a proper tree which contains more capacitor-branches than the maximum proper tree.

Then the property i) holds.

2) Case of $A'_{SC}C_1F_{CL}G_0^{-1}F_{CC}R_0^{-1}e_{E1}\neq 0$
There exists a link-capacitor-branch in $CL(C)$. There exists a gyrator-branch in $LC(S)$. There exists a voltage source-branch in loop (the pair branch of the gyrator). Then in the similar manner to 1), we can prove that this property i) holds.

3) Case of $A_{SC}^{-1} F_{CL} G_0^{-1} j \Sigma \neq 0$

There exists a link-capacitor-branch in $CL(C)$. There exists a twig-gyrator-branch in $LC(C)$. There exists a current source-branch in cut-set (the pair branch of the twig-gyrator). Then this property ii) holds.

Q.E.D.

Proof of Property 4-4.3

If the rank of the matrix $F_{EC}$ is less than the number of the twig-capacitor-branches for the maximum proper tree, then the matrix $F_{EC}^T R_0^{-1} F_{EC}$ is singular and the matrix $A$ is singular.

When the rank of $F_{EC}$ is less than the number of the twig-capacitor-branches for the maximum proper tree, a proper tree which has minimum $|T_p(C)|$ contains at least one capacitor-branch. Then if there exists no proper tree in $G$ (all the capacitor-branches), the matrix $A$ is singular.

Q.E.D.
APPENDIX VI ALGORITHM FOR A P-TREE

I) Node-labeling I
1-1 Label all the nodes $v_0$ adjacent to the r-node $i$ by $i$ in $G\{\text{all r-branches}\}$.
1-2 Label all the nodes $v_I$ adjacent to $v_0$ by $i$ in $G\{\text{all r-branches}\}$.
1-3 Repeat Step 2 until no more labels can be assigned.
Do Node-labeling I for all the r-branches.

II) Branch-labeling I
2-1 Label a branch by the union of the labels on the two nodes touching the branch.

III) Simplifying graph
3-1 Divide the graph into nonseparable components.
   (If a component contains less than three r-branches, Theorem 6-2.1 and 6-2.2 can be applied.)
3-2 Delete branches whose labels correspond to only one r-branch.
3-3 Transpose parallel branches which contain no r-branch into a branch.
3-4 Transpose series branches which contain no r-branches into a branch.
3-5 Transpose $d-p(i,j)$ into a branch touching the nodes $i$ and $j$.
3-6 Transpose series three branches, whose center-branch is an r-branch, and where others are not r-branches, into series two branches, one of which is the r-branch and the other of which is not an r-branch.

This routine does not affect the existence of p-trees.
This routine is applied to simplify graphs after deleting and/or
contracting branches in the following routines.

IV) Check d-path

Even if the number of r-branches is more than two, there
is no p-tree when there exists a d-path described in Theorem 6-2.1
and 6-2.2.
1) In the case where the r-node $i$ is identical to the r-node $j$,
   if there exists $d-p(i',j')$, there is no p-tree and stop.
2) If there exist $d-p(i,j)$, $d-p(i',j)$, $d-p(i,j')$ and $d-p(i',j')$,
   there is no p-tree and stop.

V) e-tree and m-tree

Definition

An e-tree is a tree which contains no r-branch (denoted by $T_e$).

Theorem

If there is no e-tree, there is no p-tree.

Theorem

If there is no branch, which is not an r-branch, in cut-set(a
twig $t$) for $T_e$, $t$ is contained of every p-tree if there is p-
tree.

Definition

A node which is not an r-node is called a v-node.

Procedure

5-1 $ST_e = \{ \phi \}$. 189
5-2 Choose a most labeled node $v$, which is not in $ST_e$.
5-3 Choose a minimum set of branches, such as the union of the set and $ST_e$ is a subtree of $T_e$ and the paths from $v$ to the $r$-nodes corresponding to the label of $v$ on the union.

Set $ST_e$ = the union.
5-4 Repeat Step 2 and 3 until there is no node as mentioned in Step 1.
5-5 If $ST_e$ is not an e-tree, choose a set of branches such as the union of the set and $ST_e$ is an e-tree.

Definition

An e-tree obtained in the above procedure is called an m-tree and denoted by $T_m$.

VI) Node-labeling II
6-1 Label a v-node adjacent to the r-node $i$ on $T_m$ by $i$.
6-2 Label a v-node adjacent to the v-node on $T_m$, which is labeled by $i$.
6-3 Repeat Step 2 until no more labels can be assigned.
Do Node-labeling II for all r-nodes.

VII) Branch-labeling II
7-1 Label a branch by the union of the labels on the two nodes touching the branch.

VIII) r-branch and f-branch

Definition

Among the twigs of $T_m$, a twig such that $cut-set(a\ twig)$
contains more than one \( r \)-branch is called a \( \tau \)-branch.

Theorem

If there exists no \( \tau \)-branch, it does not affect the existence of the \( p \)-tree that a twig in loop (a link which is not an \( r \)-branch) for \( T_m \) is contracted.

Definition

Assume that \( t_1 \) and \( t_2 \) are \( \tau \)-branches such that \( \text{cut-set}(t_1) \) contains \( r \)-branches \( r_1, \ldots, r_n \) but \( \text{cut-set}(t_2) \) contains neither of \( r_1, \ldots, r_n \). Let an \( r \)-branch in \( \text{cut-set}(t_2) \) be \( r \). If there exist paths, on \( T_m\text{-loop}(r) \), from the node touching \( r \) to the nodes touching \( r_1, \ldots, r_n \), then the branches on the paths are called \( f \)-branches for \( t_1 \).

Definition

Let the number of the \( r \)-branches in \( \text{cut-set}(t) \) which is a \( \tau \)-branch be \( N(t/T_m) \). If there is no \( \tau \)-branch \( t_1 \) such as \( N(t_1/T_m) > N(t/T_m) \), \( t \) is called a maximum \( \tau \)-branch.

IX) Choosing p-tree

9-1 Choose an \( m \)-tree \( T_m \).

9-2 If there exists a \( \tau \)-branch, go to Step 3. Otherwise go to Step 5.

9-3 i) If there exists a \( \tau \)-branch \( t \) such as \( \text{cut-set}(t) \) contains \( t \) and \( r \)-branches only, contract \( t \) and go to Node-labeling I. Otherwise go to Step 3-ii).

ii) If there exists no such a \( \tau \)-branch, neglect \( t \) such as
the branches in loop(l) have the same labels and do
Step 3-i). Otherwise go to Step 4.

9-4 Choose a maximum r-branch t such as the deference between
the set of the r-nodes touching the r-branches in cut-set(t)
and the set of the r-nodes corresponding to the lables of
the r-branches is a maximum. Assume that cut-set(t) contains
r_i, r_j, ... . If cut-set(t) contains d-p(i, j), go to Step 4-i).
Otherwise go to 4-ii).

i) If the union of d-p(i, j), the f-braches for t and r-
braches contains no loop, contract t and go to Node-labeling I.
Otherwise go to Step 4-ii). 

ii) Contract t and go to Node-labeling I.

9-5 Contract a twig in loop(a link which is not an r-branch)
and go to Node-labeling I.

We obtain a p-tree by doing Node-labeling I, Branch-labeling I,
Simplifying graph, Check d-graph, Node-labeling II, Branch-
labeling II and Choosing p-tree routines, successsively.
APPENDIX VII  SUBMATRICES IN Eq. (6-5.17)

\[
A_{11} = -(1 + F_{r_2}^t F_{r_2})^{-1} ((C_1^{-1} + F_{r_2} C_2^{-1} F_{r_2}) r^{-1} + F_{r_2} C_2^{-1} y (1 + F_{r_2}^t F_{r_2})^{-1} \\
\times (F_{r_2} C_1^{-1} - C_2^{-1} F_{r_2}) (C_1^{-1} + F_{r_2} C_2^{-1} F_{r_2})^{-1} + F_{r_2} C_2^{-1} y) (1 + F_{r_2}^t F_{r_2})
\]

\[
A_{12} = -(1 + F_{r_2}^t F_{r_2})^{-1} F_{r_2} C_2^{-1} y
\]

\[
A_{13} = (1 + F_{r_2}^t F_{r_2})^{-1} F_{r_2} C_2^{-1} H
\]

\[
A_{14} = -(1 + F_{r_2}^t F_{r_2})^{-1} F_{r_2} C_2^{-1} H ((1 + F_{1g}^t F_{1g})^{-1} (L_1^{-1} F_{1g} F_{1g} L_2^{-1} \\
\times (L_2^{-1} + F_{1g}^t L_1^{-1} F_{1g})^{-1} (1 + F_{1g}^t F_{1g}) + F_{1g}))
\]

\[
A_{21} = (1 + F_{r_2}^t F_{r_2})^{-1} ((F_{r_2} C_1^{-1} - C_2^{-1} F_{r_2}) (C_1^{-1} + F_{r_2} C_2^{-1} F_{r_2})^{-1} F_{r_2} + 1) C^{-1} y \\
\times (1 + F_{r_2}^t F_{r_2})^{-1} (F_{r_2} C_1^{-1} - C_2^{-1} F_{r_2}) (C_1^{-1} + F_{r_2} C_2^{-1} F_{r_2})^{-1} (1 + F_{r_2}^t F_{r_2})
\]

\[
A_{22} = -(1 + F_{r_2}^t F_{r_2})^{-1} ((F_{r_2} C_1^{-1} - C_2^{-1} F_{r_2}) (C_1^{-1} + F_{r_2} C_2^{-1} F_{r_2})^{-1} F_{r_2} + 1) \\
\times C_2^{-1} y
\]

\[
A_{23} = (1 + F_{r_2}^t F_{r_2})^{-1} ((F_{r_2} C_1^{-1} - C_2^{-1} F_{r_2}) (C_1^{-1} + F_{r_2} C_2^{-1} F_{r_2})^{-1} F_{r_2} + 1) \\
\times C_2^{-1} H
\]

\[
A_{24} = -(1 + F_{r_2}^t F_{r_2})^{-1} ((F_{r_2} C_1^{-1} - C_2^{-1} F_{r_2}) (C_1^{-1} + F_{r_2} C_2^{-1} F_{r_2})^{-1} F_{r_2} + 1) \\
\times C_2^{-1} H ((1 + F_{1g}^t F_{1g})^{-1} (L_1^{-1} F_{1g} F_{1g} L_2^{-1} (L_2^{-1} + F_{1g}^t L_1^{-1} F_{1g})^{-1} \\
\times (1 + F_{1g}^t F_{1g}) + F_{1g})
\]

\[
A_{31} = -(1 + F_{1g}^t F_{1g})^{-1} ((L_1^{-1} F_{1g} F_{1g} L_2^{-1} (L_2^{-1} + F_{1g}^t L_1^{-1} F_{1g})^{-1} F_{1g}^{-1}) \\
\times L_1^{-1} H (F_{r_2} - (1 + F_{r_2}^t F_{r_2})^{-1} (F_{r_2} C_1^{-1} - C_2^{-1} F_{r_2}) (C_1^{-1} + F_{r_2} C_2^{-1} F_{r_2})^{-1} \\
\times (1 + F_{r_2}^t F_{r_2}))
\]

\[
A_{32} = (1 + F_{1g}^t F_{1g})^{-1} ((L_1^{-1} F_{1g} F_{1g} L_2^{-1} (L_2^{-1} + F_{1g}^t L_1^{-1} F_{1g})^{-1} F_{1g}^{-1}) \\
\times L_1^{-1} H^t
\]

\[
A_{33} = (1 + F_{1g}^t F_{1g})^{-1} ((L_1^{-1} F_{1g} F_{1g} L_2^{-1} (L_2^{-1} + F_{1g}^t L_1^{-1} F_{1g})^{-1} F_{1g}^{-1}) \\
\times L_1^{-1} z
\]

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\[ A_{34} = -(1 + F_{1g'1g})^{-1} \left( (L_{1}^{-1} F_{1g'}^{-1} - F_{1g'1g})^{-1} (L_{2}^{-1} + F_{1g'1g} L_{1}^{-1} F_{1g'}^{-1})^{-1} F_{1g}^{-1} \right) \]
\[ \times (1 + F_{1g'1g})^{-1} (L_{1}^{-1} F_{1g'}^{-1} - F_{1g'1g})^{-1} (L_{2}^{-1} + F_{1g'1g} L_{1}^{-1} F_{1g'}^{-1})^{-1} (1 + F_{1g'1g}) \]

\[ A_{41} = (1 + F_{1g'1g})^{-1} F_{1g'}^{-1} L_{2}^{-1} H' \left( F_{r2'}^{-1} \left( 1 + F_{r2'1g'} \right) \right) \]
\[ \times (C_{1}^{-1} + F_{r2'2} C_{2}^{-1} F_{r2'})^{-1} (1 + F_{r2'1g'}) \]

\[ A_{42} = (1 + F_{1g'1g})^{-1} F_{1g'}^{-1} L_{1}^{-1} H' \]

\[ A_{43} = (1 + F_{1g'1g})^{-1} F_{1g'}^{-1} L_{1}^{-1} F_{1g'}^{-1} \]

\[ A_{44} = -(1 + F_{1g'1g})^{-1} \left( F_{1g'}^{-1} L_{1}^{-1} F_{1g}^{-1} (1 + F_{1g'1g}^{-1} L_{1}^{-1} F_{1g'}^{-1})^{-1} \right) \]
\[ \times (L_{2}^{-1} + F_{1g'1g} L_{1}^{-1} F_{1g'})^{-1} (L_{2}^{-1} + F_{1g'1g} L_{1}^{-1} F_{1g'})^{-1} (1 + F_{1g'1g}) \]

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APPENDIX VIII. SUPPLEMENT OF SECTION 6-6

Lemma

If \( d^D > 0 \) and \( t_1 < t_2 \), the following inequalities hold.

\[
f(1) > 0
\]

\[
0 < a\{\cos \beta t_1 \cos \beta t_2 - (1 + \frac{2}{\beta^2 t^2}) \sin \beta t_1 \sin \beta t_2\} < 1.
\]

(A-8.1) \hspace{1cm} (A-8.2)

Proof: Since \( a > 0 \) and \( a > 1 \), we obtain

\[
a\{\cos \beta t_1 \cos \beta t_2 - (1 + \frac{2}{\beta^2 t^2}) \sin \beta t_1 \sin \beta t_2\} > 0.
\]

The following equality holds.

\[
a\{\cos \beta t_1 \cos \beta t_2 - (1 + \frac{2}{\beta^2 t^2}) \sin \beta t_1 \sin \beta t_2\}
= e^{(t_1 - t_2)/\tau} \{(1 + \frac{1}{\beta^2 t^2}) \cos \beta (t_1 + t_2) - \frac{1}{\beta^2 t^2} \cos \beta (t_1 - t_2)\}.
\]

We define \( g(x) \) as

\[
g(x) = e^{x/\tau} \{(1 + \frac{1}{\beta^2 t^2}) \cos \beta (t_1 + t_2) - \frac{1}{\beta^2 t^2} \cos \beta x\},
\]

where \( x = t_1 - t_2 \).

Let us define \( g_1(x) \) as

\[
g_1(x) = e^{x/\tau} \{(1 + \frac{1}{\beta^2 t^2}) - \frac{1}{\beta^2 t^2} \cos \beta x\},
\]

then we obtain

\( g_1(x) > g(x) \) and \( g_1(0) = 1 \).

Differentiating \( g_1(x) \) with respect to \( x \), we obtain

\[
g_1'(x) = e^{x/\tau} \{(1 + \frac{1}{\beta^2 t^2}) - \frac{1}{\beta^2 t^2} \sqrt{(1 + \frac{1}{\beta^2 t^2}) \cos (\beta x + \theta)},
\]

where \( \tan \theta = \beta t \).

There is no value of \( x \) such that \( g_1'(x) = 0 \). Since \( g_1'(0) > 0 \), we obtain \( g_1'(x) > 0 \). Then we obtain

\( g_1(x) < 1 \) for \( x < 0 \).
Then the inequality (A-8.2) holds.

Next we consider the inequality (A-8.1). We define $h(x)$ as

$$h(x) = f(1) = 1 - 2\left(1 + \frac{1}{\beta^2 - \tau^2}\right) \cos \beta (t_1 + t_2) - \frac{1}{\beta^2 - \tau^2} \cos \beta x \right) e^{x/\tau} + e^{2x/\tau},$$

where $x = t_1 - t_2$.

Let us define $h_1(x)$ as

$$h_1(x) = 1 - 2\left(1 + \frac{1}{\beta^2 - \tau^2}\right) - \frac{1}{\beta^2 - \tau^2} \cos \beta x \right) e^{x/\tau} + e^{2x/\tau}.$$

Then we obtain

$$h_1(x) < h(x) \text{ and } h_1(0) > 1.$$

Differentiating $h_1(x)$ with respect to $x$, we obtain

$$h_1'(x) = \frac{2}{\tau} e^{x/\tau} \{e^{x/\tau} - \left(1 + \frac{1}{\beta^2 - \tau^2}\right) + \frac{1}{\beta^2 - \tau^2} \cos (\beta x + \theta)\},$$

where $\tan \theta = \beta\tau$.

We obtain $x_1$ such that $e^{x/\tau} + \frac{1}{\beta^2 - \tau^2} \cos (\beta x + \theta)$ is maximum as

$$e^{x_1/\tau} = \left(1 + \frac{1}{\beta^2 - \tau^2}\right) \sin (\beta x_1 + \theta).$$

Then we obtain $h_1'(x_1) < 0$. Therefore we obtain

$$h_1(x) > 1 \text{ for } x < 0.$$  

Consequently the inequality (A-8.1) holds. Q.E.D.
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